Review

Chain Rule: \[ \frac{d}{dx} (f(g(x))) = f'(g(x)) \cdot g'(x) \]

a.k.a. \[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

Example

\[ y = \cos \left( \sqrt{\sin(\tan(\pi x))} \right) \]

\[ \frac{dy}{dx} = -\sin \left( \sqrt{\sin(\tan(\pi x))} \right) \cdot \frac{1}{2\sqrt{\sin(\tan(\pi x))}} \cdot \cos(\tan(\pi x)) \cdot \sec^2(\pi x) \cdot \pi \]

We used:

\[ \frac{d}{dx} \cos x = -\sin x \]

\[ \frac{d}{dx} \sin x = \cos x \]

\[ \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}} \]

\[ \frac{d}{dx} \tan x = \sec^2 x \]

Ex.

\[ h(\theta) = (\theta^2 + 1)^{3}(\theta + 5)^9 \]

\[ h'(\theta) = 3(\theta^2 + 1)^2 \cdot 2\theta (2\theta + 5)^8 + (\theta^2 + 1)^3 \cdot 10(2\theta + 5)^9 \cdot 2 \]

\[ = 6(\theta^2 + 1)^2 (2\theta + 5)^8 + 20(\theta^2 + 1)^3 (2\theta + 5)^9 \]

Ex.

\[ y = \sin(kx) \]

\[ \frac{dy}{dx} = (3 \sin(kx)) \cdot \cos(kx) - k \]
Problem: Suppose \( y \) and \( x \) are related by
\[
y^2 + x^2 = 4.
\]

Find instantaneous rate of change of \( y \) with respect to \( x \) at point \( x = 1 \), \( y = \sqrt{3} \).

* Not given \( y \) as a function of \( x \)
Near \( x = 1 \), \( y \) is a function of \( x \).

\[
y = \sqrt{4 - x^2}
\]

\[
\frac{dy}{dx} = \frac{1}{2} (4 - x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{4 - x^2}}
\]

\[
\left. \frac{dy}{dx} \right|_{x=1} = \frac{-1}{\sqrt{3}}
\]

Implicit Differentiation

1. Given relation involving (for example) \( x \) and \( y \),
   want \( \frac{dy}{dx} \).

2. Assume \( y \) is a function of \( x \), \( y = y(x) \), even if we can't solve for it.

3. Take derivative of both sides with respect to \( x \).

4. Solve for \( \frac{dy}{dx} \).
Removal: The y was not relevant.

This proves that the line is perpendicular to this.

\[
\text{Slope} = -\frac{1}{x}
\]

\[
\text{Slope} = y
\]

\[
\frac{y}{x} = \frac{k}{x}
\]

\[
x^2 + y^2 = 0 \quad \text{Think: } x + y(x) = h
\]

Example: x + y = h
Example

Find \( \frac{dy}{dx} \)

\[
\cos(xy) + x + y = 5
\]

\[
-\sin(xy) \cdot (y + x \frac{dy}{dx}) + 1 + \frac{dy}{dx} = 0
\]

\[
-y \sin(xy) - \frac{dy}{dx} x \sin(xy) + 1 + \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = \frac{-1 + y \sin(xy)}{-x \sin(xy) + 1}
\]

"OK that \( y \)'s are in the answer, we can't solve for \( y \) anymore.

Example

Find eq of tangent line to the cardioid

\[
x^2 + y^2 = (2x^2 + y^2 - x)^2 \text{ at } (0, \frac{1}{2})
\]

\[
2x + 2y \frac{dy}{dx} = 2(2x^2 + 2y^2 - x)(4xy + 4yy' - 1) \quad (y' = \frac{dy}{dx})
\]

"Easier to plug in before solving for \( y \)"

\[
0 + y' = 2(0 + \frac{1}{2})(0 + 2y' - 1)
\]

\[
y' = 2y' - 1
\]

\[
y' = 1
\]

\[
y - \frac{1}{2} = x
\]
\[
\frac{x - y \sqrt{1 - \sin^2 x}}{y} = \frac{\sin \frac{x}{p}}{1 - \sin^2 x}
\]

\[
\frac{x}{y} = \frac{\cos \frac{x}{p}}{1 - \sin^2 x}
\]

\[
1 = \frac{\cos \frac{x}{p}}{1 - \sin^2 x}
\]

\[
\sin y = x
\]

\[
x_1 = y
\]

**Ex.** Let \( y = \sin x \) Find:

\[
\frac{\partial y}{\partial x} = \frac{\cos \frac{x}{p}}{1 - \sin^2 x} - \frac{\frac{\sin \frac{x}{p}}{p}}{y}
\]

\[
0 = \frac{\partial y}{\partial x} (f(x) - y) + x - \frac{\cos \frac{x}{p}}{p} - \frac{\sin \frac{x}{p}}{y}
\]

\[
0 = \frac{\partial y}{\partial x} (f(x) + x) + x - \cos \frac{x}{p} + \frac{\sin \frac{x}{p}}{y}
\]

\[
\frac{\partial y}{\partial x} = 1
\]

\[
\text{Final:} \quad x + \cos \frac{x}{p} = 1
\]