# INVERSE SCATTERING TRANSFORM FOR THE DEFOCUSING MANAKOV SYSTEM WITH NONZERO BOUNDARY CONDITIONS* 

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#### Abstract

The inverse scattering transform for the defocusing Manakov system with nonzero boundary conditions at infinity is rigorously studied. Several new results are obtained: (i) The analyticity of the Jost eigenfunctions is investigated, and precise conditions on the potential that guarantee such analyticity are provided. (ii) The analyticity of the scattering coefficients is established. (iii) The behavior of the eigenfunctions and scattering coefficients at the branch points is discussed. (iv) New symmetries are derived for the analytic eigenfunctions (which differ from those in the scalar case). (v) These symmetries are used to obtain a rigorous characterization of the discrete spectrum and to rigorously derive the symmetries of the associated norming constants. (vi) The asymptotic behavior of the Jost eigenfunctions is derived systematically. (vii) A general formulation of the inverse scattering problem as a Riemann-Hilbert problem is presented. (viii) Precise results guaranteeing the existence and uniqueness of solutions of the Riemann-Hilbert problem are provided. (ix) Explicit relations among all reflection coefficients are given, and all entries of the scattering matrix are determined in the case of reflectionless solutions. (x) A compact, closed-form expression is presented for general soliton solutions, including any combination of dark-dark and dark-bright solitons. (xi) A consistent framework is formulated for obtaining solutions corresponding to double zeros of the analytic scattering coefficients, leading to double poles in the Riemann-Hilbert problem, and such solutions are constructed explicitly.


Key words. nonlinear Schrödinger equations, Lax pairs, inverse scattering transform, solitons, integrable systems, nonzero boundary conditions

AMS subject classifications. 35, 35Q55, 35Q15, 35Q51
DOI. 10.1137/130943479

1. Introduction. Scalar and vector nonlinear Schrödinger (NLS) equations are universal models for the evolution of weakly nonlinear dispersive wave trains. As such, they appear in many physical contexts, such as deep water waves, nonlinear optics, acoustics, and Bose-Einstein condensation (e.g., see [5, 23, 35, 39] and references therein). Many of these equations are also completely integrable infinite-dimensional Hamiltonian systems, and as such they possess a remarkably rich mathematical structure. As a consequence, they have been the object of considerable research over the last fifty years (e.g., see $[3,5,8,19,21,30]$ and references therein). In particular, it is well known that for the integrable cases, the initial value problem can in principle be solved by the inverse scattering transform (IST), a nonlinear analogue of the Fourier transform.

This work is concerned with the Manakov system, namely, the two-component vector NLS equation

$$
\begin{equation*}
i \mathbf{q}_{t}+\mathbf{q}_{x x}+2 \sigma\left(q_{o}^{2}-\|\mathbf{q}\|^{2}\right) \mathbf{q}=\mathbf{0} \tag{1.1}
\end{equation*}
$$

with the following nonzero boundary conditions (NZBC) at infinity:

$$
\begin{equation*}
\lim _{x \rightarrow \pm \infty} \mathbf{q}(x, t)=\mathbf{q}_{ \pm}=\mathbf{q}_{o} \mathrm{e}^{i \theta_{ \pm}} \tag{1.2}
\end{equation*}
$$

[^0]Hereafter, $\mathbf{q}=\mathbf{q}(x, t)$ and $\mathbf{q}_{o}$ are two-component vectors, $\|\cdot\|$ is the standard Euclidean norm, $\theta_{ \pm}$are real numbers, $q_{o}=\left\|\mathbf{q}_{o}\right\|>0$, and subscripts $x$ and $t$ denote partial differentiation throughout. The extra term $q_{o}^{2}$ in (1.1) is added so that the asymptotic values of the potential are independent of time. We discuss (1.1) in the defocusing case ( $\sigma=1$ ).

The IST for the scalar NLS equation (i.e., the one-component reduction of (1.1)) was developed by Zakharov and Shabat in [41] for the focusing case and in [42] for the defocusing case (see also $[1,5,6,19]$ ). The IST for (1.1) in the case with zero boundary conditions (ZBC) (i.e., when $\mathbf{q}_{o}=\mathbf{0}$ ) was derived in [29] and revisited and generalized to an arbitrary number of components in [3]. On the other hand, the IST for the Manakov system (1.1) with NZBC remained an open problem for a long time. A successful approach to the IST for this problem was recently presented in [31], but several issues were not addressed. (Indeed, several questions remain open even in the scalar case with NZBC; e.g., see [11, 18].)

The first purpose of this work is to develop the IST for the defocusing Manakov system with NZBC in a rigorous way. Several new results are obtained: (i) Precise conditions on the potential that guarantee the analyticity of the Jost eigenfunctions are provided. (ii) The analyticity of the scattering coefficients is established. (iii) The behavior of the eigenfunctions and scattering coefficients at the branch points is elucidated. (iv) New symmetries are derived for the analytic eigenfunctions, which differ from the symmetries of the scalar case. (v) These symmetries are used to obtain a rigorous characterization of the discrete spectrum and to rigorously derive the symmetries of the associated norming constants. (vi) The asymptotic behavior of the Jost eigenfunctions is derived systematically. (vii) A general formulation of the inverse scattering problem as a Riemann-Hilbert problem is presented. (viii) Explicit relations among all reflection coefficients are given, and all entries of the scattering matrix are determined in the case of reflectionless solutions. (ix) A compact, closedform expression is presented for general soliton solutions, including any combination of dark-dark and dark-bright solitons.

The second purpose of this work is to use the above results to derive novel solutions of the defocusing Manakov system. For most integrable nonlinear partial differential equations (PDEs), solitons are associated with the zeros of the analytic scattering coefficients. In the development of the IST, it is commonly assumed for simplicity that such zeros are simple. On the other hand, in some cases, the analytic scattering coefficients are allowed to have double zeros. Indeed, it is well known that solutions corresponding to such double zeros exist for the scalar focusing NLS equation [41]. On the contrary, for the scalar defocusing NLS, no such solutions exist since one can prove that the zeros of the analytic scattering coefficients are always simple [19] (as in the case of the Korteweg-de Vries equation). On the other hand, the proof does not generalize to the defocusing vector system. Here, we use the rigorous formulation of the IST described above to write down a consistent framework for obtaining solutions corresponding to double zeros of the analytic scattering coefficients, leading to double poles in the Riemann-Hilbert problem, and we construct such "double-pole solutions" explicitly. To the best of our knowledge, such solutions are new.

The outline of this work is the following: In section 2, we formulate the direct problem (taking into account automatically the time evolution). In section 2.5, we characterize the discrete spectrum. In section 3, we formulate the inverse problem. In section 3.6, we discuss the soliton solutions, and in section 4, we present novel double-pole solutions. The proofs of all theorems, lemmas, and corollaries in the text
are given in the appendix. Throughout, an asterisk denotes complex conjugation, and superscripts $T$ and $\dagger$ denote, respectively, matrix transpose and matrix adjoint. Also, we denote, respectively, with $\mathbf{A}_{d}, \mathbf{A}_{o}, \mathbf{A}_{b d}$, and $\mathbf{A}_{b o}$ the diagonal, off-diagonal, block diagonal, and block off-diagonal parts of a $3 \times 3$ matrix $\mathbf{A}$.
2. Direct scattering. As usual, the IST for an integrable nonlinear PDE is based on its formulation in terms of a Lax pair. The $3 \times 3$ Lax pair associated with the Manakov system (1.1) is

$$
\begin{equation*}
\phi_{x}=\mathbf{X} \phi, \quad \phi_{t}=\mathbf{T} \phi, \tag{2.1a}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathbf{X}(x, t, k)=-i k \mathbf{J}+\mathbf{Q}, \quad \mathbf{T}(x, t, k)=2 i k^{2} \mathbf{J}-i \mathbf{J}\left(\mathbf{Q}_{x}-\mathbf{Q}^{2}+q_{o}^{2}\right)-2 k \mathbf{Q}  \tag{2.1b}\\
\mathbf{J}=\left(\begin{array}{cc}
1 & \mathbf{0}^{T} \\
\mathbf{0} & -\mathbf{I}
\end{array}\right), \quad \mathbf{Q}(x, t)=\left(\begin{array}{cc}
0 & \mathbf{r}^{T} \\
\mathbf{q} & \mathbf{0}
\end{array}\right) \tag{2.1c}
\end{gather*}
$$

$\mathbf{r}(x, t)=\mathbf{q}^{*}$, and $\mathbf{I}$ and $\mathbf{0}$ are the appropriately sized identity matrix and zero matrix, respectively. That is, (1.1) is the compatibility condition $\phi_{x t}=\phi_{t x}$ (as is easily verified by direct calculation and noting that $\mathbf{J Q}=-\mathbf{Q J}$ ). As usual, the first half of (2.1a) is referred to as the scattering problem, $k$ as the scattering parameter, and $\mathbf{q}(x, t)$ as the scattering potential. The direct problem in IST consists of characterizing the eigenfunctions and scattering data based on the knowledge of the scattering potential. Unlike the usual approach to IST for the defocusing NLS equation and the Manakov system with NZBC, here we formulate the IST in a way that allows the reduction $q_{o} \rightarrow 0$ to be taken explicitly throughout. Also, it will be convenient to consider $\phi(x, t, k)$ as a $3 \times 3$ matrix. Some basic symmetry properties of the scattering problem are discussed in Appendix A.1. We should point out that, unlike [31], the direct problem is done here without assuming that $\mathbf{q}_{+}$is parallel to $\mathbf{q}_{-}$.
2.1. Riemann surface and uniformization. One can expect that, as $x \rightarrow$ $\pm \infty$, the solutions of the scattering problem are approximated by those of the asymptotic scattering problems

$$
\begin{equation*}
\phi_{x}=\mathbf{X}_{ \pm} \phi \tag{2.2}
\end{equation*}
$$

where $\mathbf{X}_{ \pm}=-i k \mathbf{J}+\mathbf{Q}_{ \pm}=\lim _{x \rightarrow \pm \infty} \mathbf{X}$. The eigenvalues of $\mathbf{X}_{ \pm}$are $i k$ and $\pm i \lambda$, where

$$
\begin{equation*}
\lambda(k)=\left(k^{2}-q_{o}^{2}\right)^{1 / 2} \tag{2.3}
\end{equation*}
$$

As in the scalar case [42], these eigenvalues have branching. To deal with this, as in $[19,31]$, we introduce the two-sheeted Riemann surface defined by (2.3). The branch points are the values of $k$ for which $\lambda(k)=0$, i.e., $k= \pm q_{o}$. As in [31], we take the branch cut on $\left(-\infty,-q_{o}\right] \cup\left[q_{o}, \infty\right)$, and we define $\lambda(k)$ so that $\operatorname{Im} \lambda \geq 0$ on sheet I and $\operatorname{Im} \lambda(k) \leq 0$ on sheet II (see [31] for further details). Next, we introduce the uniformization variable by defining

$$
\begin{equation*}
z=k+\lambda \tag{2.4}
\end{equation*}
$$

The inverse transformation is

$$
\begin{equation*}
k=\left(z+q_{o}^{2} / z\right) / 2, \quad \lambda=\left(z-q_{o}^{2} / z\right) / 2 . \tag{2.5}
\end{equation*}
$$

We can then express all $k$-dependence of eigenfunctions and scattering data (including the one resulting from $\lambda$ ) in terms of $z$, thereby eliminating all square roots. The
branch cuts on the two sheets of the Riemann surface are mapped onto the real $z$ axis, $\mathbb{C}_{I}$ is mapped onto the upper half plane of the complex $z$-plane, $\mathbb{C}_{I I}$ is mapped onto the lower half plane of the complex $z$-plane, $z\left(\infty_{\mathrm{I}}\right)=\infty$ if $\operatorname{Im}(k)>0, z\left(\infty_{\mathrm{I}}\right)=0$ if $\operatorname{Im}(k)<0, z\left(\infty_{\mathrm{II}}\right)=0$ if $\operatorname{Im}(k)>0, z\left(\infty_{\mathrm{II}}\right)=\infty$ if $\operatorname{Im}(k)<0, z\left(k, \lambda_{\mathrm{I}}\right) z\left(k, \lambda_{\mathrm{II}}\right)=q_{o}^{2}$, $|k| \rightarrow \infty$ in the upper half plane of $\mathbb{C}_{\mathrm{I}}$ corresponds to $z \rightarrow \infty$ in the upper half $z$ plane, $|k| \rightarrow \infty$ in the lower half plane of $\mathbb{C}_{\text {II }}$ corresponds to $z \rightarrow \infty$ in the lower half $z$-plane, $|k| \rightarrow \infty$ in the lower half plane of $\mathbb{C}_{\mathrm{I}}$ corresponds to $z \rightarrow 0$ in the upper half $z$-plane, and $|k| \rightarrow \infty$ in the upper half plane of $\mathbb{C}_{\mathrm{II}}$ corresponds to $z \rightarrow 0$ in the lower half $z$-plane. Finally, the segments $k \in\left[-q_{o}, q_{o}\right]$ in each sheet correspond, respectively, to the upper half and lower half of the circle $C_{o}$ of radius $q_{o}$ centered at the origin in the complex $z$-plane. Throughout this work, subscripts $\pm$ will denote normalization as $x \rightarrow-\infty$ or as $x \rightarrow \infty$, respectively, whereas superscripts $\pm$ will denote analyticity (or, more generally, meromorphicity) in the upper or lower half of the $z$-plane, respectively.
2.2. Jost solutions and scattering matrix. The continuous spectrum consists of all values of $k$ (in either sheet) such that $\lambda(k) \in \mathbb{R}$; that is, $k \in \mathbb{R} \backslash\left(-q_{o}, q_{o}\right)$. In the complex $z$-plane, the corresponding set is the whole real axis. For any twocomponent complex-valued vector $\mathbf{v}=\left(v_{1}, v_{2}\right)^{T}$, we define its orthogonal vector as $\mathbf{v}^{\perp}=\left(v_{2},-v_{1}\right)^{\dagger}$ so that $\mathbf{v}^{\dagger} \mathbf{v}^{\perp}=\left(\mathbf{v}^{\perp}\right)^{\dagger} \mathbf{v}=0$. (Note that this definition differs from that of [31].) We may then write the eigenvalues and the corresponding eigenvector matrices of the asymptotic scattering problem (2.2) as

$$
i \boldsymbol{\Lambda}(z)=\operatorname{diag}(-i \lambda, i k, i \lambda), \quad \mathbf{E}_{ \pm}(z)=\left(\begin{array}{ccc}
1 & 0 & -i q_{o} / z  \tag{2.6}\\
i \mathbf{q}_{ \pm} / z & \mathbf{q}_{ \pm}^{\perp} / q_{o} & \mathbf{q}_{ \pm} / q_{o}
\end{array}\right)
$$

respectively, so that

$$
\begin{equation*}
\mathbf{X}_{ \pm} \mathbf{E}_{ \pm}=\mathbf{E}_{ \pm} i \boldsymbol{\Lambda} \tag{2.7}
\end{equation*}
$$

This normalization is a generalization of the one recently used in [11] for the scalar case. One could employ the invariances of the Manakov system to fix the asymptotic polarization vectors $\mathbf{q}_{ \pm} / q_{o}$ so as to obtain a simpler eigenfactor matrix. (The transformation of the Jost solutions and scattering matrix under each of the invariances of the Manakov system is discussed in Appendix A.1.) However, it will not be necessary to do so. It will be useful to note that

$$
\operatorname{det} \mathbf{E}_{ \pm}(z)=1-q_{o}^{2} / z^{2}:=\gamma(z), \quad \mathbf{E}_{ \pm}^{-1}(z)=\frac{1}{\gamma(z)}\left(\begin{array}{cc}
1 & i \mathbf{q}_{ \pm}^{\dagger} / z  \tag{2.8}\\
0 & \gamma(z)\left(\mathbf{q}_{ \pm}^{\perp}\right)^{\dagger} / q_{o} \\
-i q_{o} / z & \mathbf{q}_{ \pm}^{\dagger} / q_{o}
\end{array}\right)
$$

Let us now discuss the asymptotic time dependence. As $x \rightarrow \pm \infty$, we expect that the time evolution of the solutions of the Lax pair will be asymptotic to

$$
\begin{equation*}
\phi_{t}=\mathbf{T}_{ \pm} \phi \tag{2.9}
\end{equation*}
$$

where $\mathbf{T}_{ \pm}=2 i k^{2} \mathbf{J}+\mathbf{H}_{ \pm}$and $\mathbf{H}_{ \pm}=i \mathbf{J} \mathbf{Q}_{ \pm}^{2}-i q_{o}^{2} \mathbf{J}-2 k \mathbf{Q}_{ \pm}$. The eigenvalues of $\mathbf{T}_{ \pm}$are $-i\left(k^{2}+\lambda^{2}\right)$ and $\pm 2 i k \lambda$. Since the boundary conditions are constant, the consistency of the Lax pair (2.1a) implies $\left[\mathbf{X}_{ \pm}, \mathbf{T}_{ \pm}\right]=\mathbf{0}$, so $\mathbf{X}_{ \pm}$and $\mathbf{T}_{ \pm}$admit common eigenvectors. Namely,

$$
\begin{equation*}
\mathbf{T}_{ \pm} \mathbf{E}_{ \pm}=-i \mathbf{E}_{ \pm} \boldsymbol{\Omega} \tag{2.10}
\end{equation*}
$$

where $\boldsymbol{\Omega}(z)=\operatorname{diag}\left(-2 k \lambda, k^{2}+\lambda^{2}, 2 k \lambda\right)$. Then for all $z \in \mathbb{R}$, we can define the Jost solutions $\phi_{ \pm}(x, t, z)$ as the simultaneous solutions of both parts of the Lax pair satisfying the boundary conditions

$$
\begin{equation*}
\phi_{ \pm}(x, t, z)=\mathbf{E}_{ \pm}(z) \mathrm{e}^{i \boldsymbol{\Theta}(x, t, z)}+o(1), \quad x \rightarrow \pm \infty \tag{2.11}
\end{equation*}
$$

where $\Theta(x, t, z)$ is the $3 \times 3$ diagonal matrix

$$
\begin{equation*}
\boldsymbol{\Theta}(x, t, z)=\boldsymbol{\Lambda}(z) x-\boldsymbol{\Omega}(z) t=\operatorname{diag}\left(\theta_{1}(x, t, z), \theta_{2}(x, t, z),-\theta_{1}(x, t, z)\right) \tag{2.12}
\end{equation*}
$$

The advantage of introducing simultaneous solutions of both parts of the Lax pair is that the scattering coefficients will be independent of time. For comparison purposes, we note that the definition of the Jost solutions in this work differs from that in [31]. More precisely, the matrix $\phi_{ \pm}(x, t, z)$ defined by (2.11) equals the matrix Jost eigenfunction in [31] multiplied by $\operatorname{diag}\left(1 / z, i / q_{o}, i / q_{o}\right)$. A similar change applies to the Jost eigenfunctions of the adjoint problem.

To make the above definitions rigorous, we factorize the asymptotic behavior of the potential and rewrite the first part of the Lax pair (2.1a) as

$$
\begin{equation*}
\left(\phi_{ \pm}\right)_{x}=\mathbf{X}_{ \pm} \phi_{ \pm}+\Delta \mathbf{Q}_{ \pm} \phi_{ \pm} \tag{2.13}
\end{equation*}
$$

where $\Delta \mathbf{Q}_{ \pm}=\mathbf{Q}-\mathbf{Q}_{ \pm}$. We remove the asymptotic exponential oscillations and introduce modified eigenfunctions,

$$
\begin{equation*}
\mu_{ \pm}(x, t, z)=\phi_{ \pm}(x, t, z) \mathrm{e}^{-i \boldsymbol{\Theta}(x, t, z)} \tag{2.14}
\end{equation*}
$$

so that

$$
\begin{equation*}
\lim _{x \rightarrow \pm \infty} \mu_{ \pm}(x, t, z)=\mathbf{E}_{ \pm}(z) \tag{2.15}
\end{equation*}
$$

Introducing the integrating factor $\psi_{ \pm}(x, t, z)=\mathrm{e}^{-i \boldsymbol{\Theta}(x, t, z)} \mathbf{E}_{ \pm}^{-1}(z) \mu_{ \pm}(x, t, z) \mathrm{e}^{i \boldsymbol{\Theta}(x, t, z)}$, we can then formally integrate the ODE for $\mu_{ \pm}(x, t, z)$ to obtain
$\mu_{-}(x, t, z)=\mathbf{E}_{-}(z)+\int_{-\infty}^{x} \mathbf{E}_{-}(z) \mathrm{e}^{i(x-y) \boldsymbol{\Lambda}(z)} \mathbf{E}_{-}^{-1}(z) \Delta \mathbf{Q}_{-}(y, t) \mu_{-}(y, t, z) \mathrm{e}^{-i(x-y) \boldsymbol{\Lambda}(z)} \mathrm{d} y$,
$\mu_{+}(x, t, z)=\mathbf{E}_{+}(z)-\int_{x}^{\infty} \mathbf{E}_{+}(z) \mathrm{e}^{i(x-y) \boldsymbol{\Lambda}(z)} \mathbf{E}_{+}^{-1}(z) \Delta \mathbf{Q}_{+}(y, t) \mu_{+}(y, t, z) \mathrm{e}^{-i(x-y) \boldsymbol{\Lambda}(z)} \mathrm{d} y$.
One can now rigorously define the Jost eigenfunctions as the solutions of the integral equations (2.16). In fact, in Appendix A.2, we prove the following.

Theorem 2.1. If $\mathbf{Q}(\cdot, t)-\mathbf{Q}_{-} \in L^{1}(-\infty, a)$ or, correspondingly, $\mathbf{Q}(\cdot, t)-\mathbf{Q}_{+} \in$ $L^{1}(a, \infty)$ for any constant $a \in \mathbb{R}$, the following columns of $\mu_{-}(x, t, z)$ or, correspondingly, $\mu_{+}(x, t, z)$ can be analytically extended onto the corresponding regions of the complex z-plane:
$\mu_{-, 1}(x, t, z), \mu_{+, 3}(x, t, z): \operatorname{Im} z>0, \quad \mu_{-, 3}(x, t, z), \mu_{+, 1}(x, t, z): \operatorname{Im} z<0$.
Equation (2.14) implies that the same analyticity and boundedness properties also hold for the columns of $\phi_{ \pm}(x, t, z)$.

We now introduce the scattering matrix. If $\phi(x, t, z)$ solves (2.1a), we have $\partial_{x}(\operatorname{det} \phi)=\operatorname{tr} \mathbf{X} \operatorname{det} \phi$ and $\partial_{t}(\operatorname{det} \phi)=\operatorname{tr} \mathbf{T} \operatorname{det} \phi$. Since $\operatorname{tr} \mathbf{X}=i k$ and $\operatorname{tr} \mathbf{T}=-i\left(k^{2}+\right.$ $\lambda^{2}$ ), we have

$$
\frac{\partial}{\partial x} \operatorname{det}\left(\phi_{ \pm}(x, t, z) \mathrm{e}^{-i \boldsymbol{\Theta}(x, t, z)}\right)=\frac{\partial}{\partial t} \operatorname{det}\left(\phi_{ \pm}(x, t, z) \mathrm{e}^{-i \boldsymbol{\Theta}(x, t, z)}\right)=0 .
$$

Then (2.11) implies

$$
\begin{equation*}
\operatorname{det} \phi_{ \pm}(x, t, z)=\gamma(z) \mathrm{e}^{i \theta_{2}(x, t, z)}, \quad(x, t) \in \mathbb{R}^{2}, \quad z \in \mathbb{R} \backslash\left\{ \pm q_{o}\right\} . \tag{2.18}
\end{equation*}
$$

That is, $\phi_{-}$and $\phi_{+}$are two fundamental matrix solutions of the Lax pair, so there exists a $3 \times 3$ matrix $\mathbf{A}(z)$ such that

$$
\begin{equation*}
\phi_{-}(x, t, z)=\phi_{+}(x, t, z) \mathbf{A}(z), \quad z \in \mathbb{R} \backslash\left\{ \pm q_{o}\right\} . \tag{2.19}
\end{equation*}
$$

As usual, $\mathbf{A}(z)=\left(a_{i j}(z)\right)$ is referred to as the scattering matrix. Note that with our conventions, $\mathbf{A}(z)$ is independent of time. Moreover, (2.18) and (2.19) imply

$$
\begin{equation*}
\operatorname{det} \mathbf{A}(z)=1, \quad z \in \mathbb{R} \backslash\left\{ \pm q_{o}\right\} . \tag{2.20}
\end{equation*}
$$

It is also convenient to introduce $\mathbf{B}(z):=\mathbf{A}^{-1}(z)=\left(b_{i j}(z)\right)$. In the scalar case, the analyticity of the diagonal scattering coefficients follows from their representations as Wronskians of analytic eigenfunctions. This approach, however, is not applicable to the vector case [31]. Nonetheless, using an alternative integral representation for the eigenfunctions in Appendix A. 3 (which generalizes the ideas developed in [18] for the defocusing scalar case), a straightforward application of the Neumann series (as in Appendix A.3) yields the following.

Lemma 2.2. The analytic modified eigenfunctions $\mu_{ \pm, 1}(x, t, z)$ and $\mu_{ \pm, 3}(x, t, z)$ remain bounded for all $x \in \mathbb{R}$ and for all $z$ in their corresponding regions of analyticity.

This result will be important to the classification of the discrete spectrum (discussed in section 2.5), as it will allow one to characterize the appropriate domains for the discrete eigenvalues. Then, in Appendix A. 4 we prove the following.

Theorem 2.3. Under the same hypotheses as in Theorem 2.1, the following scattering coefficients can be analytically extended off of the real $z$-axis in the following regions:

$$
\begin{equation*}
a_{11}(z), b_{33}(z): \quad \operatorname{Im} z>0, \quad a_{33}(z), b_{11}(z): \quad \operatorname{Im} z<0 . \tag{2.21}
\end{equation*}
$$

Note how, in contrast to the ZBC case [3], nothing can be proved about the remaining entries of the scattering matrix. Note that, as in the scalar case, the scattering matrix at the branch points becomes singular. The behavior of eigenfunctions and scattering matrix at the branch points is discussed in section 2.5.

It is important to note that the results in Lemma 2.2 and Theorem 2.3 were not present in [31].
2.3. Adjoint problem and auxiliary eigenfunctions. A complete set of analytic eigenfunctions is needed to solve the inverse problem, but $\phi_{ \pm, 2}$ are nowhere analytic in general. To obviate this problem, as in [31], we consider the so-called adjoint Lax pair (using the terminology of [26]):

$$
\begin{equation*}
\tilde{\phi}_{x}=\tilde{\mathbf{X}} \tilde{\phi}, \quad \tilde{\phi}_{t}=\tilde{\mathbf{T}} \tilde{\phi}, \tag{2.22}
\end{equation*}
$$

where $\tilde{\mathbf{X}}=i k \mathbf{J}+\mathbf{Q}^{*}$ and $\tilde{\mathbf{T}}=-2 i k^{2} \mathbf{J}+i \mathbf{J Q}_{x}^{*}-i \mathbf{J}\left(\mathbf{Q}^{*}\right)^{2}+i q_{o}^{2} \mathbf{J}-2 k \mathbf{Q}^{*}$. Hereafter, tildes will denote that a quantity is defined for the adjoint problem (2.22) instead of the original one (2.1a). Note that $\tilde{\mathbf{X}}=\mathbf{X}^{*}$ and $\tilde{\mathbf{T}}=\mathbf{T}^{*}$ for all $z \in \mathbb{R}$.

Proposition 2.4. If $\tilde{\mathbf{v}}(x, t, z)$ and $\tilde{\mathbf{w}}(x, t, z)$ are two arbitrary solutions of the adjoint problem (2.22), then

$$
\begin{equation*}
\mathbf{u}(x, t, z)=\mathrm{e}^{i \theta_{2}(x, t, z)} \mathbf{J}[\tilde{\mathbf{v}} \times \tilde{\mathbf{w}}](x, t, z), \tag{2.23}
\end{equation*}
$$

where " $\times$ " denotes the usual cross product, is a solution of the Lax pair (2.1a).
The first half of Proposition 2.4 (corresponding to the scattering problem) was obtained in [31]. We use this result to construct two additional analytic eigenfunctions, one in each half plane. We do so by constructing Jost eigenfunctions for the adjoint problem. The eigenvalues of $\tilde{\mathbf{X}}_{ \pm}$are $-i k$ and $\pm i \lambda$. Denoting the corresponding eigenvalue matrix as $-i \boldsymbol{\Lambda}(z)=\operatorname{diag}(i \lambda,-i k,-i \lambda)$, we can choose the eigenvector matrix as $\tilde{\mathbf{E}}_{ \pm}(z)=\mathbf{E}_{ \pm}^{*}(z)$. Note that $\operatorname{det} \tilde{\mathbf{E}}_{ \pm}(z)=\gamma(z)$. As $x \rightarrow \pm \infty$, we expect that the solutions of the second half of (2.22) will be asymptotic to those of $\tilde{\phi}_{t}=\tilde{\mathbf{T}}_{ \pm} \tilde{\phi}$. The eigenvalues of $\tilde{\mathbf{T}}_{ \pm}$are $i\left(k^{2}+\lambda^{2}\right)$ and $\pm 2 i k \lambda$, and (2.10) imply $\tilde{\mathbf{T}}_{ \pm} \tilde{\mathbf{E}}_{ \pm}=\tilde{\mathbf{E}}_{ \pm} i \boldsymbol{\Omega}$. As before, for all $z \in \mathbb{R}$, we then define the Jost solutions of the adjoint problem as the simultaneous solutions $\tilde{\phi}_{ \pm}$of (2.22) such that

$$
\begin{equation*}
\tilde{\phi}_{ \pm}(x, t, z)=\tilde{\mathbf{E}}_{ \pm}(z) \mathrm{e}^{-i \boldsymbol{\Theta}(x, t, z)}+o(1), \quad x \rightarrow \pm \infty \tag{2.24}
\end{equation*}
$$

Introducing modified eigenfunctions $\tilde{\mu}_{ \pm}(x, t, z)=\tilde{\phi}_{ \pm}(x, t, z) \mathrm{e}^{i \boldsymbol{\Theta}(x, t, z)}$ as before, we find that the following columns of $\tilde{\mu}_{ \pm}(x, t, z)$ can be extended into the complex plane:

$$
\tilde{\mu}_{-, 3}(x, t, z), \tilde{\mu}_{+, 1}(x, t, z): \operatorname{Im} z>0, \quad \tilde{\mu}_{-, 1}(x, t, z), \tilde{\mu}_{+, 3}(x, t, z): \operatorname{Im} z<0
$$

But the columns $\tilde{\mu}_{ \pm, 2}$ cannot be extended in general. As before, $\tilde{\phi}_{ \pm}$are both fundamental matrix solutions of the same problem, and therefore there exists an invertible $3 \times 3$ matrix $\tilde{\mathbf{A}}(z)$ such that

$$
\begin{equation*}
\tilde{\phi}_{-}(x, t, z)=\tilde{\phi}_{+}(x, t, z) \tilde{\mathbf{A}}(z) \tag{2.25}
\end{equation*}
$$

The same techniques used for the original scattering matrix show that for suitable potentials, the following coefficients can be analytically extended into the following regions:

$$
\begin{equation*}
\tilde{a}_{11}(z), \tilde{b}_{33}(z): \quad \operatorname{Im} z<0, \quad \tilde{a}_{33}(z), \tilde{b}_{11}(z): \quad \operatorname{Im} z>0 \tag{2.26}
\end{equation*}
$$

where $\tilde{\mathbf{B}}(z)=\tilde{\mathbf{A}}^{-1}(z)$. In light of these results, we can define two new solutions of the original Lax pair (2.1a):

$$
\begin{align*}
\chi(x, t, z) & =-\mathrm{e}^{i \theta_{2}(x, t, z)} \mathbf{J}\left[\tilde{\phi}_{-, 3} \times \tilde{\phi}_{+, 1}\right](x, t, z) / \gamma(z)  \tag{2.27a}\\
\bar{\chi}(x, t, z) & =-\mathrm{e}^{i \theta_{2}(x, t, z)} \mathbf{J}\left[\tilde{\phi}_{-, 1} \times \tilde{\phi}_{+, 3}\right](x, t, z) / \gamma(z) \tag{2.27b}
\end{align*}
$$

By construction, we have the following.
Lemma 2.5. Under the same hypotheses as in Theorem 2.1, $\chi(x, t, z)$ is analytic for $\operatorname{Im} z>0$, while $\bar{\chi}(x, t, z)$ is analytic for $\operatorname{Im} z<0$.

For comparison purposes, note that the auxiliary eigenfunctions defined in [31] equal the ones defined in (2.27) times $-i q_{o} z \gamma(z)$.

Following [31], we now establish a relation between the adjoint Jost eigenfunctions and the eigenfunctions of the original Lax pair (2.1a) (see proofs in Appendix A.5).

Lemma 2.6. For $z \in \mathbb{R}$ and for all cyclic indices $j$, $\ell$, and $m$,

$$
\begin{align*}
& \phi_{ \pm, j}(x, t, z)=-\mathrm{e}^{i \theta_{2}(x, t, z)} \mathbf{J}\left[\tilde{\phi}_{ \pm, \ell} \times \tilde{\phi}_{ \pm, m}\right](x, t, z) / \gamma_{j}(z),  \tag{2.28a}\\
& \tilde{\phi}_{ \pm, j}(x, t, z)=-\mathrm{e}^{-i \theta_{2}(x, t, z)} \mathbf{J}\left[\phi_{ \pm, \ell} \times \phi_{ \pm, m}\right](x, t, z) / \gamma_{j}(z), \tag{2.28b}
\end{align*}
$$

where

$$
\begin{equation*}
\gamma_{1}(z)=-1, \quad \gamma_{2}(z)=\gamma(z), \quad \gamma_{3}(z)=1 . \tag{2.28c}
\end{equation*}
$$

Corollary 2.7. The scattering matrices $\mathbf{A}(z)$ and $\tilde{\mathbf{A}}(z)$ are related by

$$
\begin{equation*}
\tilde{\mathbf{A}}(z)=\boldsymbol{\Gamma}(z)\left(\mathbf{A}^{-1}(z)\right)^{T} \boldsymbol{\Gamma}^{-1}(z), \tag{2.29}
\end{equation*}
$$

where $\boldsymbol{\Gamma}(z)=\operatorname{diag}(-1, \gamma(z), 1)$.
Corollary 2.8. For all $z \in \mathbb{R}$, the nonanalytic Jost eigenfunctions have the following decompositions:

$$
\begin{equation*}
\phi_{-, 2}(x, t, z)=\frac{1}{a_{33}(z)}\left[a_{32}(z) \phi_{-, 3}(z)-\bar{\chi}(z)\right]=\frac{1}{a_{11}(z)}\left[a_{12}(z) \phi_{-, 1}(z)+\chi(z)\right], \tag{2.30a}
\end{equation*}
$$

$$
\begin{equation*}
\phi_{+, 2}(x, t, z)=\frac{1}{b_{11}(z)}\left[b_{12}(z) \phi_{+, 1}(z)-\bar{\chi}(z)\right]=\frac{1}{b_{33}(z)}\left[b_{32}(z) \phi_{+, 3}(z)+\chi(z)\right], \tag{2.30b}
\end{equation*}
$$

where the $(x, t)$-dependence was omitted from the right-hand side for simplicity.
The use of the adjoint eigenfunctions will be instrumental in obtaining many of the results in the following sections.

In addition, similarly to the Jost eigenfunctions it will be useful to remove the exponential oscillations of $\bar{\chi}$ and $\chi$ and define the modified auxiliary eigenfunctions as

$$
\begin{equation*}
\bar{m}(x, t, z)=\bar{\chi}(x, t, z) \mathrm{e}^{-i \theta_{2}(x, t, z)}, \quad m(x, t, z)=\chi(x, t, z) \mathrm{e}^{-i \theta_{2}(x, t, z)} . \tag{2.31}
\end{equation*}
$$

Then, using Lemma 2.2 and (2.27), it is straightforward to characterize the asymptotic behavior of the auxiliary eigenfunctions as $x \rightarrow \pm \infty$.

Lemma 2.9. As $x \rightarrow \pm \infty$, the modified auxiliary eigenfunctions remain bounded in their corresponding domains of analyticity.

These results will be key to the full characterization of the discrete spectrum (cf. section 2.5), as we will see that the eigenfunctions behave differently for large $|x|$ depending on whether a given point $z \in \mathbb{C}$ is inside or outside of $C_{0}$.
2.4. Symmetries. For the NLS equation and the Manakov system with ZBC, the only symmetry of the scattering problem is the mapping $k \mapsto k^{*}$. For the same equations with NZBC, however, the symmetries are complicated by the presence of a Riemann surface with the need to keep track of each sheet. Correspondingly, the problem admits two symmetries. The symmetries are also complicated by the fact that, after removing the asymptotic oscillations, the Jost solutions do not tend to the identity matrix. Recall that $\lambda_{\mathrm{II}}(k)=-\lambda_{\mathrm{I}}(k), z=k+\lambda, q_{o}^{2} / z=k-\lambda, \lambda=\left(z-q_{o}^{2} / z\right) / 2$, and $k=\left(z+q_{o}^{2} / z\right) / 2$.
2.4.1. First symmetry. Consider the transformation $z \mapsto z^{*}$ (upper/lower half plane), which implies $(k, \lambda) \mapsto\left(k^{*}, \lambda^{*}\right)$.

Proposition 2.10. If $\phi(x, t, z)$ is a fundamental matrix solution of the Lax pair (2.1a), so is $\mathbf{w}(x, t, z)=\mathbf{J}\left(\phi^{\dagger}\left(x, t, z^{*}\right)\right)^{-1}$.

Proposition 2.10 is proved in Appendix A.6. There, we also show that, as a consequence, we have the following.

Lemma 2.11. For all $z \in \mathbb{R}$, the Jost eigenfunctions satisfy the symmetry

$$
\begin{equation*}
\mathbf{J}\left(\phi_{ \pm}^{\dagger}(x, t, z)\right)^{-1} \mathbf{C}(z)=\phi_{ \pm}(x, t, z) \tag{2.32}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{C}(z)=-\gamma(z) \boldsymbol{\Gamma}^{-1}(z)=\operatorname{diag}(-\gamma(z), 1, \gamma(z)) \tag{2.33}
\end{equation*}
$$

It will also be convenient to note also that

$$
\left(\phi_{ \pm}^{-1}(x, t, z)\right)^{T}=\frac{1}{\operatorname{det} \phi_{ \pm}(x, t, z)}\left(\phi_{ \pm, 2} \times \phi_{ \pm, 3}, \phi_{ \pm, 3} \times \phi_{ \pm, 1}, \phi_{ \pm, 1} \times \phi_{ \pm, 2}\right)(x, t, z)
$$

Then substituting (2.30) into (2.32) and using the Schwarz reflection principle yields the following.

Lemma 2.12. The analytic Jost eigenfunctions obey the following symmetry relations:

$$
\begin{align*}
\phi_{-, 1}^{*}\left(x, t, z^{*}\right)=-\frac{1}{a_{33}(z)} \mathbf{J}\left[\bar{\chi} \times \phi_{-, 3}\right](x, t, z) \mathrm{e}^{-i \theta_{2}(x, t, z)}, \quad \operatorname{Im} z \leq 0  \tag{2.34a}\\
\phi_{+, 1}^{*}\left(x, t, z^{*}\right)=\frac{1}{b_{33}(z)} \mathbf{J}\left[\chi \times \phi_{+, 3}\right](x, t, z) \mathrm{e}^{-i \theta_{2}(x, t, z)}, \quad \operatorname{Im} z \geq 0  \tag{2.34b}\\
\phi_{-, 3}^{*}\left(x, t, z^{*}\right)=\frac{1}{a_{11}(z)} \mathbf{J}\left[\chi \times \phi_{-, 1}\right](x, t, z) \mathrm{e}^{-i \theta_{2}(x, t, z)}, \quad \operatorname{Im} z \geq 0  \tag{2.34c}\\
\phi_{+, 3}^{*}\left(x, t, z^{*}\right)=-\frac{1}{b_{11}(z)} \mathbf{J}\left[\bar{\chi} \times \phi_{+, 1}\right](x, t, z) \mathrm{e}^{-i \theta_{2}(x, t, z)}, \quad \operatorname{Im} z \leq 0 \tag{2.34d}
\end{align*}
$$

Moreover, using (2.32) in the scattering relation (2.19), we conclude as follows.
Lemma 2.13. The scattering matrix and its inverse satisfy the symmetry relation

$$
\begin{equation*}
(\mathbf{A}(z))^{\dagger}=\boldsymbol{\Gamma}^{-1}(z) \mathbf{B}(z) \boldsymbol{\Gamma}(z) . \quad z \in \mathbb{R} \tag{2.35}
\end{equation*}
$$

Componentwise, for $z \in \mathbb{R}$, (2.35) yields

$$
\begin{align*}
b_{11}(z) & =a_{11}^{*}(z), \quad b_{12}(z)=-\frac{1}{\gamma(z)} a_{21}^{*}(z), \quad b_{13}(z)=-a_{31}^{*}(z)  \tag{2.36a}\\
b_{21}(z) & =-\gamma(z) a_{12}^{*}(z), \quad b_{22}(z)=a_{22}^{*}(z), \quad b_{23}(z)=\gamma(z) a_{32}^{*}(z)  \tag{2.36b}\\
b_{31}(z) & =-a_{13}^{*}(z), \quad b_{32}(z)=\frac{1}{\gamma(z)} a_{23}^{*}(z), \quad b_{33}(z)=a_{33}^{*}(z) \tag{2.36c}
\end{align*}
$$

The Schwarz reflection principle then allows us to conclude that

$$
\begin{equation*}
b_{11}(z)=a_{11}^{*}\left(z^{*}\right), \quad \operatorname{Im} z \leq 0, \quad b_{33}(z)=a_{33}^{*}\left(z^{*}\right), \quad \operatorname{Im} z \geq 0 \tag{2.37}
\end{equation*}
$$

We can also obtain similar symmetry relations for the auxiliary eigenfunctions.
COROLLARY 2.14. The auxiliary analytic eigenfunctions satisfy the following symmetry relations:

$$
\begin{array}{ll}
\bar{\chi}(x, t, z)=-\mathrm{e}^{i \theta_{2}(x, t, z)} \mathbf{J}\left[\phi_{-, 1}^{*} \times \phi_{+, 3}^{*}\right]\left(x, t, z^{*}\right) / \gamma(z), & \operatorname{Im} z<0 \\
\chi(x, t, z)=-\mathrm{e}^{i \theta_{2}(x, t, z)} \mathbf{J}\left[\phi_{-, 3}^{*} \times \phi_{+, 1}^{*}\right]\left(x, t, z^{*}\right) / \gamma(z), & \operatorname{Im} z>0 \tag{2.38b}
\end{array}
$$

In addition, the proof of Corollary 2.14 and (2.28) yield

$$
\begin{equation*}
\phi_{ \pm, j}^{*}(x, t, z)=-\mathrm{e}^{-i \theta_{2}(x, t, z)} \mathbf{J}\left[\phi_{ \pm, \ell} \times \phi_{ \pm, m}\right](x, t, z) / \gamma_{j}(z) \tag{2.39}
\end{equation*}
$$

where $j, \ell$, and $m$ are cyclic indices and $z \in \mathbb{R}$.
2.4.2. Second symmetry. Consider the transformation $z \mapsto q_{o}^{2} / z$ (outside/ inside the circle of radius $q_{o}$ centered at 0 ), implying $(k, \lambda) \mapsto(k,-\lambda)$. We use this symmetry to relate the values of the eigenfunctions on the two sheets (particularly, across the cuts), where $k$ is arbitrary but fixed (on either sheet). It is easy to show the following.

Proposition 2.15. If $\phi(x, t, z)$ is a solution of the Lax pair, so is

$$
\mathbf{W}(x, t, z)=\phi\left(x, t, q_{o}^{2} / z\right)
$$

In Appendix A. 6 we then show that, as a consequence, we have the following.
Lemma 2.16. The Jost eigenfunctions satisfy the following symmetry relations:

$$
\begin{equation*}
\phi_{ \pm}(x, t, z)=\phi_{ \pm}\left(x, t, q_{o}^{2} / z\right) \Pi(z), \quad z \in \mathbb{R} \tag{2.40}
\end{equation*}
$$

where

$$
\boldsymbol{\Pi}(z)=\left(\begin{array}{ccc}
0 & 0 & -i q_{o} / z  \tag{2.41}\\
0 & 1 & 0 \\
i q_{o} / z & 0 & 0
\end{array}\right)
$$

As before, the analyticity properties of the eigenfunctions then allow us to extend some of the above relations:

$$
\begin{gather*}
\phi_{ \pm, 3}(x, t, z)=-\frac{i q_{o}}{z} \phi_{ \pm, 1}\left(x, t, q_{o}^{2} / z\right), \quad \operatorname{Im} z \gtrless 0  \tag{2.42a}\\
\phi_{ \pm, 2}(x, t, z)=\phi_{ \pm, 2}\left(x, t, q_{o}^{2} / z\right), \quad z \in \mathbb{R} \tag{2.42b}
\end{gather*}
$$

We again use (2.19) to obtain the following lemma.
Lemma 2.17. The scattering matrix satisfies the symmetry relation

$$
\begin{equation*}
\mathbf{A}\left(q_{o}^{2} / z\right)=\boldsymbol{\Pi}(z) \mathbf{A}(z) \boldsymbol{\Pi}^{-1}(z), \quad z \in \mathbb{R} \tag{2.43}
\end{equation*}
$$

Componentwise, we have

$$
\begin{gather*}
a_{11}(z)=a_{33}\left(q_{o}^{2} / z\right), \quad a_{12}(z)=-\frac{i z}{q_{o}} a_{32}\left(q_{o}^{2} / z\right), \quad a_{13}(z)=-a_{31}\left(q_{o}^{2} / z\right)  \tag{2.44a}\\
a_{21}(z)=\frac{i q_{o}}{z} a_{23}\left(q_{o}^{2} / z\right), \quad a_{22}(z)=a_{22}\left(q_{o}^{2} / z\right), \quad a_{23}(z)=-\frac{i q_{o}}{z} a_{21}\left(q_{o}^{2} / z\right),  \tag{2.44b}\\
a_{31}(z)=-a_{13}\left(q_{o}^{2} / z\right), \quad a_{32}(z)=\frac{i z}{q_{o}} a_{12}\left(q_{o}^{2} / z\right), \quad a_{33}(z)=a_{11}\left(q_{o}^{2} / z\right) \tag{2.44c}
\end{gather*}
$$

An identical set of equations holds for the elements of $\mathbf{B}(z)$. The analyticity properties of the scattering matrix entries then allow us to conclude that

$$
\begin{equation*}
a_{11}(z)=a_{33}\left(q_{o}^{2} / z\right), \quad b_{33}(z)=b_{11}\left(q_{o}^{2} / z\right), \quad \operatorname{Im} z \geq 0 \tag{2.45}
\end{equation*}
$$

Finally, we combine (2.42) with (2.38) to conclude the following.
Lemma 2.18. The auxiliary eigenfunctions satisfy the symmetry relation

$$
\begin{equation*}
\chi(x, t, z)=-\bar{\chi}\left(x, t, q_{o}^{2} / z\right), \quad \operatorname{Im} z \geq 0 \tag{2.46}
\end{equation*}
$$

2.4.3. Combined symmetry and reflection coefficients. Of course, one can combine the above two symmetries to obtain relations between eigenfunctions and scattering coefficients evaluated at $z$ and at $q_{o}^{2} / z^{*}$. We omit these relations for brevity.

The following reflection coefficients will appear in the inverse problem:

$$
\begin{equation*}
\rho_{1}(z)=\frac{b_{13}(z)}{b_{11}(z)}=-\frac{a_{31}^{*}(z)}{a_{11}^{*}(z)}, \quad \rho_{2}(z)=\frac{a_{21}(z)}{a_{11}(z)}=-\gamma(z) \frac{b_{12}^{*}(z)}{b_{11}^{*}(z)} \tag{2.47}
\end{equation*}
$$

Using the symmetries of the scattering coefficients, we can also express the reflection coefficients as

$$
\begin{equation*}
\rho_{1}\left(q_{o}^{2} / z\right)=-\frac{b_{31}(z)}{b_{33}(z)}=\frac{a_{13}^{*}(z)}{a_{33}^{*}(z)}, \quad \rho_{2}\left(q_{o}^{2} / z\right)=\frac{i z}{q_{o}} \frac{a_{23}(z)}{a_{33}(z)}=\gamma(z) \frac{i z}{q_{o}} \frac{b_{32}^{*}(z)}{b_{33}^{*}(z)} . \tag{2.48}
\end{equation*}
$$

On the other hand, unlike the scalar case [42, 11, 18], the symmetries of the scattering coefficients do not result in any symmetry relations among these reflection coefficients. Once the trace formulae for the analytic scattering coefficients are obtained in section 3.3, we will show that one can combine all of the above symmetries to reconstruct the entire scattering matrix.
2.5. Discrete spectrum. Recall that in the $2 \times 2$ scattering problem for the NLS equation with NZBC, there is a one-to-one correspondence between zeros of the analytic scattering coefficients and discrete eigenvalues, each of which corresponds to the presence of a bound state. Moreover, the self-adjointness of the scattering problem implies that such discrete eigenvalues $k$ must be real, and one can show that no discrete eigenvalues can arise inside the continuous spectrum. Thus, in the $z$-plane, the discrete eigenvalues are confined to the circle $C_{o}$. The scattering problem in (2.1a) for the Manakov system is also self-adjoint, and a similar constraint as for the scalar NLS equation exists for the proper eigenvalues of the scattering problem.

Lemma 2.19 (see [31]). Let $\mathbf{v}(x, t, z)$ be a nontrivial solution of the scattering problem in (2.1a). If $\mathbf{v}(x, t, z) \in L^{2}(\mathbb{R})$, then $z \in C_{o}$.

Nonetheless, it was shown in [31] that in order to fully characterize the inverse problem, one needs to also consider zeros of the analytic scattering coefficients off the circle $C_{o}$. This does not contradict Lemma 2.19 since, as discussed below, the zeros of the analytic scattering coefficients off $C_{o}$ do not lead to bound states. More precisely, we will see that zeros of $a_{11}(z)$ inside $C_{o}$ are allowed, and that these zeros lead to eigenfunctions that do not decay at both space infinities.

In light of the analyticity properties of the eigenfunctions, to characterize the discrete spectrum it is convenient to introduce the following $3 \times 3$ matrices:

$$
\begin{gather*}
\mathbf{\Phi}^{+}(x, t, z)=\left(\phi_{-, 1}(x, t, z), \chi(x, t, z), \phi_{+, 3}(x, t, z)\right)  \tag{2.49a}\\
\mathbf{\Phi}^{-}(x, t, z)=\left(\phi_{+, 1}(x, t, z),-\bar{\chi}(x, t, z), \phi_{-, 3}(x, t, z)\right) \tag{2.49b}
\end{gather*}
$$

which are analytic for $\operatorname{Im} z>0$ and $\operatorname{Im} z<0$, respectively. Using the decompositions (2.30) we obtain

$$
\begin{array}{ll}
\operatorname{det} \boldsymbol{\Phi}^{+}(x, t, z)=a_{11}(z) b_{33}(z) \gamma(z) \mathrm{e}^{i \theta_{2}(x, t, z)}, & \operatorname{Im} z \geq 0 \\
\operatorname{det} \boldsymbol{\Phi}^{-}(x, t, z)=a_{33}(z) b_{11}(z) \gamma(z) \mathrm{e}^{i \theta_{2}(x, t, z)}, & \operatorname{Im} z \leq 0 \tag{2.50b}
\end{array}
$$

(As customary, (2.50) are first obtained along the real $z$-axis, where the decompositions (2.30) hold, and then extended to the respective domains of analyticity by a continuation principle.) Thus, the columns of $\boldsymbol{\Phi}^{+}(x, t, z)$ become linearly dependent
at the zeros of $a_{11}(z)$ and $b_{33}(z)$ in the upper half plane, and those of $\boldsymbol{\Phi}^{-}(x, t, z)$ at the zeros of $a_{33}(z)$ and $b_{11}(z)$ in the lower half plane. Even though such zeros do not lead to bound states when $z \notin C_{o}$, they nonetheless must be included as part of the discrete spectrum in the inverse problem.

LEMMA 2.20 (see [31]). Suppose $a_{11}(z)$ has a zero $z_{n}$ in the upper half $z$-plane. Then

$$
\begin{equation*}
a_{11}\left(z_{n}\right)=0 \Leftrightarrow b_{11}\left(z_{n}^{*}\right)=0 \Leftrightarrow a_{33}\left(q_{o}^{2} / z_{n}\right)=0 \Leftrightarrow b_{33}\left(q_{o}^{2} / z_{n}^{*}\right)=0 . \tag{2.51}
\end{equation*}
$$

Lemma 2.20 implies that discrete eigenvalues $\zeta_{n}$ lying on the circle $C_{o}$ appear in complex conjugate pairs $\left\{\zeta_{n}, \zeta_{n}^{*}\right\}$, whereas discrete eigenvalues $z_{n}$ off $C_{o}$ appear in symmetric quartets

$$
\left\{z_{n}, z_{n}^{*}, q_{o}^{2} / z_{n}, q_{o}^{2} / z_{n}^{*}\right\} .
$$

The following lemmas are also instrumental in the characterization of the discrete spectrum.

Lemma 2.21. If $\operatorname{Im} z_{o}>0$ and $z_{o} \notin C_{o}$, then $\chi\left(x, t, z_{o}\right) \neq \mathbf{0}$.
Lemma 2.22. Suppose $\operatorname{Im} z_{o}>0$. Then the following statements are equivalent:
(i) $\chi\left(x, t, z_{o}\right)=\mathbf{0}$.
(ii) $\bar{\chi}\left(x, t, q_{o}^{2} / z_{o}\right)=\mathbf{0}$.
(iii) $\chi\left(x, t, q_{o}^{2} / z_{o}^{*}\right)=\mathbf{0}$.
(iv) $\bar{\chi}\left(x, t, z_{o}^{*}\right)=\mathbf{0}$.
(v) There exists a constant $b_{o}$ such that $\phi_{-, 3}\left(x, t, z_{o}^{*}\right)=b_{o} \phi_{+, 1}\left(x, t, z_{o}^{*}\right)$.
(vi) There exists a constant $\tilde{b}_{o}$ such that $\phi_{-, 1}\left(x, t, q_{o}^{2} / z_{o}^{*}\right)=\tilde{b}_{o} \phi_{+, 3}\left(x, t, q_{o}^{2} / z_{o}^{*}\right)$.
(vii) There exists a constant $\hat{b}_{o}$ such that $\phi_{-, 1}\left(x, t, z_{o}\right)=\hat{b}_{o} \phi_{+, 3}\left(x, t, z_{o}\right)$.
(viii) There exists a constant $\check{b}_{o}$ such that $\phi_{-, 3}\left(x, t, q_{o}^{2} / z_{o}\right)=\check{b}_{o} \phi_{+, 1}\left(x, t, q_{o}^{2} / z_{o}\right)$.

We are are now finally ready to characterize the behavior of the eigenfunctions in correspondence of the discrete spectrum. The two theorems that follow are proved in Appendix A. 7 without assuming that the off-diagonal scattering coefficients can be extended off the real $z$-axis (as was done in [31] instead).

THEOREM 2.23. Let $\zeta_{n}$ be a zero of $a_{11}(z)$ in the upper half plane with $\left|\zeta_{n}\right|=q_{o}$. Then $\chi\left(x, t, \zeta_{n}\right)=\bar{\chi}\left(x, t, \zeta_{n}^{*}\right)=\mathbf{0}$. As a result, there exist constants $c_{n}$ and $\bar{c}_{n}$ such that

$$
\begin{equation*}
\phi_{-, 1}\left(x, t, \zeta_{n}\right)=c_{n} \phi_{+, 3}\left(x, t, \zeta_{n}\right), \quad \phi_{-, 3}\left(x, t, \zeta_{n}^{*}\right)=\bar{c}_{n} \phi_{+, 1}\left(x, t, \zeta_{n}^{*}\right) \tag{2.52}
\end{equation*}
$$

THEOREM 2.24. Let $z_{n}$ be a zero of $a_{11}(z)$ in the upper half plane with $\left|z_{n}\right| \neq q_{o}$. Then $\left|z_{n}\right|<q_{o}$ and $b_{33}\left(z_{n}\right) \neq 0$. Moreover, there exist constants $d_{n}, \check{d}_{n}, \hat{d}_{n}$, and $\bar{d}_{n}$ such that
$\phi_{-, 1}\left(x, t, z_{n}\right)=d_{n} \chi\left(x, t, z_{n}\right) / b_{33}\left(z_{n}\right), \quad \phi_{-, 3}\left(x, t, q_{o}^{2} / z_{n}\right)=\check{d}_{n} \bar{\chi}\left(x, t, q_{o}^{2} / z_{n}\right)$,

$$
\begin{equation*}
\chi\left(x, t, q_{o}^{2} / z_{n}^{*}\right)=\hat{d}_{n} \phi_{+, 3}\left(x, t, q_{o}^{2} / z_{n}^{*}\right), \quad \bar{\chi}\left(x, t, z_{n}^{*}\right)=\bar{d}_{n} \phi_{+, 1}\left(x, t, z_{n}^{*}\right) \tag{2.53b}
\end{equation*}
$$

We should remark on the importance of these results. Recall from Lemma 2.19 that the only points in the discrete spectrum corresponding to bound states arise for real values of $k$, corresponding to $z \in C_{o}$. Indeed, the results of Theorem 2.23 imply that each zero of $a_{11}(z)$ on $C_{o}$ does indeed correspond to a bound state. On the other hand, it is Lemma 2.19 that leads to the constraint $\left|z_{n}\right|<q_{o}$ in Theorem 2.24. This is because, if $\left|z_{n}\right|>q_{o}$, the first of (2.53a), combined with the asymptotics in

Lemma 2.2, implies that $\phi_{-, 1}\left(x, t, z_{n}\right)$ vanishes as $x \rightarrow \pm \infty$. This is, of course, a bound state, which would contradict Lemma 2.19. Conversely, if $\left|z_{n}\right|<q_{o}$, the eigenfunction $\phi_{-, 1}\left(x, t, z_{n}\right)$ grows exponentially as $x \rightarrow \infty$, which does not contradict Lemma 2.19 (since the eigenfunction is not in $L^{2}(\mathbb{R})$ ), and this case is therefore allowed. Indeed, as we will see in section 3.6, this case leads to dark-bright soliton solutions of the Manakov system.

Lemma 2.25. Assume that $a_{11}(z)$ has simple zeros $\left\{\zeta_{n}\right\}_{n=1}^{N_{1}}$ on $C_{o}$. Then the norming constants in (2.52) obey the following symmetry relations:

$$
\begin{equation*}
\bar{c}_{n}=-c_{n}, \quad c_{n}^{*}=\frac{b_{11}^{\prime}\left(\zeta_{n}^{*}\right)}{a_{33}^{\prime}\left(\zeta_{n}^{*}\right)} \bar{c}_{n}, \quad n=1, \ldots, N_{1} \tag{2.54}
\end{equation*}
$$

Lemma 2.26. Assume that $a_{11}(z)$ has zeros $\left\{z_{n}\right\}_{n=1}^{N_{2}}$ off the circle $C_{o}$. (Note that now it is not necessary to assume that such zeros are simple.) Then the norming constants in (2.53) obey the following symmetry relations for $n=1, \ldots, N_{2}$ :

$$
\begin{equation*}
\check{d}_{n}=\frac{i z_{n}}{q_{o}} \frac{d_{n}}{b_{33}\left(z_{n}\right)}, \quad \bar{d}_{n}=-\frac{d_{n}^{*}}{\gamma\left(z_{n}^{*}\right)}, \quad \hat{d}_{n}=\frac{i q_{o}}{z_{n}^{*}} \frac{d_{n}^{*}}{\gamma\left(z_{n}^{*}\right)} . \tag{2.55}
\end{equation*}
$$

2.6. Asymptotic behavior as $z \rightarrow \infty$ and $\boldsymbol{z} \rightarrow \mathbf{0}$. To normalize the Riemann-Hilbert problem (RHP), it will be necessary to examine the asymptotic behavior both as $z \rightarrow \infty$ and as $z \rightarrow 0$. Consider the following formal expansion for $\mu_{+}(x, t, z)$ :

$$
\begin{equation*}
\mu_{+}(x, t, z)=\sum_{n=0}^{\infty} \mu_{n}(x, t, z) \tag{2.56a}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu_{n+1}(x, t, z)=-\int_{x}^{\infty} \mathbf{E}_{+}(z) \mathrm{e}^{i(x-y) \boldsymbol{\Lambda}(z)} \mathbf{E}_{+}^{-1}(z) \Delta \mathbf{Q}_{+}(y, t) \mu_{n}(y, t, z) \mathrm{e}^{-i(x-y) \boldsymbol{\Lambda}(z)} \mathrm{d} y \tag{2.56c}
\end{equation*}
$$

Recall that subscripts " $b d$ " and "bo" denote, respectively, the block diagonal and block off-diagonal parts of a given matrix. Using (2.56a), in Appendix A. 8 we prove the following.

Lemma 2.27. For all $m \geq 0$, (2.56a) provides an asymptotic expansion for the columns of $\mu_{+}(x, t, z)$ as $z \rightarrow \infty$ in the appropriate region of the complex $z$-plane, with

$$
\begin{array}{cc}
{\left[\mu_{2 m}\right]_{b d}=O\left(1 / z^{m}\right),} & {\left[\mu_{2 m}\right]_{b o}=O\left(1 / z^{m+1}\right)} \\
{\left[\mu_{2 m+1}\right]_{b d}=O\left(1 / z^{m+1}\right),} & {\left[\mu_{2 m+1}\right]_{b o}=O\left(1 / z^{m+1}\right)} \tag{2.57b}
\end{array}
$$

Lemma 2.28. For all $m \geq 0$, (2.56a) provides an asymptotic expansion for the columns of $\mu_{+}(x, t, z)$ as $z \rightarrow 0$ in the appropriate region of the complex $z$-plane, with

$$
\begin{array}{cc}
{\left[\mu_{2 m}\right]_{b d}=O\left(z^{m}\right),} & {\left[\mu_{2 m}\right]_{b o}=O\left(z^{m-1}\right)} \\
{\left[\mu_{2 m+1}\right]_{b d}=O\left(z^{m}\right),} & {\left[\mu_{2 m+1}\right]_{b o}=O\left(z^{m}\right)} \tag{2.58b}
\end{array}
$$

Then, evaluating explicitly the first few terms in (2.56a), we obtain the following.

Corollary 2.29. As $z \rightarrow \infty$ in the appropriate regions of the complex plane,

$$
\begin{aligned}
& \mu_{ \pm, 1}(x, t, z)=\binom{1}{(i / z) \mathbf{q}(x, t)}+O\left(1 / z^{2}\right) \\
& \mu_{ \pm, 3}(x, t, z)=\binom{-i \mathbf{q}^{\dagger}(x, t) \mathbf{q}_{ \pm} /\left(q_{o} z\right)}{\mathbf{q}_{ \pm} / q_{o}}+O\left(1 / z^{2}\right)
\end{aligned}
$$

Similarly, as $z \rightarrow 0$ in the appropriate regions of the complex plane,

$$
\begin{align*}
& \mu_{ \pm, 1}(x, t, z)=\binom{\mathbf{q}^{\dagger}(x, t) \mathbf{q}_{ \pm} / q_{o}^{2}}{(i / z) \mathbf{q}_{ \pm}}+O(z)  \tag{2.60}\\
& \mu_{ \pm, 3}(x, t, z)=\binom{-i q_{o} / z}{\mathbf{q}(x, t) / q_{o}}+O(z)
\end{align*}
$$

We now compute the asymptotic behavior of the auxiliary eigenfunctions $\chi(x, t, z)$ and $\bar{\chi}(x, t, z)$. We recall the definition of the modified auxiliary eigenfunctions (2.31) and combine the above asymptotics with (2.38) to obtain the following.

Lemma 2.30. As $z \rightarrow \infty$ in the appropriate regions of the complex plane,

$$
\begin{aligned}
& m(x, t, z)=\binom{-i \mathbf{q}^{\dagger}(x, t) \mathbf{q}_{-}^{\perp} /\left(q_{o} z\right)}{\mathbf{q}_{-}^{\perp} / q_{o}}+O\left(1 / z^{2}\right) \\
& \bar{m}(x, t, z)=\binom{i \mathbf{q}^{\dagger}(x, t) \mathbf{q}_{+}^{\perp} /\left(q_{o} z\right)}{-\mathbf{q}_{+}^{\perp} / q_{o}}+O\left(1 / z^{2}\right) .
\end{aligned}
$$

Similarly, as $z \rightarrow 0$ in the appropriate regions of the complex plane,

$$
m(x, t, z)=\binom{0}{\mathbf{q}_{+}^{\perp} / q_{o}}+O(z), \quad \bar{m}(x, t, z)=\binom{0}{-\mathbf{q}_{-}^{\perp} / q_{o}}+O(z)
$$

Next, we find the asymptotic behavior of the scattering matrix entries.
Corollary 2.31. As $z \rightarrow \infty$ in the appropriate regions of the complex plane,

$$
\begin{align*}
& a_{11}(z)=1+O(1 / z), \quad b_{33}(z)=\frac{1}{q_{o}^{2}} \mathbf{q}_{-}^{\dagger} \mathbf{q}_{+}+O(1 / z),  \tag{2.61a}\\
& a_{33}(z)=\frac{1}{q_{o}^{2}} \mathbf{q}_{+}^{\dagger} \mathbf{q}_{-}+O(1 / z), \quad b_{11}(z)=1+O(1 / z) \tag{2.61~b}
\end{align*}
$$

Similarly, as $z \rightarrow 0$ in the appropriate regions of the complex plane,

$$
\begin{align*}
& a_{11}(z)=\frac{1}{q_{o}^{2}} \mathbf{q}_{+}^{\dagger} \mathbf{q}_{-}+O(z), \quad b_{33}(z)=1+O(z)  \tag{2.62a}\\
& a_{33}(z)=1+O(z), \quad b_{11}(z)=\frac{1}{q_{o}^{2}} \mathbf{q}_{-}^{\dagger} \mathbf{q}_{+}+O(z) \tag{2.62b}
\end{align*}
$$

Finally, we find the asymptotic behavior of the off-diagonal scattering matrix entries.

Corollary 2.32. As $z \rightarrow \infty$ on the real $z$-axis,

$$
\left[\mathbf{A}^{ \pm 1}(z)\right]_{o}=\frac{1}{q_{o}^{2}}\left(\begin{array}{ccc}
0 & 0 & 0  \tag{2.63}\\
0 & 0 & -\left(\mathbf{q}_{\mp}^{\frac{1}{\mp}}\right)^{\dagger} \mathbf{q}_{ \pm} \\
0 & \mathbf{q}_{ \pm}^{\dagger} \mathbf{q}_{\mp}^{\perp} & 0
\end{array}\right)+O(1 / z)
$$

$$
\begin{equation*}
a_{22}(z)=\frac{1}{q_{o}^{2}} \mathbf{q}_{-}^{\dagger} \mathbf{q}_{+}+O(1 / z), \quad b_{22}(z)=\frac{1}{q_{o}^{2}} \mathbf{q}_{+}^{\dagger} \mathbf{q}_{-}+O(1 / z) \tag{2.64}
\end{equation*}
$$

Similarly, as $z \rightarrow 0$ on the real $z$-axis,

$$
\begin{gather*}
{\left[\mathbf{A}^{ \pm 1}(z)\right]_{o}=\frac{i q_{o}}{z}\left(\begin{array}{ccc}
0 & 0 & 0 \\
\left(\mathbf{q}_{\mp}^{\perp}\right)^{\dagger} \mathbf{q}_{ \pm} & 0 & 0 \\
0 & 0 & 0
\end{array}\right)+O(1),}  \tag{2.65}\\
a_{22}(z)=\frac{1}{q_{o}^{2}} \mathbf{q}_{-}^{\dagger} \mathbf{q}_{+}+O(z), \quad b_{22}(z)=\frac{1}{q_{o}^{2}} \mathbf{q}_{+}^{\dagger} \mathbf{q}_{-}+O(z) . \tag{2.66}
\end{gather*}
$$

Note that, unlike what happens in the scalar case and in the case with ZBC, not all off-diagonal entries of the scattering matrix vanish as $z \rightarrow \infty$. As we will see, however, this does not complicate the inverse problem since all the reflection coefficients will still vanish as $z \rightarrow \infty$.
2.7. Behavior at the branch points. We now discuss the behavior of the Jost eigenfunctions and the scattering matrix at the branch points $k= \pm q_{o}$. The complication there is due to the fact that $\lambda\left( \pm q_{o}\right)=0$, and therefore, at $z= \pm q_{o}$, the two exponentials $\mathrm{e}^{ \pm i \lambda x}$ reduce to the identity. Correspondingly, at $z= \pm q_{o}$, the matrices $\mathbf{E}_{ \pm}(z)$ are degenerate. Nonetheless, the term $\mathbf{E}_{ \pm}(z) \mathrm{e}^{i(x-y) \boldsymbol{\Lambda}(z)} \mathbf{E}_{ \pm}^{-1}(z)$ appearing in the integral equations for the Jost eigenfunctions remains finite as $z \rightarrow$ $\pm q_{o}$ :

$$
\lim _{z \rightarrow \pm q_{o}} \mathbf{E}_{ \pm}(z) \mathrm{e}^{i \xi \boldsymbol{\Lambda}(z)} \mathbf{E}_{ \pm}^{-1}(z)=\left(\begin{array}{cc}
1 \mp i q_{o} \xi & \xi \mathbf{q}_{ \pm}^{\dagger}  \tag{2.67}\\
\xi \mathbf{q}_{ \pm} & \frac{1}{q_{o}^{2}} \mathbf{U}_{ \pm}(\xi)
\end{array}\right)
$$

where $\xi=x-y$ and $\mathbf{U}_{ \pm}(\xi)=\left(1 \pm i q_{o} \xi\right) \mathbf{q}_{ \pm} \mathbf{q}_{ \pm}^{\dagger}+\mathrm{e}^{ \pm i q_{o} \xi} \mathbf{q}_{ \pm}^{\perp}\left(\mathbf{q}_{ \pm}^{\perp}\right)^{\dagger}$. Thus, if $\mathbf{q} \rightarrow \mathbf{q}_{ \pm}$ sufficiently fast as $x \rightarrow \pm \infty$, the integrals in (2.16) are also convergent at $z= \pm q_{o}$, and the Jost solutions admit a well-defined limit at the branch points. Nonetheless, $\operatorname{det} \phi_{ \pm}\left(x, t, \pm q_{o}\right)=0$ for all $(x, t) \in \mathbb{R}^{2}$. Thus, the columns of $\phi_{ \pm}\left(x, t, q_{o}\right)$ (as well as those of $\left.\phi_{ \pm}\left(x, t,-q_{o}\right)\right)$ are linearly dependent. Comparing the asymptotic behavior of the columns of $\phi_{ \pm}\left(x, t, \pm q_{o}\right)$ as $x \rightarrow \pm \infty$, we obtain

$$
\begin{equation*}
\phi_{ \pm, 1}\left(x, t, q_{o}\right)=i \phi_{ \pm, 3}\left(x, t, q_{o}\right), \quad \phi_{ \pm, 1}\left(x, t,-q_{o}\right)=-i \phi_{ \pm, 3}\left(x, t,-q_{o}\right) \tag{2.68}
\end{equation*}
$$

Next, we characterize the limiting behavior of the scattering matrix near the branch points. It is easy to express all entries of the scattering matrix $\mathbf{A}(z)$ as Wronskians:

$$
\begin{equation*}
a_{j \ell}(z)=\frac{z^{2}}{z^{2}-q_{o}^{2}} W_{j \ell}(x, t, z) \mathrm{e}^{-i \theta_{2}(x, t, z)} \tag{2.69a}
\end{equation*}
$$

where

$$
\begin{equation*}
W_{j \ell}(x, t, z)=\operatorname{det}\left(\phi_{-, \ell}(x, t, z), \phi_{+, j+1}(x, t, z), \phi_{+, j+2}(x, t, z)\right), \tag{2.69b}
\end{equation*}
$$

and $j+1$ and $j+2$ are calculated modulo 3 . We then have the following Laurent series expansions about $z= \pm q_{o}$ :

$$
\begin{equation*}
a_{i j}(z)=\frac{a_{i j, \pm}}{z \mp q_{o}}+a_{i j, \pm}^{(o)}+O\left(z \mp q_{o}\right), \quad z \in \mathbb{R} \backslash\left\{ \pm q_{o}\right\}, \tag{2.70}
\end{equation*}
$$

where, for example,

$$
\begin{equation*}
a_{11, \pm}= \pm \frac{q_{o}}{2} W_{11}\left(x, t, \pm q_{o}\right) \mathrm{e}^{\mp i q_{o}\left(x \mp q_{o} t\right)} \tag{2.71a}
\end{equation*}
$$

$$
\begin{equation*}
a_{11, \pm}^{(o)}= \pm\left.\frac{q_{o}}{2} \frac{\mathrm{~d}}{\mathrm{~d} z} W_{11}(x, t, z)\right|_{z= \pm q_{o}} \mathrm{e}^{\mp i q_{o}\left(x \mp q_{o} t\right)}+W_{11}\left(x, t, \pm q_{o}\right) \mathrm{e}^{\mp i q_{o}\left(x \mp q_{o} t\right)} \tag{2.71b}
\end{equation*}
$$

Summarizing, the asymptotic expansion of $\mathbf{A}(z)$ in a neighborhood of the branch point is

$$
\begin{equation*}
\mathbf{A}(z)=\frac{1}{z \mp q_{o}} \mathbf{A}_{ \pm}+\mathbf{A}_{ \pm}^{(o)}+O\left(z \mp q_{o}\right) \tag{2.72}
\end{equation*}
$$

where $\mathbf{A}_{ \pm}^{(o)}=\left(a_{i j, \pm}^{(o)}\right)$,

$$
\mathbf{A}_{ \pm}=a_{11, \pm}\left(\begin{array}{ccc}
1 & 0 & \mp i  \tag{2.73}\\
0 & 0 & 0 \\
\mp i & 0 & -1
\end{array}\right)+a_{12, \pm}\left(\begin{array}{ccc}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & \mp i & 0
\end{array}\right)
$$

and $a_{12, \pm}= \pm\left(q_{o} / 2\right) W_{12}\left(x, t, \pm q_{o}\right) \mathrm{e}^{\mp i q_{o}\left(x \mp q_{o} t\right)}$. Note that the second row of $\mathbf{A}_{ \pm}$is identically zero because $a_{2 j, \pm}= \pm\left(q_{o} / 2\right) W_{2 j}\left(x, t, \pm q_{o}\right) \mathrm{e}^{\mp i q_{o}\left(x \mp q_{o} t\right)}$, which is zero by virtue of (2.68). Finally, it is straightforward to see from (2.73) and the symmetry (2.35) that

$$
\begin{equation*}
\lim _{z \rightarrow \pm q_{o}} \rho_{1}(z)=\mp i, \quad \lim _{z \rightarrow \pm q_{o}} \rho_{2}(z)=0 \tag{2.74}
\end{equation*}
$$

where the reflection coefficients $\rho_{1}(z)$ and $\rho_{2}(z)$ are as defined in (2.47).
3. Inverse problem. As usual, the inverse scattering problem is formulated in terms of an appropriate RHP. To this end, one needs a suitable jump condition that relates eigenfunctions that are meromorphic in the upper half $z$-plane to eigenfunctions that are meromorphic in the lower half $z$-plane. For simplicity, in the development of the inverse problem in this section and the next one we will restrict ourselves to the class of potentials such that $\mathbf{q}_{+}$is parallel to $\mathbf{q}_{-}$.
3.1. Riemann-Hilbert problem. The starting point for the formulation of the inverse problem is the scattering relation (2.19), which will lead to a jump condition for the RHP. The derivation, however, is considerably more involved than in the scalar case. The reason is that some of the Jost eigenfunctions are not analytic, and therefore (2.19) must be reformulated in terms of the fundamental analytic eigenfunctions $\boldsymbol{\Phi}^{ \pm}(x, t, z)$ defined in (2.49). Proceeding in this way, in Appendix A. 9 we prove the following lemma.

Lemma 3.1. The meromorphic matrices $\mathbf{M}^{ \pm}(x, t, z)=\left(m_{1}^{ \pm}, m_{2}^{ \pm}, m_{3}^{ \pm}\right)$, defined as

$$
\begin{equation*}
\mathbf{M}^{+}(x, t, z)=\boldsymbol{\Phi}^{+} \mathrm{e}^{-i \boldsymbol{\Theta}} \operatorname{diag}\left(\frac{1}{a_{11}}, \frac{1}{b_{33}}, 1\right)=\left(\frac{\mu_{-, 1}}{a_{11}}, \frac{m}{b_{33}}, \mu_{+, 3}\right), \quad \operatorname{Im} z>0 \tag{3.1a}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{M}^{-}(x, t, z)=\boldsymbol{\Phi}^{-} \mathrm{e}^{-i \boldsymbol{\Theta}} \operatorname{diag}\left(1, \frac{1}{b_{11}}, \frac{1}{a_{33}}\right)=\left(\mu_{+, 1},-\frac{\bar{m}}{b_{11}}, \frac{\mu_{-, 3}}{a_{33}}\right), \quad \operatorname{Im} z<0 \tag{3.1b}
\end{equation*}
$$

satisfy the jump condition

$$
\begin{equation*}
\mathbf{M}^{+}(x, t, z)=\mathbf{M}^{-}(x, t, z)\left(\mathbf{I}-\mathrm{e}^{-i \mathbf{K} \boldsymbol{\Theta}(x, t, z)} \mathbf{L}(z) \mathrm{e}^{i \mathbf{K} \boldsymbol{\Theta}(x, t, z)}\right), \quad z \in \mathbb{R} \tag{3.2}
\end{equation*}
$$

where $\mathbf{K}=\operatorname{diag}(-1,1,-1)$ as before and

$$
\mathbf{L}(z)=\left(\begin{array}{ccc}
\frac{\left|\rho_{2}\right|^{2}}{\gamma}-\rho_{1}^{*}\left[\hat{\rho}_{1}^{*}+\frac{i q_{o}}{z \gamma} \rho_{2}^{*} \hat{\rho}_{2}\right] & \frac{\rho_{2}^{*}}{\gamma}+\frac{q_{o}^{2}}{z^{2} \gamma^{2}} \rho_{2}^{*}\left|\hat{\rho}_{2}\right|^{2}-\frac{i q_{o}}{z \gamma} \hat{\rho}_{1}^{*} \hat{\rho}_{2}^{*} & \frac{i q_{o}}{z \gamma} \rho_{2}^{*} \hat{\rho}_{2}+\hat{\rho}_{1}^{*} \\
-\rho_{2}+\frac{i q_{o}}{z} \rho_{1}^{*} \hat{\rho}_{2} & -\frac{q_{o}^{2}}{z^{2} \gamma}\left|\hat{\rho}_{2}\right|^{2} & -\frac{i q_{o}}{z} \hat{\rho}_{2} \\
\rho_{1}^{*} & \frac{i q_{o}}{z \gamma} \hat{\rho}_{2}^{*} & 0
\end{array}\right)
$$

where for brevity we denoted $\rho_{j}=\rho_{j}(z)$ and $\hat{\rho}_{j}=\rho_{j}\left(q_{o}^{2} / z\right)$ for $j=1,2$.
Note the appearance of the matrix $\mathbf{K}$ in the jump condition, which can be traced to the use of (2.30) to eliminate the nonanalytic eigenfunctions. In order for the above RHP to admit a unique solution, one must also specify a suitable normalization condition. In this case, this condition is provided by the leading-order asymptotic behavior of $\mathbf{M}^{ \pm}$as $z \rightarrow \infty$ and the pole contribution at 0 to help regularize the RHP (3.2). Using the information from section 2.6 , we have the following.

Lemma 3.2. The matrices $\mathbf{M}^{ \pm}(x, t, z)$ defined in (3.1) have the following asymptotic behavior:

$$
\begin{array}{lr}
\mathbf{M}^{ \pm}(x, t, z)=\mathbf{M}_{\infty}+O(1 / z), & z \rightarrow \infty, \\
\mathbf{M}^{ \pm}(x, t, z)=(i / z) \mathbf{M}_{0}+O(1), & z \rightarrow 0,  \tag{3.3b}\\
\operatorname{Im} z \gtrless 0,
\end{array}
$$

where

$$
\mathbf{M}_{\infty}=\left(\begin{array}{ccc}
1 & 0 & 0  \tag{3.3c}\\
\mathbf{0} & \mathbf{q}_{+}^{\perp} / q_{o} & \mathbf{q}_{+} / q_{o}
\end{array}\right), \quad \mathbf{M}_{0}=\left(\begin{array}{ccc}
0 & 0 & -q_{o} \\
\mathbf{q}_{+} & \mathbf{0} & \mathbf{0}
\end{array}\right)
$$

Note that both behaviors are expressed in terms of the value of the potential as $x \rightarrow \infty$ (instead of that as $x \rightarrow-\infty$ ). This is because (2.19) breaks the symmetry between $\mu_{-}$and $\mu_{+}$.

In addition to the asymptotics in Lemma 3.2, to fully specify the RHP (3.2) one must also specify residue conditions. This is done using the characterization of the discrete spectrum obtained in section 2.5. For the remainder of this section, we assume that the zeros $\left\{\zeta_{n}\right\}_{n=1}^{N_{1}}$ and $\left\{z_{n}\right\}_{n=1}^{N_{2}}$ of $a_{11}(z)$ of the analytic scattering coefficients are all simple. Then in Appendix A. 9 we prove the following.

Lemma 3.3. The meromorphic matrices defined in Lemma 3.1 satisfy the following residue conditions:
$\left[\operatorname{Res}_{z=\zeta_{n}} \mathbf{M}^{+}\right](x, t)=C_{n}\left(m_{3}^{+}\left(\zeta_{n}\right), \mathbf{0}, \mathbf{0}\right),\left[\operatorname{Res}_{z=\zeta_{n}^{*}} \mathbf{M}^{-}\right](x, t)=\bar{C}_{n}\left(\mathbf{0}, \mathbf{0}, m_{1}^{-}\left(\zeta_{n}^{*}\right)\right)$,
$\left[\operatorname{Res}_{z=z_{n}} \mathbf{M}^{+}\right](x, t)=D_{n}\left(m_{2}^{+}\left(z_{n}\right), \mathbf{0}, \mathbf{0}\right),\left[\operatorname{Res}_{z=q_{o}^{2} / z_{n}} \mathbf{M}^{-}\right](x, t)=-\check{D}_{n}\left(\mathbf{0}, \mathbf{0}, m_{2}^{-}\left(q_{o}^{2} / z_{n}\right)\right)$,
$\left[\operatorname{Res}_{z=z_{n}^{*}} \mathbf{M}^{-}\right](x, t)=-\bar{D}_{n}\left(\mathbf{0}, m_{1}^{-}\left(z_{n}^{*}\right), \mathbf{0}\right),\left[\operatorname{Res}_{z=q_{o}^{2} / z_{n}^{*}} \mathbf{M}^{+}\right](x, t)=\hat{D}_{n}\left(\mathbf{0}, m_{3}^{+}\left(q_{o}^{2} / z_{n}^{*}\right), \mathbf{0}\right)$,
with

$$
\begin{gathered}
C_{n}(x, t)=\frac{c_{n}}{a_{11}^{\prime}\left(\zeta_{n}\right)} \mathrm{e}^{-2 i \theta_{1}\left(\zeta_{n}\right)}, \quad \bar{C}_{n}(x, t)=\frac{\bar{c}_{n}}{a_{33}^{\prime}\left(\zeta_{n}^{*}\right)} \mathrm{e}^{-2 i \theta_{1}\left(\zeta_{n}\right)}, \\
D_{n}(x, t)=\frac{d_{n}}{a_{11}^{\prime}\left(z_{n}\right)} \mathrm{e}^{-i\left(\theta_{1}-\theta_{2}\right)\left(z_{n}\right)}, \quad \check{D}_{n}(x, t)=\frac{\tilde{d}_{n} b_{33}\left(z_{n}\right)}{a_{33}^{\prime}\left(q_{o}^{2} / z_{n}\right)} \mathrm{e}^{-i\left(\theta_{1}-\theta_{2}\right)\left(z_{n}\right)}, \\
\hat{D}_{n}(x, t)=\frac{\hat{d}_{n}}{b_{33}^{\prime}\left(q_{o}^{2} / z_{n}^{*}\right)} \mathrm{e}^{i\left(\theta_{1}-\theta_{2}\right)\left(z_{n}^{*}\right)}, \quad \bar{D}_{n}(x, t)=\frac{\bar{d}_{n}}{b_{11}^{\prime}\left(z_{n}^{*}\right)} \mathrm{e}^{i\left(\theta_{1}-\theta_{2}\right)\left(z_{n}^{*}\right)},
\end{gathered}
$$

$n=1, \ldots, N_{1}$ for equations involving $\zeta_{n}$, and $n=1, \ldots, N_{2}$ for equations involving $z_{n}$.

As usual, the formulation of the RHP involves "continuous spectral data" (namely, the reflection coefficients, which determine the matrix $\mathbf{L}(z)$ for all $z \in \mathbb{R}$ ) "discrete spectral data" (namely, the discrete eigenvalues and norming constants, which determine the residue conditions), and the normalization as $z \rightarrow \infty$ (the normalization as $z \rightarrow 0$ is then determined via the symmetries). Moreover, the norming constants appearing in Lemma 3.3 are related by the following equations.

LEMMA 3.4 (symmetries of the residues). The functions $C_{n}, \bar{C}_{n}, D_{n}, \check{D}_{n}, \hat{D}_{n}$, and $\bar{D}_{n}$ defined in Theorem 3.7 obey the following symmetry relations:

$$
\begin{equation*}
\check{D}_{n}(x, t)=-\frac{i q_{o}}{z_{n}} D_{n}(x, t), \quad \bar{D}_{n}(x, t)=-\frac{D_{n}^{*}(x, t)}{\gamma\left(z_{n}^{*}\right)}, \quad \hat{D}_{n}(x, t)=-\frac{i q_{o}^{3}}{\left(z_{n}^{*}\right)^{3}} \frac{D_{n}^{*}(x, t)}{\gamma\left(z_{n}^{*}\right)} . \tag{3.5b}
\end{equation*}
$$

Correspondingly, the minimal set of spectral data is composed of the continuous reflection coefficients $\rho_{1}(z)$ and $\rho_{2}(z)$, the boundary condition $\mathbf{q}_{+}$, the discrete eigenvalues $\left\{\zeta_{n}\right\}_{n=1}^{N_{1}}$ and $\left\{z_{n}\right\}_{n=1}^{N_{2}}$, and the norming constants $\left\{C_{n}(x, t)\right\}_{n=1}^{N_{1}}$ and $\left\{D_{n}(x, t)\right\}_{n=1}^{N_{2}}$. Moreover, note that (3.5a) in Lemma 3.4 immediately implies the following.

Corollary 3.5. The norming constants $C_{n}$ for the discrete eigenvalues on the circle satisfy the constraint $\arg \left(C_{n}\right)=\arg \left(\zeta_{n}\right)$ for $n=1, \ldots, N_{1}$.

This result is not surprising, since it is the same as in the scalar case [19]. On the other hand, no such constraint exists for the eigenvalues off the circle. As we will see later, this difference will translate into the number of degrees of freedom of the corresponding soliton solutions generated by each eigenvalue (cf. section 3.5).

Remark 3.6. Summarizing, the RHP for the inverse problem is formulated as follows. Given
(i) the boundary condition $\mathbf{q}_{+}$;
(ii) the reflection coefficients $\rho_{1}(z)$ and $\rho_{2}(z)$ for $z \in\left(-\infty,-q_{o}\right) \cup\left(q_{o}, \infty\right)$;
(iii) the discrete eigenvalues $\left\{\zeta_{n}\right\}_{n=1}^{N_{1}}$ and $\left\{z_{n}\right\}_{n=1}^{N_{2}}$ on and off the circle, respectively, and the corresponding norming constants $\left\{C_{n}(x, t)\right\}_{n=1}^{N_{1}}$ and $\left\{D_{n}(x, t)\right\}_{n=1}^{N_{2}}$;
(iv) the symmetries (2.48) and (3.5),
find a sectionally meromorphic function $\mathbf{M}(x, t, z)$ satisfying the jump condition (3.2), the normalization conditions (3.3), and the residue conditions (3.4).

Recall that the symmetries (2.48) yield the reflection coefficients for $z \in\left(-q_{o}, q_{o}\right)$. Also, $\lim _{z \rightarrow \infty} \rho_{j}(z)=0$ for $j=1,2$ (cf. (2.63)), while $\lim _{z \rightarrow \pm q_{o}} \rho_{1}(z)=\mp i$ and $\lim _{z \rightarrow \pm q_{\circ}} \rho_{2}(z)=0($ cf. (2.74)).
3.2. Formal solution of the RHP and reconstruction formula. The RHP defined in the previous section consists of finding a sectionally meromorphic matrix $\mathbf{M}(x, t, z)$ which equals $\mathbf{M}^{ \pm}(x, t, z)$ for $\operatorname{Im} z \gtrless 0$ and satisfies the jump condition (3.2) as well as the asymptotics and residue conditions in Lemmas 3.2 and 3.3. The solution of this RHP can be expressed in terms of a mixed system of algebraic-integral equations, which are obtained by subtracting the asymptotic behavior at infinity, by regularizing (i.e., subtracting any pole contributions from the discrete spectrum) and then applying Cauchy projectors. Specifically, in Appendix A.9, we prove the following.

Theorem 3.7. The solution of the RHP defined by Lemmas 3.1, 3.2, and 3.3 is given by

$$
\begin{align*}
& \mathbf{M}(x, t, z)=\mathbf{M}_{\infty}+(i / z) \mathbf{M}_{0}- \frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{\mathbf{M}^{-}(\zeta)}{\zeta-z} \mathrm{e}^{-i \mathbf{K} \boldsymbol{\Theta}(\zeta)} \mathbf{L}(\zeta) \mathrm{e}^{i \mathbf{K} \boldsymbol{\Theta}(\zeta)} \mathrm{d} \zeta  \tag{3.6}\\
&+\sum_{i=1}^{N_{1}}\left(\frac{\operatorname{Res}_{z=\zeta_{i}} \mathbf{M}^{+}}{z-\zeta_{i}}+\frac{\operatorname{Res}_{z=\zeta_{i}^{*}} \mathbf{M}^{-}}{z-\zeta_{i}^{*}}\right)+\sum_{j=1}^{N_{2}}\left(\frac{\operatorname{Res}_{z=z_{j}^{*}} \mathbf{M}^{-}}{z-z_{j}^{*}}+\frac{\operatorname{Res}_{z=z_{j}} \mathbf{M}^{+}}{z-z_{j}}\right) \\
&+ \sum_{j=1}^{N_{2}}\left(\frac{\operatorname{Res}_{z=q_{o}^{2} / z_{j}} \mathbf{M}^{-}}{z-q_{o}^{2} / z_{j}}+\frac{\operatorname{Res}_{z=q_{o}^{2} / z_{j}^{*}} \mathbf{M}^{+}}{z-q_{o}^{2} / z_{j}^{*}}\right),
\end{align*}
$$

where $\mathbf{M}(x, t, z)=\mathbf{M}^{ \pm}(x, t, z)$ for $\operatorname{Im} z \gtrless 0$, respectively.
Moreover, the eigenfunctions in the residue conditions in Lemma 3.3 are given by

$$
\begin{align*}
& m_{1}^{-}(x, t, w)=\binom{1}{(i / w) \mathbf{q}_{+}}-\frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{\left(\mathbf{M}^{-} \mathrm{e}^{-i \mathbf{K} \boldsymbol{\Theta}} \mathbf{L} \mathrm{e}^{i \mathbf{K} \boldsymbol{\Theta}}\right)_{1}(\zeta)}{\zeta-w} \mathrm{~d} \zeta  \tag{3.7a}\\
& +\sum_{i=1}^{N_{1}}\left[\frac{C_{i}}{w-\zeta_{i}} m_{3}^{+}\left(\zeta_{i}\right)\right]+\sum_{j=1}^{N_{2}}\left[\frac{D_{j}}{w-z_{j}} m_{2}^{+}\left(z_{j}\right)\right], \quad w=\zeta_{n}^{*}, z_{n}^{*}, \\
& m_{3}^{+}(x, t, w)=\binom{-i q_{o} / w}{\mathbf{q}_{+} / q_{o}}-\frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{\left(\mathbf{M}^{-} \mathrm{e}^{-i \mathbf{K} \boldsymbol{\Theta}} \mathbf{L e} \mathrm{e}^{i \mathbf{K} \boldsymbol{\Theta}}\right)_{3}(\zeta)}{\zeta-w} \mathrm{~d} \zeta  \tag{3.7b}\\
& +\sum_{i=1}^{N_{1}}\left[\frac{\bar{C}_{i}}{w-\zeta_{i}^{*}} m_{1}^{-}\left(\zeta_{i}^{*}\right)\right]-\sum_{j=1}^{N_{2}}\left[\frac{\check{D}_{j}}{w-q_{o}^{2} / z_{j}} m_{2}^{-}\left(q_{o}^{2} / z_{j}\right)\right], \quad w=\zeta_{n}, q_{o}^{2} / z_{n}^{*}, \\
& m_{2}^{-}\left(x, t, q_{o}^{2} / z_{j^{\prime}}\right)=\binom{0}{\mathbf{q}_{+}^{\frac{1}{2}} / q_{o}}-\frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{\left(\mathbf{M}^{-} \mathrm{e}^{-i \mathbf{K} \Theta} \mathbf{L e} \mathrm{e}^{i \mathbf{K} \boldsymbol{\Theta}}\right)_{2}(\zeta)}{\zeta-q_{o}^{2} / z_{j^{\prime}}} \mathrm{d} \zeta  \tag{3.7c}\\
& -\sum_{j=1}^{N_{2}}\left[\frac{\bar{D}_{j}}{q_{o}^{2} / z_{j^{\prime}}-z_{j}^{*}} m_{1}^{-}\left(z_{j}^{*}\right)\right]+\sum_{j=1}^{N_{2}}\left[\frac{\hat{D}_{j}}{q_{o}^{2} / z_{j^{\prime}}-q_{o}^{2} / z_{j}^{*}} m_{3}^{+}\left(q_{o}^{2} / z_{j}^{*}\right)\right], \\
& m_{2}^{+}\left(x, t, z_{j^{\prime}}\right)=\binom{0}{\mathbf{q}_{+}^{+} / q_{o}}-\frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{\left(\mathbf{M}^{-} \mathrm{e}^{-i \mathbf{K} \boldsymbol{\Theta}} \mathbf{L} \mathrm{e}^{i \mathbf{K} \boldsymbol{} \boldsymbol{\Theta}}\right)_{2}(\zeta)}{\zeta-z_{j^{\prime}}} \mathrm{d} \zeta  \tag{3.7d}\\
& -\sum_{j=1}^{N_{2}}\left[\frac{\bar{D}_{j}}{z_{j^{\prime}}-z_{j}^{*}} m_{1}^{-}\left(z_{j}^{*}\right)\right]+\sum_{j=1}^{N_{2}}\left[\frac{\hat{D}_{j}}{z_{j^{\prime}}-q_{o}^{2} / z_{j}^{*}} m_{3}^{+}\left(q_{o}^{2} / z_{j}^{*}\right)\right] .
\end{align*}
$$

Throughout, the $(x, t)$-dependence was omitted from the right-hand side of all equations for simplicity.

As usual, once the solution of the RHP has been obtained, one can reconstruct the potential in terms of the norming constants and scattering coefficients by comparing the resulting asymptotics of the eigenfunctions to that obtained from the direct scattering problem. In this way, in Appendix A. 9 we prove the following.

Theorem 3.8 (reconstruction formula). Let $\mathbf{M}(x, t, z)$ be the solution of the RHP in Theorem 3.7. The corresponding solution $\mathbf{q}(x, t)=\left(q_{1}(x, t), q_{2}(x, t)\right)^{T}$ of the defocusing Manakov system with NZBC (1.2) is reconstructed as

$$
\begin{align*}
q_{k}(x, t)=q_{+, k} & -\frac{1}{2 \pi} \int_{\mathbb{R}}\left(\mathbf{M}^{-} \mathrm{e}^{-i \mathbf{K} \Theta} \mathbf{L} \mathrm{e}^{i \mathbf{K} \boldsymbol{\Theta}}\right)_{(k+1) 1}(x, t, \zeta) \mathrm{d} \zeta  \tag{3.8}\\
& -i \sum_{i=1}^{N_{1}} C_{i}(x, t) m_{(k+1) 3}^{+}\left(x, t, \zeta_{i}\right)-i \sum_{j=1}^{N_{2}} D_{j}(x, t) m_{(k+1) 2}^{+}\left(x, t, z_{j}\right) .
\end{align*}
$$

3.3. Trace formula and asymptotic phase difference. The last task of the inverse problem is that of reconstructing the analytic scattering coefficients from the scattering data (i.e., the discrete eigenvalues and the reflection coefficients). In Appendix A.9, we prove the following.

Lemma 3.9 (trace formula). The analytic scattering coefficient $a_{11}(z)$ defined in (2.19) is given by

$$
\begin{align*}
a_{11}(z)=\prod_{n=1}^{N_{1}} & \frac{z-\zeta_{n}}{z-\zeta_{n}^{*}} \prod_{n=1}^{N_{2}} \frac{z-z_{n}}{z-z_{n}^{*}}  \tag{3.9}\\
& \quad \times \exp \left[-\frac{1}{2 \pi i} \int_{\mathbb{R}} \log \left(1-\left|\rho_{1}(\zeta)\right|^{2}-\frac{\zeta^{2}}{\zeta^{2}-q_{o}^{2}}\left|\rho_{2}(\zeta)\right|^{2}\right) \frac{\mathrm{d} \zeta}{\zeta-z}\right] .
\end{align*}
$$

Expressions for the other analytic coefficients follow immediately from the symmetries (2.35) and (2.43). Explicitly, $b_{11}(z)=b_{33}\left(q_{o}^{2} / z\right)=a_{11}^{*}\left(z^{*}\right)=a_{33}^{*}\left(q_{o}^{2} / z^{*}\right)$. Comparing (3.9) with the asymptotic behavior of $a_{11}(z)$ as $z \rightarrow 0$ in (2.62a) yields the following.

Corollary 3.10. The asymptotic phase difference $\Delta \theta=\theta_{+}-\theta_{-}$between the limiting values of the potential is given by the expression

$$
\begin{equation*}
\Delta \theta=-2 \sum_{n=1}^{N_{1}} \arg \zeta_{n}-2 \sum_{n=1}^{N_{2}} \arg z_{n}-\frac{1}{2 \pi} \int_{\mathbb{R}} \log \left(1-\left|\rho_{1}(\zeta)\right|^{2}-\frac{\zeta^{2}}{\zeta^{2}-q_{o}^{2}}\left|\rho_{2}(\zeta)\right|^{2}\right) \frac{\mathrm{d} \zeta}{\zeta} . \tag{3.10}
\end{equation*}
$$

Equation (3.10) is the generalization of the so-called theta condition that was obtained in [19] for the scalar case (i.e., for the NLS equation). Note, however, that (3.10) does not imply that there exists an additional constraint on the spectral data. Rather, (3.10) simply means that the asymptotic phase shift is determined uniquely by the spectral data as part of the inverse problem, and therefore one cannot prescribe it independently.

Finally, we note that one can reconstruct the entire scattering matrix in terms of the trace formulae and reflection coefficients. Explicitly, combining (2.47) and (2.48) with the definition $\mathbf{B}(z)=\mathbf{A}^{-1}(z)$ yields the following for $z \in \mathbb{R}$ :

$$
\begin{gather*}
a_{12}(z)=a_{11}^{*}(z) a_{11}^{*}\left(q_{o}^{2} / z\right)\left[\left(i q_{o} / z\right) \rho_{1}(z) \rho_{2}^{*}\left(q_{o}^{2} / z\right)+\rho_{2}^{*}(z)\right] / \gamma(z),  \tag{3.11a}\\
a_{22}(z)=a_{11}^{*}(z) a_{11}^{*}\left(q_{o}^{2} / z\right)\left[1+\rho_{1}(z) \rho_{1}\left(q_{o}^{2} / z\right)\right],  \tag{3.11b}\\
a_{32}(z)=a_{11}^{*}(z) a_{11}^{*}\left(q_{o}^{2} / z\right)\left[\rho_{1}\left(q_{o}^{2} / z\right) \rho_{2}^{*}(z)-\left(i q_{o} / z\right) \rho_{2}^{*}\left(q_{o}^{2} / z\right)\right] / \gamma(z) . \tag{3.11c}
\end{gather*}
$$

(Recall that $a_{13}(z)$ and $a_{23}(z)$ can be obtained directly in terms of the reflection coefficients and the analytic scattering coefficients via (2.48).)
3.4. Existence and uniqueness of the solution of the RHP. The representation (3.6) of the solution of the RHP was derived under the assumption of existence. An obvious and important issue is whether rigorous results can be obtained about existence and uniqueness of solutions. In Appendix A. 10 we show that (restricting ourselves for simplicity to the case in which no discrete spectrum is present) the issue of uniqueness can be answered in a straightforward way.

Theorem 3.11. Suppose that no discrete spectrum is present. If the RHP defined by Lemmas 3.1, 3.2, and 3.3 admits a solution, this solution is unique.

On the other hand, the issue of existence is much more subtle. One can reduce the question of existence of a solution of the RHP to one of the existence of a solution of an appropriately formulated integral equation, which we introduce next. Let us denote by $I$ the identity operator on $L^{2}(\mathbb{R})$ and define the Cauchy projection operators as

$$
\begin{equation*}
\left(P^{ \pm} f\right)(s)=\frac{1}{2 \pi i} \lim _{\epsilon \rightarrow 0^{+}} \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta-(s \pm i \epsilon)} \mathrm{d} \zeta, \tag{3.12a}
\end{equation*}
$$

which are also well defined in $L^{2}(\mathbb{R})$, and recall that $\left(P^{ \pm} f\right)(s)=\lim _{z \rightarrow s}(P f)(z)$, where $P$ denotes the Cauchy-type integral

$$
\begin{equation*}
(P f)(z)=\frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta-z} \mathrm{~d} \zeta, \quad z \notin \mathbb{R} \tag{3.12b}
\end{equation*}
$$

and the limit is taken from the upper or lower half plane, respectively. We begin by rewriting the jump condition (3.2) as

$$
\mathbf{M}^{+}(x, t, s)=\mathbf{M}^{-}(x, t, s) \mathbf{V}(x, t, s), \quad s \in \mathbb{R}
$$

where the jump matrix $\mathbf{V}(x, t, s)$ is

$$
\mathbf{V}(x, t, s)=\mathbf{I}-\mathrm{e}^{-i \mathbf{K} \boldsymbol{\Theta}(x, t, s)} \mathbf{L}(x, t, s) \mathrm{e}^{i \mathbf{K} \boldsymbol{\Theta}(x, t, s)}, \quad s \in \mathbb{R}
$$

Hereafter, for simplicity we omit the dependence on $x$ and $t$ for the remainder of this section. Without loss of generality we may decompose the jump matrix as

$$
\begin{equation*}
\mathbf{V}(s)=\mathbf{V}_{+}^{-1}(s) \mathbf{V}_{-}(s), \quad s \in \mathbb{R} \tag{3.13}
\end{equation*}
$$

where $\mathbf{V}_{ \pm}(s)$ are, respectively, upper/lower triangular matrices. (Note that here the subscripts $\pm$ do not indicate normalization as $x \rightarrow \pm \infty$ as in the rest of this work.) Next we define

$$
\begin{equation*}
\mathbf{W}_{ \pm}= \pm\left(\mathbf{I}-\mathbf{V}_{ \pm}\right), \quad \mathbf{W}=\mathbf{W}_{+}+\mathbf{W}_{-} \tag{3.14}
\end{equation*}
$$

where, for brevity, we have omitted the $s$ dependence. Finally, we use these quantities to define a new operator $P_{\mathbf{w}}$ in $L^{2}(\mathbb{R})$ by (3.15):

$$
\begin{equation*}
P_{\mathbf{w}} f=P^{+}\left(f \mathbf{W}_{+}\right)+P^{-}\left(f \mathbf{W}_{-}\right) \tag{3.15}
\end{equation*}
$$

In Appendix A. 10 we then follow the approach of $[6,7,14]$ to prove the following.
THEOREM 3.12. Suppose that no discrete spectrum is present. If $\mathbf{L}(\cdot) \in L^{2}(\mathbb{R}) \cap$ $L^{\infty}(\mathbb{R})$ and $I-P_{\mathbf{w}}$ has Fredholm index zero, the RHP defined by Lemmas 3.1, 3.2, and 3.3 admits a unique solution.

Note that stronger results could be obtained. For example, using techniques similar to those in $[6,7,15]$, one can show that if $\mathbf{q}(x, t)-\mathbf{q}_{ \pm}$decay sufficiently rapidly as $x \rightarrow \pm \infty$, the scattering coefficients are infinitely differentiable functions, and therefore the condition $\mathbf{L}(\cdot) \in L^{\infty}(\mathbb{R})$ can be removed, since it is automatically satisfied. Similarly, one can show that the asymptotic behavior in Corollaries 2.31 and 2.32 implies $\mathbf{L}(\cdot) \in L^{2}(\mathbb{R})$. Similarly, the possible presence of a discrete spectrum can be taken into account without much difficulty. Essentially, in this case the inverse problem can be reduced to the inversion of a linear operator of the form $T=I+T_{1}+T_{2}$,
where $T_{1}$ has small norm and $T_{2}$ is compact [7]. For brevity, however, we omit a proof of these results.

A more subtle issue is the requirement in Theorem 3.12 that the Fredholm index of the operator $I-P_{\mathbf{w}}$ be zero. One can again use the methods of $[6,7,14]$ to show that this is a consequence of the properties of the scattering data. A detailed proof of this result, however, is nontrivial, and it is therefore omitted for simplicity. We refer the reader to $[6,7,14]$ for a discussion of this issue in related contexts.
3.5. Reflectionless potentials and pure soliton solutions. We now look at potentials $\mathbf{q}(x, t)$ for which there is no jump from $\mathbf{M}^{+}$to $\mathbf{M}^{-}$across the continuous spectrum. In this case, the reflection coefficients from (2.47) vanish identically, and the inverse problem reduces to an algebraic system whose solution yields the soliton solutions of the integrable nonlinear equation. Note first that the scattering matrices contain off-diagonal elements that do not appear in the definition of the reflection coefficients. Nonetheless, the first and second symmetries combined with the fact that $\mathbf{B}(z)=\mathbf{A}^{-1}(z)$ allow us to conclude the following.

LEMMA 3.13. The scattering matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$ are diagonal in the reflectionless case.

By virtue of Corollary 3.5, we can parametrize the functions $C_{n}(x, t)$ in Theorem 3.7 as follows:

$$
C_{n}(x, t) \mathrm{e}^{2 i \theta_{1}\left(x, t, \zeta_{n}\right)}=2\left|\lambda\left(\zeta_{n}\right)\right| \mathrm{e}^{2\left|\lambda\left(\zeta_{n}\right)\right| \xi_{n}+i \varphi_{n}}, \quad n=1, \ldots, N_{1}
$$

where $\xi_{n}$ and $\varphi_{n}$ are real parameters and where $\varphi_{n}=\arg \left(\zeta_{n}\right)+m \pi(m=0,1)$ were found by comparing the first and second equations of (3.5a). We will see that $m=0,1$ for singular and for regular soliton solutions, respectively.

Theorem 3.14. In the reflectionless case, the solution (3.8) of the defocusing Manakov system with NZBC may be written

$$
\mathbf{q}(x, t)=\frac{1}{\operatorname{det} \mathbf{G}}\binom{\operatorname{det} \mathbf{G}_{1}^{\text {aug }}}{\operatorname{det} \mathbf{G}_{2}^{\text {aug }}}, \quad \mathbf{G}_{n}^{\text {aug }}=\left(\begin{array}{cc}
q_{+, n} & \mathbf{Y}^{T}  \tag{3.16}\\
\mathbf{B}_{n} & \mathbf{G}
\end{array}\right), \quad n=1,2,
$$

where

$$
\begin{aligned}
& \mathbf{B}_{n}=\left(B_{n 1}, \ldots, B_{n\left(N_{1}+N_{2}\right)}\right)^{T}, \quad \mathbf{G}=\mathbf{I}+\mathbf{F}, \quad \mathbf{Y}=\left(Y_{1}, \ldots, Y_{N_{1}+N_{2}}\right)^{T}, \\
& B_{n i^{\prime}}= \begin{cases}q_{+, n} / q_{o}, & i^{\prime}=1, \ldots, N_{1}, \\
(-1)^{n+1} \frac{r_{+, \pi}}{q_{o}}-\sum_{j=1}^{N_{2}} \frac{i q_{+, n}}{z_{j}^{*}} d_{j i^{\prime}}, & i^{\prime}=N_{1}+1, \ldots, N_{1}+N_{2},\end{cases} \\
& \begin{cases}-\frac{i \zeta_{k}}{q_{o}} d_{k}^{(2)}\left(\zeta_{j}\right), & j, k=1, \ldots, N_{1}, \\
d_{k-N_{1}}^{(4)}\left(\zeta_{j}\right), & j=1, \ldots, N_{1}, \quad k=N_{1}+1, \ldots, N_{1}+N_{2},\end{cases} \\
& F_{j k}= \begin{cases}\sum_{\ell=1}^{N_{2}} d_{\ell j} d_{k}^{(1)}\left(z_{\ell}^{*}\right), & j=N_{1}+1, \ldots, N_{1}+N_{2}, \quad k=1, \ldots, N_{1}, \\
\sum_{\ell=1}^{N_{2}} d_{\ell j} d_{k-N_{1}}^{(3)}\left(z_{\ell}^{*}\right), & j, k=N_{1}+1, \ldots, N_{1}+N_{2},\end{cases} \\
& d_{j k}=d_{j}^{(5)}\left(z_{k-N_{1}}\right)+\frac{i z_{j}^{*}}{q_{o}} d_{j}^{(6)}\left(z_{k-N_{1}}\right), \quad Y_{n}= \begin{cases}i C_{n}(x, t), & n=1, \ldots, N_{1}, \\
i D_{n-N_{1}}(x, t), & n=N_{1}+1, \ldots, N_{1}+N_{2},\end{cases} \\
& \text { and } \bar{n}=n+(-1)^{n+1} \text {. }
\end{aligned}
$$

Recall that the norming constants $C_{n}$ associated with discrete eigenvalues on the circle satisfy the constraint in Corollary 3.5, whereas no such constraint exists for the norming constants $D_{n}$ associated with discrete eigenvalues off the circle. (Also recall that discrete eigenvalues on the circle are parametrized by one real constant, while those off the circle are parametrized by two real constants.) Correspondingly, discrete eigenvalues on the circle generate the usual dark soliton solutions, which have two real degrees of freedom: the soliton depth (governed by the discrete eigenvalue) and the soliton position offset (governed by the norming constant). In contrast, discrete eigenvalues off the circle generate dark-bright soliton solutions, which have two additional degrees of freedom in addition to those of dark solitons: the amplitude of the bright component (also governed by the discrete eigenvalue) and its phase (also governed by the norming constant).
3.6. Explicit solutions. In the case of just one discrete eigenvalue in each fundamental domain (circle or disk), the resulting soliton solutions assume a particularly simple form.

For example, considering a single pair of eigenvalues on the circle $\left(N_{1}=1\right.$ and $N_{2}=0$ ) and parametrizing the discrete eigenvalue and norming constant as

$$
\zeta_{1}=q_{o} \mathrm{e}^{i \alpha}, \quad c_{1}=\mathrm{e}^{\xi+i(\alpha+\pi / 2+(m-1) \pi)}, \quad 0<\alpha<\pi, \quad \xi \in \mathbb{R}, \quad m=0,1
$$

from (3.16) one obtains the following singular/regular dark soliton solution of the Manakov system (corresponding to $m=0$ and $m=1$, respectively):

$$
\mathbf{q}(x, t)=\mathrm{e}^{i \alpha}\left[\cos \alpha-i \sin \alpha\left[\tanh \left[q_{o} \sin \alpha\left(x-2 q_{o} t \cos \alpha\right)-\xi / 2\right]\right]^{(-1)^{m+1}}\right] \mathbf{q}_{+} .
$$

Similarly, considering a single quartet of eigenvalues off the circle $\left(N_{1}=0\right.$ and $N_{2}=1$ ) and introducing the parametrizations

$$
z_{1}=Z \mathrm{e}^{i \alpha}, \quad d_{1}=\mathrm{e}^{\xi+i \phi}, \quad 0<Z<q_{o}, \quad 0<\alpha<\pi, \quad \xi, \phi \in \mathbb{R}
$$

from (3.16) one obtains the following dark-bright soliton solution of the Manakov system:
$\mathbf{q}(x, t)=\mathrm{e}^{i \alpha}[\cos \alpha-i \sin \alpha \tanh U(x, t)] \mathbf{q}_{+}+V_{o} \mathrm{e}^{i\left[Z x \cos \alpha-Z^{2} t \cos (2 \alpha)\right]} \operatorname{sech} U(x, t) \mathbf{q}_{+}^{\perp}$,
where

$$
\begin{gathered}
U(x, t)=Z \sin \alpha(x-2 Z t \cos \alpha)-\xi-\frac{1}{2} \log \left(\frac{Z^{2}\left(q_{o}^{2}-Z^{2}\right)}{q_{o}^{4}+Z^{4}-2 Z^{2} q_{o}^{2} \cos (2 \alpha)}\right), \\
V_{o}=\frac{i \sqrt{q_{o}^{2}-Z^{2}}\left(q_{o}^{2} \mathrm{e}^{2 i \alpha}-Z^{2}\right)\left(1-\mathrm{e}^{2 i \alpha}\right)}{2 q_{o} \sqrt{q_{o}^{4}+Z^{4}-2 Z^{2} q_{o}^{2} \cos (2 \alpha)}} \mathrm{e}^{i[\phi-3 \alpha]}
\end{gathered}
$$

Note how the bright soliton part (which is aligned with $\mathbf{q}_{+}^{\perp}$ ) is always along an orthogonal polarization to that of the dark soliton part (which is aligned with $\mathbf{q}_{+}$).

Of course, (3.16) allow one to produce multisoliton solutions just as easily (but the resulting expressions will be more complicated). For example, Figure 1 shows a two-dark soliton solution ( $N_{1}=2, N_{2}=0$ ), Figure 2 shows a two-dark-bright soliton solution $\left(N_{1}=0, N_{2}=2\right)$, and Figure 3 shows a two-dark, two-dark-bright soliton solution $\left(N_{1}=N_{2}=2\right)$.


Fig. 1. A two-dark soliton solution of the defocusing Manakov system obtained by taking $N_{1}=2, N_{2}=0, \mathbf{q}_{+}=(1,0)^{T}, \zeta_{1}=\mathrm{e}^{i \pi / 2}, \zeta_{2}=\mathrm{e}^{i \pi / 4}, \xi_{1}=\xi_{2}=0$.



FIG. 2. A two-dark-bright soliton solution of the defocusing Manakov system obtained by taking $N_{1}=0, N_{2}=2, \mathbf{q}_{+}=(1,0)^{T}, z_{1}=0.5 \mathrm{e}^{i \pi / 2}, z_{2}=0.75 \mathrm{e}^{i \pi / 4}$.


Fig. 3. A two-dark, two-dark-bright soliton solution of the defocusing Manakov system obtained by taking $N_{1}=2, N_{2}=2, \mathbf{q}_{+}=(1,0)^{T}, \zeta_{1}=\mathrm{e}^{i \pi / 2}, \zeta_{2}=\mathrm{e}^{i \pi / 5}, \xi_{1}=\xi_{2}=0, z_{1}=0.5 \mathrm{e}^{i \pi / 2}$, $z_{2}=0.75 \mathrm{e}^{i \pi / 4}$.

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4. Double-pole solutions. In this section, we present novel solutions of the Manakov system obtained when the analytic scattering coefficients have a double zero. Recall that such a situation is allowed in the focusing NLS equation, even with ZBC [41], but is not possible in the defocusing NLS equation [19]. For brevity, we refer to the corresponding solutions as "double-pole" solutions of the Manakov system, but it should be clear that it is only the meromorphic matrices in the RHP that possess double poles, while the solutions of the Manakov system are regular in the whole $x t$-plane.

Suppose that $a_{11}\left(z_{o}\right)=a_{11}^{\prime}\left(z_{o}\right)=0$ and $a_{11}^{\prime \prime}\left(z_{o}\right) \neq 0$, with $\left|z_{o}\right|<q_{o}$. As before, in order to regularize the RHP (3.2), one must subtract the residue contributions. As we will see, however, the principal part of the Laurent series expansion of the meromorphic matrices contains additional terms, which must also be subtracted. As a result, the derivatives of the eigenfunctions with respect to $z$ will appear as additional unknowns in the RHP. In turn, this will result in the presence of additional norming constants, whose symmetries must also be properly characterized. The proofs of all the results presented in this section are collected in Appendix A. 12.
4.1. Behavior of the eigenfunctions at a double pole. For brevity, we suppress the $(x, t)$-dependence of the eigenfunctions on the right-hand sides of equations throughout this and the following section when doing so introduces no confusion.

LEMMA 4.1. Suppose that $a_{11}\left(z_{o}\right)=a_{11}^{\prime}\left(z_{o}\right)=0$ and $a_{11}^{\prime \prime}\left(z_{o}\right) \neq 0$, with $\left|z_{o}\right|<q_{o}$. There exist constants $d_{o}, \hat{d}_{o}, \check{d}_{o}, \bar{d}_{o}, f_{o}, \hat{f}_{o}, \check{f}_{o}, \bar{f}_{o}, g_{o}, \hat{g}_{o}, \check{g}_{o}$, and $\bar{g}_{o}$ such that

$$
\begin{gather*}
\phi_{-, 1}^{\prime}\left(x, t, z_{o}\right)=d_{o} \chi^{\prime}\left(z_{o}\right)+f_{o} \chi\left(z_{o}\right)+g_{o} \phi_{+, 3}\left(z_{o}\right)  \tag{4.1a}\\
\chi^{\prime}\left(x, t, q_{o}^{2} / z_{o}^{*}\right)=\hat{d}_{o} \phi_{+, 3}^{\prime}\left(q_{o}^{2} / z_{o}^{*}\right)+\hat{f}_{o} \phi_{+, 3}\left(q_{o}^{2} / z_{o}^{*}\right)+\hat{g}_{o} \phi_{-, 1}\left(q_{o}^{2} / z_{o}^{*}\right),  \tag{4.1b}\\
\phi_{-, 3}^{\prime}\left(x, t, q_{o}^{2} / z_{o}\right)=\check{d}_{o} \bar{\chi}^{\prime}\left(q_{o}^{2} / z_{o}\right)+\check{f}_{o} \bar{\chi}\left(q_{o}^{2} / z_{o}\right)+\check{g}_{o} \phi_{+, 1}\left(q_{o}^{2} / z_{o}\right),  \tag{4.1c}\\
\bar{\chi}^{\prime}\left(x, t, z_{o}^{*}\right)=\bar{d}_{o} \phi_{+, 1}^{\prime}\left(z_{o}^{*}\right)+\bar{f}_{o} \phi_{+, 1}\left(z_{o}^{*}\right)+\bar{g}_{o} \phi_{-, 3}\left(z_{o}^{*}\right) . \tag{4.1~d}
\end{gather*}
$$

Note that $d_{o}, \ldots, \bar{d}_{o}$ are the same constants appearing in the relations (2.53) for a single eigenvalue, whereas $f_{o}, \ldots, \bar{f}_{o}$ and $g_{o}, \ldots, \bar{g}_{o}$ appear as a result of the double multiplicity. It will be useful to express (4.1) in terms of the modified eigenfunctions:

$$
\begin{align*}
\mu_{-, 1}^{\prime}\left(x, t, z_{o}\right)=-i \theta_{1}^{\prime}\left(z_{o}\right) \mu_{-, 1}\left(z_{o}\right) & +\left(i d_{o} \theta_{2}^{\prime}\left(z_{o}\right)+f_{o}\right) m\left(z_{o}\right) \mathrm{e}^{i\left(\theta_{2}-\theta_{1}\right)\left(z_{o}\right)}  \tag{4.2a}\\
& +d_{o} m^{\prime}\left(z_{o}\right) \mathrm{e}^{i\left(\theta_{2}-\theta_{1}\right)\left(z_{o}\right)}+g_{o} \mu_{+, 3}\left(z_{o}\right) \mathrm{e}^{-2 i \theta_{1}\left(z_{o}\right)}
\end{align*}
$$

$$
\begin{align*}
& m^{\prime}\left(x, t, q_{o}^{2} / z_{o}^{*}\right)=-i \theta_{2}^{\prime}\left(q_{o}^{2} / z_{o}^{*}\right) m\left(q_{o}^{2} / z_{o}^{*}\right)+\hat{g}_{o} \mu_{-, 1}\left(q_{o}^{2} / z_{o}^{*}\right) \mathrm{e}^{i\left(\theta_{1}-\theta_{2}\right)\left(q_{o}^{2} / z_{o}^{*}\right)}  \tag{4.2b}\\
& \quad+\left[\left(-i \hat{d}_{o} \theta_{1}^{\prime}\left(q_{o}^{2} / z_{o}^{*}\right)+\hat{f}_{o}\right) \mu_{+, 3}\left(q_{o}^{2} / z_{o}^{*}\right)+\hat{d}_{o} \mu_{+, 3}^{\prime}\left(q_{o}^{2} / z_{o}^{*}\right)\right] \mathrm{e}^{-i\left(\theta_{1}+\theta_{2}\right)\left(q_{o}^{2} / z_{o}^{*}\right)}
\end{align*}
$$

$$
\begin{align*}
\mu_{-, 3}^{\prime}\left(x, t, q_{o}^{2} / z_{o}\right) & =i \theta_{1}^{\prime}\left(q_{o}^{2} / z_{o}\right) \mu_{-, 3}\left(q_{o}^{2} / z_{o}\right)+\check{g}_{o} \mu_{+, 1}\left(q_{o}^{2} / z_{o}\right) \mathrm{e}^{2 i \theta_{1}\left(q_{o}^{2} / z_{o}\right)}  \tag{4.2c}\\
& +\left[\left(i \check{d}_{o} \theta_{2}^{\prime}\left(q_{o}^{2} / z_{o}\right)+\check{f}_{o}\right) \bar{m}\left(q_{o}^{2} / z_{o}\right)+\check{d}_{o} \bar{m}^{\prime}\left(q_{o}^{2} / z_{o}\right)\right] \mathrm{e}^{i\left(\theta_{1}+\theta_{2}\right)\left(q_{o}^{2} / z_{o}\right)}
\end{align*}
$$

$$
\begin{align*}
& \bar{m}^{\prime}\left(x, t, z_{o}^{*}\right)=\left[\left(i \bar{d}_{o} \theta_{1}^{\prime}\left(z_{o}^{*}\right)+\bar{f}_{o}\right) \mu_{+, 1}\left(z_{o}^{*}\right)+\bar{d}_{o} \mu_{+, 1}^{\prime}\left(z_{o}^{*}\right)\right] \mathrm{e}^{i\left(\theta_{1}-\theta_{2}\right)\left(z_{o}^{*}\right)}  \tag{4.2d}\\
&-i \theta_{2}^{\prime}\left(z_{o}^{*}\right) \bar{m}\left(z_{o}^{*}\right)+\bar{g}_{o} \mu_{-, 3}\left(z_{o}^{*}\right) \mathrm{e}^{-i\left(\theta_{1}+\theta_{2}\right)\left(z_{o}^{*}\right)}
\end{align*}
$$

These expressions will allow us to obtain the generalization of the residue relations.

Let $\left.P_{-2}[F]\right|_{z=z_{o}}$ denote the coefficient of $1 /\left(z-z_{o}\right)^{2}$ in the Laurent series expansion of a meromorphic function $F(z)$ near $z=z_{o}$. The following result is trivial.

Proposition 4.2. Suppose that $f(z)$ and $a(z)$ are analytic functions of $z$ in a neighborhood of $z=z_{o}$. Also, suppose $a(z)$ has a double zero at $z=z_{o}$ and $f\left(z_{o}\right) \neq 0$. Then

$$
\begin{equation*}
\operatorname{Res}_{z=z_{o}}\left(\frac{f}{a}\right)=\frac{2 f^{\prime}\left(z_{o}\right)}{a^{\prime \prime}\left(z_{o}\right)}-\frac{2}{3} \frac{f\left(z_{o}\right) a^{\prime \prime \prime}\left(z_{o}\right)}{\left(a^{\prime \prime}\left(z_{o}\right)\right)^{2}}, \quad P_{z=z_{o}}\left(\frac{f}{a}\right)=\frac{2 f\left(z_{o}\right)}{a^{\prime \prime}\left(z_{o}\right)} \tag{4.3}
\end{equation*}
$$

Corollary 4.3. Under the hypotheses of Lemma 4.1, we have

$$
\begin{gathered}
P_{z=z_{o}}\left(\frac{\mu_{-, 1}}{a_{11}}\right)(x, t)=K_{o} m\left(z_{o}\right), \quad \underset{z=q_{o}^{2} / z_{o}}{P_{-2}}\left(\frac{\mu_{-, 3}}{a_{33}}\right)(x, t)=\check{K}_{o} \bar{m}\left(q_{o}^{2} / z_{o}\right), \\
P_{z=z_{o}^{*}}\left(-\frac{\bar{m}}{b_{11}}\right)(x, t)=-\bar{K}_{o} \mu_{+, 1}\left(z_{o}^{*}\right), \quad \underset{z=q_{o}^{2} / z_{o}^{*}}{P_{o}}\left(\frac{m}{b_{33}}\right)(x, t)=\hat{K}_{o} \mu_{+, 3}\left(q_{o}^{2} / z_{o}^{*}\right), \\
\operatorname{Res}_{z=z_{o}}\left(\frac{\mu_{-, 1}}{a_{11}}\right)(x, t)=K_{o} m^{\prime}\left(z_{o}\right)+K_{o}\left[F_{o}+i\left(x-2 z_{o} t\right)\right] m\left(z_{o}\right)+G_{o} \mu_{+, 3}\left(z_{o}\right), \\
\operatorname{Res}_{z=q_{o}^{2} / z_{o}}\left(\frac{\mu_{-, 3}}{a_{33}}\right)(x, t)=\check{K}_{o} \bar{m}^{\prime}\left(q_{o}^{2} / z_{o}\right)+\check{K}_{o}\left[\check{F}_{o}-\frac{i z_{o}^{2}}{q_{o}^{2}}\left(x-2 z_{o} t\right)\right] \bar{m}\left(q_{o}^{2} / z_{o}\right) \\
\\
+\check{G}_{o} \mu_{+, 1}\left(q_{o}^{2} / z_{o}\right), \\
\operatorname{Res}_{z=z_{o}^{*}}\left(-\frac{\bar{m}}{b_{11}}\right)(x, t)=-\bar{K}_{o} \mu_{+, 1}^{\prime}\left(z_{o}^{*}\right)-\bar{K}_{o}\left[\bar{F}_{o}-i\left(x-2 z_{o}^{*} t\right)\right] \mu_{+, 1}\left(z_{o}^{*}\right)-\bar{G}_{o} \mu_{-, 3}\left(z_{o}^{*}\right), \\
\operatorname{Res}_{z=q_{o}^{2} / z_{o}^{*}}\left(\frac{m}{b_{33}}\right)(x, t)=\hat{K}_{o} \mu_{+, 3}^{\prime}\left(q_{o}^{2} / z_{o}^{*}\right)+\hat{K}_{o}\left[\hat{F}_{o}+\frac{i\left(z_{o}^{*}\right)^{2}}{q_{o}^{2}}\left(x-2 z_{o}^{*} t\right)\right] \mu_{+, 3}\left(q_{o}^{2} / z_{o}^{*}\right) \\
+\hat{G}_{o} \mu_{-, 1}\left(q_{o}^{2} / z_{o}^{*}\right),
\end{gathered}
$$

where

$$
\begin{gathered}
K_{o}(x, t)=\frac{2 d_{o}}{a_{11}^{\prime \prime}\left(z_{o}\right)} \mathrm{e}^{-i\left(\theta_{1}-\theta_{2}\right)\left(z_{o}\right)}, \quad \check{K}_{o}(x, t)=\frac{2 \check{d}_{o}}{a_{33}^{\prime \prime}\left(q_{o}^{2} / z_{o}\right)} \mathrm{e}^{-i\left(\theta_{1}-\theta_{2}\right)\left(z_{o}\right)} \\
\bar{K}_{o}(x, t)=\frac{2 \bar{d}_{o}}{b_{11}^{\prime \prime}\left(z_{o}^{*}\right)} \mathrm{e}^{i\left(\theta_{1}-\theta_{2}\right)\left(z_{o}^{*}\right)}, \quad \hat{K}_{o}(x, t)=\frac{2 \hat{d}_{o}}{b_{33}^{\prime \prime}\left(q_{o}^{2} / z_{o}^{*}\right)} \mathrm{e}^{i\left(\theta_{1}-\theta_{2}\right)\left(z_{o}^{*}\right)} \\
F_{o}=\frac{f_{o}}{d_{o}}-\frac{a_{11}^{\prime \prime \prime}\left(z_{o}\right)}{3 a_{11}^{\prime \prime}\left(z_{o}\right)}, \quad G_{o}(x, t)=\frac{2 g_{o}}{a_{11}^{\prime \prime}\left(z_{o}\right)} \mathrm{e}^{-2 i \theta_{1}\left(z_{o}\right)} \\
\check{F}_{o}=\frac{\check{f}_{o}}{\check{d}_{o}}-\frac{a_{33}^{\prime \prime \prime}\left(q_{o}^{2} / z_{o}\right)}{3 a_{33}^{\prime \prime}\left(q_{o}^{2} / z_{o}\right)}, \quad \check{G}_{o}(x, t)=\frac{2 \check{g}_{o}}{a_{33}^{\prime \prime}\left(q_{o}^{2} / z_{o}\right)} \mathrm{e}^{-2 i \theta_{1}\left(z_{o}\right)} \\
\bar{F}_{o}=\frac{\bar{f}_{o}}{\bar{d}_{o}}-\frac{b_{11}^{\prime \prime \prime}\left(z_{o}^{*}\right)}{3 b_{11}^{\prime \prime}\left(z_{o}^{*}\right)}, \quad \bar{G}_{o}(x, t)=\frac{2 \bar{g}_{o}}{b_{11}^{\prime \prime}\left(z_{o}^{*}\right)} \mathrm{e}^{-i\left(\theta_{1}+\theta_{2}\right)\left(z_{o}^{*}\right)} \\
\hat{F}_{o}=\frac{\hat{f}_{o}}{\hat{d}_{o}}-\frac{b_{33}^{\prime \prime \prime}\left(q_{o}^{2} / z_{o}^{*}\right)}{3 b_{33}^{\prime \prime}\left(q_{o}^{2} / z_{o}^{*}\right)}, \quad \hat{G}_{o}(x, t)=\frac{2 \hat{g}_{o}}{b_{33}^{\prime \prime}\left(q_{o}^{2} / z_{o}^{*}\right)} \mathrm{e}^{-i\left(\theta_{1}+\theta_{2}\right)\left(z_{o}^{*}\right)}
\end{gathered}
$$

4.2. Symmetries with double poles. The symmetries of the eigenfunctions and scattering coefficients are also more involved than in the case of simple zeros.

LEMMA 4.4. Suppose that $a_{11}\left(z_{o}\right)=a_{11}^{\prime}\left(z_{o}\right)=0$ and $a_{11}^{\prime \prime}\left(z_{o}\right) \neq 0$, with $\left|z_{o}\right|<q_{o}$. The analytic scattering coefficients obey the following symmetry relations:

$$
a_{11}^{(n)}\left(z_{o}\right)=\left.\left(b_{11}^{(n)}(z)\right)^{*}\right|_{z=z_{o}^{*}}, \quad b_{33}^{(n)}\left(z_{o}\right)=\left.\left(a_{33}^{(n)}(z)\right)^{*}\right|_{z=z_{o}^{*}}, \quad n=2,3,
$$

$$
\begin{gathered}
b_{33}^{\prime \prime}\left(q_{o}^{2} / z_{o}^{*}\right)=\left.\frac{\left(z_{o}^{*}\right)^{4}}{q_{o}^{4}} b_{11}^{\prime \prime}(z)\right|_{z=z_{o}^{*}}, \quad a_{11}^{\prime \prime}\left(z_{o}\right)=\left.\frac{q_{o}^{4}}{z_{o}^{4}}\left(a_{33}^{\prime \prime}(z)\right)\right|_{z=q_{o}^{2} / z_{o}} \\
b_{33}^{\prime \prime \prime}\left(q_{o}^{2} / z_{o}^{*}\right)=-\left.\frac{\left(z_{o}^{*}\right)^{5}}{q_{o}^{6}}\left(6 b_{11}^{\prime \prime}(z)+z_{o}^{*} b_{11}^{\prime \prime \prime}(z)\right)\right|_{z=z_{o}^{*}} \\
a_{11}^{\prime \prime \prime}\left(z_{o}\right)=-\left.\frac{q_{o}^{4}}{z_{o}^{5}}\left(6 a_{33}^{\prime \prime}(z)+\frac{q_{o}^{2}}{z_{o}} a_{33}^{\prime \prime \prime}(z)\right)\right|_{z=q_{o}^{2} / z_{o}}
\end{gathered}
$$

LEMMA 4.5. Suppose that $a_{11}\left(z_{o}\right)=a_{11}^{\prime}\left(z_{o}\right)=0$ and $a_{11}^{\prime \prime}\left(z_{o}\right) \neq 0$, with $\left|z_{o}\right|<q_{o}$. The eigenfunctions obey the following symmetry relations:

$$
\begin{gather*}
\phi_{+, 1}^{\prime}\left(x, t, z_{o}^{*}\right)=-\frac{i q_{o}}{\left(z_{o}^{*}\right)^{2}}\left(\phi_{+, 3}\left(x, t, q_{o}^{2} / z_{o}^{*}\right)+\frac{q_{o}^{2}}{z_{o}^{*}} \phi_{+, 3}^{\prime}\left(x, t, q_{o}^{2} / z_{o}^{*}\right)\right),  \tag{4.4a}\\
\phi_{-, 1}^{\prime}\left(x, t, z_{o}\right)=-\frac{i q_{o}}{z_{o}^{2}}\left(\phi_{-, 3}\left(x, t, q_{o}^{2} / z_{o}\right)+\frac{q_{o}^{2}}{z_{o}} \phi_{-, 3}^{\prime}\left(x, t, q_{o}^{2} / z_{o}\right)\right),  \tag{4.4b}\\
\phi_{+, 3}^{\prime}\left(x, t, z_{o}\right)=\frac{i q_{o}}{z_{o}^{2}}\left(\phi_{+, 1}\left(x, t, q_{o}^{2} / z_{o}\right)+\frac{q_{o}^{2}}{z_{o}} \phi_{+, 1}^{\prime}\left(x, t, q_{o}^{2} / z_{o}\right)\right),  \tag{4.4c}\\
\phi_{-, 3}^{\prime}\left(x, t, z_{o}^{*}\right)=\frac{i q_{o}}{\left(z_{o}^{*}\right)^{2}}\left(\phi_{-, 1}\left(x, t, q_{o}^{2} / z_{o}^{*}\right)+\frac{q_{o}^{2}}{z_{o}^{*}} \phi_{-, 1}^{\prime}\left(x, t, q_{o}^{2} / z_{o}^{*}\right)\right),  \tag{4.4d}\\
\chi^{\prime}\left(x, t, z_{o}\right)=\frac{q_{o}^{2}}{z_{o}^{2}} \bar{\chi}^{\prime}\left(x, t, q_{o}^{2} / z_{o}\right), \quad \bar{\chi}^{\prime}\left(x, t, z_{o}^{*}\right)=\frac{q_{o}^{2}}{\left(z_{o}^{*}\right)^{2}} \chi^{\prime}\left(x, t, q_{o}^{2} / z_{o}^{*}\right) . \tag{4.4e}
\end{gather*}
$$

Lemma 4.6. Under the hypotheses of Lemma 4.5, the norming constants in Lemma 4.1 obey the following symmetry relations:

$$
\begin{gather*}
\check{d}_{o}=\frac{i z_{o}}{q_{o}} d_{o}, \quad \bar{d}_{o}=-\frac{\left[b_{33}\left(z_{o}\right)\right]^{*}}{\gamma\left(z_{o}^{*}\right)} d_{o}^{*}, \quad \hat{d}_{o}=\frac{i q_{o}}{z_{o}^{*}} \frac{\left[b_{33}\left(z_{o}\right)\right]^{*}}{\gamma\left(z_{o}^{*}\right)} d_{o}^{*}  \tag{4.5a}\\
f_{o}=-\frac{\gamma\left(z_{o}\right)}{b_{33}\left(z_{o}\right)} \bar{f}_{o}^{*}-\left.\bar{d}_{o}^{*}\left(\frac{\gamma(z)}{b_{33}(z)}\right)^{\prime}\right|_{z=z_{o}} \quad, \quad f_{o}=\frac{i q_{o}}{z_{o}}\left[\frac{\check{d}_{o}}{z_{o}}+\frac{q_{o}^{2}}{z_{o}^{2}} \check{f}_{o}\right],  \tag{4.5b}\\
\bar{f}_{o}=\frac{i z_{o}^{*}}{q_{o}}\left[\frac{\hat{d}_{o}}{z_{o}^{*}}-\frac{q_{o}^{2}}{\left(z_{o}^{*}\right)^{2}} \hat{f}_{o}\right], \quad g_{o}=\check{g}_{o}=\bar{g}_{o}=\hat{g}_{o}=0 \tag{4.5c}
\end{gather*}
$$

In turn, the symmetries in Lemma 4.6 yield

$$
\begin{gathered}
\check{K}_{o}(x, t)=\frac{i q_{o}^{3}}{z_{o}^{3}} K_{o}(x, t), \quad \bar{K}_{o}(x, t)=-\frac{\left[b_{33}\left(z_{o}\right)\right]^{*}}{\gamma\left(z_{o}^{*}\right)} K_{o}^{*}(x, t), \\
\hat{K}_{o}(x, t)=\frac{i q_{o}^{5}}{\left(z_{o}^{*}\right)^{5}} \frac{\left[b_{33}\left(z_{o}\right)\right]^{*}}{\gamma\left(z_{o}^{*}\right)} K_{o}^{*}(x, t), \quad \hat{F}_{o}=-\frac{\left(z_{o}^{*}\right)^{2}}{q_{o}^{2}} \bar{F}_{o}+\frac{3 z_{o}^{*}}{q_{o}^{2}}, \\
F_{o}=\bar{F}_{o}^{*}+\left.\frac{b_{33}\left(z_{o}\right)}{\gamma\left(z_{o}\right)}\left(\frac{\gamma(z)}{b_{33}(z)}\right)^{\prime}\right|_{z=z_{o}}, \\
\check{F}_{o}=-\frac{z_{o}^{2}}{q_{o}^{2}} \bar{F}_{o}^{*}+\frac{z_{o}}{q_{o}^{2}}-\left.\frac{z_{o}^{2}}{q_{o}^{2}} \frac{b_{33}\left(z_{o}\right)}{\gamma\left(z_{o}\right)}\left(\frac{\gamma(z)}{b_{33}(z)}\right)^{\prime}\right|_{z=z_{o}}, \\
G_{o}(x, t)=\check{G}_{o}(x, t)=\bar{G}_{o}(x, t)=\hat{G}_{o}(x, t)=0 .
\end{gathered}
$$

It is then straightforward to combine the notation of this section with the definition
of the meromorphic matrices (3.1) to obtain the residue conditions:
(4.6a)

$$
\mathbf{M}_{-2, z_{o}}^{+}(x, t)=\left(\underset{z=z_{o}}{P_{-2}}\left(\frac{\mu_{-, 1}}{a_{11}}\right), \mathbf{0}, \mathbf{0}\right), \mathbf{M}_{-2, q_{o}^{2} / z_{o}}^{-}(x, t)=\left(\mathbf{0}, \mathbf{0}, \underset{z=q_{o}^{2} / z_{o}}{P_{-2}}\left(\frac{\mu_{-, 3}}{a_{33}}\right)\right)
$$

$$
\mathbf{M}_{-2, z_{o}^{*}}^{-}(x, t)=\left(\begin{array}{c}
\mathbf{0}, P_{-2}  \tag{4.6~b}\\
z=z_{o}
\end{array}\left(-\frac{\bar{m}}{b_{11}}\right), \mathbf{0}\right), \mathbf{M}_{-2, q_{o}^{2} / z_{o}^{*}}^{+}(x, t)=\left(\underset{z=q_{o}^{2} / z_{o}^{*}}{P_{-2}}\left(\frac{m}{b_{33}}\right), \mathbf{0}\right)
$$

$$
\begin{equation*}
\mathbf{M}_{-1, z_{o}}^{+}(x, t)=\left(\operatorname{Res}_{z=z_{o}}\left(\frac{\mu_{-, 1}}{a_{11}}\right), \mathbf{0}, \mathbf{0}\right), \mathbf{M}_{-1, q_{o}^{2} / z_{o}}^{-}(x, t)=\left(\mathbf{0}, \mathbf{0}, \operatorname{Res}_{z=q_{o}^{2} / z_{o}}\left(\frac{\mu_{-, 3}}{a_{33}}\right)\right) \tag{4.6c}
\end{equation*}
$$

$\mathbf{M}_{-1, z_{o}^{*}}^{-}(x, t)=\left(\mathbf{0}, \operatorname{Res}_{z=z_{o}^{*}}\left(-\frac{\bar{m}}{b_{11}}\right), \mathbf{0}\right), \mathbf{M}_{-1, q_{o}^{2} / z_{o}^{*}}^{+}(x, t)=\left(\mathbf{0}, \operatorname{Res}_{z=q_{o}^{2} / z_{o}^{*}}\left(\frac{m}{b_{33}}\right), \mathbf{0}\right)$,
where the individual columns are given in Corollary 4.3.
4.3. Inverse problem and reflectionless solutions with double poles. Both the RHP and the reconstruction formula are affected by the appearance of the derivatives of the eigenfunctions. For simplicity, from now on we restrict our attention to situations in which $a_{11}(z)$ has just one double zero and is nonzero everywhere else. The methodology, however, is easily extended to include any combination of single and double zeros, as well as to zeros of higher order.

Like in the case of simple zeros, the RHP consists of finding a sectionally meromorphic matrix $\mathbf{M}(x, t, z)$ which equals $\mathbf{M}^{ \pm}(x, t, z)$ for $\operatorname{Im} z \gtrless 0$ and satisfies the jump condition (3.2) as well as the asymptotics (3.3). On the other hand, the residue conditions are now different and are given by (4.6). As in section 3.2, the solution of this RHP can be expressed in terms of a mixed system of algebraic-integral equations, which are obtained by subtracting the asymptotic behavior at infinity, regularizing (i.e., subtracting any pole contributions from the discrete spectrum), and then applying Cauchy projectors. Specifically, in Appendix A. 12 we prove the following.

THEOREM 4.7. Suppose that $a_{11}\left(z_{o}\right)=a_{11}^{\prime}\left(z_{o}\right)=0$ and $a_{11}^{\prime \prime}\left(z_{o}\right) \neq 0$, with $\left|z_{o}\right|<q_{o}$, and $a_{11}(z) \neq 0$ for $z \neq z_{o}$. The solution of the RHP defined by Lemmas 3.1 and 3.2 with residue conditions (4.6) is given by

$$
\begin{align*}
\mathbf{M}(x, t, z)= & \mathbf{M}_{\infty}+(i / z) \mathbf{M}_{0}-\frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{\mathbf{M}^{-}(\zeta)}{\zeta-z} \mathrm{e}^{-i \mathbf{K} \boldsymbol{\Theta}(\zeta)} \mathbf{L}(\zeta) \mathrm{e}^{i \mathbf{K} \boldsymbol{\Theta}(\zeta)} \mathrm{d} \zeta  \tag{4.7}\\
& +\sum_{n=1}^{2}\left(\frac{\mathbf{M}_{-n, z_{o}}^{+}}{\left(z-z_{o}\right)^{n}}+\frac{\mathbf{M}_{-n, q_{o}^{2} / z_{o}^{*}}^{+}}{\left(z-q_{o}^{2} / z_{o}^{*}\right)^{n}}+\frac{\mathbf{M}_{-n, z_{o}^{*}}^{-}}{\left(z-z_{o}^{*}\right)^{n}}+\frac{\mathbf{M}_{-n, q_{o}^{2} / z_{o}}^{-}}{\left(z-q_{o}^{2} / z_{o}\right)^{n}}\right)
\end{align*}
$$

Moreover, the eigenfunctions in the residue conditions in (4.6) are given by

$$
\begin{align*}
\mu_{+, 1}\left(x, t, z_{o}^{*}\right)= & \binom{1}{\left(i / z_{o}^{*}\right) \mathbf{q}_{+}}+\frac{K_{o}}{z_{o}^{*}-z_{o}} m^{\prime}\left(z_{o}\right)  \tag{4.8a}\\
& -\frac{1}{2 \pi i}
\end{align*} \int_{\mathbb{R}} \frac{\left(\mathbf{M}^{-} \mathrm{e}^{-i \mathbf{K} \Theta} \mathbf{L} \mathrm{Le}^{i \mathbf{K} \boldsymbol{\Theta}}\right)_{1}(\zeta)}{\zeta-z_{o}^{*}} \mathrm{~d} \zeta,
$$

(4.8b)

$$
\begin{align*}
& \frac{m\left(x, t, z_{o}\right)}{b_{33}\left(z_{o}\right)}=\binom{0}{\mathbf{q}_{+}^{\perp} / q_{o}}+\left[-\frac{\bar{K}_{o}}{z_{o}-z_{o}^{*}}+\frac{i\left(z_{o}^{*}\right)^{3}}{q_{o}^{3}} \frac{\hat{K}_{o}}{z_{o}-q_{o}^{2} / z_{o}^{*}}\right] \mu_{+, 1}^{\prime}\left(z_{o}^{*}\right) \\
& +\left\{-\frac{i z_{o}^{*}}{q_{o}} \frac{\hat{K}_{o}}{\left(z_{o}-q_{o}^{2} / z_{o}^{*}\right)^{2}}\left[1+\left(z_{o}-q_{o}^{2} / z_{o}^{*}\right)\left(\hat{F}_{o}+\frac{i\left(z_{o}^{*}\right)^{2}}{q_{o}^{2}}\left(x-2 z_{o}^{*} t\right)-\frac{z_{o}^{*}}{q_{o}^{2}}\right)\right]\right. \\
& \left.-\frac{\bar{K}_{o}}{\left(z_{o}-z_{o}^{*}\right)^{2}}\left[1+\left(z_{o}-z_{o}^{*}\right)\left(\bar{F}_{o}-i\left(x-2 z_{o}^{*} t\right)\right)\right]\right\} \mu_{+, 1}\left(z_{o}^{*}\right) \\
& -\frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{\left(\mathbf{M}^{-} \mathrm{e}^{-i \mathbf{K} \Theta} \mathbf{L} \mathrm{e}^{i \mathbf{K} \Theta}\right)_{2}(\zeta)}{\zeta-z_{o}} \mathrm{~d} \zeta,  \tag{4.8c}\\
& \mu_{+, 1}^{\prime}\left(x, t, z_{o}^{*}\right)=-\binom{0}{i \mathbf{q}_{+} /\left(z_{o}^{*}\right)^{2}}-\frac{K_{o}}{\left(z_{o}^{*}-z_{o}\right)^{2}} m^{\prime}\left(z_{o}\right)  \tag{4.8d}\\
& -\frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{\left(\mathbf{M}^{-} \mathrm{e}^{-i \mathbf{K} \boldsymbol{\Theta}} \mathbf{L e} \mathrm{e}^{i \mathbf{K} \boldsymbol{\Theta}}\right)_{1}(\zeta)}{\left(\zeta-z_{o}^{*}\right)^{2}} \mathrm{~d} \zeta \\
& -\frac{K_{o}}{\left(z_{o}^{*}-z_{o}\right)^{3}}\left[2+\left(z_{o}^{*}-z_{o}\right)\left(F_{o}+i\left(x-2 z_{o} t\right)\right)\right] m\left(z_{o}\right) .
\end{align*}
$$

Throughout, the $(x, t)$-dependence was omitted from the right-hand side of all equations for simplicity.

Similarly to the case of simple zeros, the norming constants can be appropriately redefined in such a way that the above residue conditions are formulated only in terms of the columns of the meromorphic matrix $\mathbf{M}(x, t, z)$-plus its derivatives in this case. For simplicity, however, we omit the relevant calculations. Also, similarly to the case of simple zeros, from the asymptotic behavior of the solution of the RHP we can reconstruct the solution of the Manakov system.

Theorem 4.8. Under the same hypotheses as those of Theorem 4.7, the solution of the defocusing Manakov system with NZBC (1.2) is reconstructed as

$$
\begin{align*}
q_{k}(x, t)=q_{+, k}- & i K_{o}(x, t)\left\{m_{k+1}^{\prime}\left(x, t, z_{o}\right)+\left[F_{o}+i\left(x-2 z_{o} t\right)\right] m_{k+1}\left(x, t, z_{o}\right)\right\}  \tag{4.9}\\
& -\frac{1}{2 \pi} \int_{\mathbb{R}}\left(\mathbf{M}^{-} \mathrm{e}^{-i \mathbf{K} \Theta} \mathbf{L e} \mathrm{e}^{i \mathbf{K} \Theta}\right)_{(k+1) 1}(x, t, \zeta) \mathrm{d} \zeta, \quad k=1,2 .
\end{align*}
$$

The trace formula is also different in the presence of double poles. Indeed, one can use an approach analogous to that used in deriving (3.9) to prove

$$
a_{11}(z)=\left(\frac{z-z_{o}}{z-z_{o}^{*}}\right)^{2} \exp \left\{-\frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{\log \left(1-\left|\rho_{1}(\zeta)\right|^{2}-\frac{\zeta^{2}}{\zeta^{2}-q_{o}^{2}}\left|\rho_{2}(\zeta)\right|^{2}\right)}{\zeta-z} \mathrm{~d} \zeta\right\} .
$$

As before, the trace formula for $b_{33}(z)$ is obtained from the above using the symmetries (2.35) and (2.43).

Similarly to before, we now restrict our attention to the reflectionless case.
Lemma 4.9. Under the same hypotheses as those in Theorem 4.7, the defocusing Manakov system with $N Z B C$ (1.2) is given by following system of linear equations:

$$
\begin{align*}
& \mu_{+, 1}\left(x, t, z_{o}^{*}\right)=\binom{1}{\left(i / z_{o}^{*}\right) \mathbf{q}_{+}}+  \tag{4.10a}\\
& \frac{K_{o}}{z_{o}^{*}-z_{o}} m^{\prime}\left(z_{o}\right) \\
&+\left[\frac{K_{o}}{\left(z_{o}^{*}-z_{o}\right)^{2}}+\frac{K_{o}}{z_{o}^{*}-z_{o}}\left(F_{o}+i\left(x-2 z_{o} t\right)\right)\right] m\left(z_{o}\right),  \tag{4.10b}\\
& \begin{aligned}
\frac{m\left(x, t, z_{o}\right)}{b_{33}\left(z_{o}\right)}= & \binom{0}{\mathbf{q}_{+}^{\perp} / q_{o}}+\left[-\frac{\bar{K}_{o}}{z_{o}-z_{o}^{*}}+\frac{i\left(z_{o}^{*}\right)^{3}}{q_{o}^{3}} \frac{\hat{K}_{o}}{z_{o}-q_{o}^{2} / z_{o}^{*}}\right] \mu_{+, 1}^{\prime}\left(z_{o}^{*}\right) \\
+\{ & -\frac{i z_{o}^{*}}{q_{o}} \frac{\hat{K}_{o}}{\left(z_{o}-q_{o}^{2} / z_{o}^{*}\right)^{2}}\left[1+\left(z_{o}-q_{o}^{2} / z_{o}^{*}\right)\left(\hat{F}_{o}+\frac{i\left(z_{o}^{*}\right)^{2}}{q_{o}^{2}}\left(x-2 z_{o}^{*} t\right)-\frac{z_{o}^{*}}{q_{o}^{2}}\right)\right]
\end{aligned} \\
&\left.\quad-\frac{\bar{K}_{o}}{\left(z_{o}-z_{o}^{*}\right)^{2}}\left[1+\left(z_{o}-z_{o}^{*}\right)\left(\bar{F}_{o}-i\left(x-2 z_{o}^{*} t\right)\right)\right]\right\} \mu_{+, 1}\left(z_{o}^{*}\right)
\end{align*}
$$

$$
\begin{align*}
& \frac{m^{\prime}\left(x, t, z_{o}\right)}{b_{33}\left(z_{o}\right)}=\left\{\frac{\bar{K}_{o}}{\left(z_{o}-z_{o}^{*}\right)^{3}}\left[2+\left(z_{o}-z_{o}^{*}\right)\left(\bar{F}_{o}-i\left(x-2 z_{o}^{*} t\right)\right)\right]\right.  \tag{4.10c}\\
&\left.\left.-\frac{i z_{o}^{*}}{q_{o}} \frac{\hat{K}_{o}}{\left(z_{o}-q_{o}^{2} / z_{o}^{*}\right)^{2}}\left[-\frac{2}{z_{o}-q_{o}^{2} / z_{o}^{*}}+\frac{z_{o}^{*}}{q_{o}^{2}}-\hat{F}_{o}-\frac{i\left(z_{o}^{*}\right)^{2}}{q_{o}^{2}}\left(x-2 z_{o}^{*} t\right)\right)\right]\right\} \mu_{+, 1}\left(z_{o}^{*}\right) \\
&+\left[\frac{\bar{K}_{o}}{\left(z_{o}-z_{o}^{*}\right)^{2}}-\frac{i\left(z_{o}^{*}\right)^{3}}{q_{o}^{3}} \frac{\hat{K}_{o}}{\left(z_{o}-q_{o}^{2} / z_{o}^{*}\right)^{2}}\right] \mu_{+, 1}^{\prime}\left(z_{o}^{*}\right)+\frac{b_{33}^{\prime}\left(z_{o}\right)}{\left(b_{33}\left(z_{o}\right)\right)^{2}} m\left(z_{o}\right)
\end{align*}
$$

$$
\begin{align*}
\mu_{+, 1}^{\prime}\left(x, t, z_{o}^{*}\right)=- & \binom{0}{i \mathbf{q}_{+} /\left(z_{o}^{*}\right)^{2}}-\frac{K_{o}}{\left(z_{o}^{*}-z_{o}\right)^{2}} m^{\prime}\left(z_{o}\right)  \tag{4.10~d}\\
& -\frac{K_{o}}{\left(z_{o}^{*}-z_{o}\right)^{3}}\left[2+\left(z_{o}^{*}-z_{o}\right)\left(F_{o}+i\left(x-2 z_{o} t\right)\right)\right] m\left(z_{o}\right)
\end{align*}
$$

where, as above, the ( $x, t$ )-dependence was omitted from the right-hand side of each equation for simplicity.

Note that the scattering coefficient $b_{33}(z)$ appearing in the above system is known in closed form in the reflectionless case thanks to the trace formulae.
4.4. Double-pole dark-bright solitons of the Manakov system. We now present an explicit solution of the defocusing Manakov system with NZBC corresponding to double zeros of the analytic scattering coefficients. Consider, for simplicity, the case of a discrete eigenvalue along the imaginary axis. Without loss of generality, we can parametrize the background state, the discrete eigenvalue, and the norming constants as

$$
\mathbf{q}_{+}=(1,0)^{T}, \quad z_{o}=i Z, \quad d_{o}=\frac{\kappa^{3 / 2}}{Z \nu} \mathrm{e}^{Z \xi+i \phi}, \quad \bar{f}_{o}=\kappa^{6} \nu n \mathrm{e}^{Z \xi+i(f-\phi)}
$$

where


Fig. 4. A solution of the defocusing Manakov system obtained when the analytic scattering coefficient $a_{11}(z)$ has a double zero at $z_{o}$, with $\mathbf{q}_{+}=(1,0)^{T}, z_{o}=.5 \mathrm{e}^{i \pi / 2}, d_{o}=i, \bar{f}_{o}=1$.

$$
\nu=1-Z^{2}, \quad \kappa=1+Z^{2},
$$

and with $\xi, \phi, n$, and $f$ arbitrary real constants and $0<Z<1$. Solving the linear system of equations (4.8) and inserting these expressions into (4.9), we obtain the double-pole dark-bright soliton solution

$$
\begin{equation*}
\mathbf{q}(x, t)=\frac{1}{q_{d}(x, t)}\binom{q_{1 n}(x, t)}{q_{2 n}(x, t)} \tag{4.11}
\end{equation*}
$$

with

$$
\begin{gathered}
q_{1 n}(x, t)=\kappa^{2} \cosh [2 \Xi(x)]+p_{1 Z}(x, t)+p_{1 f}(x, t) \\
q_{2 n}(x, t)=4 \mathrm{e}^{i\left(\phi+Z^{2} t\right)}\left\{\left(Z \kappa^{3 / 2} \nu x+s_{o}\right) \cosh [\Xi(x)]-\left(2 i Z^{2} \kappa^{3 / 2} \nu t-s_{1}\right) \sinh [\Xi(x)]\right\} \\
q_{d}(x, t)=\kappa^{2} \cosh [2 \Xi(x)]+p_{2 Z}(x, t)+p_{2 f}(x, t)
\end{gathered}
$$

and

$$
\begin{gathered}
\Xi(x)=Z(x-\xi), \quad p_{1 Z}(x, t)=-p_{2 Z}(x, t)+8 i Z^{2} \kappa \nu t+\left(2-4 Z^{4}+2 Z^{8}\right) / \kappa \nu, \\
p_{2 Z}(x, t)=2 Z^{2} \kappa \nu x^{2}+4 Z\left(1-3 Z^{2}\right) x+8 Z^{4} \kappa \nu t^{2} \\
+\left[3-Z^{2}\left(10-20 Z^{2}-2 Z^{4}-Z^{6}\right)\right] / \kappa \nu \\
p_{1 f}(x, t)=-p_{2 f}(x, t)-4 i \kappa^{1 / 2} \nu n \cos f, \\
p_{2 f}(x, t)=4 \kappa^{-1 / 2} n\left[-2 Z^{2} \kappa \nu t \cos f+\left(1-3 Z^{2}+Z \kappa \nu x\right) \sin f\right]+2 \nu n^{2} \\
s_{o}=\kappa \nu n \sin f+\left(1-3 Z^{2}\right) \kappa^{1 / 2}, \quad s_{1}=i \kappa \nu n \cos f+Z^{2} \kappa^{3 / 2}
\end{gathered}
$$

Figure 4 shows a typical profile of this solution. The center of mass of the solution is localized at $x=\xi$. The solution appears to be similar to a superposition of two darkbright solitons with the same velocity (which is zero in this case since the eigenvalue is along the imaginary axis). On the other hand, the two solitons attract each other and diverge logarithmically, as is evident from the figure. This behavior is very similar to that of double-pole solutions of the scalar focusing NLS with ZBC, although in that case the corresponding solutions are just bright solitons.

An especially simple expression is obtained when $n=0$ (i.e., by setting $\bar{f}_{o}=0$ ), in which case (4.11) reduces to

$$
q_{1 n}(x, t)=\kappa^{2} \cosh [2 \Xi(x)]+p_{1 Z}(x, t),
$$



Fig. 5. A solution of the defocusing Manakov system obtained when the analytic scattering coefficient $a_{11}(z)$ has a double zero at $z_{o}$, with $\mathbf{q}_{+}=(1,0)^{T}, z_{o}=.5 \mathrm{e}^{2 \pi i / 5}, d_{o}=2+i, \bar{f}_{o}=3 i$.

$$
\begin{gathered}
q_{2 n}(x, t)=4 \kappa^{1 / 2} \mathrm{e}^{i\left(\phi+Z^{2} t\right)}\left\{\left(1-3 Z^{2}+Z \kappa \nu x\right) \cosh [\Xi(x)]+Z^{2} \kappa(1-2 i \nu t) \sinh [\Xi(x)]\right\} \\
q_{d}(x, t)=\kappa^{2} \cosh [2 \Xi(x)]+p_{2 Z}(x, t)
\end{gathered}
$$

with $p_{1 Z}(x, t), p_{2 Z}(x, t)$, and $\Xi(x)$ as above. Note, however, that no choice of norming constants exists for which the two solitons are stationary with respect to each other. This situation is similar to the double-pole solutions of the scalar focusing NLS with ZBC. Also, in the limit $Z \rightarrow 1^{-}$, (4.11) reduces to the constant background solution.

Of course, double-pole solutions with discrete eigenvalues off the imaginary axis can also be easily obtained. An example of such a solution is shown in Figure 5. Modulo the nonzero velocity, the behavior of this solution is similar to that of the stationary solution. We emphasize, however, that while in the case of ZBC the moving solutions can be obtained from the stationary ones simply by applying a Galilean transformation, this is not the case with NZBC. This difference can be understood both from a physical and a spectral point of view. For the former, note that both the stationary and the moving solutions satisfy the same constant boundary conditions $\mathbf{q}(x, t) \rightarrow \mathbf{q}_{ \pm}$as $x \rightarrow \pm \infty$, whereas Galilean-boosted stationary solutions would have an oscillating phase with respect to $x$ as $x \rightarrow \pm \infty$. From a spectral point of view, note that for the Galilean-boosted stationary solution, the real part of the discrete eigenvalue is along the imaginary axis, i.e., above the midpoint of the branch cut. In contrast, for the traveling solution, the discrete eigenvalue does not lie directly above the branch cut. The same difference applies to soliton solutions obtained from simple zeros of the scattering coefficients.
5. Conclusions. As we have seen in the previous sections, unlike the case of ZBC, the IST for the Manakov system with NZBC presents significant differences from the IST for the scalar case (i.e., the NLS equation). The most obvious of these are (i) the need to introduce the adjoint problem to obtain auxiliary eigenfunctions and complete the eigenfunction bases; (ii) the more complicated symmetries among the eigenfunctions involving said auxiliary eigenfunctions; (iii) the existence of eigenvalues off of $C_{o}$ and the corresponding dark-bright soliton solutions; (iv) the existence of double-pole solutions.

Another important difference between the scalar case and the Manakov system is that for the latter, one cannot exclude the possibility of zeros along the continuous spectrum. Indeed, a direct expansion of the determinant of the scattering matrix
yields, for all $z \in \mathbb{R} \backslash\left\{ \pm q_{o}\right\}$,

$$
\operatorname{det} \mathbf{A}(z)=b_{11}(z) a_{11}(z)+b_{12}(z) a_{21}(z)+b_{13}(z) a_{31}(z),
$$

which, applying (2.35), yields

$$
\begin{equation*}
\left|a_{11}(z)\right|^{2}=1+\left|a_{21}(z)\right|^{2} / \gamma(z)+\left|a_{31}(z)\right|^{2} . \tag{5.1}
\end{equation*}
$$

Since $\gamma(z)<0$ for $z \in\left(-q_{o}, q_{o}\right)$, we cannot exclude possible zeros of $a_{11}(z)$ in the interval $\left(-q_{o}, q_{o}\right)$. Similar results follow for $a_{33}(z), b_{11}(z)$, and $b_{33}(z)$. Thus, this situation is more similar to that of the focusing NLS equation with ZBC, which is known to admit zeros of the analytic scattering coefficients along the real $k$-axis [5, 43].

It should be mentioned that in the scalar case, no area theorem is possible with NZBC [9,11]. That is, a class of potentials can be produced for which discrete eigenvalues exist for arbitrarily small deviations from the uniform background. Since every solution of the scalar NLS equation can be trivially extended to a solution of the Manakov system, these results imply that no area theorem is possible for the latter as well. (This is in contrast to the scalar case with ZBC, where a precise lower bound can be found for the $L^{1}$ norm of the potential for the existence of discrete eigenvalues [5, 28].) Similarly, (3.10) indicates that the continuous spectrum can provide a nonzero contribution to the asymptotic phase difference. Specific examples illustrating such situations were provided in $[10,11]$ for the scalar case, and of course extend trivially to the Manakov system.

From an applied point of view, we expect the results of this paper to be useful in characterizing recent experiments in nonlinear optics $[12,20,36]$ and Bose-Einstein condensation [22, 40]. Conversely, from a theoretical point of view, the results in this paper pave the way for studying several open problems: (i) an investigation of the possible existence of double zeros on $C_{o}$ (see [19] for a proof of the nonexistence of such zeros in the scalar case); (ii) an investigation of the possible existence of real spectral singularities (known to exist in the scalar focusing case $[4,25,43]$ and known not to exist in the scalar defocusing case [19]); (iii) a study of the long-time asymptotics using the Deift-Zhou method [16, 17] (see [24, 37, 38] for the scalar case); (iv) the development of an appropriate perturbation theory (see [2, 27] for the scalar case); (v) the extension of the present approach to the $N$-component case. (In this regard, we remark that the $N$-component case was recently studied in [32] using the approach of [7], but the results of [32] were incomplete due to the lack of a proper characterization of the symmetries of the analytic eigenfunctions. We believe that the novel approach to the symmetries presented in this work will provide the missing link to resolve this difficulty and allow the construction of nontrivial explicit multicomponent solutions.)

Finally, we reiterate that here the direct problem was developed without requiring that the asymptotic polarizations $\mathbf{q}_{+}$and $\mathbf{q}_{-}$be collinear. The advantage of imposing the condition $\left|\mathbf{q}_{+}^{\dagger} \mathbf{q}_{-}\right|=q_{o}$ is that it ensures that the limits of the analytic scattering coefficients as $z \rightarrow 0, \infty$ are at most a phase (cf. Corollary 2.31). In turn, this simplifies the asymptotic behavior of the meromorphic matrices $\mathbf{M}^{ \pm}(x, t, z)$ appearing in the RHP, as defined in (3.1) (cf. Lemma 3.2). On the other hand, there are no obstacles to formulating the inverse problem with general $\mathbf{q}_{ \pm}$, as long as these asymptotic vectors are not orthogonal (i.e., as long as $\mathbf{q}_{+}^{\dagger} \mathbf{q}_{-} \neq 0$ ). Doing so might be important as it might lead to new and possibly physically relevant exact solutions. Conversely, the case $\mathbf{q}_{+}^{\dagger} \mathbf{q}_{-}=0$ requires some additional care, as in this case the matrices $\mathbf{M}^{ \pm}(x, t, z)$ acquire additional poles as $z \rightarrow \infty$ and/or $z \rightarrow 0$, which then need to be subtracted in order to regularize the resulting RHP.

We plan to study some of the above problems in the near future.

Appendix. The following sections contain the proofs of all the results presented in the main text.
A.1. IST and the invariances of the Manakov system. Recall that if $q(x, t)$ is any solution of the scalar defocusing NLS equation, $\mathbf{q}_{\mathrm{nls}}(x, t)=(0, q(x, t))^{T}$ is a solution of the Manakov system. Also, if $\mathbf{q}(x, t)$ is any solution of the Manakov system and $\alpha, v, x_{o}$, and $t_{o}$ are any real constants, $\overline{\mathbf{q}}(x, t)=\mathrm{e}^{i \alpha} \mathbf{q}(x, t)$ and $\hat{\mathbf{q}}(x, t)=$ $\mathbf{q}\left(x-x_{o}, t-t_{o}\right)$ are solutions of the Manakov system as well. Finally, for any constant unitary $2 \times 2$ matrix $\mathbf{U}$ (i.e., $\mathbf{U U}^{\dagger}=\mathbf{U}^{\dagger} \mathbf{U}=\mathbf{I}$ ), $\check{\mathbf{q}}(x, t)=\mathbf{U q}(x, t)$ is also a solution. We next show how each transformation affects the IST.

Lemma A.1. Let $\phi_{ \pm, \mathrm{nls}}(x, t, z)$ and $\mathbf{A}_{\mathrm{nls}}(z)=\left(a_{i j, \mathrm{nls}}(z)\right)$ denote the $2 \times 2$ Jost solutions and scattering matrix of the IST for the scalar case, respectively. We have

$$
\begin{gathered}
\phi_{ \pm}(x, t, z)=\left(\begin{array}{ccc}
\phi_{ \pm, 11, \mathrm{nls}} & 0 & \mathrm{e}^{i \theta_{ \pm} \phi_{ \pm, 12, \mathrm{nls}}} \\
0 & \mathrm{e}^{-i \theta_{ \pm}} \mathrm{e}^{i \theta_{2}(x, t, z)} & 0 \\
\phi_{ \pm, 21, \mathrm{nls}} & 0 & \mathrm{e}^{i \theta_{ \pm} \phi_{ \pm, 22, \mathrm{nls}}}
\end{array}\right) \\
\mathbf{A}(z)=\left(\begin{array}{ccc}
a_{11, \mathrm{nls}}(z) & 0 & \mathrm{e}^{i \theta_{-}} a_{12, \mathrm{nls}}(z) \\
0 & \mathrm{e}^{i\left(\theta_{+}-\theta_{-}\right)} & 0 \\
\mathrm{e}^{-i \theta_{+}} a_{21, \mathrm{nls}}(z) & 0 & \mathrm{e}^{-i\left(\theta_{+}-\theta_{-}\right)} a_{22, \mathrm{nls}}(z)
\end{array}\right),
\end{gathered}
$$

where the normalization in [11] for the IST in the scalar case was used.
Lemma A.2. Let $\bar{\phi}_{ \pm}(x, t, z)$ and $\overline{\mathbf{A}}(z)$ be the Jost solutions and scattering matrix corresponding to $\overline{\mathbf{q}}(x, t)$. We have

$$
\begin{align*}
& \phi_{ \pm}(x, t, z)=\mathrm{e}^{i(\alpha / 2) \mathbf{J}} \bar{\phi}_{ \pm}(x, t, z) \operatorname{diag}\left(\mathrm{e}^{-i \alpha / 2}, \mathrm{e}^{3 i \alpha / 2}, \mathrm{e}^{-i \alpha / 2}\right)  \tag{A.2a}\\
& \mathbf{A}(z)=\operatorname{diag}\left(\mathrm{e}^{-i \alpha / 2}, \mathrm{e}^{3 i \alpha / 2}, \mathrm{e}^{-i \alpha / 2}\right) \overline{\mathbf{A}}(z) \operatorname{diag}\left(\mathrm{e}^{i \alpha / 2}, \mathrm{e}^{-3 i \alpha / 2}, \mathrm{e}^{i \alpha / 2}\right) \tag{A.2b}
\end{align*}
$$

Proof. Since $\mathrm{e}^{i(\alpha / 2) \mathbf{J}} \bar{\phi}_{ \pm}(x, t, z)$ solves the asymptotic scattering problem (2.2), we may choose $\overline{\mathbf{E}}_{ \pm}(z)=\mathrm{e}^{-i(\alpha / 2) \mathbf{J}} \mathbf{E}_{ \pm}(z) \operatorname{diag}\left(\mathrm{e}^{i \alpha / 2}, \mathrm{e}^{-3 i \alpha / 2}, \mathrm{e}^{i \alpha / 2}\right)$. Then since $\mathrm{e}^{i(\alpha / 2) \mathbf{J}} \bar{\phi}_{ \pm}$ and $\phi_{ \pm}$are both fundamental matrix solutions of the asymptotic scattering problem, there exists an invertible $3 \times 3$ matrix $\overline{\mathbf{C}}(z)$ such that $\phi_{ \pm}(x, t, z)=\mathrm{e}^{i(\alpha / 2) \mathbf{J}} \bar{\phi}_{ \pm}(x, t, z) \overline{\mathbf{C}}(z)$. Comparing the asymptotics as $x \rightarrow \pm \infty$ of $\phi_{ \pm}$with those of $\mathrm{e}^{i(\alpha / 2) \mathbf{J}} \overline{\bar{\phi}}_{ \pm} \overline{\mathbf{C}}$ yields $\overline{\mathbf{C}}(z)=\operatorname{diag}\left(\mathrm{e}^{-i \alpha / 2}, \mathrm{e}^{3 i \alpha / 2}, \mathrm{e}^{-i \alpha / 2}\right)$. Combining (A.2a) with the fact that $\bar{\phi}_{-}=\bar{\phi}_{+} \overline{\mathbf{A}}$ yields (A.2b).

The proofs of the remaining lemmas in this section are omitted since they are similar to the proof of Lemma A.2.

Lemma A.3. Let $\hat{\phi}_{ \pm}(x, t, z)$ and $\hat{\mathbf{A}}(z)$ be the Jost solutions and scattering matrix corresponding to $\hat{\mathbf{q}}(x, t)$. We have

$$
\phi_{ \pm}(x, t, z)=\hat{\phi}_{ \pm}(x, t, z) \mathrm{e}^{i \boldsymbol{\Theta}\left(x_{o}, t_{o}, z\right)}, \quad \mathbf{A}(z)=\mathrm{e}^{-i \boldsymbol{\Theta}\left(x_{o}, t_{o}, z\right)} \hat{\mathbf{A}}(z) \mathrm{e}^{i \boldsymbol{\Theta}\left(x_{o}, t_{o}, z\right)}
$$

Lemma A.4. Let $\check{\phi}_{ \pm}(x, t, z)$ and $\check{\mathbf{A}}(z)$ be the Jost solutions and scattering matrix corresponding to $\check{\mathbf{q}}(x, t)$. We have

$$
\begin{gathered}
\phi_{ \pm}(x, t, z)=\operatorname{diag}\left(1, \mathbf{U}^{\dagger}\right) \check{\phi}_{ \pm}(x, t, z) \operatorname{diag}\left(1, \mathrm{e}^{i u}, 1\right), \\
\mathbf{A}(z)=\operatorname{diag}\left(1, \mathrm{e}^{-i u}, 1\right) \check{\mathbf{A}}(z) \operatorname{diag}\left(1, \mathrm{e}^{i u}, 1\right),
\end{gathered}
$$

where $\operatorname{det} \mathbf{U}=\mathrm{e}^{i u}$, with $u \in \mathbb{R}$.
Finally, note that the Manakov system also possesses a Galilean invariance; i.e., if $\mathbf{q}(x, t)$ is a solution, so is $\breve{\mathbf{q}}(x, t)=\mathrm{e}^{i v(x-v t)} \mathbf{q}(x-2 v t, t)$. Note, however, that if $\mathbf{q}(x, t)$ does not vanish as $x \rightarrow \pm \infty, \breve{\mathbf{q}}(x, t)$ is outside the class of potentials for which the

IST can be applied. Therefore, no simple correspondence between the Jost solutions and scattering matrices can be established.

Definition A.5. We say that $\mathbf{q}(x, t)$ is a reducible solution of the Manakov system if there exists a constant unitary $2 \times 2$ matrix $\mathbf{U}$ such that $\mathbf{q}(x, t)=\mathbf{U}\left(0, q_{s}(x, t)\right)^{T}$, where $q_{s}(x, t)$ is a solution of the scalar defocusing NLS.

Note that there is no loss in generality in assuming that the zero in the above vector is in the first entry. We can then combine Lemma A. 1 with Lemma A. 4 to obtain the following.

Lemma A.6. If $\mathbf{q}(x, t)$ is reducible,
$\mathbf{A}(z)=\operatorname{diag}\left(1, \mathrm{e}^{-i u}, 1\right)\left(\begin{array}{ccc}a_{11, \mathrm{nls}}(z) & 0 & \mathrm{e}^{i \theta_{-}-a_{12, \mathrm{nls}}(z)} \\ 0 & \mathrm{e}^{i\left(\theta_{+}-\theta_{-}\right)} & 0 \\ \mathrm{e}^{-i \theta_{+}} a_{21, \mathrm{nls}}(z) & 0 & \mathrm{e}^{-i\left(\theta_{+}-\theta_{-}\right)} a_{22, \mathrm{nls}}(z)\end{array}\right) \operatorname{diag}\left(1, \mathrm{e}^{i u}, 1\right)$,
where $\operatorname{det} \mathbf{U}=\mathrm{e}^{i u}$, with $u \in \mathbb{R}$, and the $a_{i j, \mathrm{nls}}(z)$ are defined as in Lemma A.1.
A comparison with solutions of the scalar defocusing NLS [11] immediately yields the following.

Corollary A.7. If $\mathbf{q}(x, t)$ is reducible, the analytic scattering coefficients can only have zeros on $C_{o}$.

The converse of Corollary A. 7 is, however, not true due to the presence of radiation. Also, an immediate consequence of Definition A. 5 is the following.

Corollary A.8. If the analytic scattering coefficients have double zeros on $C_{o}$, then the corresponding solution of the defocusing Manakov system is not reducible.

## A.2. Analyticity of the eigenfunctions.

Proof of Theorem 2.1. We start by rewriting the first of the integral equations (2.16) that define the Jost eigenfunctions:
$\mu_{-}(x, t, z)=\mathbf{E}_{-}(z)\left[\mathbf{I}+\int_{-\infty}^{x} \mathrm{e}^{i(x-y) \boldsymbol{\Lambda}(z)} \mathbf{E}_{-}^{-1}(z) \Delta \mathbf{Q}_{-}(y, t) \mu_{-}(y, t, z) \mathrm{e}^{-i(x-y) \boldsymbol{\Lambda}(z)} \mathrm{d} y\right]$.
The limits of integration imply that $x-y$ is always positive for $\mu_{-}$(and always negative for $\mu_{+}$). Also, note that the matrix products in the right-hand side of (A.3) operate columnwise. In particular, letting $W(x, z)=\mathbf{E}_{-}^{-1} \mu_{-}$, for the first column $w$ of $W$, one has

$$
w(x, t, z)=\left(\begin{array}{l}
1  \tag{A.4}\\
0 \\
0
\end{array}\right)+\int_{-\infty}^{x} \mathbf{G}(x-y, z) \Delta \mathbf{Q}_{-}(y, t) \mathbf{E}_{-}(z) w(y, t, z) \mathrm{d} y
$$

where

$$
\begin{equation*}
\mathbf{G}(\xi, z)=\operatorname{diag}\left(1, \mathrm{e}^{i(k(z)+\lambda(z)) \xi}, \mathrm{e}^{2 i \lambda(z) \xi}\right) \mathbf{E}_{-}^{-1}(z) \tag{A.5}
\end{equation*}
$$

Now, we introduce a Neumann series representation for $w$ :

$$
\begin{equation*}
w(x, z)=\sum_{n=0}^{\infty} w^{(n)} \tag{A.6a}
\end{equation*}
$$

with

$$
w^{(0)}=\left(\begin{array}{l}
1  \tag{A.6b}\\
0 \\
0
\end{array}\right), \quad w^{(n+1)}(x, t, z)=\int_{-\infty}^{x} \mathbf{C}(x, y, t, z) w^{(n)}(y, t, z) \mathrm{d} y
$$

and where $\mathbf{C}(x, y, t, z)=\mathbf{G}(x-y, z) \Delta \mathbf{Q}(y, t) \mathbf{E}_{-}(z)$. Introducing the $L^{1}$ vector norm $\|w\|=\left|w_{1}\right|+\left|w_{2}\right|+\left|w_{3}\right|$ and the corresponding subordinate matrix norm $\|C\|$, we then have

$$
\begin{equation*}
\left\|w^{(n+1)}(x, t, z)\right\| \leq \int_{-\infty}^{x}\|\mathbf{C}(x, y, t, z)\|\left\|w^{(n)}(y, t, z)\right\| \mathrm{d} y \tag{A.7}
\end{equation*}
$$

Note that $\left\|\mathbf{E}_{ \pm}\right\| \leq 1+q_{o} /|z|$ and $\left\|\mathbf{E}_{ \pm}^{-1}\right\| \leq\left(1+q_{o} /|z|\right) /|\gamma(z)|$. The properties of the matrix norm imply

$$
\begin{align*}
\|\mathbf{C}(x, y, t, z)\| & \leq\left\|\operatorname{diag}\left(1, \mathrm{e}^{i(k+\lambda)(x-y)}, \mathrm{e}^{2 i \lambda(x-y)}\right)\right\|\left\|\mathbf{E}_{-}(z)\right\|\|\Delta \mathbf{Q}(y, t)\|\left\|\mathbf{E}_{-}^{-1}(z)\right\|  \tag{A.8}\\
& \leq c(z)\left(1+\mathrm{e}^{-\left(k_{\mathrm{im}}(z)+\lambda_{\mathrm{im}}(z)\right)(x-y)}+\mathrm{e}^{-2 \lambda_{\mathrm{im}}(z)(x-y)}\right)\left\|\mathbf{q}(y, t)-\mathbf{q}_{-}\right\|,
\end{align*}
$$

where $\lambda_{\mathrm{im}}(z)=\operatorname{Im} \lambda(z), k_{\mathrm{im}}(z)=\operatorname{Im} k(z)$, and $c(z)=\left(1+q_{o} /|z|\right)^{2} /|\gamma(z)|$. Now, recall that $\operatorname{Im} \lambda(z)>0$ for $z$ in $\mathbb{C}_{I}$. On the other hand, $c(z) \rightarrow \infty$ as $z \rightarrow \pm q_{o}$. Thus, given $\epsilon>0$, we restrict our attention to the domain $\left(\mathbb{C}_{I}\right)_{\epsilon}=\mathbb{C}_{I} \backslash\left(B_{\epsilon}\left(q_{o}\right) \cup B_{\epsilon}\left(-q_{o}\right)\right)$, where $B_{\epsilon}\left(z_{o}\right)=\left\{z \in \mathbb{C}:\left|z-z_{o}\right|<\epsilon q_{o}\right\}$. It is straightforward to show that $c_{\epsilon}=$ $\max _{z \in\left(\mathbb{C}_{I}\right)_{\epsilon}} c(z)=2+2 / \epsilon$. Next, we prove that for all $z \in\left(\mathbb{C}_{I}\right)_{\epsilon}$ and for all $n \in \mathbb{N}$,

$$
\begin{equation*}
\left\|w^{(n)}(x, t, z)\right\| \leq \frac{M^{n}(x, t)}{n!} \tag{A.9a}
\end{equation*}
$$

where

$$
\begin{equation*}
M(x, t)=2 c_{\epsilon} \int_{-\infty}^{x}\left\|\mathbf{q}(y, t)-\mathbf{q}_{-}\right\| \mathrm{d} y \tag{A.9b}
\end{equation*}
$$

We will prove the result by induction, following [3]. The claim is trivially true for $n=$ 0 . Also, note that for all $z \in \overline{\mathbb{C}_{I}}$ and for all $y \leq x$, one has $1+\mathrm{e}^{-\left(k_{\mathrm{im}}(z)+\lambda_{\mathrm{im}}(z)\right)(x-y)}+$ $\mathrm{e}^{-2 \lambda_{\mathrm{im}}(x-y)} \leq 3$. Then, if (A.9a) holds for $n=j$, (A.7) implies

$$
\begin{equation*}
\left\|w^{(j+1)}(x, t, z)\right\| \leq \frac{3 c_{\epsilon}}{j!} \int_{-\infty}^{x}\|\mathbf{q}(y, t)-\mathbf{q}-\| M^{j}(y, t) \mathrm{d} y=\frac{1}{j!(j+1)} M^{j+1}(x, t) \tag{A.10}
\end{equation*}
$$

proving the induction step (namely, that the validity of (A.9a) for $n=j$ implies its validity for $n=j+1)$. Thus, if $\mathbf{q}(x, t)-\mathbf{q}_{-} \in L^{1}(-\infty, a]$ for all finite $a \in \mathbb{R}$ and for all $\epsilon>0$, then the Neumann series converges absolutely and uniformly with respect to $x \in(-\infty, a)$ and to $z \in\left(\mathbb{C}_{I}\right)_{\epsilon}$. Similar results hold for $\mu_{+}(x, t, z)$. Since a uniformly convergent series of analytic functions converges to an analytic function, this demonstrates the validity of (2.17). Note that since $\mathbf{q}_{+} \neq \mathbf{q}_{-}$in general, $\mathbf{q}(x, t)-\mathbf{q}_{-} \notin$ $L_{1}(\mathbb{R})$, and therefore one cannot take $a=\infty$. This problem can be resolved using an approach similar to that of [32] or alternatively by deriving a different set of integral equations for the Jost eigenfunctions, as discussed in the following section. Note also that, as in the scalar case, additional conditions need to be imposed on the potential to establish convergence of the Neumann series at the branch points [18].
A.3. Alternative integral representation for the Jost eigenfunctions. In order to derive the analyticity properties of the scattering coefficients, we found it necessary to introduce an alternative integral representation for the Jost eigenfunctions. While the resulting equations are slightly more complicated than the standard integral equations (2.16), this representation has the advantage of allowing one to prove explicitly that $\mu_{ \pm}(x, t, z)$ remain bounded for all $x \in \mathbb{R}$ in their regions of analyticity.

We follow an approach similar to that used in [18] for the scalar case. Since the scattering matrix is time-independent, it is sufficient to do the calculations at $t=0$. With this understanding, we omit the time dependence from the potential and the eigenfunctions throughout this subsection.

We first note that the scattering problem (2.13) is equivalent to the problem

$$
\begin{equation*}
\phi_{x}=\overline{\mathbf{X}}(x, z) \phi+\left(\mathbf{Q}(x)-\mathbf{Q}_{f}(x)\right) \phi \tag{A.11}
\end{equation*}
$$

where

$$
\begin{equation*}
\overline{\mathbf{X}}(x, z)=H(x) \mathbf{X}_{+}(z)+H(-x) \mathbf{X}_{-}(z), \quad \mathbf{Q}_{f}(x)=H(x) \mathbf{Q}_{+}+H(-x) \mathbf{Q}_{-} \tag{A.12}
\end{equation*}
$$

and $H(x)$ denotes the Heaviside function (namely, $H(x)=1$ if $x \geq 0$ and $H(x)=0$ otherwise). The advantage of using (A.11) instead of (2.13) is that the "forcing" term $\mathbf{Q}-\mathbf{Q}_{f}$ vanishes both as $x \rightarrow-\infty$ and as $x \rightarrow \infty$, which leads to integral equations that are better behaved. (Correspondingly, the factorized problem (A.11) is now the same for both $\phi_{-}$and $\phi_{+}$. ) For $z \in \mathbb{R}$, we introduce fundamental eigenfunctions $\bar{\phi}_{ \pm}(x, z)$ as square matrix solutions of (A.11) satisfying

$$
\begin{equation*}
\bar{\phi}_{ \pm}(x, z)=\mathrm{e}^{x \mathbf{X}_{ \pm}(z)}[\mathbf{I}+o(1)], \quad x \rightarrow \pm \infty \tag{A.13}
\end{equation*}
$$

By solving (A.11) in a way similar to that of (2.16), we obtain

$$
\begin{align*}
\bar{\phi}_{-}(x, z) & =\mathbf{G}_{f}(x, 0, z)+\int_{-\infty}^{x} \mathbf{G}_{f}(x, y, z)\left[\mathbf{Q}(y)-\mathbf{Q}_{f}(y)\right] \bar{\phi}_{-}(y, z) \mathrm{d} y  \tag{A.14a}\\
\bar{\phi}_{+}(x, z) & =\mathbf{G}_{f}(x, 0, z)-\int_{x}^{\infty} \mathbf{G}_{f}(x, y, z)\left[\mathbf{Q}(y)-\mathbf{Q}_{f}(y)\right] \bar{\phi}_{+}(y, z) \mathrm{d} y \tag{A.14b}
\end{align*}
$$

where $\mathbf{G}_{f}(x, y, z)$ is the special solution of the homogeneous problem, i.e., $\mathbf{G}_{x}(x, y, z)=$ $\overline{\mathbf{X}}(x, z) \mathbf{G}(x, y, z)$, satisfying the "initial conditions" $\mathbf{G}(x, x, z)=\mathbf{I}$. Namely,

$$
\mathbf{G}_{f}(x, y, z)= \begin{cases}\mathrm{e}^{(x-y) \mathbf{X}_{+}(z)}, & x, y \geq 0  \tag{A.15}\\ \mathrm{e}^{(x-y) \mathbf{X}_{-}(z)}, & x, y \leq 0 \\ \mathrm{e}^{x \mathbf{X}_{+}(z)} \mathrm{e}^{-y \mathbf{X}_{-}(z)}, & x,-y \geq 0 \\ \mathrm{e}^{x \mathbf{X}_{-}(z)} \mathrm{e}^{-y \mathbf{X}_{+}(z)}, & x,-y \leq 0\end{cases}
$$

Using (A.14), we conclude that

$$
\begin{equation*}
\bar{\phi}_{ \pm}(x, z)=\mathbf{G}_{f}(x, 0, z)\left[\mathbf{A}_{\mp}(z)+o(1)\right], \quad x \rightarrow \mp \infty, \quad z \in \mathbb{R} \tag{A.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{A}_{\mp}(z)=\mathbf{I} \mp \int_{\mathbb{R}} \mathbf{G}_{f}(0, y, z)\left[\mathbf{Q}(y)-\mathbf{Q}_{f}(y)\right] \bar{\phi}_{ \pm}(y, z) \mathrm{d} y \tag{A.17}
\end{equation*}
$$

Since $\mathrm{e}^{x \mathbf{X}_{ \pm}(z)}$ are bounded for $x \in \mathbb{R}$ when $z \in \mathbb{R}$, assuming that $\mathbf{Q}(x)-\mathbf{Q}_{f}(x) \in L^{1}(\mathbb{R})$ and applying Gronwall's inequality implies $\bar{\phi}_{ \pm}(x, z)$ are bounded as $x \rightarrow \mp \infty$. In addition, comparing (A.15) with the solutions of the asymptotic scattering problem (2.2) yields $\bar{\phi}_{ \pm}(x, z) \mathbf{E}_{ \pm}(z)=\phi_{ \pm}(x, z)$, so (A.14) imply

$$
\begin{equation*}
\phi_{-}(x, z)=\mathbf{G}_{f}(x, 0, z) \mathbf{E}_{-}(z)+\int_{-\infty}^{x} \mathbf{G}_{f}(x, y, z)\left[\mathbf{Q}(y)-\mathbf{Q}_{f}(y)\right] \phi_{-}(y, z) \mathrm{d} y \tag{A.18a}
\end{equation*}
$$

$$
\begin{equation*}
\phi_{+}(x, z)=\mathbf{G}_{f}(x, 0, z) \mathbf{E}_{+}(z)-\int_{x}^{\infty} \mathbf{G}_{f}(x, y, z)\left[\mathbf{Q}(y)-\mathbf{Q}_{f}(y)\right] \phi_{+}(y, z) \mathrm{d} y \tag{A.18b}
\end{equation*}
$$

Note that (A.18a) coincides with (2.16a) for all $x \leq 0$, and (A.18b) coincides with (2.16b) for all $x \geq 0$. Additionally, assuming $\mathbf{q}(x)-\mathbf{q}_{+} \in L^{1}(0, \infty)$ and $\mathbf{q}(x)-\mathbf{q}_{-} \in$ $L^{1}(-\infty, 0)$ implies $\mathbf{Q}(x)-\mathbf{Q}_{f}(x) \in L^{1}(\mathbb{R})$, so we can use this information and (A.18) to prove Theorem 2.1 as well as to establish that $\mu_{ \pm}(x, z)=\phi_{ \pm}(x, z) \mathrm{e}^{-i x \boldsymbol{\Lambda}(z)}$ remain bounded as $x \rightarrow \mp \infty$. This result will be instrumental in proving the analyticity of the entries of the scattering matrix (see Theorem 2.3 and the following section).

## A.4. Analyticity of the scattering matrix.

Proof of Theorem 2.3. We compare the asymptotics as $x \rightarrow \infty$ of $\phi_{-}(x, z)$ from (A.16) with those of $\phi_{+}(x, z) \mathbf{A}(z)$ from (2.11) to obtain

$$
\begin{equation*}
\mathbf{A}(z)=\mathbf{E}_{+}^{-1}(z) \mathbf{A}_{+}(z) \mathbf{E}_{-}(z) \tag{A.19}
\end{equation*}
$$

The expression in (A.19) simplifies to the following integral representation for the scattering matrix:

$$
\begin{align*}
\mathbf{A}(z)= & \int_{0}^{\infty} \mathrm{e}^{-i y \boldsymbol{\Lambda}(z)} \mathbf{E}_{+}^{-1}(z)\left[\mathbf{Q}(y)-\mathbf{Q}_{+}\right] \phi_{-}(y, z) \mathrm{d} y  \tag{A.20}\\
& +\mathbf{E}_{+}^{-1}(z) \mathbf{E}_{-}(z)\left[\mathbf{I}+\int_{-\infty}^{0} \mathrm{e}^{-i y \boldsymbol{\Lambda}(z)} \mathbf{E}_{-}^{-1}(z)\left[\mathbf{Q}(y)-\mathbf{Q}_{-}\right] \phi_{-}(y, z) \mathrm{d} y\right]
\end{align*}
$$

A similar expression can be found for $\mathbf{B}(z)$. We can now examine the individual entries of (A.20). In particular, the 1,1 entry of (A.20) yields an integral representation for $a_{11}(z)$, and the corresponding two integrands from (A.20) are, respectively,

$$
\begin{align*}
& \frac{1}{\gamma(z)} \mathrm{e}^{i \lambda y}\left[\frac{i}{z} \mathbf{q}_{+}^{\dagger} \Delta \mathbf{q}(y) \phi_{-, 11}(y, z)+\Delta r_{1}(y) \phi_{-, 21}(y, z)+\Delta r_{2}(y) \phi_{-, 31}(y, z)\right]  \tag{A.21a}\\
& \sum_{j=1}^{3}\left[c_{11}(z) T_{1 j}(y, z)+\right. c_{12}(z) T_{2 j}(y, z) \mathrm{e}^{-i(k+\lambda) y}  \tag{A.21b}\\
&\left.+c_{13}(z) T_{3 j}(y, z) \mathrm{e}^{-2 i \lambda y}\right] \phi_{-, j 1}(y, z) \mathrm{e}^{i \lambda y}
\end{align*}
$$

where $\Delta \mathbf{q}(x)=\mathbf{q}(x)-\mathbf{q}_{f}(x)$ (similarly for $\left.\Delta \mathbf{r}(x)\right)$ and

$$
\mathbf{E}_{+}^{-1}(z) \mathbf{E}_{-}(z)=\left(c_{i j}(z)\right), \quad \mathbf{E}_{-}^{-1}(z)\left[\mathbf{Q}(y)-\mathbf{Q}_{-}\right]=\left(T_{i j}(y, z)\right)
$$

Recall that $\phi_{-, 1}(y, z) \mathrm{e}^{i \lambda(z) y}$ is analytic for $\operatorname{Im} z>0$ and bounded over $y \in \mathbb{R}$, so each term in (A.21a) is analytic for $\operatorname{Im} z>0$ and bounded when $y>0$. Thus, the first integral in the representation (A.20) for $a_{11}(z)$ defines an analytic function for all $\operatorname{Im} z>0$. Further, recalling that $\operatorname{Im} \lambda(z)$ and $\operatorname{Im}(k(z)+\lambda(z))$ have the same sign, we conclude that each term in (A.21b) is analytic for $\operatorname{Im} z>0$ and bounded when $y<0$, so the second integral also defines an analytic function for all $\operatorname{Im} z>0$. Thus, the integral representation (A.20) for $a_{11}(z)$ can be analytically extended off the real $z$-axis onto the upper half of the $z$-plane. The remainder of Theorem 2.3 is proved similarly.

## A.5. Adjoint problem.

Proof of Proposition 2.4. The result follows by noting that, in the defocusing case, $\mathbf{Q}^{T}=\mathbf{Q}^{*}$, implying $\mathbf{Q}^{\dagger}=\mathbf{Q}$, and by using the fact that for any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{3}$ one has

$$
\begin{gathered}
{[(\mathbf{J} \mathbf{u}) \times \mathbf{v}]+[\mathbf{u} \times(\mathbf{J} \mathbf{v})]+[\mathbf{u} \times \mathbf{v}]+[(\mathbf{J} \mathbf{u}) \times(\mathbf{J} \mathbf{v})]=\mathbf{0},} \\
\mathbf{J}[\mathbf{u} \times \mathbf{v}]=(\mathbf{J} \mathbf{u}) \times(\mathbf{J} \mathbf{v}), \\
\mathbf{Q}[\mathbf{u} \times \mathbf{v}]+\left[\left(\mathbf{Q}^{T} \mathbf{u}\right) \times \mathbf{v}\right]+\left[\mathbf{u} \times\left(\mathbf{Q}^{T} \mathbf{v}\right)\right]=\mathbf{0}, \\
\mathbf{J Q}^{2}[\mathbf{u} \times \mathbf{v}]+\left[\left(\mathbf{J}\left(\mathbf{Q}^{T}\right)^{2} \mathbf{u}\right) \times \mathbf{v}\right]+\left[\mathbf{u} \times\left(\mathbf{J}\left(\mathbf{Q}^{T}\right)^{2} \mathbf{v}\right)\right]=\mathbf{0} .
\end{gathered}
$$

Proof of Lemma 2.6. We verify (2.28a) with $j=3$. Equations (2.24) and (2.23) yield

$$
\mathbf{v}_{ \pm}(x, t, z)=-\mathrm{e}^{-i \theta_{1}(x, t, z)} \mathbf{E}_{ \pm, 3}(z)+o(1), \quad x \rightarrow \pm \infty
$$

However, $\mathbf{v}_{ \pm}$must be a linear combination of the columns of $\phi_{ \pm}$, so there exist scalar functions $a_{ \pm}(z), b_{ \pm}(z)$, and $c_{ \pm}(z)$ such that $\mathbf{v}_{ \pm}(x, t, z)=a_{ \pm}(z) \phi_{ \pm, 1}(x, t, z)+$ $b_{ \pm}(z) \phi_{ \pm, 2}(x, t, z)+c_{ \pm}(z) \phi_{ \pm, 3}(x, t, z)$. Comparing the asymptotics as $x \rightarrow \pm \infty$ in (2.11) with those of $\mathbf{v}_{ \pm}$yields $a_{ \pm}(z)=b_{ \pm}(z)=0$ and $c_{ \pm}(z)=-1$. The rest of Lemma 2.6 is proved similarly.

Proof of Corollary 2.7. We suppress the $x$-, $t$-, and $z$-dependence for brevity. Combining (2.28) and (2.19) yields $\tilde{\phi}_{+, 1}=\left(b_{22} b_{33}-b_{32} b_{23}\right) \tilde{\phi}_{-, 1}+\gamma\left(b_{32} b_{13}-b_{12} b_{33}\right) \tilde{\phi}_{-, 2}+$ $\left(b_{22} b_{13}-b_{12} b_{23}\right) \tilde{\phi}_{-, 3}$. Combining this with (2.25) yields

$$
\tilde{b}_{11}=b_{22} b_{33}-b_{32} b_{23}, \quad \tilde{b}_{21}=\gamma\left(b_{12} b_{33}-b_{32} b_{13}\right), \quad \tilde{b}_{31}=b_{22} b_{13}-b_{12} b_{23}
$$

Using a similar process, we find that

$$
\begin{gathered}
\tilde{b}_{12}=\frac{1}{\gamma}\left(b_{33} b_{21}-b_{23} b_{31}\right), \quad \tilde{b}_{22}=b_{33} b_{11}-b_{13} b_{31}, \quad \tilde{b}_{32}=\frac{1}{\gamma}\left(b_{13} b_{21}-b_{23} b_{11}\right), \\
\tilde{b}_{13}=b_{31} b_{22}-b_{21} b_{32}, \quad \tilde{b}_{23}=\gamma\left(b_{31} b_{12}-b_{11} b_{32}\right), \quad \tilde{b}_{33}=b_{11} b_{22}-b_{21} b_{12}
\end{gathered}
$$

Next, note that

$$
\mathbf{A}^{T}=\left(\begin{array}{lll}
b_{22} b_{33}-b_{23} b_{32} & b_{23} b_{31}-b_{21} b_{33} & b_{21} b_{32}-b_{22} b_{31} \\
b_{13} b_{32}-b_{12} b_{33} & b_{11} b_{33}-b_{13} b_{31} & b_{12} b_{31}-b_{11} b_{32} \\
b_{12} b_{23}-b_{13} b_{22} & b_{13} b_{21}-b_{11} b_{23} & b_{11} b_{22}-b_{12} b_{21}
\end{array}\right) .
$$

Combining all this information, we finally obtain (2.29).
Proof of Corollary 2.8. Substituting (2.25) into (2.27) yields the following for $z \in \mathbb{R}$ :
(A.22a)

$$
\left.\gamma(z) \bar{\chi}(x, t, z)=\mathrm{e}^{i \theta_{2}(x, t, z)} \mathbf{J}\left[\left[\tilde{b}_{23}(z) \tilde{\phi}_{-, 2}(x, t, z)+\tilde{b}_{33}(z) \tilde{\phi}_{-, 3}(x, t, z)\right] \times \tilde{\phi}_{-, 1}(x, t, z)\right]\right]
$$

$$
\begin{equation*}
\gamma(z) \chi(x, t, z)=\mathrm{e}^{i \theta_{2}(x, t, z)} \mathbf{J}\left[\left[\tilde{b}_{11}(z) \tilde{\phi}_{-, 1}(x, t, z)+\tilde{b}_{21}(z) \tilde{\phi}_{-, 2}(x, t, z)\right] \times \tilde{\phi}_{-, 3}(x, t, z)\right] \tag{A.22b}
\end{equation*}
$$

Applying (2.28) to (A.22) yields the following for $z \in \mathbb{R}$ :

$$
\begin{equation*}
\gamma(z) \bar{\chi}(x, t, z)=\tilde{b}_{23}(z) \phi_{-, 3}(x, t, z)-\tilde{b}_{33}(z) \gamma(z) \phi_{-, 2}(x, t, z) \tag{A.23a}
\end{equation*}
$$

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$$
\begin{equation*}
\gamma(z) \chi(x, t, z)=\tilde{b}_{11}(z) \gamma(z) \phi_{-, 2}(x, t, z)+\tilde{b}_{21}(z) \phi_{-, 1}(x, t, z) \tag{A.23b}
\end{equation*}
$$

We apply (2.29) to (A.23) to obtain (2.30a). Similarly, we obtain

$$
\begin{align*}
& \gamma(z) \bar{\chi}(x, t, z)=-\tilde{a}_{11}(z) \gamma(z) \phi_{+, 2}(x, t, z)-\tilde{a}_{21}(z) \phi_{+, 1}(x, t, z)  \tag{A.24a}\\
& \gamma(z) \chi(x, t, z)=-\tilde{a}_{23}(z) \phi_{+, 3}(x, t, z)+\tilde{a}_{33}(z) \gamma(z) \phi_{+, 2}(x, t, z) \tag{A.24b}
\end{align*}
$$

We then combine (A.24) with (2.29) to obtain (2.30b).

## A.6. Symmetries.

Proof of Proposition 2.10. Let $\phi(x, t, z)$ be a nonsingular solution of the Lax pair (2.1a). Then $\phi_{x}^{\dagger}=\phi^{\dagger} \mathbf{X}^{\dagger}$ and $\phi_{t}^{\dagger}=\phi^{\dagger} \mathbf{T}^{\dagger}$. Indeed, since $\mathbf{Q}^{\dagger}=\mathbf{Q}$ and $z \in \mathbb{R}$,

$$
\begin{gathered}
\mathbf{w}_{x}=-\mathbf{J}\left(\phi^{\dagger}\right)^{-1} \phi_{x}^{\dagger}\left(\phi^{\dagger}\right)^{-1}=-\mathbf{J}(i k \mathbf{J}+\mathbf{Q}) \mathbf{J} \mathbf{w}=\mathbf{X} \mathbf{w} \\
\mathbf{w}_{t}=-\mathbf{J}\left(\phi^{\dagger}\right)^{-1} \phi_{t}^{\dagger}\left(\phi^{\dagger}\right)^{-1}=-\mathbf{J}\left(-2 i k^{2} \mathbf{J}+i \mathbf{J}\left(-\mathbf{Q}_{x}-\mathbf{Q}^{2}+q_{o}^{2}\right)-2 k \mathbf{Q}\right) \mathbf{J} \mathbf{w}=\mathbf{T} \mathbf{w}
\end{gathered}
$$

Thus, $\mathbf{w}$ is a solution of the Lax pair.
Proof of Lemma 2.11. Define

$$
\begin{equation*}
\mathbf{w}_{ \pm}(x, t, z)=\mathbf{J}\left(\phi_{ \pm}^{\dagger}(x, t, z)\right)^{-1}, \quad z \in \mathbb{R} \tag{A.25}
\end{equation*}
$$

Also, note that for all $z \in \mathbb{C}$,

$$
\begin{equation*}
\left(\mathrm{e}^{i \boldsymbol{\Theta}\left(x, t, z^{*}\right)}\right)^{\dagger}=\mathrm{e}^{-i \boldsymbol{\Theta}(x, t, z)} \tag{A.26}
\end{equation*}
$$

It is easy to see that

$$
\begin{equation*}
\mathbf{w}_{ \pm}(x, t, z)=\mathbf{J}\left(\mathbf{E}_{ \pm}^{\dagger}(z)\right)^{-1} \mathrm{e}^{i \boldsymbol{\Theta}(x, t, z)}+o(1), \quad x \rightarrow \pm \infty \tag{A.27}
\end{equation*}
$$

Since both $\mathbf{w}_{ \pm}$and $\phi_{ \pm}$are fundamental matrix solutions of the Lax pair (2.1a), there must exist an invertible $3 \times 3$ matrix $\mathbf{C}(z)$ such that (2.41) holds. Comparing the asymptotics from (A.27) to those from (2.11), we then obtain the desired result.

Proof of Lemma 2.12. Using (2.32), we obtain the following for $z \in \mathbb{R}$ :

$$
\phi_{ \pm}^{*}=\mathbf{J}\left(\left[\phi_{ \pm, 2} \times \phi_{ \pm, 3}\right],\left[\phi_{ \pm, 3} \times \phi_{ \pm, 1}\right],\left[\phi_{ \pm, 1} \times \phi_{ \pm, 2}\right]\right) \mathbf{C} / \operatorname{det} \phi_{ \pm}
$$

where we have suppressed the $x$-, $t$-, and $z$-dependence for brevity. We can then apply (2.30a) and (2.30b) to obtain

$$
\begin{align*}
\phi_{-, 1}^{*}(x, t, z) & =-\frac{1}{a_{33}(z)} \mathbf{J}\left[\bar{\chi} \times \phi_{-, 3}\right](x, t, z) \mathrm{e}^{-i \theta_{2}(x, t, z)}  \tag{A.28a}\\
\phi_{+, 1}^{*}(x, t, z) & =\frac{1}{b_{33}(z)} \mathbf{J}\left[\chi \times \phi_{+, 3}\right](x, t, z) \mathrm{e}^{-i \theta_{2}(x, t, z)} \tag{A.28b}
\end{align*}
$$

Recalling the analyticity properties of each function in (A.28) allows us to apply the Schwarz reflection principle to obtain (2.34a) and (2.34b). The rest of (2.34) is obtained in a similar manner.

Proof of Corollary 2.14. Taking into consideration the boundary conditions (2.11) and the corresponding boundary conditions for the adjoint eigenfunctions, we obtain $\phi_{ \pm}^{*}(x, t, z)=\tilde{\phi}_{ \pm}(x, t, z)$ (for $\left.z \in \mathbb{R}\right)$, and thus, by the Schwarz reflection principle,

$$
\begin{equation*}
\phi_{ \pm, 1}^{*}\left(x, t, z^{*}\right)=\tilde{\phi}_{ \pm, 1}(x, t, z), \quad \operatorname{Im} z \gtrless 0 \tag{A.29a}
\end{equation*}
$$

$$
\begin{equation*}
\phi_{ \pm, 3}^{*}\left(x, t, z^{*}\right)=\tilde{\phi}_{ \pm, 3}(x, t, z), \quad \operatorname{Im} z \lessgtr 0 \tag{A.29b}
\end{equation*}
$$

We can then combine (A.29) and (2.27) to obtain (2.38).
Proof of Lemma 2.16. For $z \in \mathbb{R}$, define $\mathbf{W}_{ \pm}(x, t, z)=\phi_{ \pm}\left(x, t, q_{o}^{2} / z\right)$. Since $\mathbf{W}_{ \pm}$ and $\phi_{ \pm}$both solve the Lax pair (2.1a), there must exist an invertible $3 \times 3$ matrix $\boldsymbol{\Pi}(z)$ satisfying (2.40). Note that

$$
\begin{equation*}
\boldsymbol{\Theta}\left(x, t, q_{o}^{2} / z\right)=\mathbf{K} \boldsymbol{\Theta}(x, t, z) \tag{A.30}
\end{equation*}
$$

where $\mathbf{K}=\operatorname{diag}(-1,1,-1)$. Comparing the asymptotics of (2.40) with the asymptotics from (2.11), we have

$$
\mathbf{E}_{ \pm}\left(q_{o}^{2} / z\right) \mathrm{e}^{i \mathbf{K} \boldsymbol{\Theta}(x, t, z)} \boldsymbol{\Pi}(z)=\mathbf{E}_{ \pm}(z) \mathrm{e}^{i \boldsymbol{\Theta}(x, t, z)}
$$

which yields (2.41).

## A.7. Discrete eigenvalues and bound states.

Proof of Lemma 2.19. It is easy to show that if $\mathbf{v}(x, t, k)=\left(v_{1}, v_{2}, v_{3}\right)^{T}$ is any nontrivial solution of the scattering problem,
(A.31) $-i\left(k-k^{*}\right) \sum_{n=1}^{3}\left|v_{n}(x, t, k)\right|^{2}=\frac{\partial}{\partial x}\left[\left|v_{1}(x, t, k)\right|^{2}-\left|v_{2}(x, t, k)\right|^{2}-\left|v_{3}(x, t, k)\right|^{2}\right]$.

Now, integrate (A.31) from $-\infty$ to $\infty$. If $\mathbf{v}(x, t, k) \in L^{2}(\mathbb{R})$, the right-hand side is zero, but since $\int_{\mathbb{R}}\|\mathbf{v}(x, t, k)\|^{2} \mathrm{~d} x \neq 0$, this implies $k^{*}=k$, i.e., $z \in \mathbb{R}$ or $z \in C_{o}$. But for $z \in \mathbb{R}$, the eigenfunctions do not decay as $x \rightarrow \pm \infty$, and therefore $\mathbf{v}(x, t, k)$ cannot belong to $L^{2}(\mathbb{R})$. Thus, the only possibility left is $z \in C_{o}$.

Proof of Lemma 2.20. Suppose $a_{11}\left(z_{n}\right)=0$, where $\operatorname{Im} z_{n}>0$. The results follow from a combination of (2.35) and (2.43).

Proof of Lemma 2.21. If $\chi\left(x, t, z_{o}\right)=\mathbf{0}$, then by (2.27), there exists a constant $b_{o}$ such that $\phi_{-, 3}\left(x, t, z_{o}^{*}\right)=b_{o} \phi_{+, 1}\left(x, t, z_{o}^{*}\right)$. However, this corresponds to a bound state, which would contradict Lemma 2.19. Therefore, $\chi\left(x, t, z_{o}\right) \neq \mathbf{0}$.

Proof of Lemma 2.22. [(i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv)] The results follow trivially from (2.46).
$\left[(\mathrm{i}) \Leftrightarrow\right.$ (iii)] Assume $\chi\left(x, t, z_{o}\right)=\mathbf{0}$. By Lemma 2.21, $\left|z_{o}\right|=q_{o}$. Then $z_{o}=q_{o}^{2} / z_{o}^{*}$, implying $\chi\left(x, t, q_{o}^{2} / z_{n}^{*}\right)=\mathbf{0}$. The converse follows by using the exact same argument.
$[(\mathrm{i}) \Leftrightarrow(\mathrm{v})]$ Assume $\chi\left(x, t, z_{o}\right)=\mathbf{0}$. Then (2.27) implies

$$
\left[\tilde{\phi}_{-, 3} \times \tilde{\phi}_{+, 1}\right]\left(x, t, z_{o}\right)=\mathbf{0}
$$

so there exists a constant $b_{o}^{*}$ such that $\tilde{\phi}_{-, 3}\left(x, t, z_{o}\right)=b_{o}^{*} \tilde{\phi}_{+, 1}\left(x, t, z_{o}\right)$. Using the symmetry (A.29) and then taking the complex conjugate yields the desired result. Conversely, assume there exists a constant $b_{o}$ such that $\phi_{-, 3}\left(x, t, z_{o}^{*}\right)=b_{o} \phi_{+, 1}\left(x, t, z_{o}^{*}\right)$. Then $\left[\phi_{-, 3}^{*} \times \phi_{+, 1}^{*}\right]\left(x, t, z_{o}^{*}\right)=\mathbf{0}$, which, together with (A.29) and (2.27), implies $\chi\left(x, t, z_{o}\right)=\mathbf{0}$.
$[(\mathrm{v}) \Leftrightarrow(\mathrm{vi})]$ Assume such a constant $b_{o}$ exists. Applying the symmetry (2.42) yields $-\left(i q_{o} / z_{o}^{*}\right) \phi_{-, 1}\left(x, t, q_{o}^{2} / z_{o}^{*}\right)=\left(i q_{o} / z_{o}^{*}\right) b_{o} \phi_{+, 3}\left(x, t, q_{o}^{2} / z_{o}^{*}\right)$. Taking $\tilde{b}_{o}=-b_{o}$ gives the desired result. The converse is proved similarly using (2.42) again.
[(vii) $\Leftrightarrow$ (viii)] Using (2.42) as in the proof that (v) $\Leftrightarrow$ (vi) will give the result.
$[(\mathrm{iv}) \Leftrightarrow$ (vii)] This is proved by using (A.29) and (2.27), as in the proof that (i) $\Leftrightarrow(\mathrm{v})$.

Proof of Theorem 2.23. Since $a_{11}\left(\zeta_{o}\right)=0,(2.34 \mathrm{c})$ seems to imply that $\phi_{-, 3}(x, t, z)$ has a pole at $z=\zeta_{o}^{*}$, but this is impossible since $\phi_{-, 3}(x, t, z)$ is analytic in the lower half plane. Hence, one of the eigenfunctions on the right-hand side of $(2.34 \mathrm{c})$ must be zero or the two eigenfunctions must be linearly dependent. Now, suppose $\chi\left(x, t, \zeta_{o}\right) \neq$ 0. Equation (2.50) implies $\operatorname{det} \boldsymbol{\Phi}^{+}\left(x, t, \zeta_{o}\right)=0$, so there must exist constants $c_{1}$ and $c_{2}$ such that $\chi\left(x, t, \zeta_{o}\right)=c_{1} \phi_{-, 1}\left(x, t, \zeta_{o}\right)+c_{2} \phi_{+, 3}\left(x, t, \zeta_{o}\right)$. But Lemma 2.22 implies that $\phi_{-, 1}\left(x, t, \zeta_{o}\right)$ and $\phi_{+, 3}\left(x, t, \zeta_{o}\right)$ are linearly independent. Thus, in order for $(2.34 \mathrm{c})$ to be finite at $z=\zeta_{o}$, one needs $c_{2}=0$. Repeating with (2.34b), one obtains $c_{1}=0$. Therefore, $\chi\left(x, t, \zeta_{o}\right)=\mathbf{0}$. Lemma 2.22 then tells us that $\bar{\chi}\left(x, t, \zeta_{o}^{*}\right)=\mathbf{0}$. Equation (2.52) then follows immediately by Lemma 2.22.

Proof of Theorem 2.24. Since $z_{n} \notin C_{o}$, Lemma 2.21 implies that $\chi\left(x, t, z_{n}\right)$, $\bar{\chi}\left(x, t, z_{n}^{*}\right), \chi\left(x, t, q_{o}^{2} / z_{n}^{*}\right)$, and $\bar{\chi}\left(x, t, q_{o}^{2} / z_{n}\right)$ are all nonzero. As in the proof of Theorem 2.23, since $a_{11}\left(z_{n}\right)=0,(2.34 \mathrm{c})$ implies $\left[\chi \times \phi_{-, 1}\right]\left(x, t, z_{n}\right)=\mathbf{0}$. This proves the existence of the constant $d_{n}$. The rest of (2.53) is proved by using (2.34) and the results from (2.51).

Note that we may write $z_{n}=\alpha_{n}+i \nu_{n}$, where $\nu_{n}>0$. Equations (2.5) then yield

$$
\begin{align*}
& k\left(z_{n}\right)=\frac{1}{2}\left[\alpha_{n}\left(1+\frac{q_{o}^{2}}{\left|z_{n}\right|^{2}}\right)+i \nu_{n}\left(1-\frac{q_{o}^{2}}{\left|z_{n}\right|^{2}}\right)\right],  \tag{A.32a}\\
& \lambda\left(z_{n}\right)=\frac{1}{2}\left[\alpha_{n}\left(1-\frac{q_{o}^{2}}{\left|z_{n}\right|^{2}}\right)+i \nu_{n}\left(1+\frac{q_{o}^{2}}{\left|z_{n}\right|^{2}}\right)\right] . \tag{A.32b}
\end{align*}
$$

We wish to show that a bound state arises from the first of (2.53a) unless $\left|z_{n}\right|<q_{o}$. We start by rewriting that expression as

$$
\phi_{-, 1}\left(x, t, z_{n}\right)=d_{n} m\left(x, t, z_{n}\right) \mathrm{e}^{i \theta_{2}\left(x, t, z_{n}\right)}
$$

Suppose $\left|z_{n}\right|>q_{o}$. We use the asymptotics from Lemma 2.9 and (A.32) to see that the left-hand side of this equation is bounded as $x \rightarrow \pm \infty$. This results in a bound state, which contradicts Lemma 2.19. When $\left|z_{o}\right|<q_{o}$, we arrive at no such contradiction, so we cannot exclude this case.

Finally, suppose $b_{33}\left(z_{n}\right)=0$. Then since $\phi_{+, 1}(x, t, z)$ is analytic at $z=z_{n}^{*},(2.34 \mathrm{~b})$ implies the existence of a constant $c_{o}$ such that $\chi\left(x, t, z_{n}\right)=c_{o} \phi_{+, 3}\left(x, t, z_{n}\right)$. However, (2.53) implies $\phi_{-, 1}\left(x, t, z_{n}\right)=d_{n} \chi\left(x, t, z_{n}\right)$. Combining these yields $\phi_{-, 1}\left(z_{n}\right)=$ $d_{n} c_{o} \phi_{+, 3}\left(x, t, z_{n}\right)$. This corresponds to a bound state, which contradicts Lemma 2.19. Therefore, $b_{33}\left(z_{n}\right) \neq 0$.

Proof of Lemma 2.25. The symmetry (2.42) yields the first of (2.54), while differentiating (2.34a) with respect to $z$, applying (2.52), and comparing the result with the derivative of $(2.34 \mathrm{~d})$ yields the second of (2.54).

Proof of Lemma 2.26. Combining (2.53) with the symmetries (2.42) and (2.46) yields the first of (2.55) and the relation $\bar{d}_{n}=\left(i z_{n}^{*} / q_{o}\right) \hat{d}_{n}$. Then, evaluating (2.34a) at $z=z_{n}^{*}$, applying the second of $(2.53 \mathrm{~b})$, and recalling the definition of $\chi(x, t, z)$ from (2.27) yields the rest of (2.55).
A.8. Asymptotics. Throughout this section, we will use the shorthand notation

$$
\mathrm{e}^{i \hat{\boldsymbol{\Lambda}}}(\mathbf{M})=\mathrm{e}^{i \boldsymbol{\Lambda}} \mathbf{M} \mathrm{e}^{-i \boldsymbol{\Lambda}}=\left(\begin{array}{ccc}
m_{11} & \mathrm{e}^{-i(k+\lambda)} m_{12} & \mathrm{e}^{-2 i \lambda} m_{13} \\
\mathrm{e}^{i(k+\lambda)} m_{21} & m_{22} & \mathrm{e}^{i(k-\lambda)} m_{23} \\
\mathrm{e}^{2 i \lambda} m_{31} & \mathrm{e}^{-i(k-\lambda)} m_{32} & m_{33}
\end{array}\right)
$$

where $\mathbf{M}$ is any $3 \times 3$ matrix. In order to prove Lemmas 2.27 and 2.28 , it will be convenient to decompose ( 2.56 c ) into block-diagonal and block-off-diagonal terms.

For example, the block diagonal and block off-diagonal terms of the scattering matrix $\mathbf{A}(z)$ are, respectively,

$$
[\mathbf{A}(z)]_{b d}=\left(\begin{array}{ccc}
a_{11}(z) & 0 & 0 \\
0 & a_{22}(z) & a_{23}(z) \\
0 & a_{32}(z) & a_{33}(z)
\end{array}\right), \quad[\mathbf{A}(z)]_{b o}=\left(\begin{array}{ccc}
0 & a_{12}(z) & a_{13}(z) \\
a_{21}(z) & 0 & 0 \\
a_{31}(z) & 0 & 0
\end{array}\right)
$$

First, note that for any $3 \times 3$ matrices $\mathbf{A}$ and $\mathbf{B}$,

$$
\begin{align*}
{[\mathbf{A B}]_{b d}=} & \mathbf{A}_{b d} \mathbf{B}_{b d}+\mathbf{A}_{b o} \mathbf{B}_{b o}, \quad[\mathbf{A B}]_{b o}=\mathbf{A}_{b d} \mathbf{B}_{b o}+\mathbf{A}_{b o} \mathbf{B}_{b d}  \tag{A.33a}\\
& {\left[\mathbf{A}_{b d} \mathbf{B}_{b d}\right]_{d}=\mathbf{A}_{d} \mathbf{B}_{d}+\left[\mathbf{A}_{b d}\right]_{o}\left[\mathbf{B}_{b d}\right]_{o} }  \tag{A.33b}\\
& {\left[\mathbf{A}_{b d} \mathbf{B}_{b d}\right]_{o}=\mathbf{A}_{d}\left[\mathbf{B}_{b d}\right]_{o}+\left[\mathbf{A}_{b d}\right]_{o} \mathbf{B}_{d} } \tag{A.33c}
\end{align*}
$$

We denote the integrand of (2.56c) as

$$
\mathbf{M}_{+}(x, y, t, z)=\mathbf{E}_{+}(z) \mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}(z)}\left(\mathbf{E}_{+}^{-1}(z) \Delta \mathbf{Q}_{+}(y, t) \mu_{n}(y, t, z)\right)
$$

We suppress $x^{-}, y^{-}, t$-, and $z$-dependence for simplicity in the following calculations when doing so introduces no confusion. Since $\mathrm{e}^{i(x-y) \boldsymbol{\Lambda}(z)}$ is a diagonal matrix, and since $\Delta \mathbf{Q}_{+}$is a block off-diagonal matrix,

$$
\begin{aligned}
& {\left[\mathbf{M}_{+}\right]_{b d}=\left[\mathbf{E}_{+}\right]_{b d} \mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(\left[\mathbf{E}_{+}^{-1}\right]_{b o} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b d}+\left[\mathbf{E}_{+}^{-1}\right]_{b d} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right)} \\
& \quad+\left[\mathbf{E}_{+}\right]_{b o} \mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(\left[\mathbf{E}_{+}^{-1}\right]_{b d} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b d}+\left[\mathbf{E}_{+}^{-1}\right]_{b o} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right)
\end{aligned}
$$

Equation (2.8) implies

$$
\left[\mathbf{E}_{ \pm}^{-1}\right]_{b d}=\frac{1}{\gamma(z)} \mathbf{D}(z)\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d}, \quad\left[\mathbf{E}_{ \pm}^{-1}\right]_{b o}=-\frac{1}{\gamma(z)}\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o}
$$

where $\mathbf{D}(z)=\operatorname{diag}(1, \gamma(z), 1)$. We then obtain

$$
\begin{aligned}
& {\left[\mathbf{M}_{+}\right]_{b d}=\frac{\left[\mathbf{E}_{+}\right]_{b d}}{\gamma} \mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(-\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b d}+\mathbf{D}\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right) } \\
&+\frac{\left[\mathbf{E}_{+}\right]_{b o}}{\gamma} \mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(\mathbf{D}\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b d}-\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right)
\end{aligned}
$$

We now discuss $\left[\mathbf{M}_{+}\right]_{b o}$. It follows that

$$
\begin{aligned}
{\left[\mathbf{M}_{+}\right]_{b o}=} & \frac{\left[\mathbf{E}_{+}\right]_{b d}}{\gamma} \mathrm{e}^{i(x-y) \hat{\Lambda}}\left(-\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}+\mathbf{D}\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b d}\right) \\
& +\frac{\left[\mathbf{E}_{+}\right]_{b o}}{\gamma} \mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(\mathbf{D}\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}-\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b d}\right)
\end{aligned}
$$

We combine (A.33) with (2.56c) to find the following for $n \geq 0$ :

$$
\begin{align*}
-\gamma\left[\mu_{n+1}\right]_{b d}= & {\left[\mathbf{E}_{+}\right]_{b d} \int_{x}^{\infty}\left[-\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{d}-\mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\left[\left[\mu_{n}\right]_{b d}\right]_{o}\right)\right] \mathrm{d} y }  \tag{A.34a}\\
& +\left[\mathbf{E}_{+}\right]_{b d} \mathbf{D} \int_{x}^{\infty}\left[\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{d}\left[\Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right]_{d}+\left[\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d}\right]_{o}\left[\Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right]_{o}\right] \mathrm{d} y
\end{align*}
$$

$$
\begin{gather*}
+\left[\mathbf{E}_{+}\right]_{b d} \int_{x}^{\infty} \mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(\mathbf{D}\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{d}\left[\Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right]_{o}+\mathbf{D}\left[\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d}\right]_{o}\left[\Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right]_{d}\right) \mathrm{d} y \\
+\left[\mathbf{E}_{+}\right]_{b o} \int_{x}^{\infty} \mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(\mathbf{D}\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b d}-\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right) \mathrm{d} y \tag{A.34b}
\end{gather*}
$$

$$
\begin{aligned}
&-\gamma\left[\mu_{n+1}\right]_{b o}= {\left[\mathbf{E}_{+}\right]_{b d} \int_{x}^{\infty} \mathrm{e}^{i(x-y) \hat{\Lambda}}\left(-\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}+\mathbf{D}\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d} \Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b d}\right) \mathrm{d} y } \\
&+ {\left[\mathbf{E}_{+}\right]_{b o} \int_{x}^{\infty}\left[\mathbf{D}\left[\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d}\right]_{o}\left[\Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right]_{o}+\mathrm{e}^{i(x-y) \hat{\Lambda}}\left(\mathbf{D}\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{d}\left[\Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right]_{o}\right)\right] \mathrm{d} y } \\
&+\left[\mathbf{E}_{+}\right]_{b o} \int_{x}^{\infty}\left[\mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(\mathbf{D}\left[\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b d}\right]_{o}\left[\Delta \mathbf{Q}_{+}\left[\mu_{n}\right]_{b o}\right]_{d}\right)-\left[\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\right]_{d}\left[\mu_{n}\right]_{d}\right] \mathrm{d} y \\
&-\left[\mathbf{E}_{+}\right]_{b o} \int_{x}^{\infty} {\left[\left[\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\right]_{o}\left[\left[\mu_{n}\right]_{b d}\right]_{o}+\mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(\left[\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\right]_{o}\left[\mu_{n}\right]_{d}\right)\right] \mathrm{d} y } \\
&- {\left[\mathbf{E}_{+}\right]_{b o} \int_{x}^{\infty} \mathrm{e}^{i(x-y) \hat{\boldsymbol{\Lambda}}}\left(\left[\left[\mathbf{E}_{ \pm}^{\dagger}\right]_{b o} \Delta \mathbf{Q}_{+}\right]_{d}\left[\left[\mu_{n}\right]_{b d}\right]_{o}\right) \mathrm{d} y . }
\end{aligned}
$$

Equations (A.34a) and (A.34b) will allow us to easily use induction to prove Lemmas 2.27 and 2.28 .

Proof of Lemma 2.27. The claims in (2.57a) are trivially true for $\mu_{0}$. Suppose the claims in (2.57) are true for some $n \geq 0$. We then use integration by parts and the facts that $k=z / 2+O(1 / z)$ and $\lambda=z / 2+O(1 / z)$ as $z \rightarrow \infty$ to see that the terms in (A.34a) are $O\left(\left[\mu_{n}\right]_{b d} / z\right), O\left(\left[\mu_{n}\right]_{b d} / z^{2}\right), O\left(\left[\mu_{n}\right]_{b o}\right), O\left(\left[\mu_{n}\right]_{b o}\right), O\left(\left[\mu_{n}\right]_{b o} / z\right), O\left(\left[\mu_{n}\right]_{b o} / z\right)$, $O\left(\left[\mu_{n}\right]_{b d} / z^{2}\right)$, and $O\left(\left[\mu_{n}\right]_{b o} / z^{3}\right)$, respectively, as $z \rightarrow \infty$.

When $n=2 m$ for some $m \in \mathbb{N}$, the first, third, and fourth terms on the righthand side of (A.34a) are $O\left(1 / z^{m+1}\right)$, the second, fifth, sixth, and seventh terms are $O\left(1 / z^{m+2}\right)$, and the eighth term is $O\left(1 / z^{m+4}\right)$ (all as $\left.z \rightarrow \infty\right)$. Then $\left[\mu_{n+1}\right]_{b d}=$ $O\left(1 / z^{m+1}\right)$ as $z \rightarrow \infty$.

When $n=2 m+1$ for some $m \in \mathbb{N}$, the third and fourth terms on the right-hand side of (A.34a) are $O\left(1 / z^{m+1}\right)$, the first, fifth, and sixth terms are $O\left(1 / z^{m+2}\right)$, the second and seventh terms are $O\left(1 / z^{m+3}\right)$, and the eighth term is $O\left(1 / z^{m+4}\right)$ (all as $z \rightarrow \infty)$. Then $\left[\mu_{n+1}\right]_{b d}=O\left(1 / z^{m+1}\right)$ as $z \rightarrow \infty$.

Similar results hold for the terms in (A.34b) using the same analysis. Also, the same results hold for $\mu_{-}(x, t, z)$ when it is expanded as a series similar to (2.56a).

Proof of Lemma 2.28. The claims in (2.58a) are trivially true for $\mu_{0}$. Suppose the claims in (2.58) are true for some $n \geq 0$. We use integration by parts and the facts that $k=O(1 / z)$ and $\lambda=O(1 / z)$ as $z \rightarrow 0$ to see that the terms on the right-hand side of (A.34a) are, respectively, $O\left(z\left[\mu_{n}\right]_{b d}\right), O\left(z^{2}\left[\mu_{n}\right]_{b d}\right), O\left(z^{2}\left[\mu_{n}\right]_{b o}\right), O\left(z^{2}\left[\mu_{n}\right]_{b o}\right)$, $O\left(z^{3}\left[\mu_{n}\right]_{b o}\right), O\left(z^{3}\left[\mu_{n}\right]_{b_{o}}\right), O\left(z^{2}\left[\mu_{n}\right]_{b d}\right)$, and $O\left(z\left[\mu_{n}\right]_{b o}\right)$ as $z \rightarrow 0$.

When $n=2 m$ for some $m \in \mathbb{N}$, the eighth term on the right-hand side of (A.34a) is $O\left(z^{m}\right)$, the first, third, and fourth terms are $O\left(z^{m+1}\right)$, and the rest are $O\left(z^{m+2}\right)$. Then $\left[\mu_{n+1}\right]_{b d}=O\left(z^{m}\right)$ as $z \rightarrow 0$.

When $n=2 m+1$ for some $m \in \mathbb{N}$, the first and eighth terms on the righthand side of (A.34a) are $O\left(z^{m+1}\right)$, the second, third, fourth, and seventh terms are $O\left(z^{m+2}\right)$, and the rest are $O\left(z^{m+3}\right)$. Then $\left[\mu_{n+1}\right]_{b d}=O\left(z^{m+1}\right)$ as $z \rightarrow 0$.

Similar results hold for the terms in (A.34b) using the same analysis. Also, the same results hold for $\mu_{-}(x, t, z)$ when it is expanded as a series similar to (2.56a).

Proof of Corollary 2.29. One obtains the results after explicitly calculating the columns of (2.56a).

Proof of Lemma 2.30. The results follow by combining (2.27) with (2.59) and (2.60).

Proof of Corollary 2.31. One obtains the results by combining the results in Corollary 2.29 with (2.19).

Proof of Corollary 2.32. Equations (2.63) and (2.65) are obtained by combining the results in Corollary 2.29 with (2.19) and the results in Lemma 2.6. The asymptotic behavior of $a_{22}(z)$ and $b_{22}(z)$ as $z \rightarrow \infty$ and $z \rightarrow 0$ in (2.64) and (2.66) is simply a consequence of (2.63) and (2.65), Corollary 2.31, and the fact that $a_{22}(z)=$ $b_{11}(z) b_{33}(z)-b_{13}(z) b_{31}(z)$ and $b_{22}(z)=a_{11}(z) a_{33}(z)-a_{13}(z) a_{31}(z)$.

## A.9. Inverse problem.

Proof of Lemma 3.1. We start by eliminating the nonanalytic eigenfunctions $\phi_{ \pm, 2}$ from (2.19) using (2.30a) and (2.30b):

$$
\begin{aligned}
\phi_{+, 3}(x, t, z)= & -\left[\frac{a_{23}(z)}{a_{33}(z)} \frac{b_{12}(z)}{b_{11}(z)}+\frac{a_{13}(z)}{a_{33}(z)}\right] \phi_{+, 1}(x, t, z)-\frac{a_{23}(z)}{a_{33}(z)}\left[-\frac{\bar{\chi}(x, t, z)}{b_{11}(z)}\right]+\frac{\phi_{-, 3}(x, t, z)}{a_{33}(z)} \\
\frac{\phi_{-, 1}(x, t, z)}{a_{11}(z)}= & {\left[1+\frac{a_{21}(z)}{a_{11}(z)} \frac{b_{12}(z)}{b_{11}(z)}\right] \phi_{+, 1}(x, t, z)+\frac{a_{21}(z)}{a_{11}(z)}\left[-\frac{\bar{\chi}(x, t, z)}{b_{11}(z)}\right]+\frac{a_{31}(z)}{a_{11}(z)} \phi_{+, 3}(x, t, z) } \\
& \frac{\chi(x, t, z)}{b_{33}(z)}=\frac{b_{12}(z)}{b_{11}(z)} \phi_{+, 1}(x, t, z)-\frac{\bar{\chi}(x, t, z)}{b_{11}(z)}-\frac{b_{32}(z)}{b_{33}(z)} \phi_{+, 3}(x, t, z) .
\end{aligned}
$$

The jump conditions (3.2) are obtained by combining the above expression for $\phi_{+, 3}(x, t, z)$ with the other two equations, recalling (2.47), and applying the symmetries of the scattering coefficients. $\square$

Proof of Lemma 3.2. One obtains the asymptotics of the columns of $\mathbf{M}^{ \pm}(x, t, z)$ by using the asymptotics of the eigenfunctions and the scattering coefficients found in section 2.6.

Proof of Lemma 3.3. The residue conditions are easily found by combining the definitions of the meromorphic matrices (3.1) with the relations in Theorems 2.23 and 2.24.

Proof of Lemma 3.4. The symmetry (2.43) implies $\left.a_{11}^{\prime}(z)\right|_{z=\zeta_{n}}=-\left.\frac{\zeta_{n}^{*}}{\zeta_{n}} a_{33}^{\prime}(z)\right|_{z=\zeta_{n}^{*}}$. Combining this information with (2.54) and (3.9) yields (3.5a). Next, combining (2.55) with the symmetries (found using (2.35) and (2.43))

$$
\begin{gathered}
\left.a_{11}^{\prime}(z)\right|_{z=z_{n}}=\left[\left.b_{11}^{\prime}(z)\right|_{z=z_{n}^{*}}\right]^{*},\left.\quad a_{33}^{\prime}(z)\right|_{z=q_{o}^{2} / z_{n}}=\left[\left.b_{33}^{\prime}(z)\right|_{z=q_{o}^{2} / z_{n}^{*}}\right]^{*} \\
\left.a_{11}^{\prime}(z)\right|_{z=z_{n}}=-\left.\frac{q_{o}^{2}}{z_{n}^{2}} a_{33}^{\prime}(z)\right|_{z=q_{o}^{2} / z_{n}}
\end{gathered}
$$

yields (3.5b).
Proof of Theorem 3.7. For brevity, we suppress $x$ - and $t$-dependence when doing so introduces no confusion. To solve (3.2), we subtract from both sides of (3.2) the quantities defined in (3.3) as well as the residue contributions from the poles inside and on the circle of radius $q_{o}$. Namely, we subtract

$$
\begin{aligned}
& \mathbf{M}_{\infty}+(i / z) \mathbf{M}_{0}+\sum_{i=1}^{N_{1}}\left(\frac{\operatorname{Res}_{z=\zeta_{i}} \mathbf{M}^{+}}{z-\zeta_{i}}+\frac{\operatorname{Res}_{z=\zeta_{i}^{*}} \mathbf{M}^{-}}{z-\zeta_{i}^{*}}\right) \\
+ & \sum_{j=1}^{N_{2}}\left(\frac{\operatorname{Res}_{z=z_{j}^{*}} \mathbf{M}^{-}}{z-z_{j}^{*}}+\frac{\operatorname{Res}_{z=z_{j}} \mathbf{M}^{+}}{z-z_{j}}\right)+\sum_{j=1}^{N_{2}}\left(\frac{\operatorname{Res}_{z=q_{o}^{2} / z_{j}} \mathbf{M}^{-}}{z-q_{o}^{2} / z_{j}}+\frac{\operatorname{Res}_{z=q_{o}^{2} / z_{j}^{*}} \mathbf{M}^{+}}{z-q_{o}^{2} / z_{j}^{*}}\right) .
\end{aligned}
$$

The left-hand side of the resulting, regularized RHP is analytic in the upper half $z$ plane and is $O(1 / z)$ as $z \rightarrow \infty$ there. Also, the right-hand side is analytic in the lower
half $z$-plane and is $O(1 / z)$ as $z \rightarrow \infty$ there. Now, recall that the definition (3.12a) of the Cauchy projectors, as well as Plemelj's formulae: If $f^{ \pm}$is analytic in the upper (resp., lower) half of the $z$-plane and $f^{ \pm}=O(1 / z)$ as $z \rightarrow \infty$ in the appropriate half plane, then $P^{ \pm} f^{ \pm}= \pm f^{ \pm}$and $P^{+} f^{-}=P^{-} f^{+}=0$. Applying (3.12a) to the regularized RHP yields (3.6).

Evaluating the first column of (3.6) at $w=\zeta_{i^{\prime}}^{*}\left(i^{\prime}=1, \ldots, N_{1}\right)$ or $w=z_{j^{\prime}}^{*}$ $\left(j^{\prime}=1, \ldots, N_{2}\right)$ yields (3.7a), evaluating the third column of (3.6) at $w=\zeta_{i^{\prime}}\left(i^{\prime}=\right.$ $\left.1, \ldots, N_{1}\right)$ or $w=q_{o}^{2} / z_{j^{\prime}}^{*}\left(j^{\prime}=1, \ldots, N_{2}\right)$ yields (3.7b), and examining the second column of (3.6) and using the symmetry (2.43) yields (3.7c) and (3.7d). From this, we will be able to explicitly find the residues in the reflectionless case.

Proof of Theorem 3.8. The asymptotics from (2.59) imply

$$
\begin{equation*}
q_{k}(x, t)=-i \lim _{z \rightarrow \infty}\left(z \mu_{+,(k+1) 1}(x, t, z)\right), \quad k=1,2 \tag{A.35}
\end{equation*}
$$

We take $\mathbf{M}=\mathbf{M}^{-}$in (3.6) and compare its 2,1 and 3,1 elements in the limit as $z \rightarrow \infty$ with the corresponding elements found in the first of (2.59) to obtain (3.8).

Proof of Lemma 3.9. Recall that $a_{11}(z)$ is analytic in the upper half $z$-plane and that it has simple zeros at the points $\left\{\zeta_{n}\right\}_{n=1}^{N_{1}}$ on the circle $C_{o}$ and the points $\left\{z_{n}\right\}_{n=1}^{N_{2}}$ inside the circle $C_{o}$. Define

$$
\begin{equation*}
\beta_{1}(z)=a_{11}(z) \prod_{n=1}^{N_{1}} \frac{z-\zeta_{n}^{*}}{z-\zeta_{n}} \prod_{n=1}^{N_{2}} \frac{z-z_{n}^{*}}{z-z_{n}}, \quad \beta_{2}(z)=b_{11}(z) \prod_{n=1}^{N_{1}} \frac{z-\zeta_{n}}{z-\zeta_{n}^{*}} \prod_{n=1}^{N_{2}} \frac{z-z_{n}}{z-z_{n}^{*}} . \tag{A.36}
\end{equation*}
$$

By construction, $\beta_{1}(z)$ is analytic in the upper half $z$-plane, it has no zeros, and $\beta(z) \rightarrow 1$ as $z \rightarrow \infty$ in the upper half $z$-plane. The same results hold for $\beta_{2}(z)$ in the lower half $z$-plane. We use (2.47) and the symmetry (2.35) to write (5.1) as

$$
\begin{equation*}
\log a_{11}(z)-\log \left(1 / b_{11}(z)\right)=\log \left[1-\left|\rho_{1}(z)\right|^{2}-\frac{z^{2}}{z^{2}-q_{o}^{2}}\left|\rho_{2}(z)\right|^{2}\right], \quad z \in \mathbb{R} \tag{A.37}
\end{equation*}
$$

Combining (A.37) with (A.36) yields

$$
\begin{equation*}
\log \beta_{1}(z)-\log \left(1 / \beta_{2}(z)\right)=\log \left[1-\left|\rho_{1}(z)\right|^{2}-\frac{z^{2}}{z^{2}-q_{o}^{2}}\left|\rho_{2}(z)\right|^{2}\right], \quad z \in \mathbb{R} \tag{A.38}
\end{equation*}
$$

Equation (A.38) is an RHP. Applying $P^{+}$from (3.12a) to (A.38) yields the desired result.

Proof of Corollary 3.10. One obtains the result by comparing (3.9) with the asymptotic behavior of $a_{11}(z)$ as $z \rightarrow 0$ in (2.62a).
A.10. Existence and uniqueness of the solution of the RHP. Since the spatial and temporal variables $x$ and $t$ only appear as parameters in the formulation of the RHP, and since their value does not affect the arguments that follow, in this section we omit the $(x, t)$-dependence for brevity.

Proof of Theorem 3.11. The proof uses a standard argument (e.g., cf. [14]). In the absence of a discrete spectrum, $\mathbf{M}(z)$ is a sectionally analytic function on $\mathbb{C} \backslash \mathbb{R}$ which satisfies the jump condition (3.2) and has the asymptotic behavior in Lemma 3.2. Letting $g(z)=\operatorname{det} \mathbf{M}(z)$ and taking the determinant of the jump condition (3.2) yields

$$
\begin{equation*}
g^{+}(z)=g^{-}(z), \quad z \in \mathbb{R} \tag{A.39}
\end{equation*}
$$

Moreover, Lemma 3.2, implies $g(z)=1+O(1 / z)$ as $z \rightarrow \infty$ and $g(z)=O(1)$ as $z \rightarrow 0$. Equation (A.39) then implies that $g(z)$ is an entire function (as there is no singularity at $z=0$ ) which is also bounded at infinity. Liouville's theorem then implies $g(z)=1$ for all $z \in \mathbb{C}$. Thus, $\mathbf{M}(z)$ is invertible, and $\mathbf{M}^{-1}(z)$ is also analytic for $\mathbb{C} \backslash \mathbb{R}$.

Now, suppose $\tilde{\mathbf{M}}(z)$ is another sectionally analytic function which satisfies the jump condition (3.2) and has the asymptotic behavior in Lemma 3.2. Introducing the matrix $\mathbf{Y}(z)=\tilde{\mathbf{M}}(z) \mathbf{M}^{-1}(z)$ and using again the jump condition (3.2), we have

$$
\begin{equation*}
\mathbf{Y}^{+}(z)=\mathbf{Y}^{-}(z), \quad z \in \mathbb{R} \tag{A.40}
\end{equation*}
$$

Lemma 3.2 implies $\mathbf{Y}(z)=\mathbf{I}+O(1 / z)$ as $z \rightarrow \infty$ and $\mathbf{Y}(z)=\mathbf{I}+O(z)$ as $z \rightarrow 0$. Thus, $\mathbf{Y}(z)$ is an entire function that is also bounded at infinity, and Liouville's theorem again allows us to conclude $\mathbf{Y}(z)=\mathbf{I}$ for all $z \in \mathbb{C}$, implying $\tilde{\mathbf{M}}(z)=\mathbf{M}(z)$.

Proof of Theorem 3.12. Again, recall that we consider the case of no discrete spectrum. Note first that, when taking the limit to $z \in \mathbb{R}$ from the appropriate directions in the complex plane, the limiting values of the Cauchy projectors $P^{ \pm}$in (3.12a) are bounded operators on $L^{2}(\mathbb{R})[34]$. Indeed, for $f \in L^{2}(\mathbb{R})$, straightforward algebra yields

$$
\int_{\mathbb{R}} \frac{f(\zeta)}{\zeta-(z \pm i \epsilon)} \mathrm{d} \zeta=\int_{\mathbb{R}} \frac{(\zeta-z) f(\zeta)}{(\zeta-z)^{2}+\epsilon^{2}} \mathrm{~d} \zeta \pm i \int_{\mathbb{R}} \frac{\epsilon f(\zeta)}{(\zeta-z)^{2}+\epsilon^{2}} \mathrm{~d} \zeta
$$

As $\epsilon \rightarrow 0^{+}$, the first integral converges to $-\pi(H f)(z)$, where $H$ is a Hilbert transform:

$$
\begin{equation*}
(H f)(z)=\lim _{\delta \rightarrow 0^{+}} \frac{1}{\pi} \int_{|\zeta-z| \geq \delta} \frac{f(\zeta)}{z-\zeta} \mathrm{d} \zeta \tag{A.41}
\end{equation*}
$$

Also, the second integral converges to $\pm i \pi f(z)$, since its integrand contains a representation of the Dirac delta. Thus,

$$
\begin{equation*}
\left(P^{ \pm} f\right)(z)= \pm \frac{1}{2} f(z)-\frac{1}{2 i}(H f)(z), \quad z \in \mathbb{R} \tag{A.42}
\end{equation*}
$$

Since $H$ is known to be a bounded operator on $L^{2}(\mathbb{R})$ [34], we conclude from (A.42) that the limiting values of $P^{ \pm}$as $z \rightarrow \mathbb{R}$ are indeed bounded operators on $L^{2}(\mathbb{R})$. Moreover, using again the properties of the Hilbert transform $H$, we find

$$
\begin{equation*}
P^{+}-P^{-}=I \tag{A.43}
\end{equation*}
$$

We now use the methods of [14] to prove the existence of the solution of the RHP. We begin by recalling the decomposition $\mathbf{V}(z)=\mathbf{V}_{+}^{-1}(z) \mathbf{V}_{-}(z)$ (3.13) of the jump condition (3.2) $\mathbf{M}^{+}(z)=\mathbf{M}^{-}(z) \mathbf{V}(z)$, where $\mathbf{V}_{ \pm}(z)$ are, respectively, upper/lower triangular matrices. (Note that here the subscripts $\pm$ do not indicate normalization as $x \rightarrow \pm \infty$ as in the rest of this work.) Thanks to the boundedness and invertibility of $\mathbf{V}(z)$, each triangular matrix on the right-hand side of $(3.13)$ is bounded in $L^{\infty}(\mathbb{R})$ and invertible. Also, $\mathbf{L}(z) \in L^{\infty}(\mathbb{R})$ implies that $\mathbf{V}_{+}(\cdot)$ and $\mathbf{V}_{-}(\cdot)$ are both in $L^{\infty}(\mathbb{R})$. Next, recalling from (3.14) that $\mathbf{W}_{ \pm}= \pm\left(\mathbf{I}-\mathbf{V}_{ \pm}\right)$and $\mathbf{W}=\mathbf{W}_{+}+\mathbf{W}_{-}$, as well the definition (3.15) of $P_{\mathbf{w}}$, it follows from the above discussion that $P_{\mathbf{w}}$ is a bounded operator in $L^{2}(\mathbb{R})$.

In Lemma A. 9 below we prove that, under the conditions of Theorem 3.12, $I-P_{\mathbf{w}}$ is an invertible operator in $L^{2}(\mathbb{R})$. Then let $\mathbf{N}$ be the unique solution of the following integral equation:

$$
\begin{equation*}
\left(\left(I-P_{\mathbf{w}}\right) \mathbf{N}\right)(s)=\mathbf{E}_{+}(s), \quad s \in \mathbb{R} \tag{A.44}
\end{equation*}
$$

Note that $\mathbf{I}-\mathbf{N} \in L^{2}(\mathbb{R})$. Then define the matrix function

$$
\begin{equation*}
\mathbf{M}_{\#}(z)=\mathbf{E}_{+}(z)+(P(\mathbf{N W}))(z), \quad z \notin \mathbb{R} \tag{A.45}
\end{equation*}
$$

where the Cauchy operator was defined in (3.12b). We next show that $\mathbf{M}_{\#}(z)$ is a solution of the RHP defined by Lemmas 3.1, 3.2, and 3.3. Note first that $\mathbf{M}_{\#}(z)$ is analytic for all $z \in \mathbb{C} \backslash \mathbb{R}$. Next we prove that $\mathbf{M}_{\#}$ satisfies the jump condition. Combining the properties of $\mathbf{N}$ with the identity (A.43) and (3.14), we obtain, for all $s \in \mathbb{R}$,

$$
\begin{align*}
& \mathbf{M}_{\#}^{+}=\mathbf{E}_{+}+P^{+}(\mathbf{N W})=\mathbf{E}_{+}+P^{+}\left(\mathbf{N} \mathbf{W}_{+}\right)+P^{+}\left(\mathbf{N} \mathbf{W}_{-}\right)  \tag{A.46}\\
& =\mathbf{E}_{+}+P^{+}\left(\mathbf{N} \mathbf{W}_{+}\right)+P^{-}\left(\mathbf{N} \mathbf{W}_{-}\right)+\mathbf{N} \mathbf{W}_{-} \\
& \quad=\mathbf{E}_{+}+P_{\mathbf{w}} \mathbf{N}+\mathbf{N} \mathbf{W}_{-}=\mathbf{N}\left(\mathbf{I}+\mathbf{W}_{-}\right)=\mathbf{N V}_{-} .
\end{align*}
$$

Similarly, we find $\mathbf{M}_{\#}^{-}=\mathbf{N} \mathbf{V}_{+}$. Hence, $\mathbf{M}_{\#}^{+}=\mathbf{M}_{\#}^{-} \mathbf{V}$, which is the jump condition (3.2). Finally, it is easy to see from the definition that $\mathbf{M}_{\#}(z)$ satisfies the asymptotic behavior from Lemma 3.2 as $z \rightarrow \infty$ and $z \rightarrow 0$. Thus, $\mathbf{M}_{\#}(z)$ solves the RHP defined by Lemmas 3.1, 3.2, and 3.3.

It remains to show that $I-P_{\mathrm{w}}$ is invertible.
Lemma A.9. Under the same hypotheses as Theorem 3.12, the operator $I-P_{\mathbf{w}}$ is invertible on $L^{2}(\mathbb{R})$.

Proof. Since $I-P_{\mathbf{w}}$ has Fredholm index zero, it is invertible if and only if $I-P_{\mathbf{w}}$ is injective $[13,33]$. So, suppose $\left(I-P_{\mathbf{w}}\right) \mathbf{G}=\mathbf{0}$ for some $\mathbf{G}(\cdot) \in L^{2}(\mathbb{R})$, and define

$$
\begin{equation*}
\mathbf{M}_{o}(z)=\frac{1}{2 \pi i} \int_{\mathbb{R}} \frac{\mathbf{G}(\zeta) \mathbf{W}(\zeta)}{\zeta-z} \mathrm{~d} \zeta, \quad z \notin \mathbb{R} \tag{A.47}
\end{equation*}
$$

We then have that $\mathbf{M}_{o}(z)$ is analytic for $z \in \mathbb{C} \backslash \mathbb{R}, \mathbf{M}_{o}^{+}(z)=\mathbf{M}_{o}^{-}(z) \mathbf{V}(z)$ for $z \in \mathbb{R}$, $\mathbf{M}_{o}(z)=O(1 / z)$ as $z \rightarrow \infty$, and $\mathbf{M}_{o}(z)=O(1)$ as $z \rightarrow 0$. Then for all $\alpha \in \mathbb{C}$, the matrix $\mathbf{M}+\alpha \mathbf{M}_{o}$ is a solution of the RHP defined by Lemmas 3.1, 3.2, and 3.3. By the above uniqueness results, however, it must then be $\mathbf{M}_{o}(z)=\mathbf{0}$ for all $z \in \mathbb{C}$, which in turn implies $P^{ \pm}(\mathbf{G W}) \equiv \mathbf{0}$ for all $z \in \mathbb{R}$ and

$$
\begin{equation*}
\mathbf{G}(z) \mathbf{W}(z)=\left(\left[P^{+}-P^{-}\right](\mathbf{G} \mathbf{W})\right)(z)=\mathbf{0}, \quad z \in \mathbb{R} \tag{A.48}
\end{equation*}
$$

Then, following the same steps as in (A.46), we obtain $\mathbf{0}=\mathbf{G} \mathbf{V}_{-}$. But since $\mathbf{V}_{-}$is invertible, this implies $\mathbf{G} \equiv \mathbf{0}$. Hence $I-P_{\mathbf{w}}$ is invertible.

## A.11. Reflectionless solutions.

Proof of Lemma 3.13. In the reflectionless case, the reflection coefficients in (2.47) are all identically zero. This, combined with the first and second symmetries of the scattering matrix, which express all off-diagonal entries of the scattering matrix in terms of the reflection coefficients, yields the result.

Proof of Theorem 3.14. For $i=1, \ldots, N_{1}$ and $j=1, \ldots, N_{2}$, define

$$
\begin{gathered}
d_{i}^{(1)}(x, t, z)=\frac{C_{i}(x, t)}{z-\zeta_{i}}, \quad d_{i}^{(2)}(x, t, z)=\frac{\bar{C}_{i}(x, t)}{z-\zeta_{i}^{*}}, \quad d_{j}^{(3)}(x, t, z)=\frac{D_{j}(x, t)}{z-z_{j}}, \\
d_{j}^{(4)}(x, t, z)=\frac{\check{D}_{j}(x, t)}{z-q_{o}^{2} / z_{j}}, \quad d_{j}^{(5)}(x, t, z)=\frac{\bar{D}_{j}(x, t)}{z-z_{j}^{*}}, \quad d_{j}^{(6)}(x, t, z)=\frac{\hat{D}_{j}(x, t)}{z-q_{o}^{2} / z_{j}^{*}} .
\end{gathered}
$$

Then in the reflectionless case, we obtain the following for $i^{\prime}=1, \ldots, N_{1}$ and $j^{\prime}=$ $1, \ldots, N_{2}$ :

$$
\begin{gathered}
m_{21}^{-}\left(x, t, \zeta_{i^{\prime}}^{*}\right)=\frac{i q_{+, 1}}{\zeta_{i^{\prime}}^{*}}+\sum_{i=1}^{N_{1}} d_{i}^{(1)}\left(\zeta_{i^{\prime}}^{*}\right) m_{23}^{+}\left(\zeta_{i}\right)+\sum_{j=1}^{N_{2}} d_{j}^{(3)}\left(\zeta_{i^{\prime}}^{*}\right) m_{22}^{+}\left(z_{j}\right), \\
m_{23}^{+}\left(x, t, \zeta_{i^{\prime}}\right)=\frac{q_{+, 1}}{q_{o}}+\sum_{i=1}^{N_{1}} d_{i}^{(2)}\left(\zeta_{i^{\prime}}\right) m_{21}^{-}\left(\zeta_{i}^{*}\right)-\sum_{j=1}^{N_{2}} d_{j}^{(4)}\left(\zeta_{i^{\prime}}\right) m_{22}^{-}\left(q_{o}^{2} / z_{j}\right), \\
m_{23}^{-}\left(x, t, z_{j^{\prime}}^{*}\right)=\frac{i q_{+, 1}^{*}}{z_{j^{\prime}}^{*}}+\sum_{i=1}^{N_{1}} d_{i}^{(1)}\left(z_{j^{\prime}}^{*}\right) m_{23}^{+}\left(\zeta_{i}\right)+\sum_{j=1}^{N_{2}} d_{j}^{(3)}\left(z_{j^{\prime}}^{*}\right) m_{22}^{+}\left(z_{j}\right), \\
m_{23}^{+}\left(x, t, q_{o}^{2} / z_{j^{\prime}}^{*}\right)=\frac{q_{+, 1}}{q_{o}}+\sum_{i=1}^{N_{1}} d_{i}^{(2)}\left(q_{o}^{2} / z_{j^{\prime}}^{*}\right) m_{21}^{-}\left(\zeta_{i}^{*}\right)-\sum_{j=1}^{N_{2}} d_{j}^{(4)}\left(q_{o}^{2} / z_{j^{\prime}}^{*}\right) m_{22}^{-}\left(q_{o}^{2} / z_{j}\right), \\
m_{22}^{-}\left(x, t, q_{o}^{2} / z_{j^{\prime}}\right)=\frac{r_{+, 2}}{q_{o}}-\sum_{j=1}^{N_{2}} d_{j}^{(5)}\left(\frac{q_{o}^{2}}{z_{j^{\prime}}}\right) m_{21}^{-}\left(z_{j}^{*}\right)+\sum_{j=1}^{N_{2}} d_{j}^{(6)}\left(\frac{q_{o}^{2}}{z_{j^{\prime}}}\right) m_{23}^{+}\left(q_{o}^{2} / z_{j}^{*}\right) \\
m_{22}^{+}\left(x, t, z_{j^{\prime}}\right)=\frac{r_{+, 2}}{q_{o}}-\sum_{j=1}^{N_{2}} d_{j}^{(5)}\left(z_{j^{\prime}}\right) m_{21}^{-}\left(z_{j}^{*}\right)+\sum_{j=1}^{N_{2}} d_{j}^{(6)}\left(z_{j^{\prime}}\right) m_{23}^{+}\left(q_{o}^{2} / z_{j}^{*}\right)
\end{gathered}
$$

However, this system of equations reduces considerably if we take into account the symmetries from (2.40) and (2.46). The reduced system is

$$
\begin{gathered}
m_{23}^{+}\left(x, t, \zeta_{i^{\prime}}\right)=\frac{q_{+, 1}}{q_{o}}+i \sum_{i=1}^{N_{1}} \frac{\zeta_{i}}{q_{o}} d_{i}^{(2)}\left(x, t, \zeta_{i^{\prime}}\right) m_{23}^{+}\left(x, t, \zeta_{i}\right)-\sum_{j=1}^{N_{2}} d_{j}^{(4)}\left(x, t, \zeta_{i^{\prime}}\right) m_{22}^{+}\left(x, t, z_{j}\right), \\
m_{22}^{+}\left(x, t, z_{j^{\prime}}\right)=\frac{r_{+, 2}}{q_{o}}-\sum_{j=1}^{N_{2}}\left[d_{j}^{(5)}\left(x, t, z_{j^{\prime}}\right)+\frac{i z_{j}^{*}}{q_{o}} d_{j}^{(6)}\left(x, t, z_{j^{\prime}}\right)\right] m_{21}^{-}\left(x, t, z_{j}^{*}\right) \\
m_{21}^{-}\left(x, t, z_{j^{\prime}}^{*}\right)=\frac{i q_{+, 1}}{z_{j^{\prime}}^{*}}+\sum_{i=1}^{N_{1}} d_{i}^{(1)}\left(x, t, z_{j^{\prime}}^{*}\right) m_{23}^{+}\left(x, t, \zeta_{i}\right)+\sum_{j=1}^{N_{2}} d_{j}^{(3)}\left(x, t, z_{j^{\prime}}^{*}\right) m_{22}^{+}\left(x, t, z_{j}\right)
\end{gathered}
$$

Substituting the third equation into the second in the reduced system yields

$$
\begin{aligned}
m_{22}^{+}\left(x, t, z_{j^{\prime}}\right) & =\frac{r_{+, 2}}{q_{o}}-\sum_{j=1}^{N_{2}} \frac{i q_{+, 1}}{z_{j}^{*}}\left[d_{j}^{(5)}\left(x, t, z_{j^{\prime}}\right)+\frac{i z_{j}^{*}}{q_{o}} d_{j}^{(6)}\left(x, t, z_{j^{\prime}}\right)\right] \\
- & \sum_{j=1}^{N_{2}} \sum_{i=1}^{N_{1}}\left[d_{j}^{(5)}\left(x, t, z_{j^{\prime}}\right)+\frac{i z_{j}^{*}}{q_{o}} d_{j}^{(6)}\left(x, t, z_{j^{\prime}}\right)\right] d_{i}^{(1)}\left(x, t, z_{j}^{*}\right) m_{23}^{+}\left(x, t, \zeta_{i}\right) \\
& -\sum_{j=1}^{N_{2}} \sum_{j^{\prime \prime}=1}^{N_{2}}\left[d_{j}^{(5)}\left(x, t, z_{j^{\prime}}\right)+\frac{i z_{j}^{*}}{q_{o}} d_{j}^{(6)}\left(x, t, z_{j^{\prime}}\right)\right] d_{j^{\prime \prime}}^{(3)}\left(x, t, z_{j}^{*}\right) m_{22}^{+}\left(x, t, z_{j^{\prime \prime}}^{\prime}\right) .
\end{aligned}
$$

The equations for $m_{23}^{+}\left(x, t, \zeta_{i^{\prime}}\right)$ and $m_{22}^{+}\left(x, t, z_{j^{\prime}}\right)$ form a closed system of $N_{1}+N_{2}$ equations with $N_{1}+N_{2}$ unknowns. We can find a similar system involving the third elements of $m_{2}^{+}\left(x, t, z_{j^{\prime}}\right)$ and $m_{3}^{+}\left(x, t, \zeta_{i^{\prime}}\right)$. These systems may be written $\mathbf{G} \mathbf{X}_{n}=\mathbf{B}_{n}$
$(n=1,2)$, where $\mathbf{X}_{n}=\left(X_{n 1}, \ldots, X_{n\left(N_{1}+N_{2}\right)}\right)^{T}$ and

$$
X_{n i^{\prime}}= \begin{cases}m_{(n+1) 3}^{+}\left(\zeta_{i^{\prime}}\right), & i^{\prime}=1, \ldots, N_{1} \\ m_{(n+1) 2}^{+}\left(z_{i^{\prime}-N_{1}}\right), & i^{\prime}=N_{1}+1, \ldots, N_{1}+N_{2}\end{cases}
$$

Using Cramer's rule, the components of the solutions of said systems are

$$
\begin{equation*}
X_{n i}=\frac{\operatorname{det} \mathbf{G}_{n i}^{\mathrm{aug}}}{\operatorname{det} \mathbf{G}}, \quad i=1, \ldots, N_{1}+N_{2}, \quad n=1,2 \tag{A.49}
\end{equation*}
$$

where $\mathbf{G}_{n i}^{\text {aug }}=\left(\mathbf{G}_{1}, \ldots, \mathbf{G}_{i-1}, \mathbf{B}_{n}, \mathbf{G}_{i+1}, \ldots, \mathbf{G}_{N_{1}+N_{2}}\right)$. Substituting the determinant form of the solution (A.49) into (3.8) yields (3.16).

## A.12. Double poles.

Proof of Lemma 4.1. Differentiating (2.50a) with respect to $z$, evaluating the result at $z=z_{o}$, and using (2.53) yields

$$
\begin{equation*}
\operatorname{det}\left(\phi_{-, 1}^{\prime}\left(x, t, z_{o}\right)-d_{o} \chi^{\prime}\left(x, t, z_{o}\right) / b_{33}\left(z_{o}\right), \chi\left(x, t, z_{o}\right), \phi_{+, 3}\left(x, t, z_{o}\right)\right)=0 \tag{A.50}
\end{equation*}
$$

Then a linear combination of the eigenfunctions in (A.50) must be zero. In other words, there exist appropriate constants (not all zero) such that

$$
p_{o}\left[\phi_{-, 1}^{\prime}\left(x, t, z_{o}\right)-d_{o} \chi^{\prime}\left(x, t, z_{o}\right) / b_{33}\left(z_{o}\right)\right]+p_{1} \chi\left(x, t, z_{o}\right)+p_{2} \phi_{+, 3}\left(x, t, z_{o}\right)=\mathbf{0}
$$

Note that the possibility $p_{1}=p_{2}=0$ does not lead to any contradictions, but we ignore this possibility for now since it will be a special case of more general results. Suppose $p_{o}=0$. Then $\chi\left(x, t, z_{o}\right)$ is proportional to $\phi_{+, 3}\left(x, t, z_{o}\right)$. This result, however, implies that $\phi_{+, 3}\left(x, t, z_{0}\right)$ is proportional to $\phi_{-, 1}\left(x, t, z_{o}\right)$, due to Theorem 2.24. As a result, we have a bound state, which contradicts Lemma 2.19. Therefore, $p_{o} \neq 0$, and we can rescale the constants to obtain the first of (4.1). The rest of (4.1) is obtained similarly. $\quad$ ㅁ

Proof of Corollary 4.3. The results are trivially obtained after combining the results of Lemma 4.1 with Proposition 4.2.

Proof of Lemma 4.4. One simply differentiates the relations for the analytic scattering coefficients in (2.35) and (2.43) with respect to $z$ to obtain the desired results.

Proof of Lemma 4.5. One obtains the results by differentiating (2.42) and (2.46) with respect to $z$.

Proof of Lemma 4.6. The symmetries (4.5a) follow trivially from Lemma 2.26. Applying (4.4) to (4.1a) and (4.1) yields $g_{o}=\left(q_{o}^{2} / z_{o}^{2}\right) \check{g}_{o}, \bar{g}_{o}=\left(i q_{o} / z_{o}^{*}\right) \hat{g}_{o}$, the second of $(4.5 \mathrm{~b})$, and the first of $(4.5 \mathrm{c})$. The first of $(4.5 \mathrm{~b})$ and the identity $g_{o}=0$ are obtained by differentiating (2.34a) with respect to $z$, evaluating the result at $z=z_{o}^{*}$, applying Theorem 2.24 and (4.1d), using (2.27), and finally comparing with (4.1a). A similar process is used to show that $\bar{g}_{o}=0$. The above symmetries then imply $\check{g}_{o}=\hat{g}_{o}=0$.

Proof of Theorem 4.7. We set up the RHP in the same way that we did in (3.2). However, as mentioned earlier, in order to normalize the RHP, we must subtract the rest of the principal parts of the Laurent series corresponding to the entries of $\mathbf{M}^{ \pm}$. In a neighborhood of $z=z_{o}$, we write

$$
\mathbf{M}^{+}(x, t, z)=\frac{\mathbf{M}_{-2, z_{o}}^{+}}{\left(z-z_{o}\right)^{2}}+\frac{\mathbf{M}_{-1, z_{o}}^{+}}{z-z_{o}}+\mathbf{M}_{0, z_{o}}^{+}
$$

where $\mathbf{M}_{0, z_{o}}^{+}(x, t, z)$ is analytic in a neighborhood of $z=z_{o}$, and similarly at the symmetric points of the discrete spectrum. We then subtract the necessary terms from (3.2) and apply the Cauchy projector from (3.12a) to obtain (4.7).

Equation (4.8a) is obtained by taking $\mathbf{M}=\mathbf{M}^{-}$in (4.7) and evaluating its first column at $z=z_{o}^{*}$, while (4.8b) is obtained by taking $\mathbf{M}=\mathbf{M}^{+}$in (4.7), evaluating its second column at $z=z_{o}$, and applying the symmetries (2.42) and (4.4). To obtain (4.8c) and (4.8d), we differentiate (4.7) with respect to $z$ to obtain

$$
\begin{aligned}
\mathbf{M}^{\prime}(x, t, z)=-\left(i / z^{2}\right) \mathbf{M}_{0} & -\frac{\mathbf{M}_{-1, z_{o}}^{+}}{\left(z-z_{o}\right)^{2}}-\frac{\mathbf{M}_{-1, q_{o}^{2} / z_{o}^{*}}^{+}}{\left(z-q_{o}^{2} / z_{o}^{*}\right)^{2}}-\frac{\mathbf{M}_{-1, z_{o}^{*}}^{-}}{\left(z-z_{o}^{*}\right)^{2}}-\frac{\mathbf{M}_{-1, q_{o}^{2} / z_{o}}^{-}}{\left(z-q_{o}^{2} / z_{o}\right)^{2}} \\
& -\frac{2 \mathbf{M}_{-2, z_{o}}^{+}}{\left(z-z_{o}\right)^{3}}-\frac{2 \mathbf{M}_{-2, q_{o}^{2} / z_{o}^{*}}^{+}}{\left(z-q_{o}^{2} / z_{o}^{*}\right)^{3}}-\frac{2 \mathbf{M}_{-2, z_{o}^{*}}^{-}}{\left(z-z_{o}^{*}\right)^{3}}-\frac{2 \mathbf{M}_{-2, q_{o}^{2} / z_{o}}^{-}}{\left(z-q_{o}^{2} / z_{o}\right)^{3}},
\end{aligned}
$$

and we evaluate the columns of $\mathbf{M}^{\prime}$ at the appropriate points.
Proof of Theorem 4.8. The result is obtained easily by examining the first column of $\mathbf{M}^{-}$from (4.7) and comparing this with (A.35).

Proof of Lemma 4.9. Simply considering the system of equations (4.8) in the reflectionless case and combining the resulting closed set of linear equations with (4.9) yields the corresponding solutions of the defocusing Manakov system with NZBC.

Acknowledgments. We thank Mark Ablowitz, Gregor Kovacic, and Barbara Prinari for many insightful discussions.

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[^0]:    *Received by the editors October 30, 2013; accepted for publication (in revised form) November 25,2014 ; published electronically February 10, 2015. The research of the authors was supported in part by the National Science Foundation under grant DMS-1311847.
    http://www.siam.org/journals/sima/47-1/94347.html
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