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Whitham modulation theory for the Zakharov–Kuznetsov equation and stability analysis of its periodic traveling wave solutions

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Abstract

We derive the Whitham modulation equations for the Zakharov–Kuznetsov equation via a multiple scales expansion and averaging two conservation laws over one oscillation period of its periodic traveling wave solutions. We then use the Whitham modulation equations to study the transverse stability of the periodic traveling wave solutions. We find that all periodic solutions traveling along the first spatial coordinate are linearly unstable with respect to purely transversal perturbations, and we obtain an explicit expression for the growth rate of perturbations in the long wave limit. We validate these predictions by linearizing the equation around its periodic solutions and solving the resulting eigenvalue problem numerically. We also calculate the growth rate of the solitary waves analytically. The predictions of Whitham modulation theory are in excellent agreement with both of these approaches. Finally, we generalize the stability analysis to periodic waves traveling in arbitrary directions and to perturbations that are not purely transversal, and we determine the resulting domains of stability and instability.

KEYWORDS
transverse stability, Whitham modulation theory, Zakharov–Kuznetsov equation

1 | INTRODUCTION

One of the most striking effects that can arise from the combination of dispersion and nonlinearity is the formation of dispersive shock waves (DSWs), which are coherent, nonstationary oscillatory structures which typically arise in the context of small dispersion problems, and which provide a dispersive counterpart to classical shock waves[47] (e.g., see the review[20] and references therein). DSWs are known to form in surface water waves (where they are known as undular bores), internal waves, nonlinear optics, the atmosphere, Bose–Einstein condensates, and beyond. Because of their ubiquity in nature, the study of DSWs continues to attract considerable interest worldwide.

A powerful tool to study small dispersion problems is Whitham modulation theory[49,50] (or Whitham theory for brevity). Looking at a DSW as a slow modulation of the periodic traveling wave solutions of the underlying partial differential equation (PDE), Whitham theory allows one to derive the so-called Whitham modulation equations (or Whitham equations for brevity), that govern the evolution of these periodic traveling wave solutions over longer spatial and temporal scales. The Whitham equations are a system of first-order, quasi-linear PDEs. For integrable equations in one spatial dimension, the inverse scattering transform (IST)[5,7,39] can also be used to study small dispersion limits (e.g., see Refs. 8, 13, 14, 33 and references therein). However, Whitham theory is more broadly applicable compared to IST, because the former does not require integrability of the original PDE, and therefore it can also be applied to nonintegrable PDEs. Moreover, even if original PDE is integrable, in many cases Whitham theory is still useful because it allows one to obtain a leading-order approximation of the solutions more easily. Because of this, Whitham theory has been applied with great success to many nonlinear wave equations in one spatial dimension (again, see Ref. 20 and references therein). Until recently, however, small dispersion limits in more than one spatial dimension had been much less studied.

Recently, one of the authors derived the Whitham modulation equations for the Kadomtsev–Petviashvili (KP) equation,[3] the Benjamin–Ono equation,[4] and a class of equations of KP type.[2] He then studied the properties of the resulting system of equations[10,11] and used it to study a variety of initial value problems of physical interest.[43–45] The Whitham modulation equations for the nonlinear Schrödinger (NLS) equation in two[6] and three[1] spatial dimensions were also recently derived. In this work, we continue this program of study, aimed at generalizing and applying Whitham modulation theory to nonlinear wave equations in two and three spatial dimensions. Specifically, we derive the Whitham modulation equations for another physically relevant model, namely, the Zakharov–Kuznetsov (ZK) equation, and we use the resulting system of equations to study the transverse stability of its periodic traveling wave solutions.

The ZK equation[52] is a physical model arising in many different contexts, including fusion plasmas and geophysical fluids,[25] magnetized plasmas,[32,52] vortex soliton theory,[40] and wave turbulence.[38] In N spatial dimensions and in semiclassical scaling, the ZK equation is written as

$$u_t + uu_{x_1} + \epsilon^2(\Delta u)_{x_1} = 0, \tag{1}$$

where $\mathbf{x} = (x_1, \dots, x_N)$ are the spatial coordinates, $\Delta = \partial_{x_1 x_1} + \cdots + \partial_{x_N x_N}$ is the Laplacian operator, and $0 < \epsilon \ll 1$ is a small parameter that quantifies the relative magnitude of dispersive effects compared to nonlinear ones. Note that the first spatial coordinate plays a special role compared to the other ones. Accordingly, for brevity we will simply write $x = x_1$ below. When solutions are independent of $x_2, \dots, x_N$, the ZK equation (1) reduces to the celebrated Korteweg–de Vries (KdV) equation. Therefore, the ZK equation is, like the KP equation[27], a multidimensional generalization of the KdV equation. Unlike the KdV and the KP equations, however, the ZK equation appears not to be integrable. (To avoid confusion, we should mention that Ref. 38 refers to (1) as the Petviashvili equation.)

The well-posedness of certain initial value problems and initial-boundary value problem for (1) was studied in Refs. 22, 24, 29, 34, 46, and the decay rate of localized solutions was studied in Refs. 35, 36. (However, these studies are concerned with solutions that vanish as $|\mathbf{x}| \to \infty$, and are therefore not directly applicable to the present work, which deals with solutions that are periodic with respect to each spatial dimension.) The stability of the solitary wave solutions of (1) was studied with various methods,[9,15,17,21,30,31,42,51] and that of its periodic solutions was studied in Ref. 26. Finally, a wave kinetic equation for (1) was derived using formal methods in Ref. 38 and rigorously in Ref. 48 for a stochastic perturbation of (1) on a lattice.

Despite its similarities with the KP equation, the ZK equation (1) is not of KP type in the sense of Ref. 2, because (1) is fully evolutionary, that is, no auxiliary field is present. Therefore the methodology presented in Ref. 2 does not apply. Specifically, the ZK equation (1) differs from the KP equation in two important respects: (i) the terms involving derivatives with respect to the transverse variables $x_2, \dots, x_N$ contain third-order derivatives, not second-order ones, and (ii) these terms involve mixed derivatives. We will see that, as a result, the parameterization of the traveling wave solutions of the ZK equation is quite different from that of the solutions of the KP equation, and in fact it has some similarities with the periodic solutions of two-dimensional NLS equation. Indeed, we will see that the Whitham modulation system for the ZK equation contains a mix of the features of the systems for the KP and NLS equations.

The main result of this work is the ZK–Whitham system (ZKWS) of modulation equations (22), or equivalently (28), as well as its use to investigate the stability of the periodic traveling wave solutions of the ZK equation (1). Specifically, we study the stability of the periodic solutions of (1) with respect to spatially periodic perturbations. In particular, we show that all periodic solutions traveling along the first spatial coordinate are linearly unstable with respect to purely transversal perturbations. We also study the stability of periodic solutions traveling in different directions with respect to arbitrary periodic perturbations, and we identify the domains of stability and instability for each traveling wave solution. For a more detailed description of these results, please see Section 3.

More generally, this document is structured as follows. In Section 2, we present the derivation of the ZKWS. In particular, in Section 2.1 we introduce the periodic traveling wave solutions and relevant conservation laws of (1), in Section 2.2 we present the multiple scales expansion used for the derivation, in Section 2.3 we present the relevant period averages, in Section 2.4 we present the calculations needed to obtain the ZKWS in its final form, and in Section 2.5 we discuss some basic symmetries and reductions of the ZKWS. In Section 3, we study the stability of the periodic traveling wave solutions. In particular, in Section 3.1 we linearize the ZKWS around a constant solution and use the resulting system to study the stability of solutions propagating along the first spatial coordinate with respect to purely transversal perturbations. In Section 3.2, we compute numerically the growth rate of transversal perturbations for the same setup by direct linearization of the

ZK equation, and in Section 3.3 we compute analytically the instability growth rate for the soliton solutions. Finally, in Section 3.4 we generalize the analysis of Section 3.1 to periodic solutions traveling in arbitrary directions and perturbations at arbitrary angles. Section 4 concludes this work with some final remarks.

2 | WHITHAM MODULATION THEORY FOR THE ZK EQUATION

2.1 | Periodic traveling wave solutions and conservation laws of the ZK equation

Recall that the Whitham equations describe modulations of periodic solutions of a nonlinear PDE. Therefore, the first step in formulating Whitham modulation theory is to write down the periodic solutions of the PDE. The ZK equation (1) admits periodic traveling wave solutions, which are most conveniently expressed by introducing Riemann-type variables $r_1 \leqslant r_2 \leqslant r_3$, similarly to what is done for the KdV, KP, and NLS equations. The derivation of these solutions is similar to that for the periodic solutions of those equations, so we omit the details for brevity. However, one can easily verify by direct substitution that (1) admits the following "cnoidal wave" solutions:

$$u(\mathbf{x}, t) = (1 + q^2)\big[(r_1 - r_2 + r_3) + 2(r_2 - r_1)\,\mathrm{cn}^2(2K_m Z, m)\big], \tag{2}$$

where $\mathrm{cn}(z, m)$ is the Jacobi elliptic cosine,[41] $K_m = K(m)$ the complete elliptic integral of the first kind,

$$m = \frac{r_2 - r_1}{r_3 - r_1} \tag{3a}$$

is the elliptic parameter,

$$Z = (\mathbf{k} \cdot \mathbf{x} - \omega t)/\epsilon, \qquad \mathbf{k} = (k_1, \dots, k_N), \qquad \mathbf{q} = (k_2, \dots, k_N)/k_1, \tag{3b}$$

$$k_1 = \frac{\sqrt{r_3 - r_1}}{2\sqrt{6}K_m}, \quad \omega = \tfrac{1}{3}(1 + q^2)(r_1 + r_2 + r_3)\,k_1, \tag{3c}$$

and $q^2 = \mathbf{q} \cdot \mathbf{q} = q_1^2 + \cdots + q_{N-1}^2$. (The calculations required to show that (2) is indeed an exact solution of (1) are very similar to those needed for the cnoidal wave solutions of the KdV equation, and are therefore omitted for brevity.) The solution (2) is uniquely determined by $N + 2$ independent parameters, $r_1, \dots, r_3$ and $q_1, \dots, q_{N-1}$, and it describes wave fronts localized along the lines $\mathbf{k} \cdot \mathbf{x} - \omega t = 2n\pi$, with unit period with respect to the variable Z and period $2K_m$ with respect to the variable x. Note the appearance of the factor $1 + q^2$ in (2) and (3c), unlike the KP equation,[3] and similarly to the NLS equation in N spatial dimensions.[1] In keeping with the notation for the first spatial coordinate, we will simply write $k_1 = k$. Also, when there are only two spatial dimensions (i.e., $N = 2$), we will simply write $y = x_2$, $l = k_2$, and $q = q_1$.

The above solutions admit two nontrivial limits: the harmonic limit, obtained when $m = 0$, corresponding to $r_2 = r_1$, and the soliton limit, obtained when $m = 1$, corresponding to $r_2 = r_3$. Specifically, recalling that $\mathrm{cn}(z, m) = \cos z + O(m)$ as $m \to 0$ and $\mathrm{cn}(z, m) = \mathrm{sech}\ z + O(1 - m)$ as $m \to 1^-$, it is trivial to see that, as $m \to 0$, the solution (2) describes vanishing-amplitude

harmonic oscillations on a nonzero background, whereas, as $m \to 1$, the solution limits to the line soliton solutions of the ZK equation. Explicitly, in two spatial dimensions,

$$u_s(\mathbf{x}, t) = (1 + q^2)\left[\bar{u} + 12c\,\mathrm{sech}^2\left(\sqrt{c}(x + qy - Vt\right)\right], \tag{4}$$

where $\bar{u} = r_1$, $c = (r_3 - r_1)/6$, and $V = (1 + q^2)(\bar{u} + 4c)$. However, we emphasize that the modulation theory presented below applies to all of the periodic solutions (2).

Recall that several methods can be used to derive the Whitham equations: multiple scales perturbation theory (as in Ref. 3), averaged Lagrangians,[49] and averaged conservation laws (as in Ref. 1). Here, we will employ the latter. Accordingly, we need the conservation laws of the ZK equation (1). The ZK equation itself can be written as a conservation law in differential form:

$$u_t + \left(\tfrac{1}{2}u^2 + \epsilon^2 \Delta u\right)_x = 0. \tag{5a}$$

Note that in this case there is no flux along the coordinates $x_2, \dots, x_N$. Moreover, the ZK equation admits an additional differential conservation law related to conservation of mass:

$$(u^2)_t + \left[\tfrac{2}{3}u^3 + 2\epsilon^2\left(u\Delta u - u_x^2 + (\boldsymbol{\nabla}_\perp u)^2\right)\right]_x - 2\epsilon^2\, \boldsymbol{\nabla}_\perp \cdot (u_x \boldsymbol{\nabla}_\perp u) = 0, \tag{5b}$$

where $\boldsymbol{\nabla}_\perp = (\partial_{x_2}, \dots, \partial_{x_N})$ is the gradient with respect to the transverse variables. As mentioned earlier, the ZK equation is not completely integrable, unlike the KdV and KP equations, so only a limited number of conservation laws are available. Nonetheless, below we will show that the above conservation laws will be sufficient for the derivation of the Whitham modulation equations.

2.2 | Multiple scales expansion

As usual in Whitham theory, we now look for modulations of the above periodic solutions. Specifically, we introduce the fast variable Z defined by

$$\boldsymbol{\nabla}_{\mathbf{X}} Z = \frac{\mathbf{k}}{\epsilon}, \qquad Z_t = -\frac{\omega}{\epsilon}, \tag{6}$$

as well as the slow variables $\mathbf{X} = \mathbf{x}$ and $T = t$, and we look for solutions

$$u(\mathbf{x}, t) = u(Z, \mathbf{X}, T), \tag{7}$$

where all of the solution's parameters are now functions of $\mathbf{X} = (X_1, \dots, X_N)$ and T. In particular, $\mathbf{k}$ and ω are now the local wavevector and the local frequency. Recall that in two spatial dimensions we have four independent parameters: r_1, r_2, r_3, and $q = q_1$. With the above multiple scales ansatz, one has

$$\boldsymbol{\nabla}_{\mathbf{x}} = \frac{\mathbf{k}}{\epsilon}\partial_Z + \boldsymbol{\nabla}_{\mathbf{X}}, \qquad \partial_t = -\frac{\omega}{\epsilon}\partial_Z + \partial_T. \tag{8}$$

Or, in two spatial dimensions, simply $\partial_x = (k/\epsilon)\partial_Z + \partial_X$, $\partial_y = (l/\epsilon)\partial_Z + \partial_Y$, and $\partial_t = -(\omega/\epsilon)\partial_Z + \partial_T$, with $X = X_1$ and $Y = X_2$. Inserting the above ansatz into (1), to leading order one recovers the periodic solutions in Section 2.1, but where the parameters r_1, r_2, r_3, and $\mathbf{q}$ are now functions of $\mathbf{X}$ and T. The Whitham modulation equations that we are seeking are precisely the PDEs that govern the spatiotemporal dynamics of these solution parameters.

It is clear from the above discussion that one needs $N+2$ equations to obtain a closed system. The first few Whitham modulation equations, referred to as "conservation of waves," are simply a consequence of the above ansatz and cross-differentiability of Z:

$$\mathbf{k}_T + \boldsymbol{\nabla}_{\mathbf{X}}\omega = 0, \tag{9a}$$

$$\boldsymbol{\nabla}_{\mathbf{X}} \wedge \mathbf{k} = 0, \tag{9b}$$

where $\mathbf{v} \wedge \mathbf{w}$ denotes the N-dimensional wedge product, which in two and three spatial dimensions can be replaced by the standard cross product.[23] In two spatial dimensions, recalling that $l = qk$, (9a) becomes

$$k_T + \omega_X = 0, \tag{10a}$$

$$(kq)_T + \omega_Y = 0, \tag{10b}$$

while (9b) becomes

$$k_Y = (kq)_X. \tag{10c}$$

Equation (9a) above provides N evolution equations, whereas, similarly to Refs. 1, 3, (9b) provides constraints on the initial values of the dependent variables (whose role will be discussed more fully below). Since we need $N+2$ modulation equations, one must therefore supplement (9a) by obtaining two additional modulation equations. The simplest way to do that is to average the first and second conservation laws over one spatial period, obtaining

$$\overline{u_T} + \overline{u u_X} + \epsilon^2 \overline{\Delta u_X} = 0, \tag{11a}$$

$$\overline{(u^2)_T} + \overline{\left[\frac{2}{3}u^3 + \epsilon^2\left(2u\Delta u - u_X^2 + (\boldsymbol{\nabla}_\perp u)^2\right)\right]_X} - 2\epsilon^2\, \overline{\boldsymbol{\nabla}_\perp \cdot (u_X \boldsymbol{\nabla}_\perp u)} = 0, \tag{11b}$$

where $\boldsymbol{\nabla}_\perp = (\partial_{X_2}, \dots, \partial_{X_N})$ is the transverse gradient in the slow variables, and where throughout this work the overbar will denote the integral of a quantity with respect to Z over the unit period. The next step in the derivation of the modulation equations is therefore to compute the above period averages.

2.3 | Period averages

Inserting the ansatz (7), the leading-order solution (2) and using (6), to leading order the averaged conservation laws (11) yield

$$(\overline{u})_T + \left(\frac{1}{2}\overline{u^2}\right)_X = 0, \tag{12a}$$

$$(\overline{u^2})_T + \left(\frac{2}{3}\overline{u^3} - k^2(3+q^2)\overline{(u_Z)^2}\right)_X - 2\,\nabla_\perp \cdot \left(k^2 \mathbf{q}\overline{(u_Z)^2}\right) = 0. \tag{12b}$$

All of the integrals appearing in the above averages can be computed exactly, yielding[16]

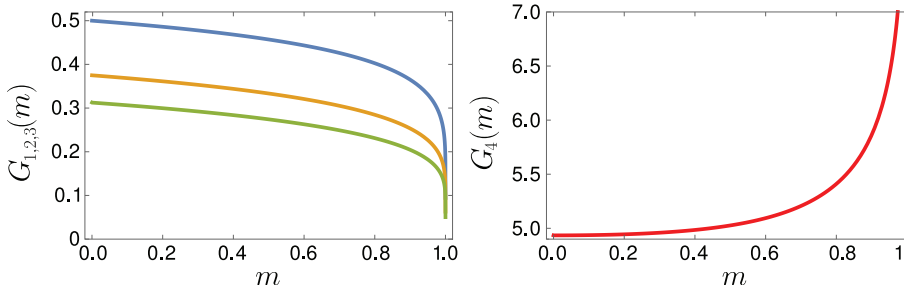

FIGURE 1 The quantities $G_1(m), \dots, G_4(m)$ in (14) (vertical axis) (in green, orange, blue, and red, respectively) as a function of m (horizontal axis).

$$\overline{u} = (1+q^2)[r_1 - r_2 + r_3 + 2(r_2 - r_1)G_1], \tag{13a}$$

$$\overline{u^2} = (1+q^2)^2\left[(r_1 - r_2 + r_3)^2 + 4(r_1 - r_2 + r_3)(r_2 - r_1)G_1 + 4(r_2 - r_1)^2 G_2\right], \tag{13b}$$

$$\begin{aligned}\overline{u^3} = (1+q^2)^3\big[&(r_1 - r_2 + r_3)^3 + 6(r_1 - r_2 + r_3)^2(r_2 - r_1)G_1 \\ &+ 12(r_1 - r_2 + r_3)(r_2 - r_1)^2 G_2 + (r_2 - r_1)^3 G_3\big],\end{aligned} \tag{13c}$$

$$\overline{(u_Z)^2} = (1+q^2)^2 4(r_2 - r_1)^2 G_4, \tag{13d}$$

where

$$G_1(m) = \int_0^1 \mathrm{cn}^2(2K_m z, m)\, dz = \frac{E_m - (1-m)K_m}{mK_m}, \tag{14a}$$

$$G_2(m) = \int_0^1 \mathrm{cn}^4(2K_m z, m)\, dz = \frac{-2(1-2m)E_m + (2 - 5m + 3m^2)K_m}{3m^2 K_m}, \tag{14b}$$

$$G_3(m) = \int_0^1 \mathrm{cn}^6(2K_m z, m)\, dz = \frac{(8 - 23m(1-m))E_m - (1-m)(8 - 19m + 15m^2)K_m}{15m^3 K_m}, \tag{14c}$$

$$\begin{aligned}G_4(m) &= 16K_m^2 \int_0^1 \mathrm{cn}^2(2K_m z, m)\mathrm{dn}^2(2K_m z, m)\,\mathrm{sn}^2(2K_m z, m)\, dz \\ &= 16K_m \frac{2(1 - m(1-m))E_m - (1-m)(2-m)K_m}{15m^2},\end{aligned} \tag{14d}$$

and $E_m = E(m)$ is the complete elliptic integral of the second kind. The behavior of these quantities as a function of m is shown in Figure 1. Their limiting values as $m \to 0$ are

$$G_1(0) = 1/2, \quad G_2(0) = 3/8, \quad G_3(0) = 5/16, \quad G_4(0) = \pi^2/2, \tag{15}$$

while their asymptotic behavior as $m \to 1$ is

$$G_1(m) = -\frac{2 + o(1)}{\log(1-m)}, \; G_2(m) = -\frac{4 + o(1)}{3\,\log(1-m)}, \; G_3(m) = -\frac{16 + o(1)}{15\,\log(1-m)}, \tag{16a}$$

$$G_1'(m) = -\frac{1+o(1)}{2(1-m)K_m^2},\ G_2'(m) = -\frac{1+o(1)}{3(1-m)K_m^2},\ G_3'(m) = -\frac{4+o(1)}{15\,(1-m)K_m^2}, \tag{16b}$$

$$G_4(m) = -\frac{32+o(1)}{15K_m}, G_4'(m) = -\frac{16+o(1)}{15(1-m)}. \tag{16c}$$

Also recall that $K_m = -\frac{1}{2}(\log(1-m) - 4\log 2) + O(1-m)$ and $K_m' = (E_m - (1-m)K_m)/(2m(1-m)) = 1/(2(1-m)) + \frac{1}{8}(\log(1-m) - 4\log 2 + 3) + O(1-m)$ as $m \to 1$.[41] These singular behaviors as $m \to 1$ imply that certain rescalings are needed in order to write the modulation equations in a convenient form, as discussed below.

2.4 | The ZKWS in two spatial dimensions

For brevity, in this section we only write down explicitly the modulation equations in detail in two spatial dimensions, but we emphasize that the calculations below are trivially generalized to any number of transverse dimensions, in a similar manner to Ref. 1. Also, for simplicity from now on we will write derivatives with respect to X, Y, and T simply as derivatives with respect to x, y, and t.

Using the averages (13), recalling the definition of k and ω in (3c), and collecting all terms, Equations (10a), (10b), (12a), and (12b) yield a system of four modulation equations. As usual, however, some manipulations are needed in order to write the resulting system in the most convenient form. We turn to this issue next.

We begin with the first conservation of waves equation, namely, (10a). Recalling (3c), multiplying (10a) by $(1-m)K_m/k$ one then obtains an expression that remains finite both as $m \to 0$ and $m \to 1$.

The second conservation of waves equation requires some additional treatment. In this case, one can first use (10a) to replace k_t, obtaining, as in Refs. 1–3, the universal transversal modulation equation

$$q_t + (D_y\omega)/k = 0, \tag{17}$$

where

$$D_y = \partial_y - q\partial_x \tag{18}$$

is the convective derivative, which will appear prominently in all modulation equations below, similarly to other modulation systems in two spatial dimensions.[1–3] Unlike the first conservation of waves equation, however, in this case in order to obtain a nontrivial equation in the limit $m \to 1$ it is necessary to use the constraint (10c), which we can rewrite so that it remains finite as $m \to 0$ and $m \to 1$ as

$$c_1 D_y r_1 + c_2 D_y r_2 + c_3 D_y r_3 + c_4 q_x = 0, \tag{19}$$

with

$$c_1 = (1-m)(K_m - E_m),\ c_2 = E_m - (1-m)K_m,\ c_3 = -mE_m,\ c_4 = 2(r_2 - r_1)(1-m)K_m. \tag{20}$$

Then, subtracting $\omega/(kK_m)$ times (19) from (17) we finally obtain the desired modulation equation.

The averaged conservation laws (12) are the most complicated, as can be seen from (13) and (14). The only manipulation needed to regularize the resulting equations, however, is just multiplication by $(1-m)K_m$. In light of (13) it is also convenient to divide (12a) by $(1+q^2)$ and (12b) by $(1+q^2)^2$, respectively. (One could also subtract a linear combination of the first conservation law and the first conservation of waves from the second conservation law to try to simplify it, but this is unnecessary for the present purposes.)

The collection of the resulting four modulation equations can be written in matrix form as

$$\tilde{C}\,\mathbf{r}_t + \tilde{A}\,\mathbf{r}_x + \tilde{B}\,\mathbf{r}_y = \mathbf{0}, \tag{21}$$

where $\mathbf{r} = \mathbf{r}(x,y,t) = (r_1, r_2, r_3, q)^T$ collects the four dependent variables. Specifically, we write the first row of (21) from (10a), the second and third rows from (12a) and (12b), respectively, and the fourth row from (10b). Hereafter, $\mathbf{0}_{m\times n}$ and $\mathbf{1}_{m\times n}$ denote matrices of size $m\times n$ with all entries equal to 0 or 1, respectively, and for brevity we will drop the size notation when it should be clear from the context.

All entries of the coefficient matrices $\tilde{C}$, $\tilde{A}$, and $\tilde{B}$ in (21) are finite for all $0<m<1$ as well as in the limits $m\to 0$ and $m\to 1$. On the other hand, their explicit expressions are fairly complicated, and are therefore omitted for brevity, since they are just an intermediate step in the derivation. At the same time, we next show how one can considerably simplify the system by suitably diagonalizing the coefficients of the temporal derivatives.

Owing to (17), the last row of $\tilde{C}$ is simply (0,0,0,1). Writing $\tilde{C}$ in block diagonal form, it is therefore convenient to introduce a partial inverse of $\tilde{C}$ as $C^{-1} = (\tilde{C}_{3\times 3}^{-1}, 1)$, where $\tilde{C}_{3\times 3}$ denotes the upper-left 3×3 block of $\tilde{C}$. Multiplying (21) from the left by C^{-1}, one can then solve the above system of modulation equations for the temporal derivatives, which yields the final *ZK-Whitham system* in matrix form as

$$\mathbf{r}_t + A\,\mathbf{r}_x + B\,\mathbf{r}_y = \mathbf{0}, \tag{22}$$

where the coefficient matrices $A = C^{-1}\tilde{A}$ and $B = C^{-1}\tilde{B}$ are

$$A = \begin{pmatrix} (1+q^2)\,V_{\text{diag}} & \frac{2}{45}q\,A_{3\times 1} \\ \mathbf{0}_{1\times 3} & (1+q^2)\,V \end{pmatrix} - qB, \qquad B = \begin{pmatrix} B_{3\times 3} & B_{3\times 1} \\ \frac{1}{3}(1+q^2)\mathbf{1}_{1\times 3} & 2q\,V \end{pmatrix}, \tag{23}$$

with

$$V_{\text{diag}} = \text{diag}(V_1, \dots, V_3), \tag{24a}$$

where $V_1, \dots, V_3$ are velocities of the KdV–Whitham system, namely,

$$V_1 = V - 2b\frac{K_m}{K_m - E_m},\ V_2 = V - 2b\frac{(1-m)K_m}{E_m - (1-m)K_m},\ V_3 = V + 2b\frac{(1-m)K_m}{mE_m}, \tag{24b}$$

where $V = \frac{1}{3}(r_1 + r_2 + r_3)$, $b = 2(r_2 - r_1)$ is the amplitude of oscillations in (2), and with

$$A_{3\times 1} = D_{3\times 3}^{-1}\mathbf{a}, \qquad D_{3\times 3} = \text{diag}(\mathbf{d}), \tag{25a}$$

$$B_{3\times 1} = \frac{4}{45}\frac{1+5q^2}{1+q^2}(r_3 - r_1)^2\, b_o\, D_{3\times 3}^{-1}\mathbf{e}, \qquad B_{3\times 3} = \frac{4q}{45mK_m}\,(r_3 - r_1)\, D_{3\times 3}^{-1}\,\mathbf{e}^T \otimes \mathbf{b}, \tag{25b}$$

where $\mathbf{v}^T \otimes \mathbf{w}$ denotes the outer product of two vectors, namely, $(\mathbf{v}^T \otimes \mathbf{w})_{i,j} = v_i w_j$, with

$$\mathbf{a} = (a_1, a_2, a_3)^T, \quad \mathbf{b} = (b_1, b_2, b_3)^T, \quad \mathbf{d} = (d_1, d_2, d_3)^T, \quad \mathbf{e} = (-1, 1, 1)^T, \tag{26}$$

and, finally, with

$$a_1 = ((1 + m(14 + m))E_m - K_m(1 - m)(1 + 7m)K_m)(r_3 - r_1)^2 + 45 d_1 r_1^2, \tag{27a}$$

$$a_2 = -((1 - m(16 + 29m))E_m - (1 - m)(1 - m(8 + 45m))K_m)(r_3 - r_1)^2 + 45 d_2 r_1 (2r_2 - r_1), \tag{27b}$$

$$a_3 = (8(2 - m)(1 - m)K_m + (29 + m(16 - m))E_m)(r_3 - r_1)^2 + 45 d_3 r_1 (2r_3 - r_1), \tag{27c}$$

$$b_o = (2 - m)(1 - m)K_m - 2(1 - m(1 - m))E_m, \tag{27d}$$

$$b_1 = 2(1 + 2m^2)E_m K_m - (1 - m)(1 + 2m)K_m^2 - (1 - m + m^2)\, E_m^2, \tag{27e}$$

$$b_2 = (1 - m)(1 - 3m)K_m^2 - 2(1 - m(2 - 3m))E_m K_m + \frac{m^2 - m + 1}{1 - m} E_m^2, \tag{27f}$$

$$b_3 = m\left(5(1 - m)K_m^2 - 2(2 - m)E_m K_m - \left(\frac{1}{1 - m} - m\right)E_m^2\right), \tag{27g}$$

$$d_1 = K_m - E_m, \;\; d_2 = E_m - (1 - m)\, K_m, \;\; d_3 = E_m. \tag{27h}$$

Equivalently, in component form, the ZKWS (22) comprises the following four PDEs:

$$r_{j,t} + (1 + q^2)V_j\, r_{j,x} + b_4\, D_y r_j + h_j\, q_x + \nu_j\, D_y q = 0, \qquad j = 1, 2, 3, \tag{28a}$$

$$q_t + (1 + q^2)V\, q_x + (1 + q^2)\, V_x + 2qV\, D_y q = 0, \tag{28b}$$

where D_y is the convective derivative introduced in (18), and

$$b_4 = \frac{4q(r_3 - r_1)e_j}{45 m K_m d_j}\, \mathbf{b} \cdot \mathbf{r}, \quad h_j = \frac{2q}{45}\frac{a_j}{d_j}, \quad \nu_j = \frac{4(1 + 5q^2)(r_3 - r_1)^2 b_o e_j}{45 d_j (1 + q^2)}, \quad j = 1, 2, 3. \tag{29}$$

The computations above can be readily performed with any computer algebra software. We also point out that the ZKWS (22) is considerably simpler than what one would obtain by multiplying (21) by the full inverse of $\tilde{C}$. More importantly, note how the above ZKWS is purely in evolution form (i.e., all four equations contain a temporal derivative), like those for the two- and three-dimensional NLS equations,[1,6] and unlike those for the KP equation,[3] two-dimensional Benjamin–Ono equation,[4] and modified KP equation[2]. This is of course a direct consequence of the fact that the ZK equation (1) does not comprise a spatial constraint like the KP equation and the two-dimensional Benjamin–Ono equation.

2.5 | Symmetries, reductions, and distinguished limits of the ZKWS

Like the Whitham modulation systems for the KdV, KP, and NLS equations, the ZKWS (22) admits a number of symmetries and reductions.

Symmetries
The ZKWS preserves some of the physical symmetries of the ZK equation, specifically, the symmetries under space-time translations and scaling:

$$u(\mathbf{x}, t) \mapsto u(\mathbf{x} - \mathbf{x}_0, t - t_0), \tag{30a}$$

$$u(\mathbf{x}, t) \mapsto a^2 u(a\mathbf{x}, a^3 t), \tag{30b}$$

respectively, where a, t_0 are arbitrary real constants, and $\mathbf{x}_0$ is an arbitrary N-component real vector. The ZK equation (1) is invariant under (30a) and (30b). Moreover, each of these transformations induces a corresponding transformation for the dependent variables $r_1, \dots, r_3, q$, namely:

$$r_j(\mathbf{x}, t) \mapsto r_j(\mathbf{x} - \mathbf{x}_0, t - t_0), \quad q(\mathbf{x}, t) \mapsto q(\mathbf{x} - \mathbf{x}_0, t - t_0), \tag{31a}$$

$$r_j(\mathbf{x}, t) \mapsto a^2 r_j(a\mathbf{x}, a^3 t), \quad q(\mathbf{x}, t) \mapsto q(a\mathbf{x}, a^3 t) \tag{31b}$$

for $j = 1, 2, 3$. It is straightforward to verify that all these transformations also leave the ZKWS (22) invariant. For brevity, we omit the details.

KdV reduction
It is straightforward to see that, when $q = 0$ and all quantities are independent of y, the ZKWS reduces to the Whitham modulation system for the KdV equation, namely,

$$r_{j,t} + V_j r_{j,x} = 0, \qquad j = 1, 2, 3, \tag{32}$$

where $V_1, \dots, V_3$ are the characteristic velocities of the KdV–Whitham system, as above.

Harmonic limit
The ZKWS system admits a self-consistent reduction in the harmonic limit $m \to 0$ (i.e., $r_2 \to r_1$). In this case, the PDEs for r_1 and r_2 coincide, and we obtain the reduced 3×3 system

$$\mathbf{w}_t + A_o \mathbf{w}_x + B_o \mathbf{w}_y = \mathbf{0} \tag{33}$$

for the three-component dependent variable $\mathbf{w}(x, y, t) = (r_1, r_3, q)$, with

$$A_o = \frac{1}{3}\begin{pmatrix} 3(2r_1 - r_3) + q^2(4r_1 - r_3) & 0 & 2q\left(4r_1^2 - 2r_3 r_1 + r_3^2\right) \\ 0 & 3(1+q^2)r_3 & 6qr_3^2 \\ -q(1+q^2) & -q(1+q^2) & (1-q^2)(2r_1 + r_3) \end{pmatrix}, \tag{34a}$$

$$B_o = \frac{1}{3}\begin{pmatrix} 2q(r_1 - r_3) & 0 & 0 \\ 0 & 0 & 0 \\ 1+q^2 & 1+q^2 & 2q(2r_1 + r_3) \end{pmatrix}. \tag{34b}$$

Soliton limit

The ZKWS system also admits a self-consistent reduction in the soliton limit $m \to 1$ (i.e., $r_2 \to r_3$). The calculations are more involved in this case, since the entries in the second and third columns of A and B diverge. As we show next, however, this is not an issue.

Recalling (3a), let $\tilde{m} = 1 - m = (r_3 - r_2)/(r_3 - r_1)$, and write $r_2 = r_3 - \tilde{m}\,(r_3 - r_1)$. The limit $m \to 1$ corresponds to $\tilde{m} \to 0$ together with $\tilde{m}_x$, $\tilde{m}_y$, and $\tilde{m}_t$. We then look at the second and third columns of A and B multiplied by (r_2, r_3). For the former, we have $a_{i,2}\, r_{2,x} + a_{i,3}\, r_{3,x} = (a_{i,2} + a_{i,3})\, r_{3,x} + a_{i,2}((r_3 - r_1)\,\tilde{m})_x$, for $i = 1, \dots, 4$ with a similar expression for the y derivatives. Since the singular parts of $a_{i,2}$ and $a_{i,3}$ are exactly equal and opposite, it is straightforward to verify that one obtains a finite expression in the limit as $\tilde{m} \to 0$. The result is the soliton modulation system

$$\mathbf{w}_t + A_1\, \mathbf{w}_x + B_1\, \mathbf{w}_y = \mathbf{0} \tag{35}$$

for the same dependent variables $\mathbf{w} = (r_1, r_3, q)$ as above, but where the coefficient matrices are now

$$A_1 = \begin{pmatrix} (1+q^2)r_1 & 0 & 2qr_1^2 \\ \frac{8}{15}q^2(r_1 - r_3) & \frac{1}{15}(5(r_1 + 2r_3) + 3q^2(6r_3 - r_1)) & 2q\frac{3r_1^2+48r_3^2-6r_1r_3+q^2\left(19r_1^2-38r_1r_3+64r_3^2\right)}{45(1+q^2)} \\ -\frac{1}{3}q(1+q^2) & -\frac{2}{3}q(1+q^2) & \frac{1}{3}(1-q^2)(r_1+2r_3) \end{pmatrix}, \tag{36a}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{8}{15}q(r_1 - r_3) & \frac{8}{15}q(r_1 - r_3) & -\frac{8(51+q^2)(r_1-r_3)^2}{45(1+q^2)} \\ \frac{1}{3}(1+q^2) & \frac{2}{3}(1+q^2) & \frac{2}{3}q(r_1+2r_3) \end{pmatrix}. \tag{36b}$$

3 | TRANSVERSE INSTABILITY OF THE PERIODIC TRAVELING WAVE SOLUTIONS OF THE ZK EQUATION

We now show how the ZK-Whitham modulation system derived in Section 2 can be applied to study the stability of the periodic traveling wave solutions of the ZK equation for all $0 \leqslant m \leqslant 1$. We will then compare the predictions of Whitham theory with a numerical evaluation of the instability growth rate, as well as with an explicit, analytical calculation of the growth rate in the soliton limit.

3.1 | Stability analysis via Whitham theory

Recall that, when r_1, r_2, r_3, and q are independent of x, y, and t, (2) is an exact periodic traveling wave solution of (1). In order to study the stability of such solutions, we therefore look for solutions of the ZKWS (22) in the form of a constant solution $\mathbf{r}^{(0)} = (r_1^{(0)}, r_2^{(0)}, r_3^{(0)}, q^{(0)})$ plus a small

perturbation, namely,

$$r_j(x,y,t) = r_j^{(0)} + \delta r_j^{(1)}(x,y,t), \quad j = 1,2,3, \qquad q(x,y,t) = q^{(0)} + \delta q^{(1)}(x,y,t), \tag{37}$$

with $0 < \delta \ll 1$. Substituting this ansatz into (22) and neglecting terms $O(\delta^2)$ and smaller, we then obtain the linearized ZKWS

$$\mathbf{r}_t^{(1)} + A^{(0)}\mathbf{r}_x^{(1)} + B^{(0)}\mathbf{r}_y^{(1)} = 0, \tag{38}$$

since $\mathbf{r}^{(0)}$ is constant with respect to x, y, and t. Here, $\mathbf{r}^{(1)} = (r_1^{(1)}, r_2^{(1)}, r_3^{(1)}, q^{(1)})$, while $A^{(0)}$ and $B^{(0)}$ denote the 4×4 matrices A and B above evaluated at $\mathbf{r} = \mathbf{r}^{(0)}$. Since (38) is a linear system of PDEs with constant coefficients, it is sufficient to study plane wave solutions of (38), which correspond to periodic perturbations of the underlying cnoidal wave solution (2). We therefore look for solutions of (38) in the form

$$\mathbf{r}^{(1)}(x,y,t) = \mathbf{R}\, \mathrm{e}^{i(Kx+Ly-Wt)}, \tag{39}$$

where $\mathbf{R}$ is a constant vector, and K, L, and W are, respectively, the perturbation wavenumbers in the x- and y-directions and the perturbation's angular frequency.

Recall that the underlying cnoidal wave solution (2) has spatial periods in x and y determined by (3). The perturbations in (39) are not required to be coperiodic with the underlying solution (2) at this point. Substituting the expression (39) into (38), the problem above is then transformed into the homogeneous linear system of equations $(-W I_4 + K A^{(0)} + L B^{(0)})\mathbf{R} = \mathbf{0}$, which is equivalent to the eigenvalue problem

$$(K A^{(0)} + L B^{(0)})\mathbf{R} = W\mathbf{R}. \tag{40}$$

The eigenvalues (corresponding to nontrivial solutions for $\mathbf{R}$) are the roots of the characteristic polynomial $p(K,L,W) = \det(K A^{(0)} + L B^{(0)} - W I_4)$, where I_4 is the 4×4 identity matrix. In turn, the condition $p(K,L,W) = 0$ determines the linearized dispersion relation $W = W(K,L)$. If $W \in \mathbb{R}$, the ZKWS system (22) is hyperbolic, and the periodic solution (2) obtained from the constant solution $\mathbf{r} = \mathbf{r}^{(0)}$ of the ZKWS is linearly stable. Otherwise, the system is elliptic and the corresponding periodic solution is unstable.

By virtue of the scaling invariance of the ZKWS (22), we can set $r_1 = 0$ and $r_3 = 1$ without loss of generality, in which case we simply have $r_2 = m$. Still, for general values of q, K, and L, finding the linearized dispersion $W(K,L)$ involves computing the roots a highly complicated quartic polynomial. On the other hand, a particularly simple scenario is obtained when $q = 0$ (which corresponds to a vertical cnoidal wave, whose period in the y-direction is therefore infinite) and $K = 0$ (which corresponds to purely transversal perturbations). In this case, the perturbations (39) are indeed coperiodic with the underlying cnoidal wave solution (2) in the x-direction. Also, in this case we simply have $(W/L)^2 = f(m)$, with

$$f(m) = \frac{4}{135}\frac{(2(1-m(1-m))E_m - (1-m)(2-m)K_m)\big((1-m)K_m^2 - 2(2-m)E_mK_m + 3E_m^2\big)}{E_m(K_m - E_m)(E_m - (1-m)K_m)}. \tag{41}$$

It is straightforward to see that $f(m) < 0$ for all $0 < m < 1$. Therefore, periodic traveling wave solutions of the ZK equation are linearly unstable with respect to transverse perturbations. More

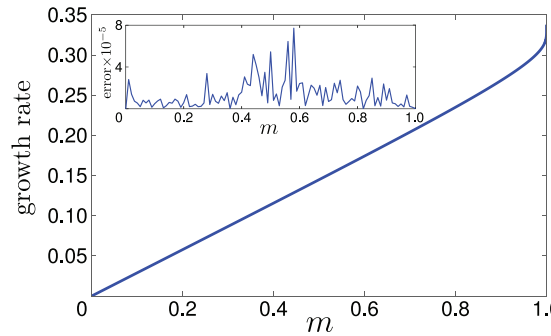

FIGURE 2 Main panel: Relative growth rate Im W/L of perturbations in the long wave limit as a function of m. Inset: Discrepancy between the analytical predictions of Whitham theory (Section 3.1) and the numerical computation of the growth rate by direct linearization of the Zakharov–Kuznetsov (ZK) equation (Section 3.2). See text for details.

precisely, the above calculations yield the relative growth rate (i.e., Im W/L) of perturbations in the long wave limit as $g(m) = \sqrt{-f(m)}$. The behavior of $g(m)$ as a function of m is shown in Figure 2. Note that $g(0) = 0$ (indicating that the constant solutions are linearly stable), and $g(m)$ increases monotonically in m, limiting to the value $g(1) = 4/(3\sqrt{15}) \simeq 0.344265$, which is the growth rate of unstable perturbations of the soliton solutions of the ZK equation in the same limit (cf. Section 3.3).

The above predictions about the instability of the periodic solutions are consistent with the results of Refs. 26, 37. The above results, however, yield a fully explicit expression for the instability growth rate, similar to Refs. 2–4. As we show next, these predictions are in excellent agreement with a numerical calculation of the growth rate (in Section 3.2) as well as with a direct perturbation theory for the soliton solutions (in Section 3.3).

The absolute growth rate, Im W, is proportional to L, and is therefore unbounded. This is because the ZKWS is a dispersionless system, like all leading-order Whitham modulation systems, and all these systems suffer from the same limitation. This is why the above calculations only apply in the long wavelength limit. To obtain the maximum growth rate over all wavenumbers L, it would be necessary to go to higher order and incorporate dispersive terms in the modulation system before computing the linearized dispersion relation. For example, this is done when studying the modulational stability or instability of a plane wave via the NLS equation. The focusing NLS equation is a modulation equation that incorporates dispersion, so its instability spectrum is nonlinear (and bounded) in the perturbation wavenumber.

3.2 | Stability analysis via linearization of the ZK equation and Floquet–Hill's method

We can validate the predictions of Whitham theory by studying numerically the linear stability of the periodic solutions of the ZK equation (1) and comparing the findings with those obtained via Whitham theory in Section 3.1.

In this case, by analogy with Section 3.1, we look for solutions of the ZK equation (1) in two spatial dimensions in the form $u(x, y, t) = u_o(x, y, t) + \delta v(x, y, t)$, where $0 < \delta \ll 1$, and where

$u_o(x, y, t)$ is an exact periodic traveling wave solution, namely,

$$\begin{aligned} u_o(x, y, t) &= (1 + q^2)((r_1 - r_2 + r_3) + 2(r_2 - r_1)\,\mathrm{cn}^2(2K_m Z, m)) \\ &= (1 + q^2)\left[r_1 - r_2 + r_3 + 2(r_2 - r_1)\,\mathrm{cn}^2\left(\sqrt{(r_3 - r_1)/6}(x + qy - Vt)/\epsilon\right)\right], \end{aligned} \tag{42}$$

$Z = k(x + qy - Vt)/\epsilon$ is the fast variable defined in (6), $V = \omega/k$ and k and ω are as in (3c). Substituting this ansatz in (1), to leading order in δ we obtain a linearized ZK equation:

$$v_t + (u_0 v)_x + \epsilon^2(\Delta v)_x = 0. \tag{43}$$

To obtain the correct balance of terms in ϵ and δ, we look for transversally modulated perturbations that are coperiodic with respect to Z, that is, we use the following ansatz for $v(x, y, t)$:

$$v(x, y, t) = w(2K_m Z)\,\mathrm{e}^{(i\zeta y + \lambda t)/\epsilon}, \tag{44}$$

with $w(2K_m Z)$ a function with unit period in Z, which implies

$$v_x = \frac{2K_m k}{\epsilon} v_Z, \quad v_y = \frac{2K_m l}{\epsilon} v_Z + \frac{i\zeta}{\epsilon} v, \quad v_t = -\frac{2K_m \omega}{\epsilon} v_Z + \frac{\lambda}{\epsilon} v, \tag{45a}$$

$$v_{yy} = \frac{4K_m^2 l^2}{\epsilon^2} v_{ZZ} + \frac{4iK_m \zeta l}{\epsilon^2} v_Z - \frac{\zeta^2}{\epsilon^2} v + O\left(\frac{1}{\epsilon}\right), \quad v_{xx} = \frac{4K_m^2 k^2}{\epsilon^2} v_{ZZ}, \tag{45b}$$

with $l = qk$ as before. (Note that the ansatz (44) corresponds precisely to the setting in Section 3.1 with $q = K = 0$.) Then, to leading order in ϵ, (43) yields

$$-2K_m \omega v_Z + \lambda v + 2K_m k(u_o v)_Z + 8K_m^3 k^3(1 + q^2) v_{ZZZ} + 8iK_m^2 \zeta k l v_{ZZ} - 2K_m k \zeta^2 v_Z = 0, \tag{46}$$

which can be written as the linear eigenvalue problem

$$\mathcal{L}_o v = \lambda v, \tag{47a}$$

with

$$\mathcal{L}_o = 2K_m \omega \partial_Z - 2K_m k\, \partial_Z u_0 - 8K_m^3 k^3(1 + q^2)\partial_Z^3 - 8iK_m^2 \zeta k^2 q \partial_Z^2 + k\zeta^2 \partial_Z. \tag{47b}$$

Explicitly, using the definition of k, (46) is

$$\begin{aligned} &-\sqrt{r_3 - r_1} V v_Z + \tilde{\lambda} v + \sqrt{r_3 - r_1}(u_o v)_Z + (r_3 - r_1)^{3/2}[(1 + q^2)/6]\, v_{ZZZ} \\ &\quad + 2iq\,[(r_3 - r_1)/\sqrt{6}]\,\zeta v_{ZZ} - \sqrt{r_3 - r_1}\,\zeta^2 v_Z = 0, \end{aligned} \tag{48}$$

where $\tilde{\lambda} = \sqrt{6}\lambda$. To compare the results of this perturbation expansion with the predictions of Whitham theory, we set $r_1 = 0$ and $r_3 = 1$, implying $r_2 = m$, and we take $q = 0$. Then, (48) yields

$$-V v_Z + \tilde{\lambda} v + (u_o v)_Z + (1/6) v_{ZZZ} - \zeta^2 v_Z = 0. \tag{49}$$

Equivalently, the eigenvalue problem (50a) becomes

$$\mathcal{L} v = \tilde{\lambda} v, \tag{50a}$$

where

$$\mathcal{L} = V\,\partial_Z - \partial_Z\, u_o - (1/6)\partial_Z^3 + \zeta^2\partial_Z. \tag{50b}$$

We compute the eigenvalues $\tilde{\lambda}$ of $\mathcal{L}$ numerically for each $0 \leqslant m < 1$ using Floquet–Hill's method.[18] The difference between the resulting values and those obtained via Whitham theory shown in the inset of Figure 2, which demonstrates excellent agreement between the two approaches. Note however that, unlike the present approach, Whitham theory yields an analytical expression for the instability growth rate. Note also that the degree of discrepancy between the two approaches depends somewhat on the value of ζ chosen, since the latter affects the accuracy of the numerical scheme. The values in Figure 2 were obtained with $\zeta = 5 \times 10^{-4}$. This is also consistent with the fact that the predictions of the ZKWS are only accurate in the long wave limit, as explained in Section 3.1.

3.3 | Analytical stability theory for soliton solutions

As a final test for the predictions of Whitham theory, we now calculate the instability growth rate for the soliton solutions analytically. That is, we look for perturbed solution in the following form:

$$u(x, y, t) = u_c(\xi) + U(\xi)\,\mathrm{e}^{i\zeta y + \lambda t}, \tag{51}$$

where $u_c(\xi)$ is the solitary wave solution [i.e., the limit $m \to 1$ of (42)], and the second term in (51) describes purely transversal perturbations. For concreteness, we choose $r_1 = 0$ and $r_2 = r_3 = 6c$ (with the specific parameterization chosen so as to simplify the calculations that follow, similarly to Ref. 42), and $q = 0$. We then have $2K_m Z = \sqrt{c}(x + qy - 4ct) = \sqrt{c}\xi$, where $\xi = x - 4ct$, and, as per (4),

$$u_c(\xi) = 12c\,\mathrm{sech}^2(\sqrt{c}\xi). \tag{52}$$

We write the ZK equation (1) in the soliton comoving reference frame (ξ, y, t), which reduces the problem to the analysis of ordinary differential equations (ODEs). We then look for a formal asymptotic expansion in ζ for λ and U near $\zeta = 0$, namely:

$$\lambda = \lambda_1\zeta + \lambda_2\zeta^2 + O(\zeta^3), \tag{53a}$$

$$U(\xi) = U_0(\xi) + \lambda_1\zeta U_1(\xi) + \lambda_2\zeta^2 U_1(\xi) + \zeta^2 U_2(\xi) + O(\zeta^3). \tag{53b}$$

We should point out the similarities and the differences between the present approach and that of Ref. 42. The perturbation expansion above is similar in spirit to that in Ref. 42. However, Ref. 42 studied the stability of solitary waves with speed close to the critical speed of propagation, whereas in this case we are studying the stability near zero transverse wavenumbers (i.e., in the limit of long wavelength perturbations).

Substituting this ansatz into the ZK equation written in the comoving reference frame, at leading order we obviously simply recover an ODE that yields the soliton solution:

$$u_c'' + \frac{1}{2}u_c^2 - 4cu_c = 0, \tag{54}$$

where primes denote differentiation with respect to ξ. Then the eigenvalue problem for λ can be written as

$$\partial_\xi(M+\zeta^2)U = \lambda U, \tag{55}$$

where

$$M = -\partial_\xi^2 + 4c - 12c\,\mathrm{sech}^2(\sqrt{c}\xi). \tag{56}$$

We can write

$$\lambda U = \zeta(\lambda_1 U_0) + \zeta^2(\lambda_2 U_0 + \lambda_1^2 U_1) + O(\zeta^3) \tag{57}$$

and

$$\partial_\xi MU = \partial_\xi MU_0 + \lambda_1\zeta\partial_\xi MU_1 + \lambda_2\zeta^2\partial_\xi MU_1 + \zeta^2\partial_\xi MU_2 + O(\zeta^3). \tag{58}$$

At $O(1)$ in ζ of the eigenvalue problem (55) we have

$$\partial_\xi(MU_0) = 0, \tag{59}$$

which yields $U_0 = u_c'(\xi)$. At $O(\zeta)$ we have

$$\partial_\xi MU_1 = U_0, \tag{60}$$

that is,

$$\left[-\partial_\xi^2 + 4c - 12c\,\mathrm{sech}^2(\sqrt{c}\xi)\right] U_1 = 12c\,\mathrm{sech}^2(\sqrt{c}\xi). \tag{61}$$

It is straightforward to see that the above ODE admits the solution

$$U_1(\xi) = \tfrac{3}{4}\,\mathrm{sech}^2(\sqrt{c}\xi)\Big(-4 + (5+4\sqrt{c}\xi)\tanh(\sqrt{c}\xi)\Big). \tag{62}$$

Then, and finally, at $O(\zeta^2)$ we have

$$\partial_\xi MU_2 + \lambda_2 U_0 + \partial_\xi U_0 = \lambda_2 U_0 + \lambda_1^2 U_1, \tag{63}$$

or equivalently

$$\partial_\xi MU_2 = \lambda_1^2 U_1 - \partial_\xi^2 u_c. \tag{64}$$

The Fredholm solvability condition requires the right-hand side of (64) to be orthogonal to the kernel of the adjoint of the operator in the left-hand side in order for (64) to admit solutions. Since M is self-adjoint, the adjoint of $\partial_\xi M$ is simply $M\partial_\xi$. The kernel in question is thus spanned by u_c. Therefore, the resulting constraint is

$$\lambda_1^2 \int_{\mathbb{R}} u_1 u_c\, d\xi = \int_{\mathbb{R}} u_c u_c''\, d\xi, \tag{65}$$

and this condition determines λ_1. The integrals in the above conditions are given by, respectively,

$$\int_{\mathbb{R}} u_1 u_c\, d\xi = -36\sqrt{c}, \quad \int_{\mathbb{R}} u_c u_c''\, d\xi = -\int (u_c')^2 d\xi = -\frac{768}{5}c^{5/2}. \tag{66}$$

Their ratio then gives λ_1 as

$$\lambda_1 = \frac{8}{\sqrt{15}} c. \tag{67}$$

In order to compare this result with Whitham theory, note that in that case we took $r_3 = 1$, implying $c = 1/6$, which then yields $\lambda_1 = 4/(3\sqrt{15})$, in perfect agreement with the results of Section 3.1.

We should note that the above formalism can be generalized in a relatively straightforward way to compute the instability growth rate for all periodic solutions of the ZK equation. However, the corresponding calculations are somewhat more involved, and at the moment they have not yet led to a closed-form result similar to (67). For brevity, they are therefore deferred to a future publication.

3.4 | Stability analysis of general periodic solutions with respect to arbitrary periodic perturbations

In Section 3.1, after presenting the general method to study the stability of periodic solutions via the ZKWS, for simplicity we only studied the stability properties of periodic solutions with $q = 0$ (i.e., with fronts parallel to the y-axis) with respect to purely transversal perturbations (i.e., with $K = 0$). In this section, we now relax both assumptions and consider the most general scenario. To do so, it is convenient to slightly reparameterize the periodic perturbations and replace the ansatz (39) with

$$\mathbf{r}^{(1)}(x, y, t) = \mathbf{R}\ \mathrm{e}^{iK(x\cos\theta + y\sin\theta) - iWt}, \tag{68}$$

so that the parameter θ identifies the directionality of the perturbation. The matrix eigenvalue problem (40) is then replaced by

$$\left(\cos\theta\, A^{(0)} + \sin\theta\, B^{(0)} - (W/K)\, I_4\right) \mathbf{R} = \mathbf{0}. \tag{69}$$

Similarly to Section 3.1, the eigenvalues of the matrix $\cos\theta\, A^{(0)} + \sin\theta\, B^{(0)}$ [i.e., the roots of the characteristic polynomial $p(W/K) = \det(\cos\theta\, A^{(0)} + \sin\theta\, B^{(0)} - (W/K)\, I_4)$] then determine the linearized dispersion relation W/K. Namely, if all four roots are real, the ZKWS (22) is hyperbolic, and the periodic solution obtained from $\mathbf{r} = \mathbf{r}^{(0)}$ is stable. Otherwise, the system is elliptic and the periodic solution is unstable.

Similarly to Section 3.1, we now use the scaling invariance of the ZK equation to set $r_1 = 0$ and $r_3 = 1$, implying $r_2 = m$, and we compute the eigenvalues of $\cos\theta\, A^{(0)} + \sin\theta\, B^{(0)}$ as functions of the elliptic parameter m, slope parameter q, and perturbation angle θ. The plots in Figure 3 show the imaginary part of the four eigenvalues as a function of θ and q for $m = 0$ (left panel), $m = 1/2$ (center panel), and $m = 0.9999$ (right panel). Note that, since the ZK equation is invariant under the transformation $y \mapsto -y$, the stability properties when $q < 0$ are identical to those when $q > 0$ and $\theta \mapsto -\theta$. Therefore, it is sufficient to only consider nonnegative values of q. Figure 4 summarizes the results of the calculations by showing, for the same values of m, the domains where the imaginary part of all four eigenvalues is zero (white regions) or where at least one eigenvalue has nonzero imaginary part (gray regions), as a function of θ (horizontal axis) and q (imaginary axis).

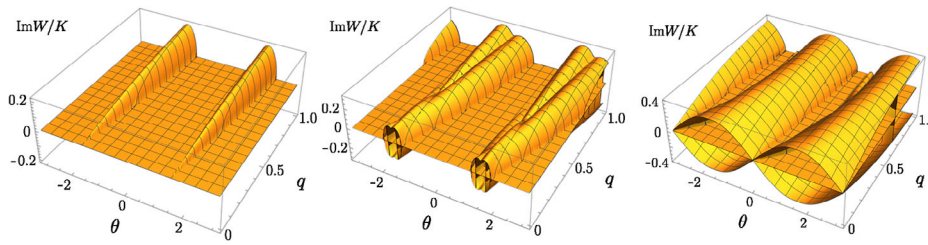

FIGURE 3 The imaginary part of the four eigenvalues of the matrix $\cos\theta\, A^{(0)} + \sin\theta\, B^{(0)}$ as a function of θ and q for $m = 0$ (left panel), $m = 1/2$ (central panel), and $m = 0.9999$ (right panel).

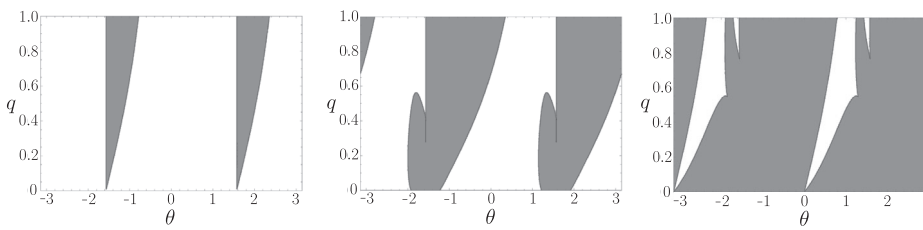

FIGURE 4 The corresponding regions of stability (white) and instability (gray) as a function of θ (horizontal axis) and q (vertical axis) for the same cases as in Figure 3.

Figure 4 shows that, for a generic nonzero value of q, and for each nonzero value of m, one can find both values of θ corresponding to stable perturbations and values of θ corresponding to unstable ones. The scenario studied in Section 3.1 (i.e., $q = 0$) corresponds to looking at the bottom of each plot in Figure 4, and is the only case in which all periodic waves associated to a nonzero value of m are unstable. Nonetheless, since for each value of q there is always a range of values of θ for which the eigenvalues are not purely real, the system (38) is not hyperbolic in general, and therefore the corresponding periodic solutions of the ZK equation are unstable to at least some periodic perturbations. Thus, we conclude that *the ZK equation (1) does not admit any periodic traveling wave solutions that are stable with respect to all periodic perturbations.*

4 | CONCLUDING REMARKS

In summary, we have derived the ZKWS, that is, the system of Whitham modulation equations for the periodic solutions of the ZK equation. The ZKWS shares some similarities with the KP–Whitham system, that is, the system of modulation equations for the KP equation. Both are first-order systems of PDEs of hydrodynamic type, and both systems involve three time evolution equations for the Riemann-type variables $r_1, \dots, r_3$ plus a fourth time evolution equation for the local slope parameter $q = k_2/k_1$. At the same time, there are some important differences between the two modulation systems. Most importantly, the fact that the ZKWS comprises only four PDEs, whereas the KP–Whitham system contains an additional PDE (which does not contain time derivatives) for an auxiliary field. (As mentioned in Ref. 3, the presence of this fifth PDE is essential for the system to correctly capture the dynamics of solutions of the KP equation.) We also studied the harmonic and soliton limits of the ZKWS, and we used the ZKWS to study the

transverse stability of the periodic traveling wave solutions, showing that all such solutions are unstable with respect to a nontrivial set of periodic perturbations.

The results of this work show that the ZK equation does not admit any exact solutions describing stable two-dimensional wave patterns. An interesting open question is whether the ZKWS can be used to study time evolution problems similarly to what was done in Refs. 43–45 for the KP equation. The situation for the ZK equation is different because its periodic solutions are unstable. Still, it is well-known that Whitham modulation equations can be very useful even when the underlying solutions of the PDE are unstable and the system is not hyperbolic (e.g., as in the case of the modulational instability of constant solutions of the focusing one-dimensional NLS equation[12,19,28]). A natural question is therefore where special solutions of the ZKWS could be useful to capture certain features of the time evolution of solutions of the ZK equation.

Obviously, it would also be interesting to study the ZKWS as a (2+1)-dimensional hydrodynamic system on its own, independently of its connection with the ZK equation. On that note, we point out that, similarly to what happens with the KP equation,[10] solutions of the ZKWS describe the modulation of solutions of the ZK equation only when the initial conditions for the ZKWS are consistent with the third conservation of waves equation, that is, the constraint $k_y = (qk)_x$. As with the KP equation,[3] it is straightforward to show that if this condition is satisfied at time zero, the ZKWS ensures that it is preserved by the time evolution. A related question concerns the possible integrability of the ZKWS. Since the ZK equation is not integrable, one would not expect the ZKWS to be integrable. Nonetheless, it is possible that certain reductions such as the harmonic limit and the soliton limit, could nonetheless be integrable.

All of these questions are left for future investigation, and it is hoped that the results of this work and the above remarks will stimulate further study on these topics.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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REFERENCES

1. Abeya A, Biondini G, Hoefer MA. Whitham modulation theory for the nonlinear Schrödinger equation in two and three spatial dimensions. *J Phys A*. 2023;56:025701.
2. Ablowitz MJ, Biondini G, Rumanov I. Whitham modulation theory for (2+1)-dimensional equations of Kadomtsev-Petviashvili type. *J Phys A*. 2018;51:215501.
3. Ablowitz MJ, Biondini G, Wang Q. Whitham modulation theory for the Kadomtsev-Petviashvili equation. *Proc R Soc A*. 2017;473:20160695.
4. Ablowitz MJ, Biondini G, Wang Q. Whitham modulation theory for the two-dimensional Benjamin-Ono equation. *Phys Rev E*. 2017;96:032225.

5. Ablowitz MJ, Clarkson PA. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press; 1991.
6. Ablowitz MJ, Cole J, Rumanov I. On the Whitham system for the (2+1)-dimensional nonlinear Schrödinger equation. *Stud Appl Math*. 2023;150:380-419.
7. Ablowitz MJ, Segur H. *Solitons and the Inverse Scattering Transform*. SIAM; 1981.
8. Bertola M, Tovbis A. Universality for the focusing nonlinear Schrödinger equation at the gradient catastrophe point: rational breathers and poles of the Tritronquée solution to Painlevé I. *Commun Pure Appl Math*. 2013;66:678-752.
9. Bettinson DC, Rowlands G. Transverse stability of plane solitons using the variational method. *J Plasma Phys*. 1998;59:543-554.
10. Biondini G, Bivolcic AJ, Hoefer MA, Moro A. Two-dimensional reductions of the Whitham modulation system for the Kadomtsev-Petviashvili equation. arXiv:2303.06436 [nlin.si], 2023.
11. Biondini G, Hoefer MA, Moro A. Integrability, exact reductions and special solutions of the KP-Whitham equations. *Nonlinearity*. 2020;33:4114-4132.
12. Biondini G, Mantzavinos D. Universal nature of the nonlinear stage of modulational instability. *Phys Rev Lett*. 2016;116:043902.
13. Biondini G, Mantzavinos D. Long-time asymptotics for the focusing nonlinear Schrödinger equation with nonzero boundary conditions at infinity and asymptotic stage of modulational instability. *Commun Pure Appl Math*. 2017;70:2300-2365.
14. Boutet de Monvel A, Lenells J, Shepelsky D. The focusing NLS equation with step-like oscillating background: scenarios of long-time asymptotics. *Commun Math Phys*. 2021;383:893-952.
15. Bridges TJ. Universal geometric condition for the transverse instability of solitary waves. *Phys Rev Lett*. 2000;84:2614-2617.
16. Byrd PF, Friedman MD. *Handbook of Elliptic Integrals for Scientists and Engineers*. Springer; 1971.
17. Cote R, Munoz C, Pilod D, Simpson G. Asymptotic stability of high-dimensional Zakharov-Kuznetsov solitons. *Arch Ration Mech Anal*. 2016;220:639-710.
18. Deconinck B, Kutz JN. Computing spectra of linear operators using the Floquet-Fourier-Hill method. *J Comput Phys*. 2006;219:296-321.
19. El GA, Gurevich AV, Khodorovskii VV, Krylov AL. Modulational instability and formation of a nonlinear oscillatory structure in a focusing medium. *Phys Lett A*. 1993;177:357-361.
20. El GA, Hoefer MA. Dispersive shock waves and modulation theory. *Physica D*. 2016;333:11-65.
21. Farah LG, Holmer J, Roudenko S, Yang K. Asymptotic stability of solitary waves of the 3D quadratic Zakharov-Kuznetsov equation. arXiv:2006.00193 [math.AP], 2020.
22. Farah LG, Molinet L. A note on the well-posedness in the energy space for the generalized ZK equation posed on $\mathbb{R} \times \mathbb{T}$. arXiv:2306.07433 [math.AP], 2023.
23. Frankel T. *The Geometry of Physics*. Cambridge; 1997.
24. Herr S, Kinoshita S. Subcritical well-posedness results for the Zakharov-Kuznetsov equation in dimension three and higher. arXiv:2001.09047, 2020.
25. Infeld E, Rowlands G. *Nonlinear Waves, Solitons and Chaos*. Cambridge University Press; 2000.
26. Johnson MA. The transverse instability of periodic waves in Zakharov-Kuznetsov type equations. *Stud Appl Math*. 2010;124:323-345.
27. Kadomtsev BB, Petviashvili VI. On the stability of solitary waves in weakly dispersive media. *Sov Phys Dokl*. 1970;15:539-541.
28. Kamchatnov AM. *Nonlinear Periodic Waves and Their Modulations*. World Scientific; 2000.
29. Kinoshita S. Global well-posedness for the Cauchy problem of the Zakharov-Kuznetsov equation in 2D. *Ann Inst Henri Poincaré C*. 2021;38:451-505.
30. Klein C, Roudenko S, Stoilov N. Numerical study of soliton stability, resolution and interactions in the 3D Zakharov-Kuznetsov equation. *Physica D*. 2021;423:132913.
31. Klein C, Saut J-C, Stoilov N. Numerical study of the transverse stability of line solitons of the Zakharov-Kuznetsov equations. *Physica D*. 2023;448:133722.
32. Lannes D, Linares F, Saut J-C. The Cauchy problem for the Euler-Poisson system and derivation of the Zakharov-Kuznetsov equation. In: *Studies in Phase Space Analysis with Applications to PDEs*. Springer; 2013:181-213.

33. Lax PD, Levermore CD. The small dispersion limit of the Korteweg-de Vries equation. I, II and III. *Commun Pure Appl Math*. 1983;36:253-290, 571-593, 809-829.
34. Linares F, Pastor A, Saut J-C. Well-posedness for the ZK equation in a cylinder and on the background of a KdV soliton. *Commun Partial Differ Equ*. 2010;35:1674-1689.
35. Mendez AJ, Muñoz C, Poblete F, Pozo J-C. On local energy decay for large solutions of the Zakharov-Kuznetsov equation. *Commun Partial Differ Equ*. 2021;46:1440-1487.
36. Mendez AJ, Riaño O. On decay properties for solutions of the Zakharov-Kuznetsov equation. arXiv:2302.11731, 2023.
37. Natali F. Transversal spectral instability of periodic traveling waves for the generalized Zakharov-Kuznetsov equation. Preprint 2023. doi:https://doi.org/10.48550/arXiv.2303.12504
38. Nazarenko S. *Wave Turbulence*. Springer; 2011.
39. Novikov SP, Manakov SV, Pitaevskii LP, Zakharov VE. *Theory of Solitons: The Inverse Scattering Method*. Plenum; 1984.
40. Nozaki K. Vortex solitons of drift waves and anomalous diffusion. *Phys Rev Lett*. 1981;46:184.
41. Olver FW, Lozier DW, Boisvert RF, Clark CW. *NIST Handbook of Mathematical Functions*. Cambridge; 2010.
42. Pelinovsky DE. Normal form for transverse instability of the line soliton with a nearly critical speed of propagation. *Math Model Nat Phenom*. 2018;13:1-20.
43. Ryskamp S, Hoefer MA, Biondini G. Oblique interactions between solitons and mean flows in the Kadomtsev-Petviashvili equation. *Nonlinearity*. 2021;34:3583-3617.
44. Ryskamp S, Hoefer MA, Biondini G. Modulation theory for soliton resonance and Mach reflection. *Proc R Soc A*. 2022;478:20210823.
45. Ryskamp S, Maiden MD, Biondini G, Hoefer MA. Evolution of truncated and bent gravity wave solitons: the mach expansion problem. *J Fluid Mech*. 2021;909:A24.
46. Saut J-C, Temam R. An initial boundary-value problem for the Zakharov-Kuznetsov equation. *Adv Differ Equ*. 2010;15:1011-1031.
47. Smoller J. *Shock Waves and Reaction Diffusion Equations*. Springer; 1994.
48. Staffilani G, Tran M-B. On the wave turbulence theory for a stochastic KdV type equation. arXiv:2106.09819 [math.AP], 2021.
49. Whitham GB. Non-linear dispersive waves. *Proc R Soc A*. 1965;283:238-261.
50. Whitham GB. *Linear and Nonlinear Waves*. Wiley; 1974.
51. Yamazaki Y. Stability for line solitary waves of Zakharov-Kuznetsov equation. *J Differ Equ*. 2017;262:4336-4389.
52. Zakharov VE, Kuznetsov EA. On three dimensional solitons. *JETP*. 1974;39:285-286.

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