Universal Behavior of Modulationally Unstable Media*

Gino Biondini[†]
Sitai Li[‡]
Dionyssios Mantzavinos[§]
Stefano Trillo[¶]

Abstract. Evidence is presented of universal behavior in modulationally unstable media. An ensemble of nonlinear evolution equations, including three partial differential equations, an integro-differential equation, a nonlocal system, and a differential-difference equation, is studied analytically and numerically. Collectively, these systems arise in a variety of applications in the physical and mathematical sciences, including water waves, optics, acoustics, Bose–Einstein condensation, and more. All these models exhibit modulational instability, namely, the property that a constant background is unstable to long-wavelength perturbations. In this work, each of these systems is studied analytically and numerically for a number of different initial perturbations of the constant background, and it is shown that, for all systems and for all initial conditions considered, the dynamics gives rise to a remarkably similar structure comprised of two outer, quiescent sectors separated by a wedge-shaped central region characterized by modulated periodic oscillations. A heuristic criterion that allows one to compute some of the properties of the central oscillation region is also given.

Key words. nonlinear evolution equations, focusing media, modulational instability, nonlinear Schrödinger systems

AMS subject classifications. 35Q55, 37K, 37K40, 74J30

DOI. 10.1137/17M1112765

1. Introduction. Modulational instability (MI), known as the Benjamin–Feir instability in water waves [9], is a ubiquitous phenomenon in focusing nonlinear media, and consists in the instability of a constant background to long-wavelength perturbations [60]. The linear stage of MI can be easily studied by linearizing the governing equations around the background and is characterized by the exponential growth of all perturbations with nonzero support in the long-wavelength portion of the Fourier spectrum. Once the size of the perturbations has become comparable with the background, however, linearization ceases to be valid, and the nonlinear stage of MI takes over.

†Departments of Mathematics and Physics, State University of New York at Buffalo, Buffalo, NY 14260 (biondini@buffalo.edu).

‡Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260 (sitaili@buffalo.edu).

§Department of Mathematics, University of Kansas, Lawrence, KS 66045 (mantzavinos@ku.edu).

¶Department of Electrical Engineering, University of Ferrara, 44122 Ferrara, Italy (trlsfn@unife.it).

The most common model used to study MI is the focusing cubic nonlinear Schrö-dinger (NLS) equation in one spatial dimension. The one-dimensional cubic NLS equation is of course a completely integrable infinite-dimensional Hamiltonian system [6, 46], and the related initial value problem is amenable to treatment by the inverse scattering transform [61, 62]. Even though certain nonintegrable NLS equations admit solutions that exhibit collapse [52], and even though certain integrable systems admit solutions which blow up in finite time [3, 47], strong numerical and theoretical evidence (see, e.g., [18, 19, 23, 56]) suggests that solutions of the focusing one-dimensional NLS equation do not exhibit any such kind of behavior, and therefore one would expect the instability to somehow saturate and enter its nonlinear stage.

Remarkably, the nonlinear stage of MI in the NLS equation with periodic bound-ary conditions is described in terms of a homoclinic structure [8, 54] characterized by two qualitatively different families of recurrent (or, more strictly, doubly-periodic) solutions, through which the perturbation is cyclically amplified and back-converted to the background. These two families are separated by the so-called Akhmediev breather, which represents the separatrix of the homoclinic structure, featuring a sin-gle cycle of conversion and back-conversion, with its unstable manifold corresponding to the MI linearized growth.

In the more general nonperiodic case (i.e., for localized perturbations of the con-stant background), however, the dynamics of MI is strikingly different. Indeed, using the inverse scattering transform for the focusing NLS equation with nonzero back-ground [13], it was shown in [12] that in this scenario (i.e., constant boundary condi-tions at infinity corresponding to localized perturbations) there exist generic perturba-tions of the constant background for which no discrete spectrum is present, and MI is mediated by the continuous spectrum of the associated scattering problem. In [15, 16], two of the authors resolved a long-standing open question about the nonlinear stage of MI on the infinite line. They did so by characterizing the long-time asymptotic behavior of solutions of the focusing NLS equation with nonzero boundary conditions. Specifically, it was shown in [15, 16] that, for a broad class of localized initial pertur-bations of the constant background, the solution of the focusing NLS equation tends to a universal asymptotic state. More precisely, given an initial condition (IC) $q(x, 0)$ representing a sufficiently localized initial perturbation of the constant background q_∞, assuming the associated scattering problem has a trivial discrete spectrum, it was proved analytically in [15, 16] that the solution $q(x, t)$ of the focusing NLS equa-tion is given by $q(x, t) = q_{\mathrm{asymp}}(x, t) + o(1)$ as $t \to \infty$, where $q_{\mathrm{asymp}}(x, t)$ has a different representation in different sectors of the xt-plane: (i) in the two "plane-wave" regions, $x < -4\sqrt{2}q_\infty t$ and $x > 4\sqrt{2}q_\infty t$, one has $|q_{\mathrm{asymp}}(x, t)| = q_\infty$, i.e., the solution has the same amplitude as the undisturbed background; (ii) in the "modulated elliptic wave" region, $-4\sqrt{2}q_\infty t < x < 4\sqrt{2}q_\infty t$, the solution is expressed by a slow modulation of the elliptic solutions of the focusing NLS equation (cf. Figure 1 below and the appendix for further details). This asymptotic behavior was also verified numerically in [14].

The methods used in [15, 16] made essential use of the integrability of the NLS equation via the inverse scattering transform and the Deift–Zhou nonlinear steepest descent method for oscillatory Riemann–Hilbert problems [27]. The actual description for the asymptotic behavior in [15, 16], however, involves only genus-0 behavior and modulated genus-1 (i.e., periodic) oscillations, or, in other words, plane-wave solutions and a modulation of the traveling wave solutions of the focusing NLS equation. The only necessary ingredient in order for such solutions to exist is the presence of a small number of suitable conservation laws (such as conservation of energy, momentum,

and Hamiltonian). The presence of such conservation laws is not a unique feature of the NLS equation, but is instead shared by many other models which exhibit MI. Therefore, it is natural to ask whether integrability is truly a necessary condition in order to observe the asymptotic behavior described in [15, 16].

The purpose of this work is to demonstrate that the answer to the question above is negative and that, instead, asymptotic behavior similar to that presented in [15, 16] is a rather general feature of modulationally unstable media. This is done by analytically and numerically studying six different models which exhibit MI, each with a variety of ICs. These models, comprising three nonlinear partial differential equations (PDEs), an integro-differential equation, a nonlocal system, and a differential-difference equation, arise in a variety of physical and mathematical settings.

2. NLS-Type Models. We study six NLS-type models, described in detail below. Each of them (except for the NLS equation itself) contains a free parameter that quantifies its departure from the NLS equation, and each of them reduces to the NLS equation in a suitable limit. Subscripts x and t denote partial differentiation with respect to space and time, and the values $\nu = \pm 1$ identify the focusing and defocusing cases, respectively. In all cases, MI is present in the focusing case ($\nu = 1$), which we consider next.

NLS Equation. The one-dimensional standard cubic NLS equation,

$$
(1) \qquad iq_t + q_{xx} + 2\nu |q|^2 q = 0 \,,
$$

is a universal model for the evolution of the envelope of weakly nonlinear dispersive wave trains [10], and it arises in deep water waves, fiber optics, and plasmas [6, 7, 40], as well as Bose–Einstein condensates in the absence of a trap [49].

Power-Law Nonlinearity. The following natural generalization of the NLS equation,

$$
(2) \qquad iq_t + q_{xx} + 2\nu |q|^{2\sigma} q = 0 \,,
$$

has been extensively used as a toy model in "hard" analysis [57, 52, 18, 19, 23, 56]. The ranges $0 \le \sigma < 2$ and $\sigma > 2$ correspond to the subcritical and supercritical cases, respectively. In the latter case, singularities may develop in finite time. Obviously, (2) reduces to (1) when $\sigma = 1$.

Saturable Nonlinearity. A frequently used model to study media with saturable nonlinearity is the PDE

$$
(3) \qquad iq_t + q_{xx} + 2\nu \frac{|q|^2}{1 + s|q|^2} q = 0 \,,
$$

which describes the propagation of laser beams in saturable media such as photorefractive materials and organic polymers [35, 22]. Here the quantity $1/s$ represents the saturation power (or intensity, depending on the context) of the medium. (3) obviously reduces to (1) when $s = 0$. In the limiting case in which s is small but not negligible, (3) yields a cubic-quintic equation of NLS-type which has been used extensively as a model of certain laser systems [44].

Thermal Media. The following NLS-type model with a nonlocal response,

$$
(4a) \qquad iq_t + q_{xx} + 2\nu mq = 0 \,,
$$

(4b) $$-s^2 m_{xx} + m = |q|^2 \,,$$

has been derived and successfully used to model soft matter such as biological tissue or, more generally, media with a significant thermal response [24, 37]. In this case the parameter s quantifies the degree of nonlocality. (4) reduces to (1) when $s = 0$.

Dispersion-Managed Fiber Systems. In optical transmission systems employing a periodic concatenation of fibers with alternating sign of dispersion, it was shown in [34, 4] that the average dynamics is governed by the following integro-differential equation, referred to as the dispersion-managed NLS (DMNLS) equation,

(5) $$i q_t + q_{xx} + 2\nu \iint_{\mathbb{R}^2} q_{(x+y)} q_{(x+z)} q^*_{(x+y+z)} R(y,z)\, \mathrm{d}y \mathrm{d}z = 0 \,,$$

with the shorthand notation $q_{(x)} = q(x,t)$; the asterisk denotes complex conjugation,

(6) $$R(y,z) = \mathrm{ci}(|yz|/s)/s = \mathcal{F}_{x_1,x_2}[r(k_1,k_2)],$$

where $\mathcal{F}_{x_1,x_2}[r(k_1,k_2)] = \int_{\mathbb{R}^2} \mathrm{e}^{i(k_1 x_1 + k_2 x_2)} r(k_1,k_2) \mathrm{d}k_1 \mathrm{d}k_2$ denotes the two-dimensional Fourier transform and where $r(k_1,k_2) = \sin(sk_1k_2)/[(2\pi)^2 sk_1k_2]$ and $\mathrm{ci}(\cdot)$ denotes the cosine integral [48]. In this case the "map strength" parameter s quantifies the magnitude of the local deviations of the fiber dispersion from its spatial average. (5) reduces to (1) as $s \to 0$.

Ablowitz–Ladik System. The Ablowitz–Ladik (AL) system [5] is the differential-difference equation

(7) $$i\dot{q}_n + \frac{1}{h^2}(q_{n+1} - 2q_n + q_{n-1}) + \nu(q_{n+1} + q_{n-1})|q_n|^2 = 0 \,,$$

with $q_n(t) = q(nh,t)$ and where h now represents the lattice spacing and the dot denotes temporal differentiation. (7), which is itself an integrable system, is a finite-difference approximation of (1), to which it reduces as $h \to 0$, and it arises in connection with orthogonal polynomials on the unit circle [50]. Note that a variant of (7) (often referred to as the standard discrete NLS equation, or DNLS equation for short) arises frequently in various physical applications [1]. The DNLS equation, however, does not support traveling-wave solutions due to the presence of a nonzero Peierls barrier, and therefore does not satisfy the minimum requirements needed to display the same dynamics as the other models. Even the solutions of the DNLS equation, however, exhibit a similar kind of behavior (see Appendix C for details).

Initial Conditions. Each of the above six models was integrated numerically in the focusing case with three different types of ICs, each representing a perturbation of the constant background: Gaussian,

(8a) $$q_{\mathrm{Gaussian}}(x,0) = 1 + i\,\mathrm{e}^{-x^2} \cos(\sqrt{2}x) \,,$$

sech-shaped,

(8b) $$q_{\mathrm{sech}}(x,0) = 1 + i\,\mathrm{sech}(10x) \,,$$

and box-like,

(8c) $$q_{\mathrm{box}}(x,0) = \begin{cases} 1 + i\cos(\pi x), & |x| < 1, \\ 1 & \text{otherwise.} \end{cases}$$
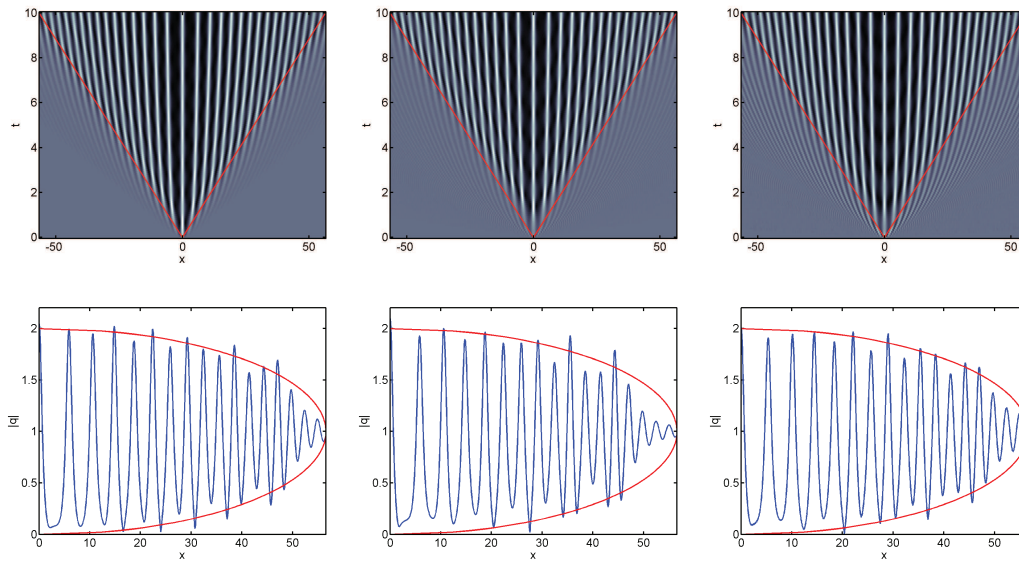
Fig. 1 *Solutions of the focusing NLS equation starting from perturbations of a constant background. Top row: Density plot of $|q(x,t)|$, together with the predicted boundary (red lines) between the modulated oscillation region and the plane wave regions from [15, 16]. Bottom row: The solution $|q(x,t)|$ at $t = 10$ (blue curve) together with the asymptotic envelope (red curve) from [15, 16]. Left column: Gaussian IC (8a); center column: sech-shaped IC (8b); right column: box-like IC (8c).*

In each case, the size of the perturbation was chosen to be comparable to the background so that the asymptotic state is reached almost immediately, minimizing transient behavior. Also, for each of the above ICs, the spectrum of the associated scattering problem is purely continuous, i.e., no discrete spectrum is present. (The factor "i" in (8) helps to ensure that this condition is satisfied and is related to the case of ICs with "large phase difference" in [12].) Note that in the NLS equation and power-law model, any background amplitude can be set to unity without loss of generality owing to the scaling invariance of the PDE. The same is not true for the other models.

Asymptotic State of MI for the NLS Equation. Before we present analytical and numerical results for all models, we begin by briefly discussing the behavior of solutions of the focusing NLS equation, for which rigorous asymptotic results are available and which therefore serves as a baseline case to compare the remaining five models (see section 4 for a brief discussion of numerical methods).

Figure 1 shows density plots as well as temporal snapshots at $t = 10$ of the numerically computed time evolution of the above ICs (in the density plots, black indicates $|q(x,t)| = 0$, white $|q(x,t)| \geq 2$, and values inbetween are coded in gray scale), together with the analytical predictions (red lines) for the boundary between the plane-wave and oscillation regions, as well as the envelope of the solution in the oscillation region, as given by the rigorous asymptotic results obtained in [16, 15] (see Appendix A for further details on the asymptotic results). Note how, despite small individual variations between the three cases, the time evolutions of the three ICs are remarkably similar and in excellent agreement with the asymptotic predictions. In fact, it can be shown that the agreement involves not only the boundary of the

oscillation region, but also the location of the peaks and the local shape of the solution near the peaks [14]. This is true even for the IC (8c), which is discontinuous at $x = \pm 1$ and would therefore be expected to give rise to Gibbs-like dispersive oscillations [29, 17]. Similar results are obtained with other kinds of localized ICs. Indeed, an instance of this behavior was recently observed numerically in [53].

3. Generalized Modulational Instability. Of all the models presented in section 2, only the NLS equation and the AL system are integrable. Nonetheless, as we discuss next, all of them are Hamiltonian systems, all of them have at least three conserved quantities (energy, momentum, and Hamiltonian), all of them possess plane wave and more general traveling-wave solutions, and all of them exhibit MI in the focusing case.

Invariances, Conserved Integrals, and Traveling-Wave Solutions. It is convenient to write all of the above evolution equations compactly as

$$(9) \qquad iq_t + L[q] + 2\nu N[q] = 0 \,.$$

The linear part is $L[q] = q_{xx}$ for all models except the AL system, for which $L[q] = (q(x+h,t) - 2q(x,t) + q(x-h,t))/h^2$ (where $x = nh$ with some abuse of notation). The nonlinear part is $N[q] = g(|q|^2)q$ for the first four models, where $g(Q) = Q^\sigma$ for the power-law model, including $\sigma = 1$ for the NLS equation as a special case, and $g(Q) = Q/(1 + sQ)$ for the saturable nonlinearity model. For the thermal media system, eliminating the nonlocal variable m yields $g(Q) = \mathcal{F}^{-1}[\mathcal{F}[Q]/(1+s^2k^2)]$, with $\mathcal{F}$ now denoting the one-dimensional Fourier transform (with the same normalizations as before), or, explicitly,

$$(10) \qquad g(Q) = \int_{\mathbb{R}} K(x - y)Q(y, t) \,\mathrm{d}y \,,$$

with $K(x) = \mathrm{e}^{-|x|/s}/(2s)$. For the DMNLS equation,

$$(11) \qquad N[q] = \iint_{\mathbb{R}^2} q_{(x+y)} q_{(x+z)} q^*_{(x+y+z)} R(y, z) \,\mathrm{d}y\mathrm{d}z \,,$$

and for the AL system, $N[q] = \frac{1}{2}|q_n|^2(q_{n+1} + q_{n-1})$.

All models admit the constant background solutions $q(x, t) = q_\infty \, \mathrm{e}^{2i\nu\Omega_\infty t}$ with $\Omega_\infty = q_\infty^2$ for the NLS and DMNLS equations, the thermal media system, and the AL system, $\Omega_\infty = q_\infty^{2\sigma}$ for the power-law model, and $\Omega_\infty = q_\infty^2/(1 + sq_\infty^2)$ for the saturable nonlinearity model. All models are also invariant under phase rotations and time translations. Additionally, the first five models are invariant under space translations and Galilean transformations (in other words, if $q(x, t)$ is any solution, so is $\mathrm{e}^{i(Vx - V^2t)}q(x - Vt, t)$). The latter invariance guarantees the existence of a one-parameter family of plane-wave solutions. Related to these invariances is the presence of conserved quantities; those of interest in our case are energy, momentum, and Hamiltonian,

$$(12) \qquad \int_{\mathbb{R}} |q|^2 \,\mathrm{d}x \,, \qquad \int_{\mathbb{R}} (q^*q_x - qq_x^*) \,\mathrm{d}x \,, \qquad \int_{\mathbb{R}} (|q_x|^2 - V[q]) \,\mathrm{d}x \,,$$

where $V[q] = \nu q^*N[q]$ for the NLS and DMNLS equations and the thermal media system, $V[q] = 2\nu\sigma|q|^{2(\sigma+1)}/(\sigma+1)$ for the power-law model, and $V[q] = (2\nu/s^2)[s|q|^2 - \ln(1+s|q|^2)]$ for the saturable nonlinearity model, all with the same Hamiltonian structure as the NLS equation [6]. (Note, however, that when considering solutions on a

nonzero background, suitable constants must be subtracted from the integrands to make the integrals convergent.) The AL system does not possess the same invariances. On the other hand, since it is a completely integrable Hamiltonian system, it has an infinite number of conserved quantities in involution, including discrete analogues of each of (12) [1].

These conservation laws point to the existence of more general spatially periodic solutions in addition to plane-wave solutions. It is such solutions that make up the "building blocks" of the nonlinear stage of MI. Indeed, in the case of the NLS equation, the structure of the asymptotic state of MI in the oscillation region is precisely a slow modulation of such periodic solutions, as was shown in [15, 16].

Linear Stage of Modulational Instability. We first study the initial stage of MI in all of the above models by linearizing each of them around the constant solution. To this end, it is convenient to perform the trivial gauge transformation $q(x,t) \mapsto \tilde{q}(x,t) = q(x,t) e^{-2i\nu\Omega_\infty t}$, which maps the background solution simply into $\tilde{q}_s(x,t) = q_\infty$ and $N(q)$ in (9) into $\tilde{N}[\tilde{q}] = N[\tilde{q}] - \Omega_\infty \tilde{q}$. For simplicity, in what follows we will drop all tildes.

Looking for a perturbed solution as $q(x,t) = q_\infty (1 + u(x,t))$ with $u(x,t) = o(1)$, one finds that, to leading order, $u(x,t)$ satisfies the linearized evolution equation

$$(13) \qquad\qquad iu_t + L[u] + 2\nu M[u, u^*] = 0 \,,$$

where

$$(14a) \qquad\qquad M_{\mathrm{nls}}[u, u^*] = q_\infty^2 (u + u^*) \,,$$

$$(14b) \qquad\qquad M_{\mathrm{power}}[u, u^*] = \sigma q_\infty^{2\sigma} (u + u^*) \,,$$

$$(14c) \qquad\qquad M_{\mathrm{saturable}}[u, u^*] = q_\infty^2 (u + u^*)/(1 + s q_\infty^2)^2 \,,$$

$$(14d) \qquad M_{\mathrm{thermal}}[u, u^*] = q_\infty^2 \int_{\mathbb{R}} K(x - y)(u(y,t) + u^*(y,t)) \, \mathrm{d}y \,,$$

$$(14e) \quad M_{\mathrm{dmnls}}[u, u^*] = q_\infty^2 \iint_{\mathbb{R}^2} (u_{(x+y)} + u_{(x+z)} - u_{(x)} + u^*_{(x+y+z)}) \, R(y, z) \, \mathrm{d}y\mathrm{d}z \,,$$

$$(14f) \qquad\qquad M_{\mathrm{al}}[u, u^*] = \frac{1}{2} q_\infty^2 (u_{n+1} + 2u_n^* + u_{n-1}) \,.$$

Rewriting (13) in terms of the dependent variables $v = u + u^*$ and $w = -i(u - u^*)$, one obtains the system

$$(15a) \qquad\qquad v_t + L[w] = 0 \,,$$

$$(15b) \qquad\qquad -w_t + L[v] + 4\nu M[v, 0] = 0$$

for the first four models, whereas for the DMNLS equation the system (15) is replaced by

$$(16a) \qquad\qquad v_t + L[w] + 2\nu q_\infty^2 J_-[w] = 0 \,,$$

$$(16b) \qquad\qquad -w_t + L[v] + 2\nu q_\infty^2 J_+[v] = 0 \,,$$

where

$$(17) \qquad J_\pm[f] = \iint_{\mathbb{R}^2} (f_{(x+y)} + f_{(x+z)} \pm f^*_{(x+y+z)} - f_{(x)}) \, R(y, z) \, \mathrm{d}y\mathrm{d}z \,,$$

and for the AL system (15) is replaced by

$$(18a) \qquad v_t + L_-[w] + \nu h^2 q_\infty^2 L_-[w] = 0 \,,$$

$$(18b) \qquad -w_t + L_-[v] + \nu h^2 q_\infty^2 L_+[v] = 0 \,,$$

with $L_\pm[f] = (f(x+h,t) \pm 2f(x,t) + f(x-h,t))/h^2$.

Looking for exponential solutions as $(v,w)^T = e^{i(kx-\omega t)}\vec{v}$ yields the homogeneous linear system $A\vec{v} = 0$, where for the first four models

$$(19) \qquad A(k) = \begin{pmatrix} i\omega & k^2 \\ 4\nu\Omega_\infty - k^2 & i\omega \end{pmatrix} \,.$$

For the DMNLS equation, the off-diagonal entries of A are instead

$$(20) \qquad A_{1,2}(k) = k^2 - 2\nu q_\infty^2 I_-(k) \,, \qquad A_{2,1}(k) = -k^2 + 2\nu q_\infty^2 I_+(k) \,,$$

with

$$(21) \qquad I_\pm(k) = 1 \pm \sin(sk^2)/(sk^2) \,,$$

and for the AL system the off-diagonal entries of A are

$$(22a) \qquad A_{1,2}(k) = 4(1 + \nu h^2 q_\infty^2)\sin^2(kh/2)/h^2 \,,$$

$$(22b) \qquad A_{2,1}(k) = 4(\nu q_\infty^2 \cos^2(kh/2) - \sin^2(kh/2)/h^2) \,.$$

Finally, looking for nontrivial solutions of the linear system yields the linearized dispersion relation as

$$(23) \qquad \omega^2(k) = -A_{1,2}(k)A_{2,1}(k) \,.$$

For all models, MI occurs for all wavenumbers $k \in \mathbb{R}$ such that $A_{1,2}(k)A_{2,1}(k) > 0$. Hereafter we restrict ourselves to the focusing case ($\nu = 1$). For the NLS equation we recover the well-known result that the unstable wavenumbers are those for which $k \in (-k_{\mathrm{mi}}, k_{\mathrm{mi}})$, with $k_{\mathrm{mi}} = 2q_\infty$. For the power-law model, the instability range is given by $k_{\mathrm{mi}} = 2\sqrt{\sigma}q_\infty^\sigma$ and may be wider or narrower than that for the NLS equation, depending on the values of σ and q_∞. For the saturable nonlinearity model, the instability threshold is $k_{\mathrm{mi}} = 2q_\infty/(1 + sq_\infty^2)$, which for all $s > 0$ is always smaller than that for the NLS equation, resulting in a reduction of MI. For the thermal media system, the threshold is $k_{\mathrm{mi}} = 2\sqrt{2}q_\infty/[1 + (1 + 16s^2q_\infty^2)^{1/2}]^{1/2}$, which is also always smaller than that for the NLS equation. For the DMNLS equation, the unstable wavenumbers are those in the ranges $|k| \in (q_\infty\sqrt{2I_-(k)}, q_\infty\sqrt{2I_+(k)})$, which once again are always narrower than for the NLS equation. Also, in this case the unstable wavenumbers are bounded away from zero. Finally, for the AL system the unstable threshold is given by $k_{\mathrm{mi}} = (2/h)\arctan(hq_\infty)$, which is again always smaller than that for the NLS equation. (For the AL system, an instability is also present in the defocusing case if $hq_\infty > 1$, which had already been observed in [47]. It is straightforward to show, however, using the same methods as above, that in this case all wavenumbers are unstable, which therefore leads to a very different behavior.)

4. Nonlinear Stage of Modulational Instability. Having established that all six models are subject to MI in the focusing case, we now study the nonlinear evolution of generic localized perturbations of a constant background (i.e., we study the dynamics past the stage described by linearization). Since no asymptotic results are available at present for any of these models except the NLS equation, we turn to numerical simulations.
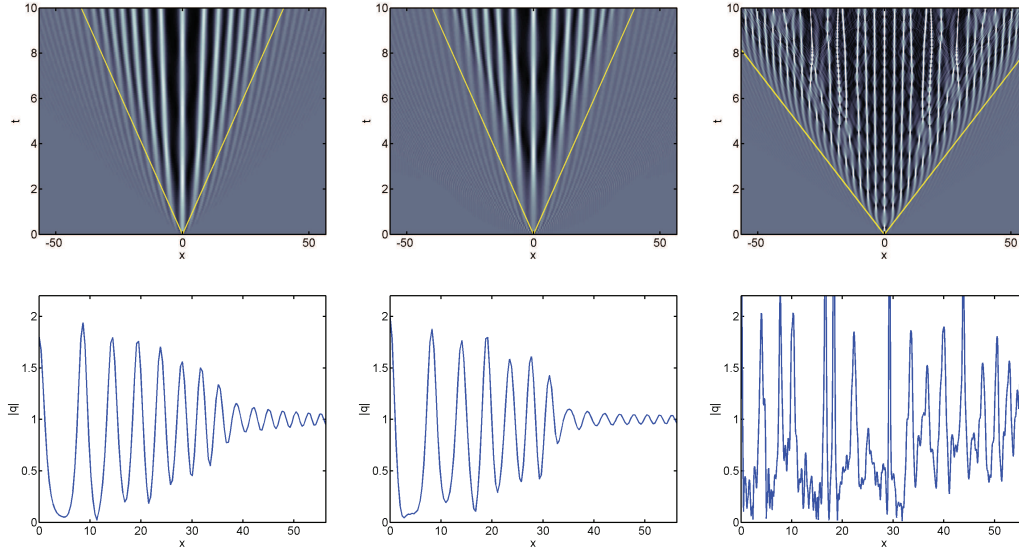
Fig. 2 *Evolution of a Gaussian perturbation* (8a) *(left column) and a sech-shaped perturbation* (8b) *(center column) of the constant background, similarly to Figure 1, but for the power-law model* (2) *with $\sigma = 1/2$, and a Gaussian IC* (8a) *with $\sigma = 3/2$ (right column). Yellow lines indicate the predicted boundary of the modulated oscillation region (see text for details).*

Numerical Methods. The three PDE models (i.e., the NLS equation itself and its power-law and saturable nonlinearity generalizations) can be effectively integrated numerically with a Fourier split-step method, owing to the ability to solve exactly the nonlinear part of the PDE [58]. An eighth-order algorithm was used in the simulations. The remaining three models (i.e., the DMNLS equation, the thermal media system, and the AL system) were integrated using a pseudospectral Fourier method in space and an integrating-factor fourth-order Runge–Kutta algorithm in time (e.g., as in [45, 51]). For the DMNLS equation (5), a computationally efficient algorithm (discussed in detail in [45, 51]) was also used to evaluate the double integral. The same ICs as for the NLS equations were considered, and periodic boundary conditions were used for all models.

Numerical Results. For brevity, here we only display results for Gaussian and sech-shaped ICs (see Appendix B for box-like ICs). Figure 2 shows the evolution for the power-law model, in particular, for $\sigma = 1/2$ and the Gaussian IC in (8a) (left column) or the sech-type IC (8b) (center column), respectively, and the right column shows how the evolution changes for the Gaussian IC when the exponent increases to $\sigma = 3/2$. Similarly, Figure 3 shows the evolution of the Gaussian IC (8a) for the saturable nonlinearity model, the thermal media system, and the DMNLS equation, respectively, all with $s = 1$. Even though individual differences are obviously present between the models (e.g., in the width of the oscillation wedge and the location and amplitude of the individual peaks), the overall dynamics is strikingly similar for all systems and for all ICs considered. Finally, the left and center columns of Figure 4 show the corresponding results for the AL system with $h = 1/2$. Even though the spatial variable in (7) is discrete, the same kind of dynamics is again clearly evident. (We should point out that a related kind of universal behavior among different dynamical models was also reported in [26].)
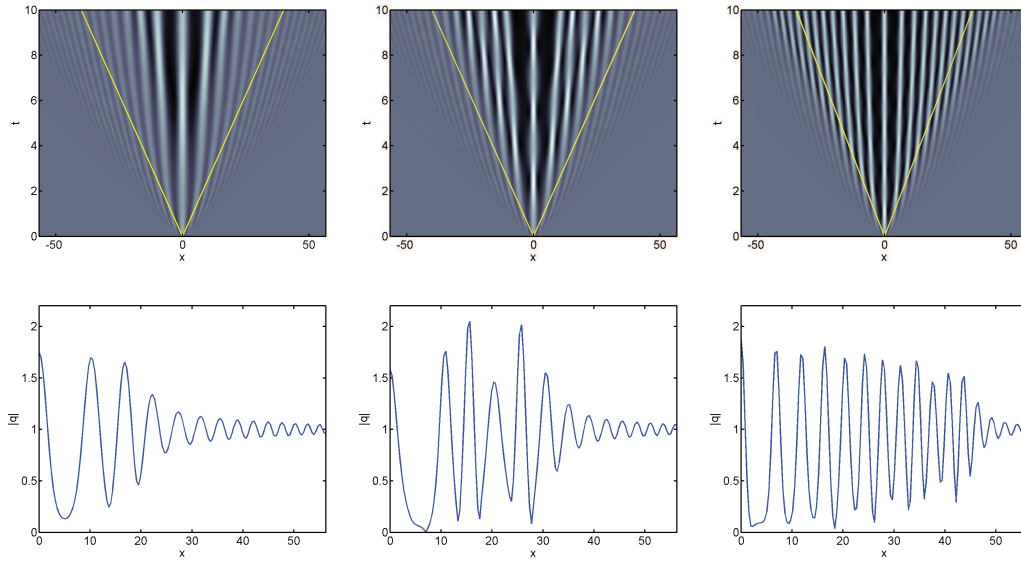
Fig. 3 *Similar to Figure* 1 *with a Gaussian IC* (8a) *for the saturable nonlinearity model (left), the thermal media system (center), and the DMNLS equation (right), all with* $s = 1$.
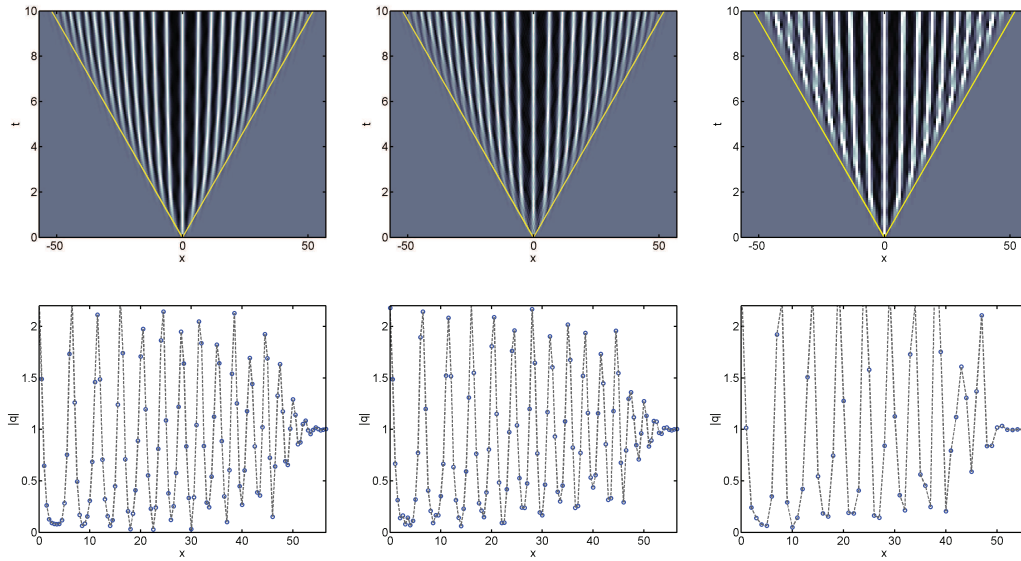

Fig. 4 *Same as Figure* 2 *for the AL* (7) *with* $h = 1/2$ *(left two columns) and with* $h = 1$ *(right column).*

The differences from the prototypical NLS behavior manifest themselves in several ways, including a different aperture angle for the oscillation wedge and a different peak amplitude at $x = 0$. Note also the appearance of periodic temporal oscillations of the peak amplitudes in the case of the saturable nonlinearity model and the
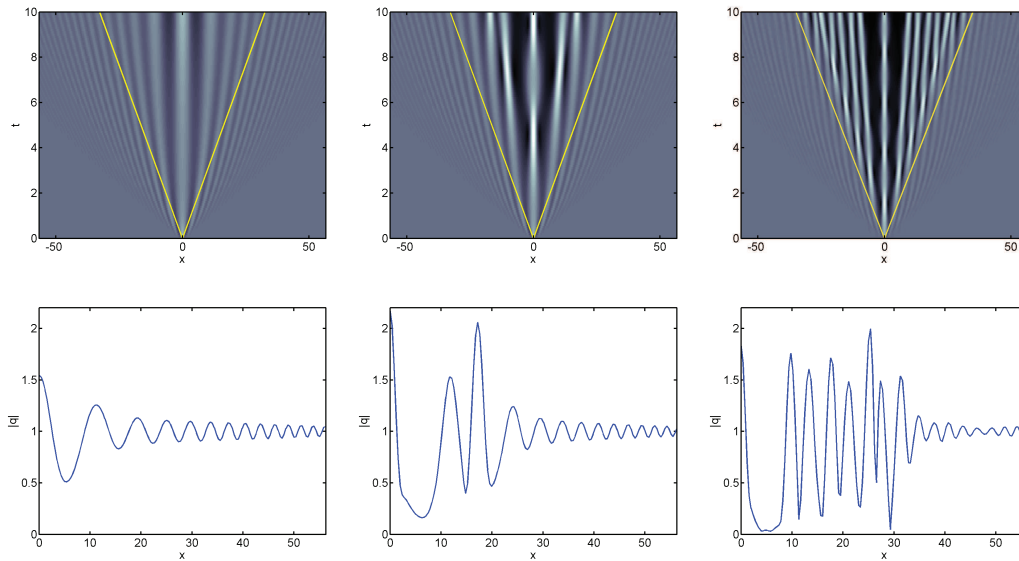
Fig. 5 *Same as Figure 3, but for s = 2.*

thermal media system. Moreover, as might be expected, these changes become more pronounced when the "deviation parameter" in each model takes on larger values. This is the case, for instance, shown in the right column of Figure 2, displaying the dynamics for the power-law model for a relatively large value of σ. Figure 5 does the same for the saturable nonlinearity model, thermal media system, and DMNLS equation, respectively, with $s = 2$, and the right column of Figure 4 for the AL system with $h = 1$. Compared to the previous cases, for the power-law model one observes not only a broader wedge, but also nontrivial interactions between the individual peaks, consistent with its nonintegrability and stronger nonlinearity compared to the NLS equation. Importantly, our simulations for the power-law model reveal that the oscillatory wedged region induced via MI also develops for values $\sigma \geq 2$ (the results are similar to those reported in the right column of Figure 2, and are omitted for brevity), at which point the model becomes critical or supercritical and hence potentially subject to competing blow-up instabilities [52]. In other words, the numerical results show that, in the supercritical regime, MI prevails over the collapse instability. This finding is compatible with the fact that the presence of the background usually restricts the conditions for the blow-up to occur [55], though further studies (beyond the scope of the present paper) are needed to deepen our understanding of the general interplay between collapse and MI. Conversely, for the DMNLS equation and the saturable nonlinearity model, the weaker nonlinearity results in a narrower oscillation wedge compared to the case of $s = 1$ and, in the case of the saturable nonlinearity model, a lower peak amplitude. Note also that when $s = 2$, the temporal oscillations of the peak amplitudes for the thermal media model become more pronounced and similar oscillations arise for the DMNLS equation (compare Figures 5 and 3). Nonetheless, we find it remarkable that the general structure of the solution remains the same as in Figure 1 in all cases (namely, quiescent sectors separated by a wedge of modulated periodic oscillations) in spite of these differences.

Boundaries of the Oscillation Region. While no analytical asymptotic results are available for any model other than the NLS equation, the boundaries between the modulated oscillation region and the plane-wave regions can still be characterized heuristically. Similar to the case of the NLS equation [30, 32, 31], one can interpret these boundaries as the paths of the slowest waves moving away from the initial perturbation. The latter are determined by the minimum of the group velocity $c_g(k) = \omega'(k)$, with $\omega(k)$ still given by (23). For the first three models one can compute such a minimum analytically, to obtain

$$(24) \quad c_{*,\mathrm{nls}} = 4\sqrt{2}q_\infty \,, \qquad c_{*,\mathrm{power}} = 4\sqrt{2\sigma}q_\infty^\sigma \,, \qquad c_{*,\mathrm{saturable}} = 4\sqrt{2}q_\infty/\sqrt{1+sq_\infty^2} \,,$$

respectively. Notice that $c_{*,\mathrm{nls}}$ coincides with the value ξ_* obtained from the asymptotic analysis of [15, 16]. For the last three models, the minimum velocity must be obtained numerically. The resulting predictions for the boundaries $x = \pm c_* t$ between the modulated region and the plane-wave regions are indicated by yellow lines in Figures 2–4, and are in good agreement with the numerical results for almost all models. The lone exception is the AL system with $h = 1$, because $c_g(k)$ has no local minima for $h > 1/2$ (and its absolute minimum is zero). Interestingly, however, the minimum group velocity for $h = 1/2$ also appears to apply for $h > 1/2$. The criterion that determines the aperture of the wedge in this latter case is unclear. Also, for all models, it is unclear whether there is any heuristic criterion that allows one to predict the amplitude of the solution at the center of the wedge.

5. Conclusions. We have shown that the asymptotic behavior of very different systems affected by MI exhibits remarkable universal properties independent of the ICs. The results of this work open up a number of interesting problems.

The fact that the behavior discussed in this work is shared among systems that arise in very different physical contexts not only means that it is a generic and robust feature of modulationally unstable media, but it also increases the likelihood that this behavior can be observed experimentally in one of the systems to which the above models apply. Of course, a natural question which remains to be explored is whether this kind of behavior is shared among an even larger variety of physical models.

It should be mentioned that MI and its nonlinear stage were also recently studied in [11, 36, 38, 59]. In particular, a claim of universality with respect to ICs was also made [11, 38]. The scenario studied in those works is very different from that in the present work: [11, 38] considered a class of ICs that are exponentially localized and tend to zero at infinity. In contrast, the present work considers a class of ICs which tend to a nonzero constant. We make no claim that the behavior presented in our work would also arise with different kinds of ICs. Indeed, the nonlinear stage of MI for the focusing NLS equation with periodic BCs is markedly different, and is described by Akhmediev breathers [8, 54]. Understanding the kind of asymptotic behavior, if any, that one can expect when given more general kinds of ICs is an interesting open problem.

Two other open issues regarding the asymptotic behavior of solutions of the focusing NLS equation are, on the one hand, whether the behavior discussed in this work also persists for solutions that are only slowly decaying to the background, and, on the other hand, what happens when the ICs give rise to a nontrivial discrete spectrum (i.e., when one or more solitons are present). One can expect that both of these questions can be effectively studied using similar techniques as in [16, 20, 21, 27].

Another interesting problem is that of identifying a set of "minimum requirements" that are needed for a dynamical model to display the kind of behavior de-

scribed in this work. At first it would seem that the existence of at least three conserved quantities, including conservation of momentum, are among these necessary requirements. (The presence of these conserved quantities should guarantee the existence of traveling-wave solutions.) On the other hand, we show in Appendix C that this kind of behavior also occurs for the DNLS equation, for which there is no conservation of momentum and which does not admit traveling-wave solutions [2, 42].

Another obvious question is how to precisely characterize the nonlinear stage of MI for other models. In this respect, since the AL system (7) is also an integrable system (like the NLS equation) [1, 5], one can look into whether the analysis of [15, 16] can be suitably modified to obtain a quantitative description of the dynamics in that model.

A much more challenging task will be to develop a mathematical theory that can characterize the nonlinear stage of MI for all models. Such a problem would be very difficult even for the NLS equation if one could not make use of the IST, since to the best of our knowledge it is not possible to compute long-time asymptotic behavior using only more general PDE methods. On the other hand, it was shown in [28] that, under appropriate conditions, the behavior of the perturbed NLS equation can be mapped in a controlled way to that of the unperturbed system. (In this sense, all such perturbed systems were termed to be integrable in [28].) Even though the situation considered in [28] is very different the one in the present work, an intriguing possibility is whether a generalization of the results of [28] could be derived that applies to the present scenario. (A key difficulty is the key requirement in [28] that $\|u\|_\infty < C/\sqrt{t}$. Such a requirement is not satisfied in our case, obviously.)

An alternative approach that can be used for all models is Whitham modulation theory, which does not require integrability. Apart from the fact that Whitham theory does not usually produce rigorous results, however, we note that no such theory is available at present for the models studied here other than the NLS equation. In fact, to the best of our knowledge no exact periodic solutions are known for the above models apart for the NLS equation and the AL system. (Such solutions provide the starting point for the derivation of genus-1 Whitham modulation equations; e.g., see [39, 33, 25].) Moreover, one expects that, as a result of MI, the resulting modulation equations for each model will be elliptic in the focusing case (as in the NLS equation), and therefore cannot serve to study the evolution of generic perturbations. Even so, one would hope that the nonlinear stage of MI is described by special solutions of such equations (similar to [30, 41]). Alternatively, one might hope to be able to evaluate analytically special limits of the Whitham modulation equations to obtain an estimate for the maximum amplitude of the solution in the oscillation region.

It is hoped that the results of this work and the above discussion will stimulate future research on these and related topics.

Note Added in Proof. We learned that the behavior described in [18, 19] was recently observed in optical fiber experiments in [43].

Appendix A. Asymptotic Stage of MI for the Focusing NLS Equation. It was shown in [15, 16] that, for a broad class of localized initial perturbations of a constant background, the solution of (1) tends to a universal asymptotic state. More precisely, for all ICs such that $q(x,0) = q - \infty + O(e^{-\epsilon|x|})$ as $x \to \infty$ for some $\epsilon > 0$ and no discrete spectrum is present, the analysis of [16], which was confirmed numerically in [14, 15], shows that

$$(25) \qquad\qquad q(x,t) = q_{\mathrm{asymp}}(x,t) + O(1/\sqrt{t}), \qquad t \to \infty,$$

with the xt-plane divided into (i) two plane-wave regions, $x < -\xi_* t$ and $x > \xi_* t$, with $\xi_* = 4\sqrt{2}q_\infty$, in which $|q_{\mathrm{asymp}}(x,t)| = q_\infty$ (i.e., the solution has the same amplitude as the undisturbed background); and (ii) a modulated elliptic wave region, $-\xi_* t < x < \xi_* t$, in which the solution is expressed by a slow modulation of the elliptic solutions of (1), namely,

$$(26) \qquad |q_{\mathrm{asymp}}(x,t)|^2 = (q_\infty + \alpha_{\mathrm{im}})^2 - 4q_\infty \alpha_{\mathrm{im}} \,\mathrm{sn}^2 \left(2C(x - 2\alpha_{\mathrm{re}}t - X_o) \,\big|\, m \right),$$

where $\mathrm{sn}(\cdot)$ is one of the Jacobian elliptic functions [48], $m \in [0,1]$ is the elliptic parameter, $C = \sqrt{q_\infty \alpha_{\mathrm{im}}/m}$, and the offset X_o depends on the IC $q(x,0)$ via the reflection coefficient. When α_{re} and α_{im} are independent of x and t, (26) is an exact traveling-wave solution of (1), with m given by (27b) below. For arbitrary ICs, however, α_{re}, α_{im}, and m are slowly varying functions of x and t which are given by the following system of three modulation equations,

$$(27a) \qquad \frac{x}{2t} = 2\alpha_{\mathrm{re}} + \frac{q_\infty^2 - \alpha_{\mathrm{im}}^2}{\alpha_{\mathrm{re}}},$$

$$(27b) \qquad m = \frac{4q_\infty \alpha_{\mathrm{im}}}{\alpha_{\mathrm{re}}^2 + (q_\infty + \alpha_{\mathrm{im}})^2},$$

$$(27c) \qquad \left(\alpha_{\mathrm{re}}^2 + (q_\infty - \alpha_{\mathrm{im}})^2\right)K(m) = (\alpha_{\mathrm{re}}^2 - \alpha_{\mathrm{im}}^2 + q_\infty^2)E(m),$$

where $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively. Importantly, it was shown in [15, 16] that these equations are universal, i.e., independent of the ICs. The implicit solution of the above system yields the envelope shown in the red curves in the bottom row of Figure 1. The structure of the asymptotic state in the modulated oscillations region was further studied in [14], where it was shown that, as $t \to \infty$, each peak in the solution profile is described by a classical, sech-shaped soliton solution of the focusing NLS equation with zero background.

It is worth emphasizing that the lines in the top row of Figure 1 are boundaries that discriminate between the different types of asymptotic limit, namely, plane wave (constant amplitude) versus oscillations (modulated elliptic solutions in the case). Thus, any apparent "crossing" of these lines in Figure 1 is not a numerical artifact, but merely a consequence of the fact that the asymptotic result is not an exact representation of the solution. On the other hand, since the error in the asymptotic estimates in (25) is $O(1/\sqrt{t})$, one would expect the agreement between asymptotics and numerics to become better as time grows.

Appendix B. Universality with Respect to Initial Conditions.   For completeness, here we present the results of numerical simulations for all models with other kinds of initial perturbations of the constant background. More precisely, we show the evolution of the sech-shaped and box-like ICs in (8b) and (8c), respectively (whereas the main text used the Gaussian ICs (8a) and a few other cases of sech-shaped ICs).

The left column of Figure 6 shows the evolution of a box-like perturbation of the constant background for the arbitrary power model with $\sigma = 1/2$, whereas the center column and the right column show, respectively, sech-shaped or box-like ICs with $\sigma = 3/2$. These plots complement Figure 2 in the main text. Similarly, Figures 7 and 8 show the evolution of the sech-shaped IC (8b) and the box-like IC (8c), respectively, for the saturable nonlinearity model, the thermal media system, and the DMNLS equation, all with $s = 1$, and they complement Figure 3 in the main text. Figures 9 and 10 do the same for all three models with both ICs but with $s = 2$, and they
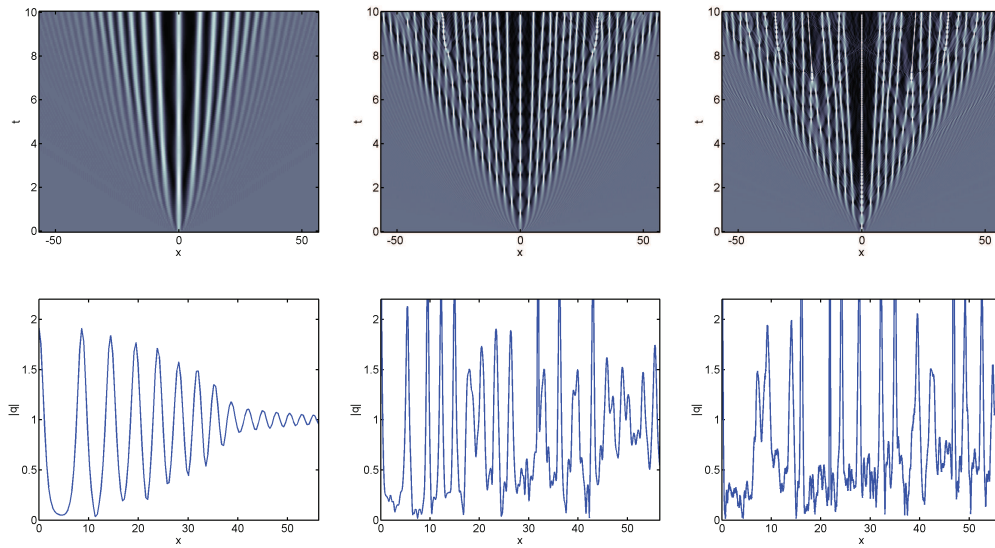
Fig. 6 *Left column: Evolution of a box-like initial perturbation (8c) of the constant background for the arbitrary power model (2), with $\sigma = 1/2$. Center column and right column: Evolution of a sech-shaped or box-like IC, respectively, for the same model with $\sigma = 3/2$.*
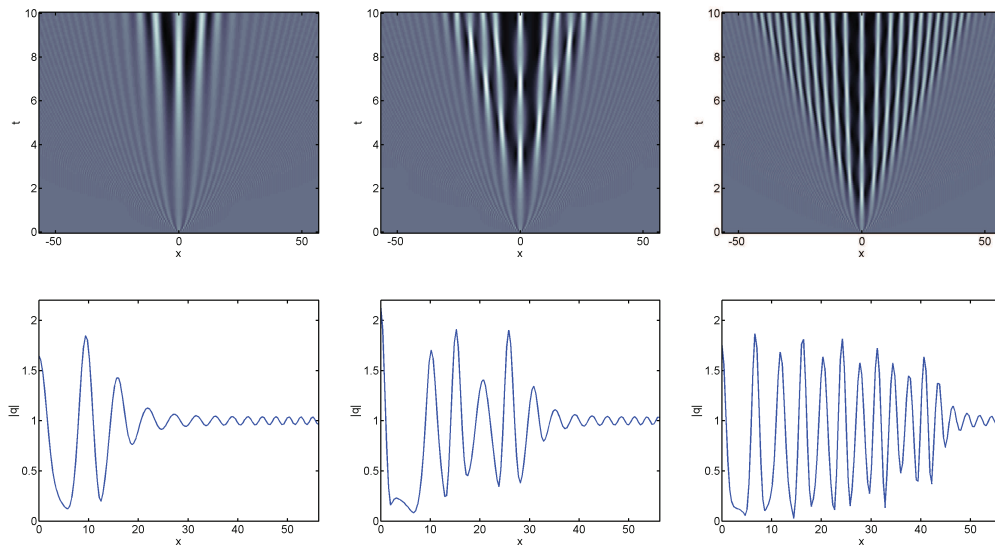

Fig. 7 *Evolution with a sech-shaped IC (8b) for the saturable nonlinearity model (left), the thermal media system (center), and the DMNLS equation (right), all with $s = 1$.*

complement Figure 5 in the main text, Finally, Figure 11 shows the same two ICs for the AL system with $h = 1/2$ and $h = 1$, and should be compared to Figure 4 in the main text.

Once more, all these figures show that, despite some individual variations across different models, (i) the evolution of localized perturbations for a given system is remarkably similar independent of the ICs, and (ii) all these models display the same
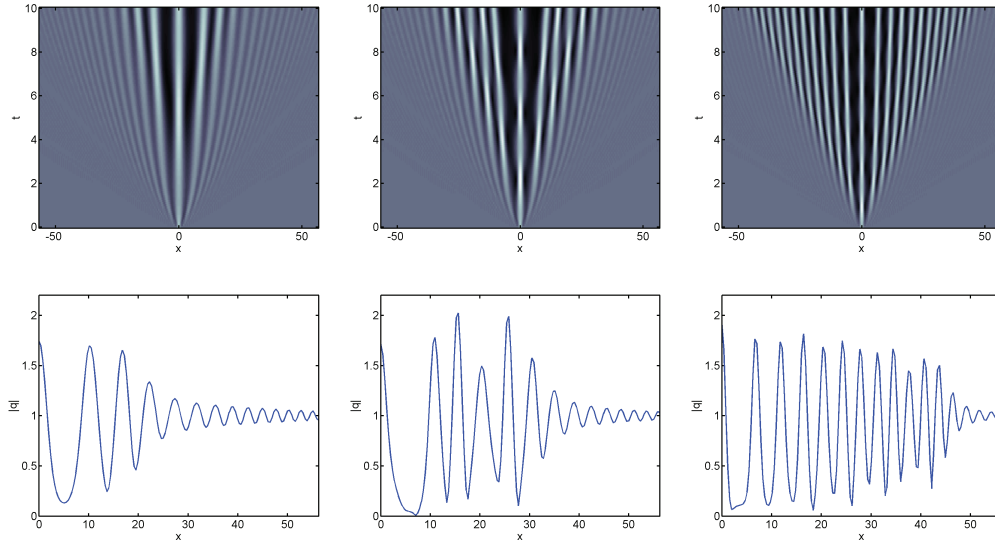
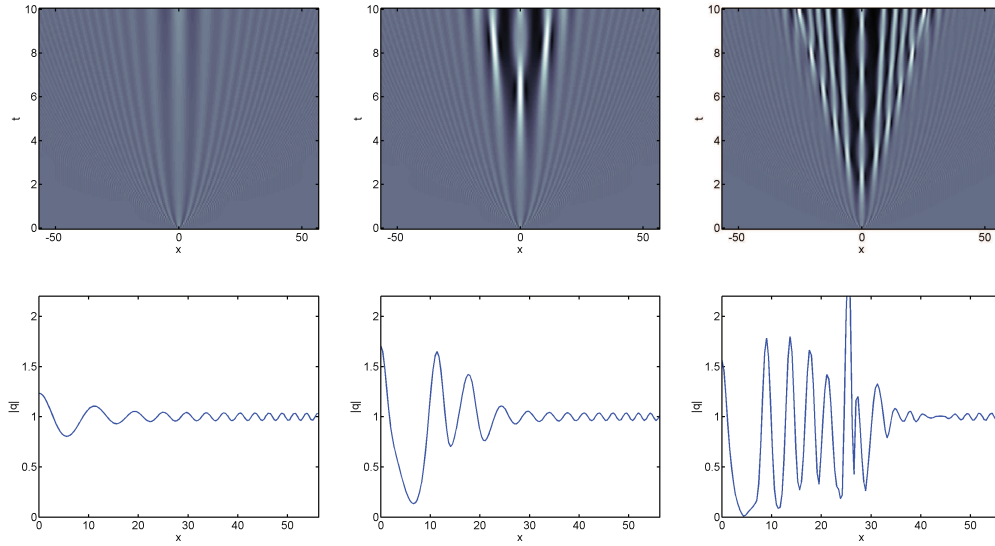**Fig. 8** *Same as Figure 7, but with a box-like IC (8c).*

Fig. 9 *Same as Figure 7, but with $s = 2$.*

kind of asymptotic behavior: two outer regions with a quiescent state, separated by a central wedge with modulated oscillations.

Appendix C. Nonlinear Stage of Modulational Instability in the Standard Discrete Nonlinear Schrödinger Equation. Another commonly used discrete version of the NLS equation is the so-called standard discrete NLS (DNLS) equation, for which one simply has $N[q] = |q_n|^2 q_n$, with the same linear part as for the AL system. Even though the DNLS equation is not integrable and does not possess traveling-wave
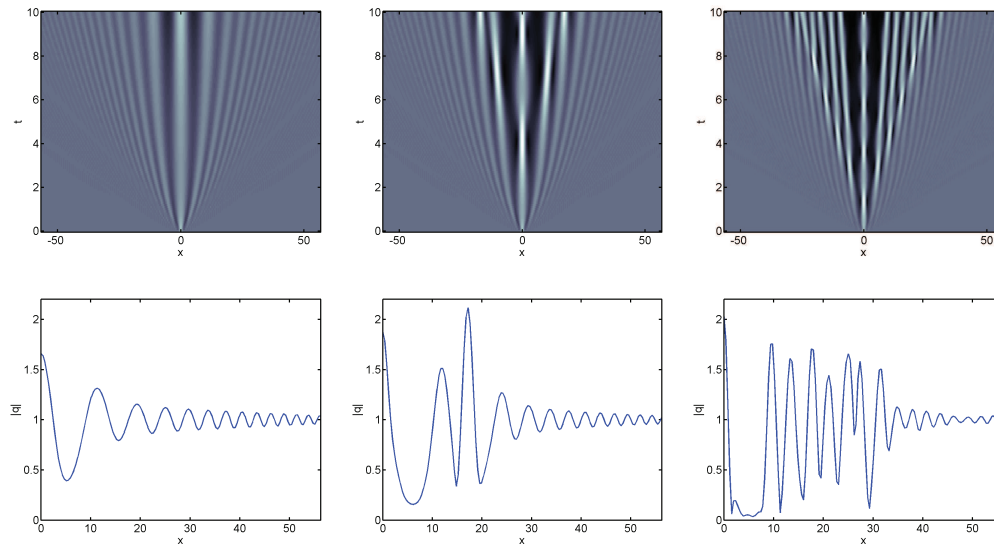
Fig. 10   *Same as Figure 8, but with $s = 2$.*
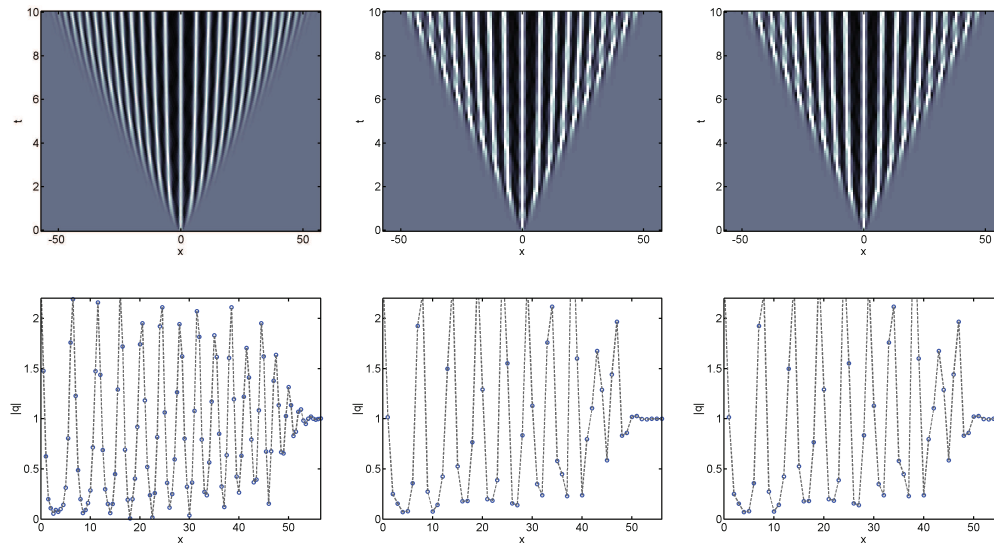

Fig. 11   *Same as Figure 6, but for the AL equation (7), with $h = 1/2$ (left column) or $h = 1$ (center and right columns).*

solutions, the evolution of a localized perturbation of a constant background still exhibits a similar kind of behavior. This is evidenced in Figure 12, which should be compared to Figure 4 for the AL system. Despite some minor differences between the DNLS equation and the AL system, these results provide further confirmation of the fact that the behavior presented in this work is a universal feature of modulationally unstable models.
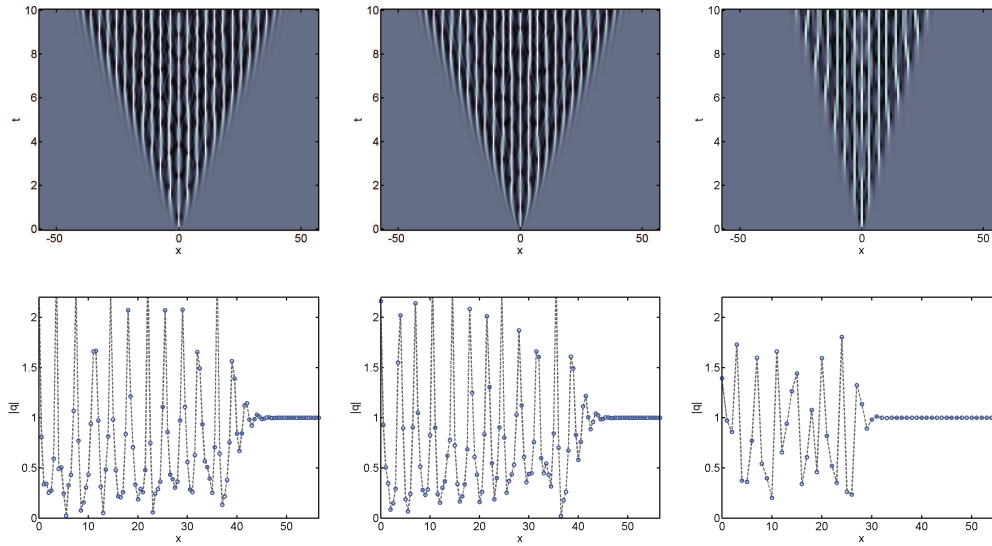
Fig. 12 *Evolution of a Gaussian (left column) and a sech-shaped (center column) initial perturbation of the constant background for the DNLS equation with h = 1/2. The right column shows the same Gaussian perturbation with h = 1.*

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