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GIBBS PHENOMENON FOR DISPERSIVE PDEs ON THE LINE*

GINO BIONDINI† AND THOMAS TROGDON‡

Abstract. We investigate the Cauchy problem for linear, constant-coefficient evolution PDEs on the real line with discontinuous initial conditions (ICs) in the small-time limit. The small-time behavior of the solution near discontinuities is expressed in terms of universal, computable special functions. We show that the leading-order behavior of the solution of dispersive PDEs near a discontinuity of the ICs is characterized by Gibbs-type oscillations and gives exactly the Wilbraham–Gibbs constants.

Key words. asymptotic expansions, dispersive PDEs, Gibbs phenomenon, steepest descent

AMS subject classifications. 35C20, 35C06, 41A55

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1. Introduction.

The Gibbs phenomenon is the well-known behavior of the Fourier series of a piecewise continuously differentiable periodic function at a jump discontinuity. Namely, the partial sums of the Fourier series have large oscillations near the jump which typically increase the maximum of the sum above that of the function itself [5, 19]. The phenomenon is the result of nonuniform convergence: the partial sums of the Fourier series are analytic but converge to a discontinuous function; hence, convergence must be nonuniform in any neighborhood of the discontinuity, which gives rise to highly oscillatory behavior of any finite truncation. In this work we give a precise characterization of the solution of dispersive PDEs with discontinuous initial conditions (ICs), displaying the Gibbs phenomenon for short times. Specifically, we consider a class of initial value problems (IVPs) for dispersive PDEs of the form

$$iq_t - \omega(-i\partial_x)q = 0, \tag{1}$$

with a real-valued, polynomial dispersion relation $\omega(k)$ and with IC

$$q(x,0) = q_o(x). \tag{2}$$

The purpose of this work is to give a quantitative description of what the solution actually looks like as $t \downarrow 0$ when q_o has discontinuities. Figure 1 shows a solution of (1) with $\omega(k) = k^5$ for short times that exhibits Gibbs-like oscillations.

It is well known that, for hyperbolic PDEs, the discontinuities of the IC travel along characteristics [7, 14]. For dispersive and diffusive PDEs, in contrast, even if the ICs are discontinuous, the solution of the IVP is typically classical for all $x \in \mathbb{R}$ as long as $t > 0$ and the IC has sufficient decay as $|x| \to \infty$. Understanding the $t \downarrow 0$ limit is useful for many reasons. For example: (i) to evaluate asymptotics for linear and nonlinear problems [40], (ii) to build/test numerical integrators, and (iii) to

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†Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260 (biondini@buffalo.edu).
‡Department of Mathematics, University of California, Irvine, Irvine, CA 92697 (ttrogdon@math.uci.edu).

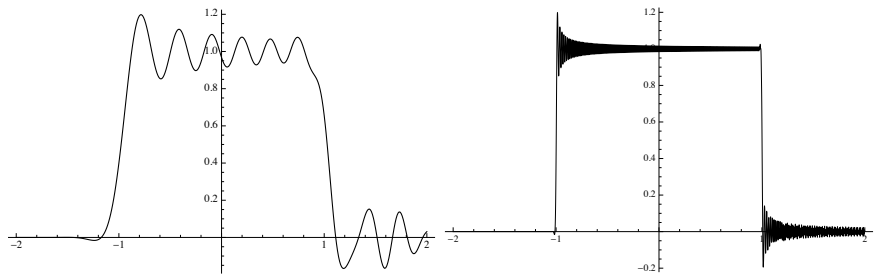

FIG. 1. *The solution of* (1) *with* $\omega(k) = k^5$ *and IC* $q_o(x) = 1$ *if* $|x| \leq 1$ *and* $q_o(x) = 0$ *otherwise. Left:* $t = 10^{-6}$. *Right:* $t = 10^{-12}$. *The solution exhibits the Gibbs phenomenon, as discussed in detail in section* 4.

understand the behavior of initial boundary value problems (IBVPs). Surprisingly, however, while the smoothing effects of diffusion are well known, this perspective on dispersive regularization is not as well characterized in the literature to the best of our knowledge. The key point here is that the slow decay of the Fourier transform of q_o as $k \to \infty$ affects the short-time asymptotics of the solution $q(x,t)$.

Let us briefly elaborate on item (iii) above. One of the original motivations for this work was the study of corner singularities in IBVPs [4, 15, 16, 17]. The issue at hand is the following. Consider (1), with $n = 2$, posed on the domain $D = (0,\infty) \times (0,T)$ so that one has to also specify boundary data at $x = 0$, say, $q(0,t) = g_0(t)$. The smoothness of $q(x,t)$ in $\overline{D}$ is restricted not only by the smoothness (and decay) of $q_o(x)$ and $g_0(t)$ but also by the compatibility of these two functions at $x = t = 0$, i.e., to first order, $q_o(0) = g_0(0)$. (Higher-order conditions are found by enforcing the PDE holds at the corner of the domain.) When compatibility fails at some order, a corner singularity is present. In order to characterize the effect of such a corner singularity on the solution of an IBVP, one must first fully understand the behavior of IVPs with discontinuous ICs.

The outline of this work is the following. In section 2 we summarize our main results concerning both the smoothness of solutions and their short-time behavior. In section 3 we perform the asymptotic analysis in the case of a single discontinuity in the IC q_o. There we identify the special functions that describe the Gibbs-like behavior. Such functions are generalizations of the classical special functions and are computable with similar numerical methods. In section 4 we display some sample solutions, we discuss their Gibbs-like behavior, we further study the properties of the special functions, and we establish a precise connection with the classical Gibbs phenomenon. In section 5 we treat the case where q_o' has one jump discontinuity. In section 6 we present our general result, which allows for multiple discontinuities in q_o itself or in any of its derivatives. A full asymptotic expansion is derived near, and away from, the singular (i.e., nonsmooth) points of q_o. Section 7 contains additional details on the analysis and numerical computation of the special functions considered here. Finally, section 8 concludes this work with a discussion of the results and some final remarks.

Further details and technical results are relegated to four appendices in the supplementary material. In Appendix SM1 we review some well-known results about the well-posedness of the IVP. In Appendix SM2 we prove the result (stated in section 2.1) concerning the classical smoothness of the solution for $t > 0$, using the

method of steepest descent for integrals. Appendix SM3 contains technical results for determining the order of the error terms in our short-time expansions. Finally, in Appendix SM4 we study the robustness of the Gibbs phenomenon by analyzing the behavior of solutions whose ICs are a small perturbation of a discontinuous function.

Note that one can always remove constant and linear terms from $\omega(k)$ by performing a phase rotation and a Galilean transformation, respectively. Thus, without loss of generality throughout this work we take the dispersion relation to be

$$\omega(k) = \sum_{j=2}^{n} \omega_j k^j. \tag{3}$$

2. Summary of results. This section contains a brief summary of our main results. Our summarized results concern regularity, the Gibbs phenomenon, and asymptotics for our special functions. Another one of our main results (Theorem 6) is not summarized here due to its complexity; it gives the full expansion of the solution of (1) for short times.

2.1. Regularity results for linear evolution PDEs. We begin this section by referring to Appendix SM1 for the required definitions and classical results concerning the well-posedness of (1) for $q_o \in L^2(\mathbb{R})$, where

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\theta(x,t,k)}\hat{q}_o(k)\mathrm{d}k, \quad \theta(x,t,k) = kx - \omega(k)t, \tag{4}$$

$$\hat{q}_0(k) = \int_{-\infty}^{\infty} e^{-ikx} q_o(x)\mathrm{d}x. \tag{5}$$

Two properties can be readily seen:

1. $q(\cdot,t) \to q(\cdot,0)$ in $L^2(\mathbb{R})$ as $t \downarrow 0$;
2. if $q_o \in H^1(\mathbb{R})$, then $q(\cdot,t) \to q(\cdot,0)$ uniformly as $t \downarrow 0$.

More delicate questions can be asked about pointwise behavior in the short-time limit, however. In particular, Sjölin [35] showed that when $\omega(k) = k^{2m}$, $m = 1, 2 \ldots$, and $q_o \in H^s(\mathbb{R})$ with compact support for $s \geq 1/4$, $\lim_{t\downarrow 0} q(x,t) = q(x,0)$ for a.e. $x \in \mathbb{R}$. This result was generalized in [27] for general $\omega(k)$ without the assumption of compact support. (See also [31, 37, 41].) The results that follow will only demonstrate a.e. convergence for a subset of $H^{1/4}(\mathbb{R})$. On the other hand, the short-time expansion that we will provide in the following sections is new.

Interesting questions related to the regularity of the solution can also be asked. As is noted in [36], when $\omega(k) = k^2,\ k^3$ the solutions are easily seen to be continuous for $t > 1$ provided $q \in L^2 \cap L^1(\mathbb{R})$. Furthermore, the $L^\infty(\mathbb{R})$ norm of $q(\cdot,t)$ decays in time. A Strichartz-type result was provided in [27], showing, in particular, the space-time estimate $\|q\|_{L^8(\mathbb{R}^2)} \leq C\|q_o\|_{L^2(\mathbb{R})}$ for $\omega(k) = k^3$. In Appendix SM2 we prove results of a classical nature concerning the regularity of the solution.

THEOREM 1 (regularity). *Let $\omega(k)$ be as in (3) and $q(x,t)$ be as in (4), with $q_o \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}, (1+|x|)^\ell \mathrm{d}x)$.*

(i) *If*

$$\ell \geq \frac{2m-n+2}{2(n-1)},$$

$q(x,t)$ is differentiable m times with respect to x for $t > 0$ and $\partial_x^m q(x,t)$ is continuous as a function of x and t.

(ii) *If*

$$\ell \geq \frac{2jn - n + 2}{2(n-1)},$$

$q(x,t)$ is differentiable j times with respect to t for $t > 0$ and $\partial_t^j q(x,t)$ is continuous as a function of x and t.

COROLLARY 1 (classical solution). *Under the same hypotheses as in Theorem 1, if*

$$\ell \geq \mathfrak{C}_n \triangleq \frac{n+2}{2(n-1)},$$

the L^2 solution of the IVP is classical (all derivatives present in the PDE exist) for $t > 0$.

The importance of these results from the perspective of this paper is that if we can guarantee that the solution is smooth for $t > 0$ and if the IC is not smooth, then we can guarantee that the limit $t \downarrow 0$ is a singular one: It forces the breakdown of smoothness. The last regularity result concerns the integrability of solutions.

COROLLARY 2 (loss of integrability). *Let $\omega(k)$ be as in (3) and $q(x,t)$ be as in (4), with $q_o \in L^1 \cap L^2(\mathbb{R})$. Assume q_o has at least one jump discontinuity.*[1] *Then $q(\cdot,t) \notin L^1(\mathbb{R})$ for any $t > 0$.*

Proof. Assume $\tilde{q}_o \triangleq q(\cdot,t) \in L^1(\mathbb{R})$ for some $t > 0$. Then take this as an initial condition for the PDE with $\omega(k)$ replaced with $-\omega(k)$ and find its solution $\tilde{q}(x,t)$. Then $\tilde{q}(x,t)$ should be continuous as a function of x by Theorem 1, but this is a contradiction as uniqueness ensures $q_o = \tilde{q}(\cdot,t)$ and q_o is discontinuous. ∎

2.2. Short-time behavior. To explain two aspects of the short-time behavior we state some theorems. Define

$$I_{\omega,0}(y,t) \triangleq \frac{1}{2\pi} \int_C \mathrm{e}^{iky - i\omega(k)t} \frac{\mathrm{d}k}{ik},$$

where C is a contour in the closed upper-half plane that runs along the real axis but avoids $k = 0$.

THEOREM 2 (leading-order universality). *Assume $q_o \in L^2(\mathbb{R})$ and there exists $c_0 = -\infty < c_1 < c_2 < \cdots < c_N < c_{N+1} = \infty$ such that the restriction $q|_{(c_i,c_{i+1})}$ has one derivative in $L^2((c_i,c_{i+1}))$ for each $i = 0,1,\ldots,N$. Then if $[q_o(c_i)] \triangleq q_o(c_i^+) - q_o(c_i^-) \neq 0$, there exists a constant q_{c_i} such that*

$$\lim_{t\downarrow 0} \frac{q(c_i + x|\omega_n t|^{1/n}, t) - q_{c_i}}{[q_o(c_i)]} = I_{\omega_n,0}(x,1), \quad \omega_n(k) = e^{i\arg(\omega_n)}k^n,$$

uniformly for x in a bounded set.

This is interpreted as a universality theorem because, after proper rescaling, the solution is the same independent of both the initial condition and the lower terms in the dispersion relation. It is proved in section 3. Because of the differential equation (40) satisfied by $I_{\omega_n,0}(x,1)$, we have that *the leading-order behavior*[2] *of the solution*

[1] To be precise, we assume $\operatorname{Re} q_o(x) = \limsup_{\delta\downarrow 0} \delta^{-1} \int_{|y-x|<\delta} \operatorname{Re} q_o(y)\mathrm{d}y$ and $\operatorname{Im} q_o(x) = \limsup_{\delta\downarrow 0} \delta^{-1} \int_{|y-x|<\delta} \operatorname{Im} q_o(y)\mathrm{d}y$.

[2] One can generalize this with appropriate scaling when any derivative of q_o is discontinuous, but we do no pursue this further here. This gives a universality statement involving $I_{\omega_n,m}$.

near a discontinuity is governed by a similarity solution expressed in terms of classical special functions.

The nonuniform convergence of $q(x,t)$ to $q_o(x)$ as $t \downarrow 0$ when $q_o(x)$ is discontinuous at $x = c$ generically results in a so-called overshoot value—the amount by which $q(x,t)$ over- (or under-) approximates $q_o(c^{\pm})$; see Figure 1. We relate the behavior of the overshoot near this region of nonuniform convergence as $t \downarrow 0$ to the Gibbs phenomenon with the following theorems. The first is a restatement of the results of Wilbraham and Gibbs (see [43] and [19]).

THEOREM 3 (Gibbs phenomenon). *Consider the Fourier series approximation of*

$$f(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{given by} \quad S_n[f](x) = \sum_{k=-n}^{n} \frac{4 \sin \frac{k\pi}{2}}{k\pi} \mathrm{e}^{\frac{\mathrm{i}kx\pi}{2}}.$$

Then for any $\delta > 0$

$$\lim_{n\to\infty} \sup_{|x\pm 1|\le\delta} S_n[f](x) = 1 + \mathfrak{g}, \quad \lim_{n\to\infty} \inf_{|x\pm 1|\le\delta} S_n[f](x) = -\mathfrak{g},$$

where

$$\mathfrak{g} = \frac{1}{\pi} \int_0^{\pi} \frac{\sin z}{z} \, \mathrm{d}z - \frac{1}{2} \approx 0.089490\ldots.$$

In this context our results give the following theorem.

THEOREM 4 (Gibbs phenomenon on the line). *Let $q_n(x,t)$ be the solution of $iq_t - (-i\partial_x)^n q = 0$ with*

$$q(x,0) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $\delta > 0$

$$\lim_{n\to\infty} \lim_{t\downarrow 0} \sup_{|x\pm 1|\le\delta} \operatorname{Re} q_n(x,t) = 1 + \mathfrak{g}, \quad \lim_{n\to\infty} \lim_{t\downarrow 0} \inf_{|x\pm 1|\le\delta} \operatorname{Re} q_n(x,t) = -\mathfrak{g},$$

$$\lim_{n\to\infty} \lim_{t\downarrow 0} \sup_{|x\pm 1|\le\delta} \operatorname{Im} q_n(x,t) = 0, \quad \lim_{n\to\infty} \lim_{t\downarrow 0} \inf_{|x\pm 1|\le\delta} \operatorname{Im} q_n(x,t) = 0.$$

One does not have to take $\omega(k) = k^n$ in the previous theorem: It follows for general $\omega(k)$ provided the coefficients are appropriately controlled. One such example is

$$\omega(k) = k^n + \sum_{j=n-m}^{n-1} c_{j,n} k^j,$$

where $-C \le c_{j,n} \le C$ are real and m is fixed. Furthermore, there is an analogue of this theorem that holds for general data as in Theorem 2. This phenomenon is explored in greater depth in section 4.4. We again emphasize that the Gibbs-like oscillations represent the real behavior of the solution of dispersive PDEs and are not a numerical artifact. In other words, Figure 4 (as well as Figure 3 and the figures in section 7) is not a result of truncation error. This fact has important consequences for the numerical solution of dispersive PDEs, particularly, in finite-volume methods where a so-called Riemann problem must be solved.

2.3. Asymptotics of $I_{\omega,m}$. The previous results rely on the asymptotic analysis of the function $I_{\omega,m}(x,t)$ as $t \downarrow 0$ or as $|x| \to \infty$ for fixed $t > 0$. We also define the kernel $K_t(x)$ by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx - i\omega(k)t}\hat{q}_o(k)\mathrm{d}k = \int_{-\infty}^{\infty} K_t(x-y)q_o(y)\mathrm{d}y.$$

In Appendix SM2 we use the method of steepest descent for integrals to derive precise asymptotics of $I_{\omega,m}$ and $K_t(x) = I_{\omega,-1}(x,t)$. First, we rescale the integral

$$\begin{aligned} I_{\omega,m}(x,t) &= \frac{1}{2\pi}\sigma^m\left(\frac{|x|}{t}\right)^{-m/(n-1)}\int_C e^{X(iz - i\omega_n\sigma^n z^n - iR_{|x|/t}(z))}\frac{\mathrm{d}z}{(iz)^{m+1}}, \\ \sigma &= \operatorname{sign}(x), \quad k = \sigma(|x|/t)^{1/(n-1)}z, \\ R_{|x|/t}(z) &\triangleq \sum_{j=2}^{n-1}\omega_j\left(\frac{|x|}{t}\right)^{\frac{j-n}{n-1}}(\sigma z)^j, \quad X \triangleq |x|\left(\frac{|x|}{t}\right)^{1/(n-1)}, \\ \Phi_{|x|/t}(z) &= iz - i\omega_n\sigma^n z^n - iR_{|x|/t}(z), \end{aligned} \tag{6}$$

and then we define $\{z_j\}_{j=1}^{N(n)}$ to be the the solutions of $\Phi'_{|x|/t}(z) = 0$ in the closed upper-half plane. Finally, we define θ_j to be the direction at which the path of steepest descent leaves z_j with increasing real part. The proof of the following can be found in Appendix SM2.

THEOREM 5. *As $|x/t| \to \infty$*

$$\begin{aligned} I_{\omega,m}(x,t) &= -i\operatorname{Res}_{k=0}\left(\frac{e^{ikx-i\omega(k)t}}{(ik)^{m+1}}\right)\chi_{(-\infty,0)}(x) \\ &\quad + \frac{\sigma^m|x|^{-1/2}}{\sqrt{2\pi}}\left(\frac{|x|}{t}\right)^{-\frac{m+1/2}{n-1}} \\ &\quad \times \sum_{j=1}^{N(n)}\frac{e^{X\Phi_{|x|/t}(z_j)+i\theta_j}}{(iz_j)^{m+1}}\frac{1}{|\Phi''_{|x|/t}(z_j)|^{1/2}}\left(1+\mathcal{O}\left(|x|^{-1}\left(\frac{|x|}{t}\right)^{-1/(n-1)}\right)\right). \end{aligned}$$

Hence, the following hold.

- *For fixed $t > 0$ as $|x| \to \infty$*

$$K_t^{(m)}(x) \le c\begin{cases} |x|^{\frac{2m-n+2}{2(n-1)}}, & n \text{ is even}, \\ |x|^{\frac{2m-n+2}{2(n-1)}}, & n \text{ is odd}, \quad \omega_n x > 0, \\ |x|^{-M} \text{ for all } M > 0, & n \text{ is odd}, \quad \omega_n x < 0, \end{cases} \tag{7}$$

  where c depends on m, t, and n.

- *For $|x| \ge \delta > 0$ and $m \ge 0$ as $t \to 0^+$*

$$I_{\omega,m}(x,t) = -i\operatorname{Res}_{k=0}\left(\frac{e^{ikx-i\omega(k)t}}{(ik)^{m+1}}\right)\chi_{(-\infty,0)}(x) + \mathcal{O}\left(t^{\frac{m+1/2}{n-1}}|x|^{-\frac{2m+2n}{2(n-1)}}\right). \tag{8}$$

3. Short-time asymptotics: Discontinuous ICs. Recall that the above representation for the weak solution (4) of the IVP is valid as long as the IC $q_o(x)$ belongs to $L^2(\mathbb{R})$. We first consider initial data with a single discontinuity. For now we will assume that q_o satisfies the following properties.

Assumption 1. Let
- $q_o \in L^2(\mathbb{R})$,
- $[q_o(c)] \triangleq q_o(c^+) - q_o(c^-) \neq 0$,
- q_o' exist on $(-\infty, c) \cup (c, \infty)$,
- $q_o' \in L^q(-\infty, c) \cap L^q(c, \infty)$ for some $1 < q < \infty$, and
- q_o be compactly supported.

In later sections we will discuss the effect of discontinuities in the derivatives of the IC, and we will remove the condition of compact support. The phenomenon we wish to investigate here is the following. The solution is classical for $t > 0$ but converges to a discontinuous function as $t \to 0$. Thus, the limit generally exists in $L^2(\mathbb{R})$ but must fail to be uniform.

To derive an expansion for the solution for short times, it is convenient to integrate the definition (5) of the Fourier transform by parts:

$$\hat{q}_o(k) = \left(\int_{-\infty}^{c} + \int_{c}^{\infty}\right) \mathrm{e}^{-ikx} q_o(x)\,\mathrm{d}x = \frac{1}{ik}\mathrm{e}^{-ikc}[q_o(c)] + \frac{1}{ik}F(k), \tag{9}$$

where

$$F(k) = \left(\int_{-\infty}^{c} + \int_{c}^{\infty}\right) \mathrm{e}^{-ikx} q_o'(x)\,\mathrm{d}x, \quad [q_o(c)] = q_o(c^+) - q_o(c^-). \tag{10}$$

In Appendix SM3 we discuss the properties of $F(k)$. Note that both terms on the right-hand side (RHS) of (9) are singular at $k = 0$, but their sum $\hat{q}_o(k)$ is not. Inserting (9) into the reconstruction formula (4) for the solution of the IVP yields

$$q(x,t) = \frac{1}{2\pi}[q_o(c)] \fint_{\mathbb{R}} \mathrm{e}^{i(k(x-c)-\omega(k)t)} \frac{\mathrm{d}k}{ik} + \frac{1}{2\pi} \fint_{\mathbb{R}} \mathrm{e}^{i\theta(x,t,k)} F(k) \frac{\mathrm{d}k}{ik}, \tag{11}$$

where $\fint$ denotes the principal value (p.v.) integral. The p.v. sign is now needed because each of the integrands in (11) is separately singular at $k = 0$. Of course, one could have chosen other ways to regularize the singularity, and the final result for $q(x,t)$ is independent of this choice.

We next show that the second term on the RHS of (11) is continuous as a function of x for all $t \geqslant 0$, while the first term yields the dominant behavior in the neighborhood of the discontinuity at short times. More precisely, we can write the p.v. integral in (11) as

$$\fint_{\mathbb{R}} f(k)\,\mathrm{d}k = \int_C f(k)\,\mathrm{d}k + \pi i \operatorname{Res}_{k=0}[f(k)], \tag{12}$$

where C is the contour shown in Figure 2. Recall that

$$I_{\omega,0}(y,t) \triangleq \frac{1}{2\pi} \int_C \mathrm{e}^{i[ky-\omega(k)t]} \frac{\mathrm{d}k}{ik}. \tag{13}$$

(The reason for the subscript "0" will become apparent later on when we generalize these results to discontinuities in the higher derivatives.) Also, define

$$q_c = \frac{1}{2}[q_o(c)] + \frac{1}{2\pi} \fint_{\mathbb{R}} \mathrm{e}^{ikc} F(k) \frac{\mathrm{d}k}{ik}, \quad q_{\mathrm{res}}(y,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{ikc} \frac{\mathrm{e}^{i\theta(y,t,k)} - 1}{ik} F(k)\,\mathrm{d}k.$$

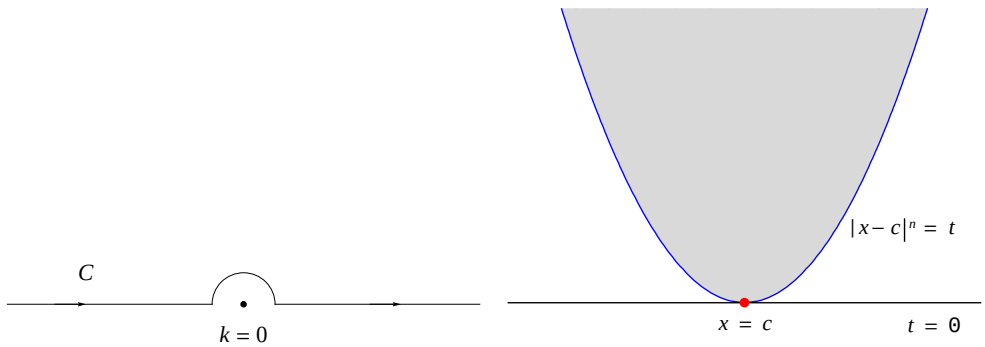

FIG. 2. *Left: The integration contour C for the evaluation of the principal value integral in* (12). *We assume the radius of the semicircle is less than* 1. *Right: The regularization region (in gray) around a discontinuity in the IC.*

Recalling that $\mathrm{Res}_{k=0}(\mathrm{e}^{i\theta(x-c,t,k)}/k) = 1$, we then write the decomposition (11) as

$$q(x,t) = q_c + [q_o(c)]I_{\omega,0}(x-c,t) + q_{\mathrm{res}}(x-c,t). \tag{14}$$

Note that the p.v. is not needed on $q_{\mathrm{res}}(y,t)$, because the integrand is continuous at $k = 0$. Note also that the above decomposition holds for an arbitrary dispersion relation $\omega(k)$.

Importantly, each of the three terms in (14) are individually a solution of the PDE (1). However, each of them provides a different type of contribution. Indeed, a closer look allows the following interpretation of these pieces:

(i) q_c represents a constant offset.

(ii) $[q_o(c)]I_{\omega,0}(y,t)$ characterizes the dominant behavior near the jump discontinuity. The detailed properties of $I_{\omega,0}(y,t)$ are discussed in Appendix SM2. In particular, Theorem 5 implies

$$\lim_{y\to\infty} I_{\omega,0}(y,t) = 0, \qquad \lim_{y\to-\infty} I_{\omega,0}(y,t) = -1. \tag{15}$$

Note also that $\lim_{t\downarrow 0} I_{\omega,0}(0,t) \neq 0$.

(iii) $q_{\mathrm{res}}(c,0) = 0$, and $q_{\mathrm{res}}(x,t)$ is Hölder continuous and vanishes at $(x,t) = (c,0)$ for $t \geq 0$.

One can look at the last item essentially as a trivial consequence of the first two, because the offset value and the jump behavior are all captured by the first and second contribution, respectively. In practice, however, the proof is done in the reverse. Namely, in Appendix SM3 we prove (iii), and we obtain precise estimates for the behavior of $q_{\mathrm{res}}(x-c,t)$ near $(x,t) = (c,0)$. More precisely, we show that, for $\|F\|_{L^p(\mathbb{R})} < \infty$,

$$q_{\mathrm{res}}(x-c,t) = \mathcal{O}(|x-c|^{1/p} + |t|^{1/(np)}). \tag{16}$$

The error term in the above short-time expansion is consistent as $t \to 0$ as long as $|x-c|^n = O(t)$. That is, the above expansion is valid in the region $|x-c|^n \leqslant Ct$ (for some $C > 0$) in the neighborhood of a discontinuity c. We call such a region the *regularization region*. Such a region is illustrated in Figure 2.

One may also wish to understand the behavior of the solution in the short-time limit away from the singularity. Of course, to leading order, we expect it to be

unaffected by the singularity and to limit pointwise to the IC. To prove that this is indeed the case, one must derive an estimate for the error term. The asymptotics of $I_{\omega,m}(x-c,t)$ can be fully characterized; see Theorem 5. The relevant behavior for the present purposes is

$$I_{\omega,0}(x-c,t) = -\chi_{(-\infty,0)}(x-c) + \mathcal{O}(t^{1/(2(n-1))})$$

as $t \to 0$ uniformly in the region $|x-c| \geq \delta > 0$. Here and below, $\chi_R(y)$ is the characteristic function of a set R. (Namely, $\chi_R(y) = 1$ for $y \in R$ and $\chi_R(y) = 0$ otherwise.) We then have

$$q(x,t) = [q_o(c)]\left(\frac{1}{2} - \chi_{(-\infty,0)}(x-c)\right) + \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\theta(x,t,k)}F(k)\frac{\mathrm{d}k}{ik} + \mathcal{O}(t^{1/(2(n-1))}).$$

The relevant tool for characterizing the limiting behavior of the rest of the solution is Lemma SM9. From that result, (9), and the above discussion it follows that

$$q_0(x) = [q_o(c)]\left(\frac{1}{2} - \chi_{(-\infty,0)}(x-c)\right) + \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}F(k)\frac{\mathrm{d}k}{ik}.$$

Therefore, for $|s-c| \geq \delta > 0$ and $\|F\|_{L^p(\mathbb{R})} < \infty$, we have

$$q(x,t) = q_0(s) + \mathcal{O}(|x-s|^{1/p} + |t|^{1/(np)} + |t|^{1/(2(n-1))}). \tag{17}$$

These observations also allow us to prove Theorem 2.

Proof of Theorem 2. Under Assumption 1,

$$\lim_{t\downarrow 0} \frac{q(c + x|\omega_n|^{1/n}t^{1/n}, t) - q_c}{[q_o(c)]} = \lim_{t\downarrow 0} I_{\omega_n,0}(x|\omega_n|^{1/n}t^{1/n}, t)$$

follows directly from (16). Then

$$I_{\omega_n,0}(x|\omega_n|^{1/n}t^{1/n}, t) = \frac{1}{2\pi}\int_C e^{i(k|\omega_n|^{1/n}t^{1/n})x - i\arg(\omega_n)(|\omega_n|^{1/n}kt^{1/n})^n - r(k)t}\frac{\mathrm{d}k}{ik},$$

where $r(k)$ is a polynomial of degree at most $n-1$. Using $kt^{1/n}|\omega_n|^{1/n} \mapsto k$ and redeforming C, we have

$$I_{\omega_n,0}(x|\omega_n|^{1/n}t^{1/n}, t) = \frac{1}{2\pi}\int_C e^{ikx - i\arg(\omega_n)k^n - r(k|\omega_n|^{-1/n}t^{-1/n})t}\frac{\mathrm{d}k}{ik}.$$

But $r(k|\omega_n|^{-1/n}t^{-1/n})t \to 0$ as $t \to 0$. To see that the limit can be passed inside the integral, deform C so that it passes along the steepest descent paths of $e^{-i\omega_n k}$; then pass the limit inside using the dominated convergence theorem, and deform back to C. From this the result follows for the case of one discontinuity, with compact support. The general case follows from Theorem 6 below. ∎

4. Gibbs phenomenon for dispersive PDEs. We now discuss the implications of decomposition (14) regarding the behavior of the solution of the IVP in the short-time limit. We have seen that, apart from a constant offset, the dominant behavior of the solution in the regularization region near a discontinuity of the IC is provided by the function $I_{\omega,0}(y,t)$. In this section we therefore examine more closely the properties of such functions. We start by discussing a simple example.

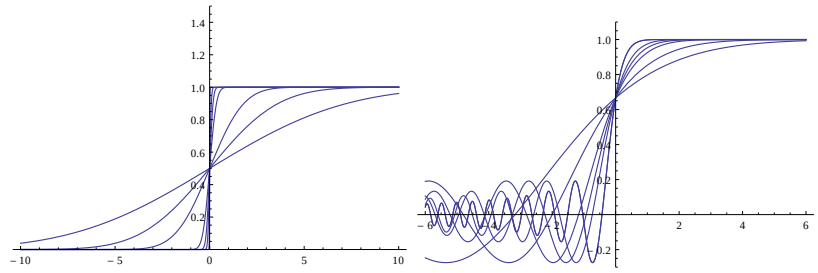

FIG. 3. *Left: The integral $I_0(x,t)+1$ (vertical axis) as a function of x (horizontal axis) for the heat equation (see* (18)*) at various values of time: $t = 0.01$, 0.05, 0.1, 0.2, 1, 2, and 4. Right: The same quantity for the Stokes equation (see* (20)*).*

4.1. Example: Heat equation. Consider the PDE

$$q_t = q_{xx}, \tag{18}$$

corresponding to $\omega(k) = -ik^2$. Let $s = y/t^{1/2}$ and $\lambda = kt^{1/2}$. Then

$$I_{\text{heat},0}(y,t) = \frac{1}{2\pi}\int_C \mathrm{e}^{i\lambda s-\lambda^2}\frac{\mathrm{d}\lambda}{i\lambda} = \frac{1}{2}\left(\operatorname{erf}(s/2)-1\right),$$

where with some abuse of notation we write $I_{\text{heat},0}(y(s),t) = I_{\text{heat},0}(s)$. Note that an easy way to compute the above integral is by using the relation

$$\frac{\mathrm{d}}{\mathrm{d}s}I_{\text{heat},0}(s) = \frac{1}{2\pi}\int_C \mathrm{e}^{i\lambda s-\lambda^2}\mathrm{d}\lambda = \frac{1}{2\sqrt{\pi}}\mathrm{e}^{-s^2/4}.$$

We will see a generalization of this later.

Figure 3 shows the value of $I_0(x,t)$ as a function of x at different times. The resulting effect is that of a diffusion-induced smoothing of the initial discontinuity. This behavior is well known and is discussed in most classical PDE books [14]. What is perhaps less known, however, is the counterpart of this behavior for dispersive PDEs, which we turn to next.

4.2. Example: Schrödinger equation. Consider now the free-particle, one-dimensional linear Schrödinger equation, namely,

$$iq_t + q_{xx} = 0, \tag{19}$$

corresponding to $\omega(k) = k^2$. In this case,

$$I_{\text{schr},0}(s) = \frac{1}{2\pi i}\int_C \mathrm{e}^{i\lambda s-i\lambda^2}\frac{\mathrm{d}\lambda}{\lambda} = \frac{1}{2}\left(\operatorname{erf}(\mathrm{e}^{-i\pi/4}s/2)-1\right).$$

The corresponding behavior is shown in Figure 4. For both PDEs, the dominant behavior near the discontinuity is expressed in terms of a *similarity solution*, depending on x and t only through the similarity variable $s = (x-c)/t^{1/2}$, as seen in Theorem 2. The solution behavior, however, is very different: While for the heat equation the integral $I_{\omega,0}(x,t)$ captures the smoothing effect of the PDE, for the Schrödinger equation, $I_{\omega,0}(x,t)$ results in oscillations.

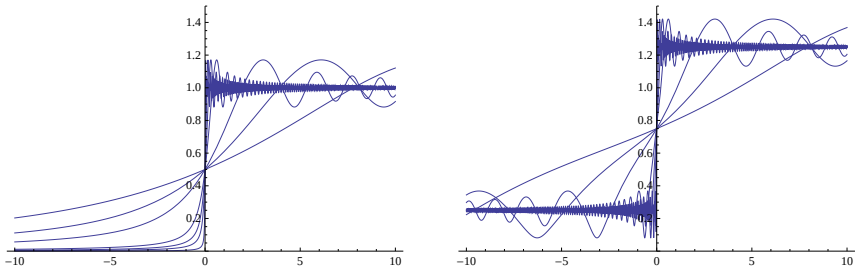

FIG. 4. *Left: Absolute value* $|I_0(x,t)+1|$ *as a function of* x *for the Schrödinger equation (see* (19)*) at the same values of* t *as in Figure* 3. *Right: Same for* $|I_0(x,t)+\frac{5}{4}|$. *Note in this last case the presence of oscillations to the left of the jump.*

4.3. Example: Stokes equation. Consider now the Stokes equations

$$q_t + q_{xxx} = 0, \tag{20}$$

corresponding to $\omega(k) = -k^3$. Letting $s = y/t^{1/3}$ and $\lambda = kt^{1/3}$, one has, using similar methods as before,

$$I_{\text{stokes},0}(y,t) = \frac{1}{2\pi i}\int_C \mathrm{e}^{i\lambda s - i\lambda^3}\frac{\mathrm{d}\lambda}{\lambda} = \int_{s/\sqrt[3]{3}}^{\infty} \mathrm{Ai}(z)\,\mathrm{d}z,$$

where $\mathrm{Ai}(z)$ is the classical Airy function (see, e.g., [30, 33]), which admits the integral representation $\mathrm{Ai}(z) = \int_{\mathbb{R}} \mathrm{e}^{i\lambda z - i\lambda^3}\mathrm{d}\lambda/(2\pi)$ [2]. The corresponding behavior is illustrated in Figure 3.

Note that, since all the PDEs considered in this work are linear, the behavior arising from a negative jump is simply the reflection with respect to the horizontal axis of that for a positive jump. On the other hand, unlike the heat and Schrödinger equation, the Stokes equation does not possess left-right symmetry. So the values of $I_{\omega,0}(y,t)$ to the left of the discontinuity are not symmetric to those to the right (as is evident from Figure 3). Note also that the results for the Stokes equation with the opposite sign of dispersion (i.e., $q_t - q_{xxx} = 0$) are obtained by simply exchanging $x-c$ with $c-x$ (i.e., y with $-y$) in the above discussion.

4.4. Gibbs-like oscillations of dispersive PDEs. The solution of the Schrödinger equation described above shares the three defining features of the Gibbs phenomenon, namely (i) nonuniform convergence of the solution of the PDE to the IC as $t \downarrow 0$ in a neighborhood of the discontinuity; (ii) spatial oscillations with increasing (in fact, unbounded) frequency as $t \downarrow 0$ (because they are governed by the similarity variable); (iii) constant overshoot in a neighborhood of the discontinuity as $t \downarrow 0$. (We will elaborate on this last issue later in this section.) Thus, the limit $t \downarrow 0$ for the solution of the PDE is perfectly analogous to the limit $n \to \infty$ in the truncation of the Fourier series.

Recall that, while q_c contributes a constant offset to the solution, the value of $q(x,t)$ at $(c,0)$ (as obtained from the reconstruction formula (14)) will differ from q_c, because, even though $q_{\text{res}}(0,0) = 0$, in general, $\lim_{t\downarrow 0} I_{\omega,0}(0,t) \neq 0$. For monomial dispersion relations, i.e., $\omega_n(k) = \omega_n k^n$, it easy to see that $I_{\omega_n,0}(0,t)$ is actually independent of time. In fact, the value of $I_{\omega_n,0}(0,t)$ can be easily obtained explicitly. From (37) we have

$$I_{\omega_n,0}(0,t) = \frac{1}{2\pi}\fint_{\mathbb{R}} \mathrm{e}^{\pm i\lambda^n}\frac{\mathrm{d}\lambda}{i\lambda} - \frac{1}{2},$$

TABLE 1

Numerically computed values for the maximum and minimum of the real part, imaginary part, and modulus of $G_n(y,t) = 1 + I_{n,0}(y,t)$ as a function of n. The overshoot converges to the Wilbraham–Gibbs constant $\mathfrak{g}$ (cf. (22)). Sample integration contours are displayed in section 7.1.

n	Max real	Min real	Max imag	Min imag	Max modulus
2	1.1702461	-0.17024605	0.24379748	-0.24379748	1.1706586
3	1.2743521	0.00000000	0.00000000	0.00000000	1.2743521
4	1.1150083	-0.11500827	0.12160263	-0.12160263	1.1060347
5	1.1982367	-0.01598413	0.00000000	0.00000000	1.1982367
6	1.1014610	-0.10146105	0.08196187	-0.08196187	1.1010280
7	1.1661061	-0.03086757	0.00000000	0.00000000	1.1661061
8	1.0962954	-0.09629541	0.06183236	-0.06183236	1.0962451
9	1.1484886	-0.04132209	0.00000000	0.00000000	1.1484886
10	1.0938431	-0.09384306	0.04962857	-0.04962857	1.0938338
11	1.1373989	-0.04878940	0.00000000	0.00000000	1.1373989
60	1.0896059	-0.08960586	0.00833110	-0.00833110	1.0896059
120	1.0895187	-0.08951866	0.00416638	-0.00416638	1.0895187
180	1.0895026	-0.08950263	0.00277769	-0.00277769	1.0895026
240	1.0894970	-0.08949704	0.00208330	-0.00208330	1.0894970
300	1.0894945	-0.08949446	0.00166665	-0.00166665	1.0894945

since $\operatorname{Res}_{\lambda=0}[\mathrm{e}^{\pm i\lambda^n}/(i\lambda)] = 1$. Now note that $\fint_{\mathbb{R}} \mathrm{e}^{\pm i\lambda^n}\,\mathrm{d}\lambda/(i\lambda) = 0$ for n even, while the same integral equals $\pm\int_{\mathbb{R}} \sin(\lambda^n)\,\mathrm{d}\lambda/\lambda = \pm\pi/n$ for n odd. Hence we have simply

$$I_{\omega_n,0}(0,t) = \begin{cases} -\frac{1}{2}\,, & n \text{ even}, \\ -\frac{1}{2}(1 \pm 1/n)\,, & n \text{ odd}. \end{cases} \tag{21}$$

One can carry the analogy with the classical Gibbs phenomenon even further and compute the "overshoot" of these special functions—namely, the ratio of the maximum difference between the value of the special function and the jump, compared to the jump size. Recall that the overshoot for the Gibbs phenomenon is given by the Wilbraham–Gibbs constant [19, 43] (see also [22]),

$$\mathfrak{g} = \frac{1}{\pi}\int_0^\pi \frac{\sin z}{z}\,\mathrm{d}z - \frac{1}{2} \approx 0.089490\ldots. \tag{22}$$

For example, the maximum value of the partial sum of the Fourier series for $\chi_{[-1,1]}(y)$ on $[-2,2]$ will converge to $1+\mathfrak{g}$, and its minimum to $-\mathfrak{g}$.

To examine the overshoot of the special functions, we look at $G_n(y,t) = I_{\omega_n,0}(y,t) + 1$, which converges pointwise to $\chi_{(0,\infty)}(y)$ for all $y \neq 0$ as $t \downarrow 0$. Specifically, we compute numerically the maximum and minimum of the real part, imaginary part, and modulus of $G_n(y,t)$. Note that, for all $t \neq 0$, all such values are independent of t. Table 1 shows these values as a function of n. Surprisingly, the table shows that *these values converge to exactly the same constants as for the Gibbs phenomenon as $n \to \infty$!*

Indeed, a simple calculation shows why this is true. Integration by parts or a simple change of variable can be used to show that, as $n \to \infty$,

$$I_{\omega_n,0}(y,1) = \frac{1}{2\pi i}\int_{C'} e^{iky - ik^n}\,\frac{\mathrm{d}k}{k} + \mathcal{O}(1/n),$$

where $C' = C \cap \{k \in \mathbb{C} : |\operatorname{Re} k| \leq 1\}$, and where without loss of generality the semicircle component of C was taken to have radius less than one. Then, by the

dominated convergence theorem we have

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{C'} e^{iky-ik^n}\frac{\mathrm{d}k}{k} = \frac{1}{2\pi i}\int_{C'} e^{iky}\frac{\mathrm{d}k}{k},$$

where convergence is uniform in y. Moreover, the integral on the RHS is easily shown to be

$$\frac{1}{2\pi i}\int_{C'} e^{iky}\frac{\mathrm{d}k}{k} = \frac{1}{2\pi}\int_{-1}^{1}\frac{\sin ky}{k}\mathrm{d}k - \frac{1}{2},$$

where the contour in the RHS was deformed back to the real axis since there is a removable singularity at $k=0$. After a simple rescaling we then have

$$\lim_{n\to\infty} I_{\omega_n,0}(y,1) = \frac{1}{\pi}\int_0^{\pi}\frac{\sin(\pi y z)}{z}\,\mathrm{d}z - \frac{1}{2},$$

uniformly in y. This integral is maximized and minimized at $y=\pm 1$, respectively, yielding

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\operatorname{Re}G_n(y,1) = 1+\mathfrak{g}, \qquad \lim_{n\to\infty}\sup_{y\in\mathbb{R}}\operatorname{Im}G_n(y,1) = 0,$$
$$\lim_{n\to\infty}\inf_{y\in\mathbb{R}}\operatorname{Re}G_n(y,1) = -\mathfrak{g}, \qquad \lim_{n\to\infty}\inf_{y\in\mathbb{R}}\operatorname{Im}G_n(y,1) = 0.$$

Note that for a fixed value of n such maxima and minima can occur on either side of the jump (e.g., cf. Figures 4 and 3).

Proof of Theorem 4. The solution $q(x,t)$ is given by

$$q(x,t) = I_{\omega,0}(x+c,t) - I_{\omega,0}(x-c,t), \quad \omega(k) = k^n.$$

Near $x=-c$ we have

$$q(y-c,t) = G_n(y,t) - (I_{\omega,0}(y-2c,t)+1), \quad y\in(-\delta,\delta), \quad 0<\delta<2c.$$

It follows from Theorem 5 that

$$|I_{\omega,0}(y-2c,t)+1| \le C_\delta t^{1/(2n-2)}, \quad C_\delta>0,$$

uniformly for all $y\in(-\infty,\delta)$. So,

$$\begin{aligned}\lim_{t\downarrow 0}\left(\sup_{|y|\le\delta}\operatorname{Re}G_n(y,t) - C_\delta t^{1/(2n-2)}\right) &\le \lim_{t\downarrow 0}\sup_{|y|\le\delta}\operatorname{Re}q(y-c,t)\\ &\le \lim_{t\downarrow 0}\left(\sup_{|y|\le\delta}\operatorname{Re}G_n(y,t) + C_\delta t^{1/(2n-2)}\right),\end{aligned}$$

and $\lim_{t\downarrow 0}\sup_{|y|\le\delta}\operatorname{Re}q(y-c,t) = \sup_{y\in\mathbb{R}}\operatorname{Re}G_n(y,1)$. From this the first claim in the theorem follows for $\delta<2c$. To allow δ to be larger, just break the analysis into an interval contained in $(-\infty,0]$ and another interval contained in $[0,\infty)$. The other claims follow from similar calculations. ∎

5. Short-time asymptotics: ICs with discontinuous derivatives. We now treat the case where one of the derivatives of q_o is discontinuous. We begin by assuming a discontinuity in the first derivative; then we treat the general case. We will further generalize the results in section 6.

Assumption 2. Let

- $q_o \in H^1(\mathbb{R})$,
- $[q_o'(c)] = q_o'(c^+) - q_o'(c^-) \neq 0$,
- q_o'' exist on $(-\infty, c) \cup (c, \infty)$,
- $q_o'' \in L^q(-\infty, c) \cap L^q(c, \infty)$ for some $1 < q < \infty$, and
- q_o be compactly supported.

Assuming compact support avoids possible complications arising from the nonexistence of some p.v. integrals (this assumption will be removed in section 6), we will show that the asymptotic behavior in the regularization region is given by integrals of the special functions considered in the previous section.

Note first that, if $F(k)$ is analytic in a neighborhood of the origin, (11) can be written as

$$q(x,t) = [q_0(c)]I_{\omega,0}(x-c,t) + \frac{1}{2\pi}\int_C e^{i\theta(x,t,k)}F(k)\frac{\mathrm{d}k}{ik},$$

with $I_{\omega,0}(y,t)$ and $F(k)$ given by (13) and (10), respectively, and with C as in Figure 2. Analyticity of F is always guaranteed if q_o has compact support. In the case that q_o is continuous but q_o' is discontinuous, we perform one more integration by parts and write

$$q(x,t) = [q_o'(c)]I_{\omega,1}(x-c,t) + \frac{1}{2\pi}\int_C e^{i\theta(x,t,k)}F_1(k)\frac{\mathrm{d}k}{(ik)^2}, \tag{23}$$

where

$$F_1(k) = \left(\int_{-\infty}^{c} + \int_c^{\infty}\right) e^{-iks}q_o''(s)\mathrm{d}s,$$

and where we have introduced the generalization of $I_{\omega,0}(y,t)$ as

$$I_{\omega,m}(y,t) = \frac{1}{2\pi}\int_C \frac{e^{iky-i\omega(k)t}}{(ik)^{m+1}}\mathrm{d}k. \tag{24}$$

As before, we now expand (23) both near and away from the singularity c. In a neighborhood of $(c,0)$, we leave $I_{\omega,1}(y,t)$ alone, and we expand $F_1(k)$. As $k \to 0$,

$$e^{ikc}e^{i\theta(x-c,t,k)} = e^{ikc}(1 + ik(x-c) + \mathcal{O}(k^2)).$$

We then have

$$q(x,t) = [q_o'(c)]I_{\omega,1}(x-c,t) + \frac{1}{2\pi}\int_C e^{ikc}\left(\frac{1+ik(x-c)}{(ik)^2}\right)F_1(k)\,\mathrm{d}k + q_{\mathrm{res},1}(x-c,t),$$

where

$$q_{\mathrm{res},1}(x-c,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikc}\left(\frac{e^{i\theta(x-c,t,k)} - 1 - ik(x-c)}{(ik)^2}\right)F_1(k)\,\mathrm{d}k.$$

We expect $q_{\mathrm{res},1}(y,t)$ to give a lower-order contribution as $(x,t) \to (c,0)$. We thus examine this expression in the regularization region $|x-c| \leq Ct^n$. Lemma SM9

indicates that $q_{\mathrm{res},1}(x,t) = \mathcal{O}(t^{1/n+1/(np)})$ because $F \in L^p(\mathbb{R})$ (where $1/p + 1/q = 1$). Therefore, $q_{\mathrm{res},1}(y,t)$ can indeed be seen as the error term.

We now examine (23) for $|x-c| \geqslant \delta > 0$ and $|s-x| \leqslant \delta/2$. We have

$$\begin{aligned} q(x,t) - q_o(s) &= [q_o'(c)](I_{\omega,1}(x-c,t) - I_{\omega,1}(s-c,0)) \\ &\quad + \frac{1}{2\pi}\int_C e^{iks}(e^{i\theta(x-s,t,k)} - 1)F_1(k)\frac{\mathrm{d}k}{(ik)^2} \\ &= [q_o'(c)](I_{\omega,1}(x-c,t) - I_{\omega,1}(s-c,0)) \\ &\quad + \frac{(x-s)}{2\pi}\int_C e^{iks}F_1(k)\frac{\mathrm{d}k}{ik} + q_{\mathrm{res},1}(x-s,t). \end{aligned}$$

Applying Theorem 5 and Lemma SM9, in the regularization region $|x-s|^n \leqslant Ct$ we have

$$\begin{aligned} q(x,t) &= q_o(s) + [q_o'(c)]((s-c)\chi_{(-\infty,c)}(s) - (x-c)\chi_{(-\infty,c)}(x)) \\ &\quad + \frac{(x-s)}{2\pi}\int_C e^{iks}F_1(k)\frac{\mathrm{d}k}{ik} + \mathcal{O}\left(t^{3/(2(n-1))} + t^{1/n+1/(np)}\right). \end{aligned}$$

This expression is simplified using $\chi_{(-\infty,c)}(s) = \chi_{(-\infty,c)}(x)$ and the relation

$$\frac{(x-s)}{2\pi}\int_C e^{iks}F_1(k)\frac{\mathrm{d}k}{ik} = -\frac{1}{2}[q_o'(c)](x-s) + \frac{(x-s)}{2\pi}\fint_C e^{iks}F_1(k)\frac{\mathrm{d}k}{ik}$$

to obtain

$$\begin{aligned} q(x,t) &= q_o(s) + [q_o'(c)](s-x)(-1/2 + \chi_{(-\infty,c)}(s)) \\ &\quad + \frac{(x-s)}{2\pi}\fint_C e^{iks}F_1(k)\frac{\mathrm{d}k}{ik} + \mathcal{O}\left(t^{3/(2(n-1))} + t^{1/n+1/(np)}\right). \end{aligned}$$

Next we generalize the above result to a discontinuity in a derivative of arbitrary order.

Assumption 3. Let

- $q_o \in H^m(\mathbb{R})$,
- $[q_o^{(m)}(c)] \neq 0$,
- $q_o^{(m+1)}$ exist on $(-\infty,c) \cup (c,\infty)$, separately,
- $q_o^{(m+1)} \in L^q(-\infty,c) \cap L^q(c,\infty)$ for some $1 < q < \infty$, and
- q_o be compactly supported.

Let $a_\ell(y,t)$ be the Taylor coefficients of $e^{i\theta(y,t,k)}$ at $k=0$. Then for $s \in \mathbb{R}$ (possibly equal to c) we find the expansion

(25)

$$\begin{aligned} q(x,t) &= [q_o^{(m)}(c)]I_{\omega,m}(x-c,t) + \frac{1}{2\pi}\int_C e^{iks}\left(\sum_{\ell=0}^{m} a_\ell(x-s,t)k^\ell\right)F_m(k)\frac{\mathrm{d}k}{(ik)^{m+1}} \\ &\quad + q_{\mathrm{res},m}(x-s,t), \end{aligned}$$

where

$$\begin{aligned} q_{\mathrm{res},m}(x-s,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{iks}\left(e^{i\theta(x-s,t,k)} - \sum_{\ell=0}^{m} a_\ell(x-s,t)k^\ell\right)F_m(k)\frac{\mathrm{d}k}{(ik)^{m+1}}, \\ F_m(k) &= \left(\int_{-\infty}^{c} + \int_c^{\infty}\right)e^{-ikx}q_o^{(m+1)}(x)\mathrm{d}x. \end{aligned}$$

Invoking Lemma SM9, this expression provides the asymptotic expansion in the regularization region $|x-s|^n \leq Ct$. Indeed, $q_{\mathrm{res},m}(x,t) = \mathcal{O}(t^{m/n+1/(pn)})$ for $1/p+1/q=1$. This expansion can be understood more thoroughly as follows. Formally, for $s \in \mathbb{R}$

(26)
$$\begin{aligned}&(-i\partial_x)^j q_o(s)\\ &= [q_o^{(m)}(c)]\operatorname{Res}_{k=0}\left(\frac{e^{ik(s-c)}}{i(ik)^{m-j+1}}\right)\chi_{(-\infty,0)}(s-c) + \frac{1}{2\pi}\int_C e^{iks}F_m(k)\frac{\mathrm{d}k}{(ik)^{m-j+1}}.\end{aligned}$$

We next show that

$$\sum_{j=0}^{M}\frac{(-it)^j}{j!}\omega(k)^j = \sum_{\ell=0}^{nM} a_\ell(0,t)k^\ell + \mathcal{O}(t^{M+1}k^{nM}), \tag{27}$$

as $|k|\to\infty$ and $t\downarrow 0$. To see this, it follows from Lemma SM9 that $a_\ell(0,t) = \mathcal{O}(t^{\ell/n})$, and then

$$e^{-i\omega(k)t} - \sum_{\ell=0}^{nM} a_\ell(0,t)k^\ell = \mathcal{O}(t^{M+1})$$

as $t\downarrow 0$, because only integer powers of t appear. Then

$$e^{-i\omega(k)t} - \sum_{j=0}^{M}\frac{(-it)^j}{j!}\omega(-i\partial_x)^j = \mathcal{O}(t^{M+1}),$$

implying

$$\sum_{j=0}^{M}\frac{(-it)^j}{j!}\omega(-i\partial_x)^j = \sum_{\ell=0}^{nM} a_\ell(0,t)k^\ell + \mathcal{O}(t^{M+1}).$$

Then (27) follows by noting that both sides have no powers of k larger than k^{nM}. In turn, (27) implies

(28)
$$\begin{aligned}\sum_{j=0}^{M}\frac{(-it)^j}{j!}\omega(-i\partial_x)^j q_o(s) &= [q_o^{(m)}(c)]\operatorname{Res}_{k=0}\left(\frac{e^{ik(s-c)}}{i(ik)^{m+1}}\sum_{\ell=0}^{nM} a_\ell(0,t)k^\ell\right)\chi_{(-\infty,0)}(s-c)\\ &\quad + \frac{1}{2\pi}\int_C e^{iks}\left(\sum_{\ell=0}^{nM} a_\ell(0,t)k^\ell\right)F_m(k)\frac{\mathrm{d}k}{(ik)^{m+1}} + \mathcal{O}(t^{M+1}).\end{aligned}$$

If $s\neq c$, then this expression is well defined and continuous for $nM\leq m$. If $s=c$, there are issues concerning the definition of the value of $q_o^{(nM)}(c)$ on the left-hand side of the equation, and we must restrict to $nM<m$.

Near the singularity. Let $M = \lfloor (m-1)/n\rfloor$. For $|x-c|^n \leq Ct$ we combine (28) and (25) to find

(29)
$$\begin{aligned}q(x,t) &= \sum_{j=0}^{M}\frac{(-it)^j}{j!}\omega(-i\partial_x)^j q_o(c) + [q_o^{(m)}(c)]I_{\omega,m}(x-c,t)\\ &\quad + \frac{1}{2\pi}\int_C e^{ikc}\left(\sum_{\ell=0}^{m}(a_\ell(x-c,t)-a_\ell(0,t))k^\ell\right)F_m(k)\frac{\mathrm{d}k}{(ik)^{m+1}} + \mathcal{O}\left(t^{\frac{m}{n}+\frac{1}{np}}\right).\end{aligned}$$

Here, the residue term in (26) vanishes at $s = c$ because $Mn < m$ and no k^{-1} term is present. It also follows (see Lemma SM9) that $a_\ell(x-c,t) = \mathcal{O}(t^{\ell/n})$ so that this is indeed a consistent expansion.

Away from the singularity. Let $M = \lfloor m/n \rfloor$. We examine the expansion for near $x = s$ for $|s-c| \geq \delta > 0$. We use the short-time asymptotics for $I_{\omega,m}$ (see Theorem 5) to find for $|x-s|^n \leq C|t|$

(30)
$$\begin{aligned} q(x,t) &= \sum_{j=0}^{M} \frac{(-it)^j}{j!}\omega(-i\partial_x)^j q_o(s) \\ &+ \frac{1}{2\pi}\int_C e^{iks}\left(\sum_{\ell=0}^{m}(a_\ell(x-s,t)-a_\ell(0,t))k^\ell\right)F_m(k)\frac{\mathrm{d}k}{(ik)^{m+1}} \\ &- i[q_o^{(m)}(c)]\,\mathrm{Res}_{k=0}\left(\frac{e^{ik(x-c)-i\omega(k)t}}{(ik)^{m+1}} - \frac{e^{ik(s-c)}}{(ik)^{m+1}}\sum_{j=0}^{M}\frac{(-i\omega(k)t)^j}{j!}\right)\chi_{(-\infty,0)}(s-c) \\ &+ \mathcal{O}\left(t^{\frac{m}{n}}\left(t^{\frac{1}{np}} + t^{\frac{n+2m}{2n(n-1)}}\right)\right). \end{aligned}$$

If we set $x = s$, then the residue term is $\mathcal{O}(t^{M+1})$ $(m/n + 1/n \leq M+1)$, and the short-time Taylor expansion

$$q(x,t) = \sum_{j=0}^{M}\frac{(-it)^j}{j!}\omega(-i\partial_x)^j q_0(x) + \mathcal{O}\left(t^{\frac{m}{n}}\left(t^{\frac{1}{np}} + t^{\frac{n+2m}{2n(n-1)}}\right)\right) \tag{31}$$

follows. Here the error term is uniform in x as x varies in the region $|x-c| \geq \delta$. Thus, in particular, if q_o vanishes identically in a neighborhood of s, then for $|x-s|^n \leq Ct$

$$q(x,t) = \mathcal{O}\left(t^{\frac{m}{n}}\left(t^{\frac{1}{np}} + t^{\frac{n+2m}{2n(n-1)}}\right)\right). \tag{32}$$

A unified formula. We now introduce some convenient and unifying notation that will be useful to combine the above results. Define

$$\begin{aligned} R_{M,m,c}(q_o;x,s) &= -i[q_o^{(m)}(c)]\,\mathrm{Res}_{k=0} \\ &\quad\times\left(\frac{e^{ik(x-c)-i\omega(k)t}}{(ik)^{m+1}} - \frac{e^{ik(s-c)}}{(ik)^{m+1}}\sum_{j=0}^{M}\frac{(-i\omega(k)t)^j}{j!}\right)\chi_{(-\infty,0)}(s-c), \\ A_m(q_o;x,s) &= \frac{1}{2\pi}\int_C e^{iks}\left(\sum_{\ell=0}^{m}(a_\ell(x-s,t)-a_\ell(0,t))k^\ell\right)F_m(k)\frac{\mathrm{d}k}{(ik)^{m+1}}. \end{aligned}$$

Note $A_m(q;x,s)$ can only be applied to functions whose Fourier transform is analytic in a neighborhood of the origin. Therefore, we have for $M = 0,\ldots,\lfloor\frac{m-1}{n}\rfloor$ and $s \in \mathbb{R}$

$$\begin{aligned} q(x,t) &= \sum_{j=0}^{M}\frac{(-it)^j}{j!}\omega(-i\partial_x)q_o(s) + A_m(q_o;x,s) \\ &+ \begin{cases} R_{M,m,c}(q_o;x,s), & s \neq c, \\ [q_o^{(m)}(c)]I_{\omega,m}(x-c,t), & s = c, \end{cases} + \mathcal{O}\left(t^{\frac{m}{n}}\left(t^{\frac{1}{np}} + t^{\frac{n+2m}{2n(n-1)}}\right)\right). \end{aligned}$$

While the formula for $s \neq c$ is also valid for $M = \lfloor m/n \rfloor$, this is a convenient form. Furthermore, when no singularity is present locally, (31) provides a cleaner formula in terms of quantities that are easier to compute. We note that A_m and $R_{M,m,c}$ $(s \neq c)$ contain terms that are analytic in x and t while $I_{\omega,m}$ encodes the dominant behavior near the singularity; i.e., it has a discontinuous derivative at some order.

6. Short-time asymptotics: ICs with multiple singular points and non-compact support. We now discuss the case of ICs with multiple points of discontinuity. The results in this section are the most general of this work regarding the short-time behavior of the solution of dispersive PDEs.

Assumption 4. For $c_0 = -\infty < c_1 < \cdots < c_N < c_{N+1} = +\infty$, let
- $q_o \in H^m(\mathbb{R}) \cap L^1((1+|x|)^\ell \mathrm{d}x)$, with $\ell \geq \mathfrak{C}_n$,
- $[q_o^{(m)}(c_i)] \neq 0$ for $i = 1, \ldots, N$,
- $q_o^{(m+1)}(x)$ exist on (c_{i-1}, c_i) for $i = 1, \ldots, N+1$, and
- $q_o^{(m+1)} \in L^2(c_{i-1}, c_i)$ for $i = 1, \ldots, N+1$.

Note that we have removed the assumption of compact support. The key in doing so is to use a Van der Corput *neutralizer* (or "bump" function) (see, e.g., [2]), namely a function that interpolates infinitely smoothly between 0 and 1. More precisely, for our purposes a neutralizer is a function $\eta_\delta(y)$ with the following properties:

(i) it possesses continuous derivatives of all orders;
(ii) $\eta_\delta(y) = 1$ for $y < \delta/2$ and $\eta_\delta(y) = 0$ for $y > \delta$;
(iii) the derivatives of $\eta_\delta(y)$ of all orders vanish at $y = \delta/2$ and $y = \delta$.

A suitable definition is given by

$$\eta_\delta(y) = n(\delta - x)/[n(y - \delta/2) + n(\delta - x)],$$

where

$$n(y) = \begin{cases} 1, & y < 0, \\ \mathrm{e}^{-1/y}, & y > 0, \end{cases}$$

but the actual form of the neutralizer is irrelevant for what follows. Then, to study the behavior near each discontinuity $(x,t) = (c_j, 0)$ for $j = 1, \ldots, N$, one can decompose the IC as

$$q_o(x) = \sum_{j=1}^{m} q_{o,j}(x) + q_{o,\mathrm{reg}}(x), \tag{33}$$

where

$$q_{o,j}(x) = q_o(x)\, \eta_\delta(|x - c_j|) \tag{34}$$

and

$$q_{o,\mathrm{reg}}(x) = q_o(x) \left(1 - \sum_{j=1}^{m} \eta_\delta(|x - c_j|)\right), \tag{35}$$

with $\delta < \min_{j=1,\ldots,m-1}(c_{j+1} - c_j)/2$. Correspondingly, the solution of the PDE is decomposed as

$$q(x,t) = \sum_{j=1}^{m} q_j(x,t) + q_{\mathrm{reg}}(x,t).$$

Note that each $q_{o,j}^{(m)}(x)$ is discontinuous but compactly supported, while $q_{o,\mathrm{reg}}^{(m)}(x)$ is noncompactly supported but continuous. Moreover, $q_{o,j}(c_{j'}) = 0$ for all $j' \neq j$, and $q_{o,\mathrm{reg}}(c_j) = 0$ for $j = 1, \ldots, m$. Importantly, it follows that $q_{o,\mathrm{reg}} \in H^{m+1}(\mathbb{R})$. Noting that $[q_{o,\mathrm{reg}}^{(m)}(c)] = 0$, with $nM \leqslant m < n(M+1)$, by (31) we have

$$q_{\mathrm{reg}}(x,t) = \sum_{j=0}^{M} \frac{(-it)^j}{j!} \omega(-i\partial_x)^j q_{o,\mathrm{reg}}(x) + \mathcal{O}(t^{m/n+1/(2n)}).$$

In the regularization region $|x - c_j|^n \leq Ct$, all derivatives of $q_{o,\mathrm{reg}}$ vanish identically so that $q_{\mathrm{reg}}(x,t) = \mathcal{O}(t^{m/n+1/(2n)}) = q_{j'}(x,t)$ for $j' \neq j$; see (32).

We state our main asymptotic result as a theorem.

THEOREM 6. *Suppose Assumption* 4 *holds.*

- *If* $|x - c_j|^n \leq C|t|$, *then for* $M = \lfloor \frac{m-1}{n} \rfloor$

$$\begin{aligned} q(x,t) = \sum_{j=0}^{M} \frac{(-it)^j}{j!} \omega(-i\partial_x)^j q_0(c_j) + [q_o^{(m)}(c_j)] I_{\omega,m}(x - c_j, t) \\ + A_m(q_{o,j}; x, c_j) + \mathcal{O}\left(t^{\frac{m}{n}} \left(t^{\frac{1}{2n}} + t^{\frac{n+2m}{2n(n-1)}}\right)\right). \end{aligned} \tag{36}$$

- *If* $|c_j - x| \geq \delta > 0$ *for all* j, *then for* $M = \lfloor \frac{m}{n} \rfloor$

$$q(x,t) = \sum_{j=0}^{M} \frac{(-it)^j}{j!} \omega(-i\partial_x)^j q_0(x) + \mathcal{O}\left(t^{\frac{m}{n}} \left(t^{\frac{1}{2n}} + t^{\frac{n+2m}{2n(n-1)}}\right)\right).$$

Proof. We use linearity. As discussed, we apply (31) and (32) so that $q_{\mathrm{reg}}(x,t) = \mathcal{O}(t^{m/n+1/(2n)})$. The first claim follows from (29) and (30). The final claim follows from (31). ∎

From (36) we conclude that near a singularity $q(x,t)$ can be written as $I_{\omega,m}$ plus lower-order and analytic terms. We have not only an asymptotic expansion but also an expansion that separates regularity properly. Furthermore, the expansion about c_j depends only on local properties of q_o through $q_{o,j}$.

7. Further analysis and computation of the special functions. It should be abundantly clear from sections 3–6 that the integrals $I_{\omega,m}(y,t)$ (defined in (24)) play a crucial role in the analysis. The detailed properties of these integrals are discussed in Appendix SM2. Here we mention some further properties of these objects, and we outline an efficient computational approach for their numerical evaluation.

Monomial dispersion relations. Recall the definition (13) of $I_{\omega,0}(y,t)$, and let $\omega(k) = \omega_n k^n$. Performing the change of variable

$$s = y/(|\omega_n|t)^{1/n}, \qquad \lambda = (|\omega_n|t)^{1/n} k,$$

with some abuse of notation we have that $I_{\omega,0}(y,t) = I_{n,0}^{\sigma}(y,t)$ is given by

$$I_{n,0}^{\sigma}(y,t) = E_{n,1}^{\sigma}(s), \tag{37}$$

with $\sigma = e^{i \arg(\omega_n)}$, and where we have defined

$$E_{n,m}^{\sigma}(s) = \frac{1}{2\pi} \int_C \mathrm{e}^{i\lambda s - \sigma i \lambda^n} \frac{\mathrm{d}\lambda}{(i\lambda)^m}. \tag{38}$$

Like their simpler counterparts $I_{\omega,0}(y,t)$, the integrals $I_{\omega,n}(y,t)$ take on a particularly simple form in the case of a monomial dispersion relation. Taking again $\omega_n \in \mathbb{R}$, we have

$$I_{n,m}(y,t) = (|\omega_n|t)^{m/n} E^{\sigma}_{n,m}(s).$$

Now,

$$E^{\mp}_{n,m}(s) = \frac{1}{2\pi}\int_C \mathrm{e}^{i\lambda s \mp i\lambda^n} \frac{d\lambda}{(i\lambda)^{m+1}}.$$

We then have

$$\frac{\mathrm{d}}{\mathrm{d}s} E^{\sigma}_{n,m}(s) = E^{\sigma}_{n,m-1}(s). \tag{39}$$

So in principle one could obtain $E^{\sigma}_{n,m}(s)$ by integrating the RHS of (39) and by fixing the integration constant appropriately. In practice, however, it is more convenient to evaluate the integral for $E^{\sigma}_{n,m}(s)$ directly, using the methods discussed below.

General dispersion relations. Following arguments from Lemma SM7, for $t > 0$, $I_{\omega,m}(y,t)$ may be deformed to a contour that is asymptotically on the path of steepest descent for $e^{-i\omega(k)}$. Let C be this contour. From this deformation, differentiability follows, and

$$\partial_y^j I_{\omega,m}(y,t) = I_{\omega,m-j}(y,t).$$

Yet more structure is present. A straightforward calculation using integration by parts shows

$$-it\omega'(-i\partial_y) I_{\omega,m}(y,t) = \frac{1}{\pi}\int_C -it\omega'(k)\frac{e^{iky-i\omega(k)t}}{(ik)^{m+1}}\mathrm{d}k = yI_{\omega,m}(y,t).$$

We thus have obtained the $(n-1)$th-order differential equation

$$\omega'(-i\partial_y) I_{\omega,m}(y,t) = \frac{iy}{t} I_{\omega,m}(y,t), \tag{40}$$

satisfied by $I_{\omega,m}(y,t)$.

Dissipative PDEs. The results in section 4 are easily modified when ω_n is not real, i.e., when one is dealing with a dissipative PDE. Recall that, for well-posedness, this can only happen when n is even, in which case $\omega_n = -i|\omega_n|$.

7.1. Numerical computation of the special functions. Next, we discuss the numerical evaluation of $I_{\omega,m}(y,t)$ for all y and t. First, introduce $\omega_t(k) = \omega(kt^{-1/n})t = \omega_n k^n + \mathcal{O}(t^{1/n}k^{n-1})$. Then

$$I_{\omega,m}(y,t) = t^{(m-1)/n} I_{\omega_t,m}(yt^{-1/n}, 1).$$

It is important that $\omega_t(k) \approx \omega_n k^n$ for t small. We consider the computation of $I_{\omega_t,m}(s,1)$ accurately for all $s \in \mathbb{R}$. The numerical method for accomplishing this follows the proof of Theorem 5. Specifically, we use quadrature along the contours Γ_j given in Appendix SM2. Since the precise paths of steepest descent do not need to be followed, we use piecewise-affine contours such that the angle of the contour that passes through each κ_j agrees with the local path of steepest descent. The routines in [34] provide a robust framework for visualizing and computing such contour integrals. In general, Clenshaw–Curtis quadrature is used on each affine component. To ensure accuracy for arbitrarily large s, the contour that passes through κ_j is chosen to be of

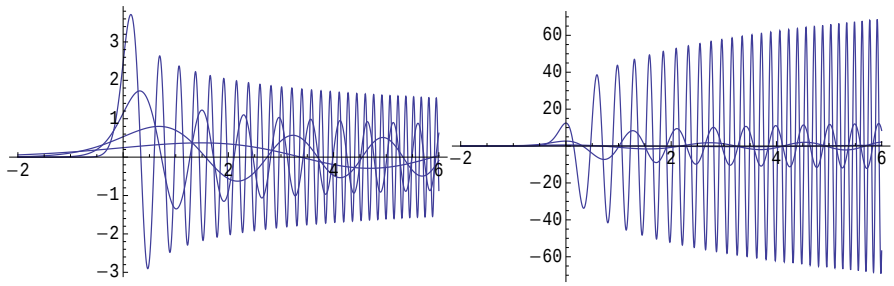

FIG. 5. *Plots of $I_{\omega,m}(y,t)$ with $\omega(k) = k^3$ versus y for $t = 1, 0.1, 0.01, 0.001$. Left: The scaled Airy function ($m = 0$). Right: The first derivative of the scaled Airy function ($m = 1$).*

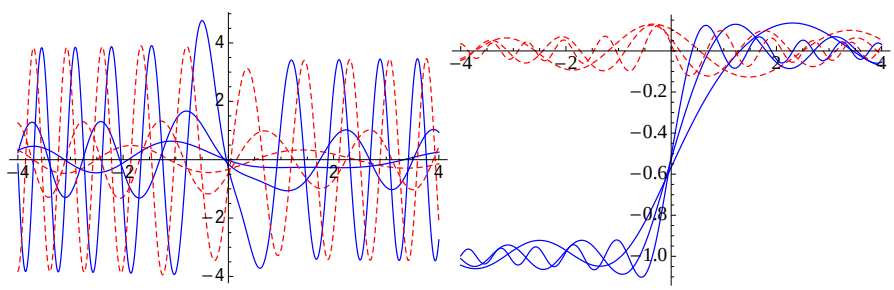

FIG. 6. *Plots of $I_{\omega,m}(y,t)$ with $\omega(k) = k^4 + 2k^3$ versus y for $t = 0.1, 0.01, 0.001$ (solid: real part; dashed: imaginary part). Left: $m = 1$. Right: $m = -1$.*

length proportional to $1/\sqrt{|s\omega_t''(\kappa_j)|}$. This ensures that the Gaussian behavior near the stationary point is captured accurately in the large s limit. If all deformations are performed correctly, with this scaling behavior, a fixed number of sample points for Clenshaw–Curtis quadrature can be used for all s. A more in-depth discussion of this idea is given in [38] and [39].

For reference purposes, the above method should be compared to a more restricted approach for the computation of generalized Airy functions presented in [6]. The authors of this paper compute special functions which correspond to $\omega(k) = k^p/p - ik^q/q$ for $m = -1, 0$; i.e., they introduce dissipation into their special functions, which corresponds to adding artificial viscosity into a finite-difference scheme for a hyperbolic system. With this artificial dissipation they are able to characterize the asymptotic behavior of finite-difference schemes in terms of these special functions.

Example: Airy function. When $\omega(k) = k^3$, the functions $I_{\omega,m}(y,t)$ are scaled derivatives and primitives of the Airy function. This function is displayed in Figure 5 for various values of t. See also Figure 3, where a primitive of the scaled Airy function ($m = -1$) is shown. (But note that in Figure 3 the dispersion relation was $\omega(k) = -k^3$, which results in a switch $y \mapsto -y$.) It is clear that while the Airy function is bounded, its derivative grows in x. This is in agreement with Theorem 5.

Example: A higher-order solution. When the dispersion relation is nonmonomial, the situation is more complicated. Consider, for example, $\omega(k) = k^4 + 2k^3$. In this case $I_{\omega,m}(y,t)$ is no longer a similarity solution. Furthermore, it has nonzero real and imaginary parts. This function is displayed in Figure 6 for various values of t.

Example: The high-order limit. Consider the dispersion relation used to demonstrate the Gibbs phenomenon in Table 1, $\omega(k) = k^n$. To produce this table, a numer-

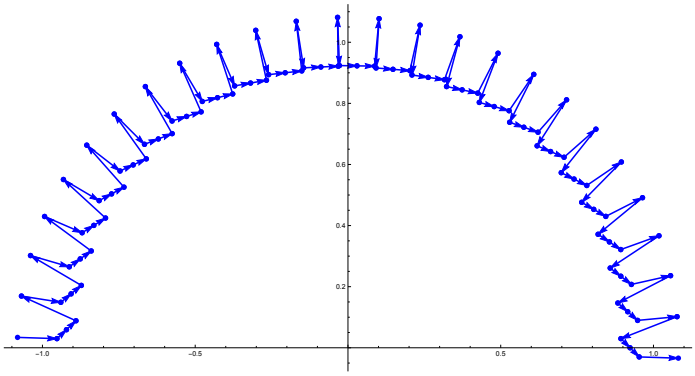

FIG. 7. *The contour in the complex-k plane along which numerical integration is performed to approximate $I_{\omega,0}(1,1)$ when $\omega(k) = k^{50}$. The path of steepest descent at each stationary phase point is approximated by an affine contour.*

ical scheme must be reliable for n large. In Figure 7 we display the contours along which quadrature is used to compute $I_{\omega,0}$ when $n = 50$.

8. Concluding remarks. We have obtained an asymptotic expansion for the short-time asymptotics of the solution of linear evolution PDEs with discontinuous ICs, including precise error estimates. The results apply to generic ICs (i.e., non-piecewise constant, noncompact support). Moreover, the results extend to arbitrary dispersion relations, multiple discontinuities, and discontinuous derivatives of the IC. In a forthcoming publication we will show that these results are also instrumental in characterizing discontinuous BCs and corner singularities in IBVPs using the unified approach presented in [18]. We end this work with a further discussion.

1. We have shown that the short-time asymptotic behavior of the solution of an evolution PDE with singular ICs is governed by similarity solutions and classical special functions. This is analogous to what happens in the long-time asymptotic behavior. In that case, however, it is the discontinuities of the Fourier transform that provide the singular points for the analysis (in addition, of course, to the stationary points or saddle points characteristic of the PDE). In turn, these are related to the slow decay of the ICs at infinity. In this sense, the short-time and long-time behaviors [42] are dual expressions of the characteristic behavior of a linear PDE.

2. We have also shown that the solutions of dispersive linear PDEs exhibit Gibbs-like behavior in the short-time limit. This Gibbs-like behavior is robust, meaning that it persists under perturbation. To explain this point, one should consider the obvious question of what happens with ICs which are a "smoothed out" discontinuity, namely, a sharp but continuous transition from one value to a different one. Such an IC can be considered to be a small perturbation of a step discontinuity in $L^2(\mathbb{R}) \cap L^1((1+|x|)^\ell)$. Thus, as long as the IVP is well-posed, the continuous dependence of the solution of the IVP on the ICs implies that a small change in the ICs will only produce a small change in the solution.

Let us briefly elaborate on this point. Obviously if the perturbed IC is continuous, the solution of the PDE will converge uniformly to it as $t \downarrow 0$. Therefore, the Gibbs phenomenon that is present for the unperturbed solution will eventually disappear in the perturbed solution in this limit. On the other hand, in Appendix SM4 we show that, if the perturbation is sufficiently small, one can still expect to observe a similar Gibbs-like effect *at finite times*.

3. The Gibbs-like behavior has been noticed in a couple of cases for nonlinear PDEs. In particular, DiFranco and McLaughlin [10] studied the behavior of the defocusing nonlinear Schrödinger (NLS) equation with box-type IC. The semiclassical focusing NLS equation was considered in [24] by Jenkins and McLaughlin. Kotlyarov and Minakov [29] studied the behavior of the Korteweg–de Vries (KdV) and modified KdV equations with Heaviside ICs. In both cases, these authors showed that the behavior of the nonlinear PDE for short times is given to leading order by the behavior of the linear PDE. And in both cases, in order to characterize the phenomenon, it was necessary to use complete integrability of the nonlinear PDEs, as well as Deift and Zhou's nonlinear analogue of the steepest descent method for oscillatory Riemann–Hilbert problems [8, 9]. But the results of this work make it clear that this behavior (i) is not a nonlinear phenomenon, and it also applies to linear PDEs; (ii) is a general phenomenon not limited to a few special PDEs.

4. At the same time it is true that for many nonlinear PDEs the nonlinear terms require $O(1)$ times in order to produce an appreciable effect on the solution. Therefore, it is reasonable to expect that the results of this work will also provide the leading-order behavior of the solution of many nonlinear PDEs for short times. Indeed, Taylor [40] studied a generalized NLS equation (which is not completely integrable), and again characterized the behavior of the solutions for short times in terms of those of the linearized PDE. It is hoped that such results can be generalized to other kinds of nonlinear PDEs.

5. Of course, for larger times the solutions of linear and nonlinear PDEs with discontinuous ICs are very different from each other: While for linear PDEs the oscillations spread out thanks to the similarity variable, for nonlinear PDEs the discontinuity gives rise to dispersive shock waves (DSWs), namely, an expanding train of modulated elliptic oscillations with a fixed spatial period, whose envelope interpolates between the values of the solution at either side of the jump. Such a nonlinear phenomenon has been known since the 1960s [21], and a large body of work has been devoted to its study (see, e.g., [3, 11, 12, 13, 20, 23, 25, 26, 28] and references therein). To the best of our knowledge, however, such behavior was never compared to the corresponding behavior for linear PDEs, unlike what was done for the long-time asymptotics (see, e.g., [1, 32]).

6. We reiterate that this Gibbs-like behavior of dispersive PDEs is not a numerical artifact of a numerical approximation to the solution of the PDE but is instead a genuine feature of the solution itself. Thus, when performing numerical simulations of dispersive PDEs, one must be careful to distinguish among spurious Gibbs features induced by the truncation of a Fourier series representation, spurious Gibbs oscillations generated by numerical dispersion (introduced by the numerical scheme used to solve the PDE), and actual Gibbs-like behavior generated by the PDE itself.

7. From a philosophical point of view, one may ask why we consider PDEs with discontinuous ICs at all. In this respect we note on one hand that, apart from any physical considerations, studying these kinds of ICs is important from a mathematical point of view to understand the properties of the PDE and its solutions. Also, on the other hand, such a study makes perfect sense physically. For example, one only need think about hyperbolic systems, for which considerable effort is devoted to the study of shock propagation. These shocks are discontinuities in the solution and describe actual physical behavior. Even though such discontinuities are only approximations of a thin boundary layer, the fact remains nonetheless that representing such situations with discontinuous solutions is a convenient mathematical representation of the actual physical behavior. More generally, while the PDE holds in the interior of the domain

$(x,t) \in \mathbb{R} \times \mathbb{R}^+$, the IC is posed on the *boundary* of this domain. In this sense $t = 0$ is always a singular limit. Indeed, the results of section 2.1 show that, generally speaking, the solution on the interior of the domain is smooth even when the IC is singular.

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