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Inverse scattering transform for the focusing nonlinear Schrödinger equation with counterpropagating flows

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Abstract
The inverse scattering transform for the focusing nonlinear Schrödinger equation is presented for a general class of initial conditions whose asymptotic behavior at infinity consists of counterpropagating waves. The formulation takes into account the branched nature of the two asymptotic eigenvalues of the associated scattering problem. The Jost eigenfunctions and scattering coefficients are defined explicitly as single-valued functions on the complex plane with jump discontinuities along certain branch cuts. The analyticity properties, symmetries, discrete spectrum, asymptotics, and behavior at the branch points are discussed explicitly. The inverse problem is formulated as a matrix Riemann-Hilbert problem with poles. Reductions to all cases previously discussed in the literature are explicitly discussed. The scattering data associated to a few special cases consisting of physically relevant Riemann problems are explicitly computed.

KEYWORDS
nonlinear waves, partial differential equations, solitons and integrable systems

1 | INTRODUCTION AND MOTIVATION

The nonlinear Schrödinger (NLS) equation, $iq_t + q_{xx} \pm 2|q|^2 q = 0$, ("+" for focusing; "−" for defocusing) is one of the most important systems in nonlinear science, since it arises as a model in deep water waves, plasmas, acoustics, optics, and Bose-Einstein condensation.[1–5] Indeed, the NLS equation is a universal model for the evolution of a complex envelope of weakly nonlinear dispersive wave trains.[6] The NLS equation is also one of the most well-known examples of an integrable nonlinear evolution equation. Infinite-dimensional integrable systems have been studied extensively due to the combination of physical relevance and rich mathematical structure.[4,7–10] In particular, for the NLS equation, the inverse scattering transform (IST) was developed by Zakharov and Shabat in 1972 to solve the initial value problem (IVP) in the case of zero boundary conditions (BCs) at infinity and of initial conditions (ICs) with sufficient smoothness.[11] Shortly after, the same authors extended the formulation of the IST to solve the IVP with symmetric nonzero boundary conditions (NZBCs) in the defocusing case.[12] The behavior of solutions in these cases has since been extensively studied and unraveled in several works, eg, see Refs. 13–28 and the references therein. In particular, the case of symmetric NZBCs in the focusing NLS equation has received renewed attention recently,[29–32] and the case of fully asymmetric NZBCs in both focusing and defocusing NLS equations was also studied.[27,33–35]

Importantly, however, all of the above works considered the case of either zero or constant BCs at infinity. In the case of the Korteweg-de Vries equation, solutions with more general kind of behavior were recently studied in Refs. 36, 37. For the NLS equation, however, only two works in the more general case of plane-wave BCs are available in the literature, one in the focusing case[38] and one in the defocusing case.[39] Nevertheless, in both of those works only a specific choice of ICs was considered, corresponding to a Riemann problem, namely, a plane wave in each of the half lines $x > 0$ and $x < 0$ with a discontinuity at the origin. The aim of this work is to develop the IST for solving the IVP for the focusing NLS equation

$$iq_t + q_{xx} + 2|q|^2 q = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

with a more general class of ICs $q(x,0)$ which reduce to plane waves only as $x \to \pm\infty$, namely,

$$q(x,0) = A_\pm \mathrm{e}^{-iV|x| \pm i\delta}(1 + o(1)), \quad x \to \pm\infty, \tag{2}$$

where $A_\pm > 0$ and $V, \delta \in \mathbb{R}$. Throughout this work, $q : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}$, and subscripts x and t denote partial differentiation. Detailed statements about the precise function spaces required for the various steps in the development of the IST will be given later. Note that one could equally well consider the seemingly more general class of ICs $q(x,0) = A_\pm \mathrm{e}^{iV_\pm x + i\delta_\pm}(1 + o(1))$ as $x \to \pm\infty$. However, there is no actual need to do so, since without loss of generality one can always reduce this latter class to the ICs (2), namely, $V_\pm = \pm V$ and $\delta_\pm = \pm\delta$, using the Galilean and phase invariances of the NLS equation. Thus, the present work encompasses the most general family of solutions of the focusing NLS equation which tends asymptotically to genus-0 (ie, constant or plane wave) behavior at infinity.

The family of ICs (2) includes those studied in all of the aforementioned works on the focusing NLS equation as special cases. In particular, the long-time asymptotics of solutions in various subcases when either A_- and/or A_+ are nonzero have been studied by various authors in recent years.[31,38,40] Here, we address the general case and show how the various subcases can be obtained as appropriate reductions, thus providing a unified framework for the study of these problems.

We also consider various Riemann problems, ie, pure step ICs. As usual, the development of the IST proceeds under the assumption of existence and uniqueness. Once a representation for the solution of the IVP has been obtained, however, one can use it as the starting point to rigorously prove the well-posedness of the problem in appropriate function spaces, eg, see Refs. 41–43.

This work is organized as follows. Section 2 introduces the Jost solutions and their properties. Section 3 introduces the scattering matrix and symmetries of the Jost solutions. Section 4 formulates the inverse problem as a matrix Riemann-Hilbert problem (RHP). Section 5 discusses various reductions as special cases, such as that of equal amplitudes, zero velocities, or one-sided BCs. Section 6 is devoted to various explicit ICs. Proofs of theorems, lemmas, and corollaries are provided in Section 7, and Section 8 ends this work with some concluding remarks.

2 | DIRECT PROBLEM: JOST SOLUTIONS AND ANALYTICITY PROPERTIES

The focusing NLS equation (1) is the compatibility condition $\phi_{xt} = \phi_{tx}$ or, equivalently, $X_t - T_x + [X, T] = 0$, of the following overdetermined linear system of ordinary differential equations (ODEs) known as a Lax pair:

$$\phi_x(x,t,k) = X(x,t,k)\phi(x,t,k), \tag{3a}$$

$$\phi_t(x,t,k) = T(x,t,k)\phi(x,t,k), \tag{3b}$$

where

$$X(x,t,k) = ik\sigma_3 + Q(x,t), \tag{4a}$$

$$T(x,t,k) = -2ik^2\sigma_3 + i\sigma_3\big(Q_x(x,t) - Q^2(x,t)\big) - 2kQ(x,t), \tag{4b}$$

and

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\overline{q(x,t)} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5}$$

with the bar denoting complex conjugation. Equation (3a) is referred to as the scattering problem, the complex-valued matrix function $\phi(x,t,k)$ is referred to as the eigenfunction, k is referred to as the scattering parameter, and $q(x,t)$ as the scattering potential. The matrix σ_1 is defined now for later use.

The IST method can be outlined as follows: first, using appropriate solutions of the Lax pair (3) known as Jost solutions, one constructs a map that associates the solution $q(x,t)$ of the NLS equation to a suitable set of "scattering data," which are independent of x and t and depend only on k. Then, inverting this map one recovers the potential in terms of said scattering data. In this section, we introduce the Jost solutions and we determine their properties. Proofs for all the results in this section are given in Section 7.1.

2.1 | Jost solutions: Formal definition

It is useful to first consider the eigenfunctions corresponding to the following two exact plane-wave solutions of the NLS equation (1):

$$q_\pm(x,t) = A_\pm \mathrm{e}^{-2if_\pm(x,t)\pm i\delta}, \tag{6}$$

with

$$f_\pm(x,t) = \frac{1}{2}\big[\pm Vx + (V^2 - 2A_\pm^2)t\big]. \tag{7}$$

Here and throughout, we use the subscripts $\pm$ to relate to behavior as $x \to \pm\infty$. (Note that the labels $q_\pm$ have been used in previous works to denote constant values, independent of x and t. This is not the case here.)

Observe that the asymptotic behavior (2) for the ICs can be written as $q(x,0) = q_\pm(x,0)\,(1 + o(1))$, $x \to \pm\infty$. Thus, as long as the IVP is well-posed, the condition (2) implies

$$q(x,t) = q_\pm(x,t)(1 + o(1)), \quad x \to \pm\infty \tag{8}$$

for all $t \in \mathbb{R}$, so that

$$Q(x,t) = Q_\pm(x,t)(1 + o(1)), \quad x \to \pm\infty, \tag{9a}$$

$$X(x,t,k) = X_\pm(x,t,k)(1 + o(1)), \quad x \to \pm\infty, \tag{9b}$$

$$T(x,t,k) = T_\pm(x,t,k)(1 + o(1)), \quad x \to \pm\infty, \tag{9c}$$

where

$$Q_\pm(x,t) = e^{-if_\pm(x,t)\sigma_3}\left(A_\pm\sigma_3 e^{\pm i\delta\sigma_3}\sigma_1\right)e^{if_\pm(x,t)\sigma_3}\,, \tag{10a}$$

$$X_\pm(x,t,k) = ik\sigma_3 + Q_\pm(x,t)\,, \tag{10b}$$

$$T_\pm(x,t,k) = -2ik^2\sigma_3 + i\sigma_3\big((Q_\pm)_x(x,t) - Q_\pm^2(x,t)\big) - 2kQ_\pm(x,t)\,. \tag{10c}$$

In Section 7.1, we derive the following simultaneous solutions $\widetilde{\phi}_\pm(x,t,k)$ to the Lax pair (3) for the exact potentials $q_\pm(x,t)$:

$$\widetilde{\phi}_\pm(x,t,k) = \mathrm{e}^{-if_\pm(x,t)\sigma_3}E_\pm(k)\mathrm{e}^{i\theta_\pm(x,t,k)\sigma_3}, \tag{11}$$

with

$$\lambda_\pm(k) = \left((k \pm V/2)^2 + A_\pm^2\right)^{1/2}, \tag{12a}$$

$$E_\pm(k) = \mathrm{I} + \frac{iA_\pm}{\lambda_\pm(k) + (k \pm V/2)} \mathrm{e}^{\pm i\delta\sigma_3}\sigma_1, \tag{12b}$$

$$\theta_\pm(x,t,k) = \lambda_\pm(k)(x - 2(k \mp V/2)t). \tag{12c}$$

Note that $\lambda_\pm(k)$ has branch points at $k = p_\pm$ and $k = \overline{p_\pm}$, where

$$p_\pm = \mp V/2 + iA_\pm. \tag{13}$$

We find that

$$D_\pm(k) := \det E_\pm(k) = \frac{2\lambda_\pm(k)}{\lambda_\pm(k) + (k \pm V/2)}. \tag{14}$$

In the special case $A_\pm = 0$, (14) reduces to $D_\pm(k) \equiv 1$, and the whole formalism reduces to the IST with zero BCs. When $A_\pm \neq 0$, $D_\pm(k)$ vanishes only at the branch points of $\lambda_\pm(k)$. Moreover, since

$$A_\pm^2 = \big(\lambda_\pm(k) + (k \pm V/2)\big)\big(\lambda_\pm(k) - (k \pm V/2)\big), \tag{15}$$

neither factor on the right-hand side is ever zero, and therefore $D_\pm(k)$ has no poles.

Motivated by (11), we define the Jost solutions $\phi_\pm(x,t,k)$ for the potential $q(x,t)$ satisfying (2) to be the simultaneous solutions of the Lax pair (3) such that

$$\phi_\pm(x,t,k) = \frac{1}{d_\pm(k)} \mathrm{e}^{-if_\pm(x,t)\sigma_3} E_\pm(k) \mathrm{e}^{i\theta_\pm(x,t,k)\sigma_3}(1 + o(1)), \quad x \to \pm\infty, \tag{16}$$

where the factor

$$d_\pm(k) := (D_\pm(k))^{1/2} \tag{17}$$

is introduced to simplify the resulting symmetries and jump matrices that will be computed (see Sections 3.3 and 4.1). Moreover, by Abel's theorem, since X and T are traceless, the determinants of $\phi_\pm(x,t,k)$ are independent of x and t and

$$\det\phi_\pm(x,t,k) = \lim_{x\to\pm\infty} \det\phi_\pm(x,t,k) = \frac{\det E_\pm(k)}{d_\pm(k)^2} = 1. \tag{18}$$

On the other hand, the factor $d_\pm(k)$ introduces poles at the branch points which will need to be considered (see Section 2.5). We will make an explicit choice of branch cut for $\lambda_\pm(k)$ and $d_\pm(k)$ in Section 2.2. A rigorous definition of the Jost solutions and their domains of existence and analyticity will be given in Section 2.3.

2.2 | Branch cuts for the asymptotic eigenvalues

To discuss the analyticity properties of the Jost solutions defined above, it is necessary to make an explicit choice of branch cut to define $\lambda_\pm(k)$ for all $k \in \mathbb{C}$. To simplify the argument, we first

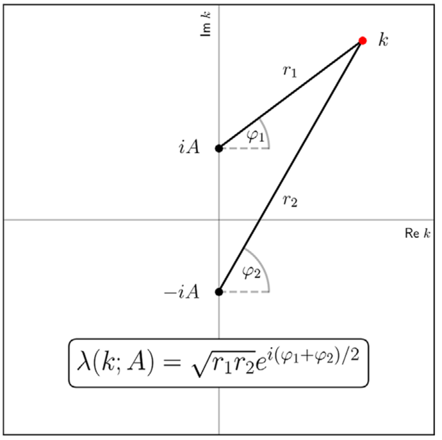

FIGURE 1 The definition (21) of the generic square root $\lambda(k;A)$ with $\phi_j \in [-\pi/2, 3\pi/2)$

define

$$\lambda(k;A) = (k^2 + A^2)^{1/2} = (k - iA)^{1/2}(k + iA)^{1/2}. \tag{19}$$

Note that $\lambda(k;A) \in \mathbb{R}$ exactly when $k \in \mathbb{R} \cup i[-A, A]$. We take the branch cut of $\lambda(k;A)$ to lie along $i[-A, A]$ oriented upward, and define $\lambda(k;A)$ to be continuous from the right. Explicitly, letting $k - iA = r_1 e^{i\varphi_1}$ and $k + iA = r_2 e^{i\varphi_2}$ with $-\pi/2 \leq \varphi_1, \varphi_2 < 3\pi/2$, we define

$$(k - iA)^{1/2} = \sqrt{r_1} e^{i\varphi_1/2}, \quad (k + iA)^{1/2} = \sqrt{r_2} e^{i\varphi_2/2}, \tag{20}$$

and

$$\lambda(k;A) = \sqrt{r_1 r_2} e^{i(\varphi_1+\varphi_2)/2}, \tag{21}$$

so that $\lambda(k;A) = k + O(1/k)$ as $k \to \infty$ in any direction (cf. Figure 1).

Lemma 1. *The function $\lambda(k;A)$ defined by (21) satisfies the following properties:*

$$\operatorname{Im} \lambda(k;A) \lessgtr 0, \quad k \in \mathbb{C}^{\mp} \setminus i[-A, A], \tag{22a}$$

$$\operatorname{Re} \lambda(k;A) \lessgtr 0, \quad k \in \mathbb{R}^{\mp} + i\mathbb{R}, \tag{22b}$$

$$\lambda(\overline{k};A) = \overline{\lambda(k;A)}, \quad k \in \mathbb{C}, \tag{22c}$$

$$\lambda(-k;A) = -\lambda(k;A), \quad k \in \mathbb{C} \setminus i[-A, A], \tag{22d}$$

$$\lambda^{\mp}(k;A) = \pm\lambda(k;A), \qquad k \in i[-A, A]. \tag{22e}$$

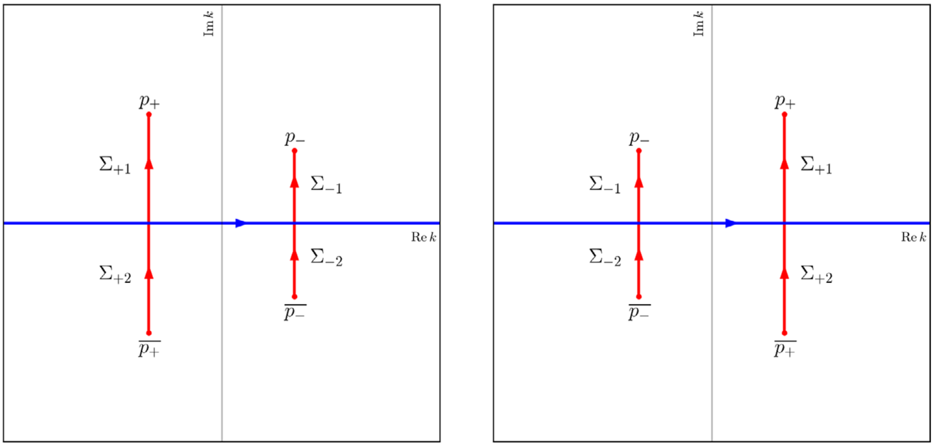

FIGURE 2 The contours $\Sigma_{\pm 1}, \Sigma_{\pm 2}$ for $V > 0$ (left) and $V < 0$ (right). Recall that $A_- < A_+$. Along the real axis (blue line), all four Jost solutions are defined. On the other hand, on $\Sigma_{\pm 1}$ and $\Sigma_{\pm 2}$ (red segments), only three of the Jost solutions are generically defined

Here and elsewhere, $\mathbb{R}^{\mp} = \{k \in \mathbb{R} : \operatorname{Re} k \lessgtr 0\}$, $\mathbb{C}^{\mp} = \{k \in \mathbb{C} : \operatorname{Im} k \lessgtr 0\}$, and the superscripts $\mp$ on functions of k denote the limit being taken from the right/left of the negative/positive side of the oriented contour, respectively. In particular, for the upward oriented contour $i[-A, A]$, the superscripts $\mp$ denote the limits from the right/left, ie,

$$\lambda^{\mp}(k; A) := \lim_{\epsilon \downarrow 0} \lambda(k \pm \epsilon; A), \quad k \in i[-A, A]. \tag{23}$$

With the above definitions, (12a) can be expressed as

$$\lambda_{\pm}(k) := \lambda(k \pm V/2; A_{\pm}). \tag{24}$$

Correspondingly, $\lambda_{\pm}(k) \in \mathbb{R}$ exactly for $k \in \mathbb{R} \cup \Sigma_{\pm}$, where

$$\Sigma_+ = [\overline{p_+}, p_+] = \Sigma_{+1} \cup \Sigma_{+2}, \quad \Sigma_- = [\overline{p_-}, p_-] = \Sigma_{-1} \cup \Sigma_{-2}, \tag{25}$$

are the upward oriented branch cuts for $\lambda_+(k)$ and $\lambda_-(k)$, respectively, and

$$\Sigma_{\pm 1} = \Sigma_{\pm} \cap (\mathbb{C}^+ \cup \{\mp V/2\}) = \mp V/2 + i[0, A_{\pm}], \tag{26a}$$

$$\Sigma_{\pm 2} = \Sigma_{\pm} \cap (\mathbb{C}^- \cup \{\mp V/2\}) = \mp V/2 + i[-A_{\pm}, 0], \tag{26b}$$

with $p_{\pm}$ as defined in (13) (cf. Figure 2). From Lemma 1, we have

$$\lambda_{\pm}^{-}(k) = \lambda_{\pm}(k), \quad k \in \Sigma_{\pm}, \tag{27a}$$

$$\lambda_{\pm}^{+}(k) = -\lambda_{\pm}(k), \quad k \in \Sigma_{\pm}. \tag{27b}$$

Hereafter, we will suppress the k-dependence of $\lambda_\pm$ when doing so does not create ambiguity. For later convenience, we also define the set

$$\Sigma = \mathbb{R} \cup \Sigma_+ \cup \Sigma_-, \tag{28}$$

which will comprise the continuous spectrum of the scattering problem (see Section 3).

Recall $D_\pm(k)$ as given in (14).

With the chosen branch cuts for $\lambda_\pm$, $D_\pm(k)$ are analytic for $k \in \mathbb{C} \setminus \Sigma_\pm$. When deriving the jump conditions in the RHP, it will also be necessary to understand the discontinuities of $D_\pm(k)$ across the branch cuts $\Sigma_\pm$. Explicitly, it is easy to show that

$$D_\pm^+(k) = \frac{4\lambda_\pm^2}{A_\pm^2}\frac{1}{D_\pm(k)}, \quad k \in \Sigma_\pm, \tag{29}$$

where, due to our choice (27) for $\lambda_\pm$, the values of $D_\pm$ on the branch cuts coincide with their limits from the right, ie, $D_\pm^-(k) = D_\pm(k)$ for all $k \in \Sigma_\pm$.

Another choice of branch cut is needed to uniquely define $d_\pm(k) = (D_\pm(k))^{1/2}$, whose discontinuities also must be understood. Explicitly, we choose

$$d_\pm(k) := \sqrt{D_\pm(k)}, \quad k \in \mathbb{C}, \tag{30}$$

where $\sqrt{\cdot}$ denotes the principal square root with branch cut along $\mathbb{R}^- \cup \{0\}$.

Lemma 2. *The function $d_\pm(k)$ is analytic in $\mathbb{C} \setminus \Sigma_\pm$ and continuous from the right on $\Sigma_\pm$ with*

$$d_\pm(k) = 1 + O(1/k), \quad k \to \infty, \tag{31a}$$

$$d_\pm^+(k) = \frac{2\lambda_\pm}{A_\pm}\frac{1}{d_\pm(k)}, \quad k \in \Sigma_\pm. \tag{31b}$$

The limiting values of the Jost solutions on the branch cuts will be discussed later in Section 3.3.

2.3 | Jost solutions: Rigorous definition, analyticity, and continuous spectrum

We now introduce integral equations that can be used to rigorously define the Jost solutions and establish their regions of existence, continuity, and analyticity.

We first remove the asymptotic oscillations that are present in (16) as well as the poles from the factor $d_\pm(k)$ by introducing the modified eigenfunctions

$$\mu_\pm(x,t,k) = d_\pm(k)\mathrm{e}^{if_\pm(x,t)\sigma_3}\phi_\pm(x,t,k)\mathrm{e}^{-i\theta_\pm(x,t,k)\sigma_3}. \tag{32}$$

The Lax pair (3) yields corresponding ODEs for the functions $\mu_\pm$. Noting that

$$X(x,t,k) = X_\pm(x,t,k) + \Delta Q_\pm(x,t), \tag{33}$$

with

$$\Delta Q_{\pm}(x,t) = Q(x,t) - Q_{\pm}(x,t), \tag{34}$$

these ODEs can be formally integrated (see Section 7.1) to obtain the integral equations

$$\mu_{\pm}(x,t,k) = E_{\pm}(k) + \int_{\pm\infty}^{x} E_{\pm}(k)\mathrm{e}^{i\lambda_{\pm}(x-y)\sigma_3} E_{\pm}^{-1}(k)\mathrm{e}^{2if_{\pm}(y,t)\sigma_3}\Delta Q_{\pm}(y,t)\mu_{\pm}(y,t,k)\mathrm{e}^{-i\lambda_{\pm}(x-y)\sigma_3}dy. \tag{35}$$

We now let $\phi_{\pm 1}$ and $\phi_{\pm 2}$ denote the first and second columns of $\phi_{\pm}$, respectively. Using the left- and right-background solutions $q_{\pm}(x,t)$ defined in (6) and the notation for $\Sigma_{\pm,1,2}$ introduced in (26) (cf. Figure 2), we then have the following:

Theorem 1. *If $(q - q_{\pm}) \in L^1_x(\mathbb{R}^{\pm})$ for all $t \in \mathbb{R}$, then*

- *$\phi_{+1}(x,t,k)$ is analytic for $k \in \mathbb{C}^+ \setminus \Sigma_{+1}$, continuous from above on $k \in \mathbb{R}$ and from the right on $k \in \Sigma^o_{+1}$, and also defined for $k \in \Sigma^o_{+2}$.*
- *$\phi_{+2}(x,t,k)$ is analytic for $k \in \mathbb{C}^- \setminus \Sigma_{+2}$, continuous from below on $k \in \mathbb{R}$ and from the right on $k \in \Sigma^o_{+2}$, and also defined for $k \in \Sigma^o_{+1}$.*
- *$\phi_{-1}(x,t,k)$ is analytic for $k \in \mathbb{C}^- \setminus \Sigma_{-2}$, continuous from below on $k \in \mathbb{R}$ and from the right on $k \in \Sigma^o_{-2}$, and also defined for $k \in \Sigma^o_{-1}$.*
- *$\phi_{-2}(x,t,k)$ is analytic for $k \in \mathbb{C}^+ \setminus \Sigma_{-1}$, continuous from above on $k \in \mathbb{R}$ and from the right on $k \in \Sigma^o_{-1}$, and also defined for $k \in \Sigma^o_{-2}$.*

Above and throughout the remaining work,

$$\Sigma^o_{+1} = \Sigma_{+1} \setminus \{p_+, -V/2\}, \quad \Sigma^o_{+2} = \Sigma_{+2} \setminus \{\overline{p_+}, -V/2\} \tag{36}$$

and similarly for Σ^o_{-1} and Σ^o_{-2}. The hypothesis of Theorem 1 does not allow us to draw any conclusions about the Jost eigenfunctions at the branch points. The behavior of the eigenfunctions at the branch points $p_{\pm}$ and $\overline{p_{\pm}}$ will be discussed in Section 2.5.

The proof of Theorem 1 proceeds nearly identically as in Ref. 29 by analyzing the Neumann series associated with the Volterra integral equation (35), and is included in Section 7.1. Moreover, the proof also implies the following:

Corollary 1. *Under the hypothesis of Theorem 1, for any $a \in \mathbb{R}$,*

$$\phi_{+1}(x,t,k) \in L^\infty_x(a,\infty), \quad k \in \mathbb{R} \cup \mathbb{C}^+ \cup \Sigma^o_{+2} \setminus \{p_+\}, \tag{37a}$$

$$\phi_{+2}(x,t,k) \in L^\infty_x(a,\infty), \quad k \in \mathbb{R} \cup \mathbb{C}^- \cup \Sigma^o_{+1} \setminus \{\overline{p_+}\}, \tag{37b}$$

$$\phi_{-1}(x,t,k) \in L^\infty_x(-\infty,a), \quad k \in \mathbb{R} \cup \mathbb{C}^- \cup \Sigma^o_{-1} \setminus \{\overline{p_-}\}, \tag{37c}$$

$$\phi_{-2}(x,t,k) \in L^\infty_x(-\infty,a), \quad k \in \mathbb{R} \cup \mathbb{C}^+ \cup \Sigma^o_{-2} \setminus \{p_-\}. \tag{37d}$$

Theorem 1 shows that $\phi_\pm(x,t,k)$ are continuous for $k \in \mathbb{R}$. Moreover, formally differentiating the Volterra integral equation (35) with respect to k and performing a similar Neumann series analysis, one can show the following:

Corollary 2. *Under the hypothesis of Theorem 1, $\phi_+(x,t,k)$ and $\phi_-(x,t,k)$ are $C^1(\mathbb{R})$, ie, continuously real-differentiable functions of k.*

2.4 | Jost solutions: Asymptotic behavior as $k \to \infty$

Understanding the asymptotic behavior of $\phi_\pm(x,t,k)$ as $k \to \infty$ is necessary to properly formulate the inverse problem, and will also allows us to recover the potential q from the scattering data.

Lemma 3. *If $(q - q_\pm) \in L^1_x(\mathbb{R}^\pm)$ and q is continuously differentiable with $(q - q_\pm)_x \in L^1_x(\mathbb{R}^\pm)$ for all $t \in \mathbb{R}$, then*

$$\mu_\pm(x,t,k) = \mathrm{I} + O(1/k), \quad k \to \infty, \tag{38}$$

within the appropriate region of the complex k-plane for each column as outlined in Theorem 1. Furthermore,

$$q(x,t) = -2i \lim_{k\to\infty} \mathrm{e}^{-2if_-(x,t)} k[\mu_-(x,t,k)]_{12}. \tag{39}$$

As a direct consequence of the above lemma,

$$\phi_\pm(x,t,k) = \mathrm{e}^{i(\theta_\pm(x,t,k)-f_\pm(x,t))\sigma_3}(\mathrm{I} + O(1/k)), \quad k \to \infty, \tag{40}$$

within the appropriate regions of the complex k-plane for each column. Observe that

$$\lambda_\pm(k) = (k \pm V/2) + \frac{A_\pm^2}{2(k \pm V/2)} + O(1/k^3), \quad k \to \infty, \tag{41}$$

and so

$$\begin{aligned} \theta_\pm(x,t,k) &= (k \pm V/2)x - \left(2k^2 - V^2/2 + A_\pm^2\right)t + O(1/k), \quad && k \to \infty \\ &= \theta_o(x,t,k) + f_\pm(x,t) + O(1/k), && k \to \infty, \end{aligned} \tag{42}$$

where we have introduced the controlling phase function for the Jost eigenfunctions in the problem with zero BCs:

$$\theta_o(x,t,k) = k(x - 2kt), \tag{43}$$

which will also be used in Section 4. Lemma 3 together with (42) imply the following:

Lemma 4. *Under the hypotheses of Lemma 3,*

$$\phi_\pm(x,t,k) = \mathrm{e}^{i\theta_o(x,t,k)\sigma_3}(\mathrm{I} + O(1/k)), \quad k \to \infty, \tag{44}$$

within the appropriate region of the complex k-plane for each column as specified by Theorem 1. Furthermore,

$$q(x,t) = -2i \lim_{k\to\infty} k\big[\phi_-(x,t,k)\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}\big]_{12}. \tag{45}$$

Note that the presence of $d_\pm(k)$ in the definition of $\phi_\pm(x,t,k)$ does not change the asymptotic k behavior at infinity due to (31a).

2.5 | Jost solutions: Behavior at the branch points

As mentioned in Section 2.3, the condition $(q - q_\pm) \in L^1_x(\mathbb{R}^\pm)$ for all $t \in \mathbb{R}$ is enough to guarantee the existence and analyticity of the Jost eigenfunctions in suitable open regions of the complex k-plane, as well as their continuity along portions of the boundary of these regions. Notably, however, these regions do not include the branch points $p_\pm$ and $\overline{p_\pm}$. On the other hand, the behavior of the eigenfunctions near the branch points must be understood to specify appropriate growth conditions for the inverse problem.

We next show that, under more strict conditions for the potential than those imposed by Theorem 1, it is possible to define the modified eigenfunctions $\mu_\pm(x,t,k)$ at the branch points. This in turn determines the behavior of the Jost solutions $\phi_\pm(x,t,k)$ near the branch points.

To do so, we introduce the weighted L^1 spaces

$$L^{1,j}(\mathbb{R}^\pm) := \left\{ f : \mathbb{R} \to \mathbb{C} \,|\, (1+|x|)^j f \in L^1(\mathbb{R}^\pm) \right\}, \quad j = 1, 2. \tag{46}$$

Lemma 5. *If $(q - q_\pm) \in L^{1,1}_x(\mathbb{R}^\pm)$ for all $t \in \mathbb{R}$, then the modified eigenfunctions $\mu_\pm(x,t,k)$ are continuous at the branch points $p_\pm, \overline{p_\pm}$. Specifically,*

$$\mu_{+1}(x,t,k) = \beta^{(0)}_{p_+}(x,t) + o(1), \quad k \to p_+, \tag{47a}$$

$$\mu_{+2}(x,t,k) = \beta^{(0)}_{\overline{p_+}}(x,t) + o(1), \quad k \to \overline{p_+}, \tag{47b}$$

$$\mu_{-1}(x,t,k) = \beta^{(0)}_{\overline{p_-}}(x,t) + o(1), \quad k \to \overline{p_-}, \tag{47c}$$

$$\mu_{-2}(x,t,k) = \beta^{(0)}_{p_-}(x,t) + o(1), \quad k \to p_- \tag{47d}$$

for some vectors $\beta^{(0)}_{p_\pm}(x,t)$, $\beta^{(0)}_{\overline{p_\pm}}(x,t)$. Moreover, $\beta^{(0)}_{p_\pm}(x,t)$ and $\beta^{(0)}_{\overline{p_\pm}}(x,t)$ are never zero.

Lemma 6. *If $(q - q_\pm) \in L_x^{1,2}(\mathbb{R}^\pm)$ for all $t \in \mathbb{R}$, then*

$$\mu_{+1}(x,t,k) = \beta_{p_+}^{(0)}(x,t) + \beta_{p_+}^{(1)}(x,t)(k-p_+)^{1/2} + o(k-p_+)^{1/2}, \quad k \to p_+, \tag{48}$$

$$\mu_{+2}(x,t,k) = \beta_{\overline{p_+}}^{(0)}(x,t) + \beta_{\overline{p_+}}^{(1)}(x,t)(k-\overline{p_+})^{1/2} + o(k-\overline{p_+})^{1/2}, \quad k \to \overline{p_+}, \tag{49}$$

$$\mu_{-1}(x,t,k) = \beta_{\overline{p_-}}^{(0)}(x,t) + \beta_{\overline{p_-}}^{(1)}(x,t)(k-\overline{p_-})^{1/2} + o(k-\overline{p_-})^{1/2}, \quad k \to \overline{p_-}, \tag{50}$$

$$\mu_{-2}(x,t,k) = \beta_{p_-}^{(0)}(x,t) + \beta_{p_-}^{(1)}(x,t)(k-p_-)^{1/2} + o(k-p_-)^{1/2}, \quad k \to p_- \tag{51}$$

for some vectors $\beta_{p_\pm}^{(1)}(x,t), \beta_{\overline{p_\pm}}^{(1)}(x,t)$ with $\beta_{p_\pm}^{(0)}(x,t), \beta_{\overline{p_\pm}}^{(0)}(x,t)$ as in Lemma 5.

Higher-order expansions in half-integer powers can be found similarly by placing further restrictions on the potential. To use the above expansions to describe the behavior of the Jost solutions $\phi_\pm(x,t,k)$ around the branch points, we now clarify the behavior of $d_\pm(k)$.

Lemma 7. *The asymptotic behavior of $d_\pm(k)$ at the branch points is given by*

$$d_\pm(k) = \left(\frac{8}{iA_\pm}\right)^{1/4} (k-p_\pm)^{1/4} + o(1), \quad k \to p_\pm, \tag{52a}$$

$$d_\pm(k) = \left(\frac{-8}{iA_\pm}\right)^{1/4} (k-\overline{p_\pm})^{1/4} + o(1), \quad k \to \overline{p_\pm}. \tag{52b}$$

The specific branch cuts for the fourth-roots appearing in the above are of little interest, since we will mainly be concerned with the rate of growth of the Jost solutions near the branch points (see Section 4.2). Nonetheless, we clarify that

$$(k-p_\pm)^{1/4} := \sqrt{(k-p_\pm)^{1/2}}, \quad (k-\overline{p_\pm})^{1/4} := \sqrt{(k-\overline{p_\pm})^{1/2}}, \tag{53}$$

where the branch cuts for $(k-p_\pm)^{1/2}$ and $(k-\overline{p_\pm})^{1/2}$ are taken analogously with the definition of $\lambda(k;A)$ in (19) so that

$$\lambda_\pm(k) = (k-p_\pm)^{1/2}(k-\overline{p_\pm})^{1/2}, \tag{54}$$

and $\sqrt{\cdot}$ is the same square root in (30).

We then have the following branch point behavior for the Jost solutions:

Corollary 3. *Under the hypothesis of Lemma 5,*

$$\phi_{+1}(x,t,k) = b_{p_+}^{(0)}(x,t)(k-p_+)^{-1/4} + o(k-p_+)^{-1/4}, \quad k \to p_+, \tag{55a}$$

$$\phi_{+2}(x,t,k) = b^{(0)}_{\overline{p_+}}(x,t)(k-\overline{p_+})^{-1/4} + o(k-\overline{p_+})^{-1/4}, \quad k \to \overline{p_+}, \tag{55b}$$

$$\phi_{-1}(x,t,k) = b^{(0)}_{\overline{p_-}}(x,t)(k-\overline{p_-})^{-1/4} + o(k-\overline{p_-})^{-1/4}, \quad k \to \overline{p_-}, \tag{55c}$$

$$\phi_{-2}(x,t,k) = b^{(0)}_{p_-}(x,t)(k-p_-)^{-1/4} + o(k-p_-)^{-1/4}, \quad k \to p_- \tag{55d}$$

for some vectors $b^{(0)}_{p_\pm}(x,t)$, $b^{(0)}_{\overline{p_\pm}}(x,t)$. Moreover, $b^{(0)}_{p_\pm}(x,t)$ and $b^{(0)}_{\overline{p_\pm}}(x,t)$ are never zero.

Corollary 4. *Under the hypothesis of Lemma 6,*

$$\phi_{+1}(x,t,k) = b^{(0)}_{p_+}(x,t)(k-p_+)^{-1/4} + b^{(1)}_{p_+}(x,t)(k-p_+)^{1/4} + o(k-p_+)^{1/4}, \quad k \to p_+, \tag{56a}$$

$$\phi_{+2}(x,t,k) = b^{(0)}_{\overline{p_+}}(x,t)(k-\overline{p_+})^{-1/4} + b^{(1)}_{\overline{p_+}}(x,t)(k-\overline{p_+})^{1/4} + o(k-\overline{p_+})^{1/4}, \quad k \to \overline{p_+}, \tag{56b}$$

$$\phi_{-1}(x,t,k) = b^{(0)}_{\overline{p_-}}(x,t)(k-\overline{p_-})^{-1/4} + b^{(1)}_{\overline{p_-}}(x,t)(k-\overline{p_-})^{1/4} + o(k-\overline{p_-})^{1/4}, \quad k \to \overline{p_-}, \tag{56c}$$

$$\phi_{-2}(x,t,k) = b^{(0)}_{p_-}(x,t)(k-p_-)^{-1/4} + b^{(1)}_{p_-}(x,t)(k-p_-)^{1/4} + o(k-p_-)^{1/4}, \quad k \to p_- \tag{56d}$$

for some vectors $b^{(1)}_{p_\pm}(x,t)$, $b^{(1)}_{\overline{p_\pm}}(x,t)$ with $b^{(0)}_{p_\pm}(x,t)$, $b^{(0)}_{\overline{p_\pm}}(x,t)$ as in Corollary 3.

3 | DIRECT PROBLEM: SCATTERING MATRIX, SYMMETRIES, AND DISCRETE EIGENVALUES

The scattering data are constructed by studying the relations between the two sets of Jost solutions ϕ_+ and ϕ_-. Proofs for all the results in this section are given in Section 7.1.

3.1 | Scattering matrix

For $k \in \mathbb{R}$, both $\phi_+(x,t,k)$ and $\phi_-(x,t,k)$ are fundamental matrix solutions of both parts of the Lax pair (3). Thus, there exists a matrix

$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}, \tag{57}$$

independent of x and t, such that

$$\phi_-(x,t,k) = \phi_+(x,t,k)S(k), \quad k \in \mathbb{R}. \tag{58}$$

The matrix $S(k)$ is known as the scattering matrix (from the right) and its entries are known as the scattering coefficients.

Note that $\det S(k) = 1$. Writing (58) columnwise, we have

$$\phi_{-1}(x,t,k) = s_{11}(k)\phi_{+1}(x,t,k) + s_{21}(k)\phi_{+2}(x,t,k), \quad k \in \mathbb{R}, \tag{59a}$$

$$\phi_{-2}(x,t,k) = s_{12}(k)\phi_{+1}(x,t,k) + s_{22}(k)\phi_{+2}(x,t,k), \quad k \in \mathbb{R}. \tag{59b}$$

Theorem 1 together with relations (59) gives the following Wronskian representations for the scattering coefficients:

Corollary 5. *Under the hypothesis of Theorem 1, the scattering coefficients admit the following Wronskian representations:*

$$s_{11}(k) = \mathrm{Wr}[\phi_{-1}(x,t,k), \phi_{+2}(x,t,k)], \quad k \in \mathbb{R} \cup \mathbb{C}^- \setminus \{\overline{p_+}, \overline{p_-}\}, \tag{60a}$$

$$s_{12}(k) = \mathrm{Wr}[\phi_{-2}(x,t,k), \phi_{+2}(x,t,k)], \quad k \in \mathbb{R} \cup \Sigma^o_{+1} \cup \Sigma^o_{-2}, \tag{60b}$$

$$s_{21}(k) = \mathrm{Wr}[\phi_{+1}(x,t,k), \phi_{-1}(x,t,k)], \quad k \in \mathbb{R} \cup \Sigma^o_{-1} \cup \Sigma^o_{+2}, \tag{60c}$$

$$s_{22}(k) = \mathrm{Wr}[\phi_{+1}(x,t,k), \phi_{-2}(x,t,k)], \quad k \in \mathbb{R} \cup \mathbb{C}^+ \setminus \{p_+, p_-\} \tag{60d}$$

with $\Sigma^o_{\pm 1}$ and $\Sigma^o_{\pm 2}$ defined by (36). Moreover, $s_{22}(k)$ and $s_{11}(k)$ are analytic in $\mathbb{C}^+ \setminus \Sigma$ and $\mathbb{C}^- \setminus \Sigma$, respectively.

Note that the Wronskian representations (60) are first defined for $k \in \mathbb{R}$, where both of the relations (59) hold. Each of them can then be extended off the real k-axis to define the corresponding scattering coefficient wherever the right-hand side of each of representations (60) is defined.

In the special case of no counterflows, ie, $V = 0$, the scattering relations (59) and Wronskian representations (60) can be further extended (see Section 5).

The Wronskian representations together with Corollary 2 and Lemma 4 give the following:

Corollary 6. *Under the hypotheses of Theorem 1, the scattering matrix $S(k)$ is $C^1(\mathbb{R})$.*

Corollary 7. *Under the hypotheses of Lemma 3,*

$$s_{11}(k) = 1 + O(1/k), \quad k \to \infty, \tag{61a}$$

$$s_{12}(k) = O(1/k), k \to \infty, \tag{61b}$$

$$s_{21}(k) = O(1/k), k \to \infty, \tag{61c}$$

$$s_{22}(k) = 1 + O(1/k), \quad k \to \infty, \tag{61d}$$

within the appropriate regions of the complex k-plane for each column as stated in Corollary 5.

Before the introduction of the scattering matrix, all calculations were symmetric upon exchanging limits as $x \to -\infty$ and as $x \to \infty$, thanks to the symmetry of the NLS equation under space reflections. The relation (58), however, breaks this symmetry. On the other hand, we can similarly write

$$\phi_+(x,t,k) = \phi_-(x,t,k)R(k), \quad k \in \mathbb{R}, \tag{62}$$

or, in column form,

$$\phi_{+1}(x,t,k) = r_{11}(k)\phi_{-1}(x,t,k) + r_{21}(k)\phi_{-2}(x,t,k), \quad k \in \mathbb{R}, \tag{63a}$$

$$\phi_{+2}(x,t,k) = r_{12}(k)\phi_{-1}(x,t,k) + r_{22}(k)\phi_{-2}(x,t,k), \quad k \in \mathbb{R} \tag{63b}$$

for some scattering matrix (from the left)

$$R(k) = \begin{pmatrix} r_{11}(k) & r_{12}(k) \\ r_{21}(k) & r_{22}(k) \end{pmatrix}. \tag{64}$$

The two scattering matrices S and R are related simply by

$$R(k) = S^{-1}(k), \quad k \in \mathbb{R}. \tag{65}$$

Moreover, Wronskian representations exist similar to those in Corollary 5:

Corollary 8. *Under the hypothesis of Theorem 1, the left scattering coefficients can be extended through the Wrosnkian representations,*

$$r_{11}(k) = \mathrm{Wr}[\phi_{+1}(x,t,k), \phi_{-2}(x,t,k)], \quad k \in \mathbb{R} \cup \mathbb{C}^+ \setminus \{p_+, p_-\}, \tag{66a}$$

$$r_{12}(k) = \mathrm{Wr}[\phi_{+2}(x,t,k), \phi_{-2}(x,t,k)], \quad k \in \mathbb{R} \cup \Sigma^o_{+1} \cup \Sigma^o_{-2}, \tag{66b}$$

$$r_{21}(k) = \mathrm{Wr}[\phi_{-1}(x,t,k), \phi_{+1}(x,t,k)], \quad k \in \mathbb{R} \cup \Sigma^o_{-1} \cup \Sigma^o_{+2}, \tag{66c}$$

$$r_{22}(k) = \mathrm{Wr}[\phi_{-1}(x,t,k), \phi_{+2}(x,t,k)], \quad k \in \mathbb{R} \cup \mathbb{C}^- \setminus \{\overline{p_+}, \overline{p_-}\}. \tag{66d}$$

Moreover, $r_{11}(k)$ and $r_{22}(k)$ are analytic in $\mathbb{C}^+ \setminus \Sigma$ and $\mathbb{C}^- \setminus \Sigma$, respectively.

From Corollaries 5 and 8, we see that

$$\begin{aligned} r_{11}(k) &= s_{22}(k), & r_{12}(k) &= -s_{12}(k), \\ r_{21}(k) &= -s_{21}(k), & r_{22}(k) &= s_{11}(k), \end{aligned} \tag{67}$$

wherever the expressions are defined.

For later use, we also define the reflection coefficients as

$$\rho(k) = s_{12}(k)/s_{22}(k), \quad k \in \mathbb{R} \cup \Sigma^o_{+1}, \tag{68a}$$

$$r(k) = 1/s_{22}(k) r_{21}(k), \quad k \in \mathbb{R} \cup \Sigma^o_{-1}. \tag{68b}$$

More precisely, $\rho(k)$ and $r(k)$ will appear in the jump matrices that define the RHP in Section 4. From Corollary 7, we see that

$$\rho(k) = O(1/k), \quad k \to \pm\infty. \tag{69}$$

One can also show that, generically, $r(k) = O(k)$ as $k \to \pm\infty$. This does not pose a problem, however, since $r(k)$ only appears in the jumps across the finite segment Σ^o_-.

As with the scattering matrix from the right, in the special case of no counterflows, ie, $V = 0$, the scattering relations (63), Wronskian representations (66), and domains for $\rho(k)$ and $r(k)$ can be further extended (see Section 5).

Corollary 6 immediately gives the following:

Corollary 9. *Under the hypothesis of Theorem 1, the reflection coefficient $\rho(k)$ is $C^1(\mathbb{R} \setminus K)$, where K is the set of zeros of $s_{22}(k)$.*

In preparation for the formulation of the inverse problem, it is convenient to introduce the following matrix:

$$\Phi(x,t,k) = \begin{cases} \left(\phi_{+1}(x,t,k), \dfrac{\phi_{-2}(x,t,k)}{s_{22}(k)}\right), & k \in \mathbb{C}^+ \setminus \Sigma, \\ \left(\dfrac{\phi_{-1}(x,t,k)}{s_{11}(k)}, \phi_{+2}(x,t,k)\right), & k \in \mathbb{C}^- \setminus \Sigma. \end{cases} \tag{70}$$

Note that $\Phi(x,t,k)$ is a simultaneous fundamental matrix solution to the Lax pair (3) that is meromorphic for $k \in \mathbb{C} \setminus \Sigma$, with $\det \Phi(x,t,k) = 1$.

3.2 | Continuous spectrum

The spectrum of the scattering problem is defined as the set of all $k \in \mathbb{C}$ for which there exist solutions to the Lax pair (3) bounded for all $x \in \mathbb{R}$. As usual, the spectrum consists of a continuum of eigenvalues, Σ^o, which we refer to as the continuous spectrum, together with a discrete set $K \cup \overline{K}$ of eigenvalues (where $\overline{K}$ is the image of K under complex conjugation) which we refer to as the discrete spectrum (discussed later). In the case of zero BCs for the potential or of symmetric BCs with zero velocity (ie, $\lim_{x\to\pm\infty} q(x,t) = 0$ or $\lim_{x\to\pm\infty} q(x,t) = \tilde{q}_\pm$ with $|\tilde{q}_+| = |\tilde{q}_-|$, respectively), the set where both columns of $\phi_-(x,t,k)$ are defined coincides with that where both columns of $\phi_+(x,t,k)$ are, and this set comprises the continuous spectrum of the scattering problem. For example, in the case of symmetric BCs with zero velocity the continuous spectrum is the set $\mathbb{R} \cup (-iA, iA)$, with $A = |\tilde{q}_\pm|$. As discussed in Section 2.3, however, this is not the case here. Specifically, $\phi_\pm(x,t,k)$ can be defined simultaneously only for $k \in \mathbb{R}$, and indeed this is the only set where the full scattering relations (58) and (62) hold.

Nonetheless, it is still possible to partially extend half of (58) and (62) along appropriate segments of $\Sigma^o_\pm$. Specifically, taking into account the regions of definition and analyticity of the Jost solutions, for all $k \in \Sigma^o_+$ (where $\lambda_+(k) \in \mathbb{R}$) one can express the analytic column of $\phi_-(x,t,k)$ as a linear combination of the columns of $\phi_+(x,t,k)$, and vice versa on Σ^o_-. Specifically:

Corollary 10. *Under the hypothesis of Theorem 1, the scattering relations (59a), (59b), (63a), and (63b) can be extended to $k \in \Sigma^o_{+2}$, Σ^o_{+1}, Σ^o_{-1} and Σ^o_{-2}, respectively. That is:*

$$\phi_{-1}(x,t,k) = s_{11}(k)\phi_{+1}(x,t,k) + s_{21}(k)\phi_{+2}(x,t,k), \quad k \in \mathbb{R} \cup \Sigma^o_{+2}, \tag{71a}$$

$$\phi_{-2}(x,t,k) = s_{12}(k)\phi_{+1}(x,t,k) + s_{22}(k)\phi_{+2}(x,t,k), \quad k \in \mathbb{R} \cup \Sigma^o_{+1}, \tag{71b}$$

$$\phi_{+1}(x,t,k) = r_{11}(k)\phi_{-1}(x,t,k) + r_{21}(k)\phi_{-2}(x,t,k), \quad k \in \mathbb{R} \cup \Sigma^o_{-1}, \tag{71c}$$

$$\phi_{+2}(x,t,k) = r_{12}(k)\phi_{-1}(x,t,k) + r_{22}(k)\phi_{-2}(x,t,k), \quad k \in \mathbb{R} \cup \Sigma^o_{-2}. \tag{71d}$$

Note that the coefficients in the right-hand side of (71) were labeled consistently with the Wronskian representations in (60). Together with Corollary 1, the expressions in Corollary 10 allow us to conclude that the corresponding eigenfunctions are bounded over all $x \in \mathbb{R}$:

Corollary 11. *Under the hypothesis of Theorem 1, for all $t \in \mathbb{R}$ we have*

$$\phi_{+1}(x,t,k) \in L^\infty_x(\mathbb{R}), \quad k \in \mathbb{R} \cup \Sigma^o_{-1}, \tag{72a}$$

$$\phi_{+2}(x,t,k) \in L^\infty_x(\mathbb{R}), \quad k \in \mathbb{R} \cup \Sigma^o_{-2}, \tag{72b}$$

$$\phi_{-1}(x,t,k) \in L^\infty_x(\mathbb{R}), \quad k \in \mathbb{R} \cup \Sigma^o_{+2}, \tag{72c}$$

$$\phi_{-2}(x,t,k) \in L^\infty_x(\mathbb{R}), \quad k \in \mathbb{R} \cup \Sigma^o_{+1}. \tag{72d}$$

In turn, defining

$$\Sigma^o := \mathbb{R} \cup \Sigma^o_+ \cup \Sigma^o_-, \tag{73}$$

Corollary 11 implies:

Corollary 12. *Under the hypothesis of Theorem 1 and for $V \neq 0$, the continuous spectrum is given by Σ^o.*

In the special case of no counterflows, ie, $V = 0$, the relations (71) can be further extended so that the regions of boundedness over $x \in \mathbb{R}$ in Corollary 11 are correspondingly extended. The continuous spectrum also requires further consideration in this case (see Section 5).

3.3 | Symmetries

The symmetries

$$\overline{X(x,t,\overline{k})} = -\sigma_* X(x,t,k)\sigma_*, \quad \overline{T(x,t,\overline{k})} = -\sigma_* T(x,t,k)\sigma_*, \quad k \in \mathbb{C}, \tag{74}$$

where

$$\sigma_* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{75}$$

lead to the following symmetry relations:

Lemma 8 (First symmetry, Jost solutions). *Under the hypothesis of Theorem 1, we have the symmetries*

$$\overline{\phi_{+1}(x,t,\overline{k})} = \quad \sigma_*\phi_{+2}(x,t,k), \quad k \in \mathbb{R}\cup\mathbb{C}^-\cup\Sigma^o_{+1}\setminus\{\overline{p_+}\}, \tag{76a}$$

$$\overline{\phi_{+2}(x,t,\overline{k})} = -\sigma_*\phi_{+1}(x,t,k), \quad k \in \mathbb{R}\cup\mathbb{C}^+\cup\Sigma^o_{+2}\setminus\{p_+\}, \tag{76b}$$

$$\overline{\phi_{-1}(x,t,\overline{k})} = \quad \sigma_*\phi_{-2}(x,t,k), \quad k \in \mathbb{R}\cup\mathbb{C}^+\cup\Sigma^o_{-2}\setminus\{p_-\}, \tag{76c}$$

$$\overline{\phi_{-2}(x,t,\overline{k})} = -\sigma_*\phi_{-1}(x,t,k), \quad k \in \mathbb{R}\cup\mathbb{C}^-\cup\Sigma^o_{-1}\setminus\{\overline{p_-}\}. \tag{76d}$$

Lemma 9 (First symmetry, scattering coefficients). *Under the hypothesis of Theorem 1, we have the symmetries*

$$\overline{s_{22}(\overline{k})} = s_{11}(k), \quad k \in \mathbb{R}\cup\mathbb{C}^-\setminus\{\overline{p_+},\overline{p_-}\}, \tag{77a}$$

$$\overline{s_{12}(\overline{k})} = -s_{21}(k), \quad k \in \mathbb{R}\cup\Sigma^o_{+2}\cup\Sigma^o_{-1}, \tag{77b}$$

$$\overline{r_{11}(\overline{k})} = r_{22}(k), \quad k \in \mathbb{R}\cup\mathbb{C}^-\setminus\{\overline{p_+},\overline{p_-}\}, \tag{77c}$$

$$\overline{r_{21}(\overline{k})} = -r_{12}(k), \quad k \in \mathbb{R}\cup\Sigma^o_{+2}\cup\Sigma^o_{-1}. \tag{77d}$$

Lemmas 8 and 9 then give the following symmetry for $\Phi(x,t,k)$ as given in (70):

Corollary 13. *Under the hypothesis of Theorem 1, we have the symmetry*

$$\Phi^\dagger(x,t,k) = \Phi(x,t,k)^{-1}, \quad k \in \mathbb{C}\setminus\Sigma, \tag{78}$$

where $\dagger$ denotes the Schwarz conjugate-transpose with respect to k.

Lemma 9 also gives the following symmetries for the reflection coefficients:

Corollary 14 (First symmetry, reflection coefficients). *Under the hypothesis of Theorem 1, we have the symmetries*

$$\overline{\rho(\bar{k})} = s_{21}(k)/s_{11}(k), \quad k \in \mathbb{R} \cup \Sigma^o_{+2}, \tag{79}$$

$$\overline{r(\bar{k})} = -1/s_{11}(k)r_{12}(k), \quad k \in \mathbb{R} \cup \Sigma^o_{-2}. \tag{80}$$

Let us again use the superscript $\pm$ to denote the left/right limits along an oriented contour in the complex k-plane, as in (27). The symmetry $\lambda^+_\pm(k) = -\lambda_\pm(k)$ for $k \in \Sigma_\pm$ leads to the following:

Lemma 10 (Second symmetry, Jost solutions). *Under the hypothesis of Theorem 1,*

$$\phi^+_{+1}(x,t,k) = -i\mathrm{e}^{-i\delta}\phi_{+2}(x,t,k), \quad k \in \Sigma^o_{+1}, \tag{81a}$$

$$\phi^+_{+2}(x,t,k) = -i\mathrm{e}^{+i\delta}\phi_{+1}(x,t,k), \quad k \in \Sigma^o_{+2}, \tag{81b}$$

$$\phi^+_{-1}(x,t,k) = -i\mathrm{e}^{+i\delta}\phi_{-2}(x,t,k), \quad k \in \Sigma^o_{-2}, \tag{81c}$$

$$\phi^+_{-2}(x,t,k) = -i\mathrm{e}^{-i\delta}\phi_{-1}(x,t,k), \quad k \in \Sigma^o_{-1}. \tag{81d}$$

Lemma 11 (Second symmetry, scattering coefficients). *Under the hypothesis of Theorem 1 and for $V \neq 0$,*

$$s^+_{22}(k) = i\mathrm{e}^{-i\delta}s_{12}(k), \qquad r^+_{11}(k) = -i\mathrm{e}^{-i\delta}r_{12}(k), \quad k \in \Sigma^o_{+1}, \tag{82a}$$

$$s^+_{22}(k) = -i\mathrm{e}^{-i\delta}s_{21}(k), \quad r^+_{11}(k) = i\mathrm{e}^{-i\delta}r_{21}(k), \quad k \in \Sigma^o_{-1}, \tag{82b}$$

$$s^+_{11}(k) = i\mathrm{e}^{+i\delta}s_{21}(k), \qquad r^+_{22}(k) = -i\mathrm{e}^{+i\delta}r_{21}(k), \quad k \in \Sigma^o_{+2}, \tag{82c}$$

$$s^+_{11}(k) = -i\mathrm{e}^{+i\delta}s_{12}(k), \quad r^+_{22}(k) = i\mathrm{e}^{+i\delta}r_{12}(k), \quad k \in \Sigma^o_{-2}. \tag{82d}$$

Note that, without the scaling factor $d_\pm(k)$ in (16) to define $\phi_\pm(x,t,k)$, the symmetries (81) and (82) would change to include factors of $iA_\pm/(\lambda_\pm + (k \pm V/2))$ (cf. Ref. 31).

In the special case of no counterflows, ie, $V = 0$, the symmetries (76), (77), (79), and (80) can be extended. The symmetries (81) are unchanged; however, the symmetries (82) must be adjusted (see Section 5).

3.4 | Discrete eigenvalues

The discrete eigenvalues of the scattering problem are those values of $k \in \mathbb{C} \setminus \Sigma$ (ie, away from the continuous spectrum and the branch points) for which there exist solutions of the Lax pair (3) bounded for all $x \in \mathbb{R}$. As usual, the discrete eigenvalues are in one-to-one correspondence with the zeros of the analytic scattering coefficients:

Lemma 12. *Under the hypothesis of Theorem 1, there exists an eigenfunction bounded for all $x \in \mathbb{R}$ satisfying the Lax pair (3) at $k = k_o \in \mathbb{C}^+ \setminus \Sigma$ [respectively, $\overline{k_o} \in \mathbb{C}^- \setminus \Sigma$] if and only if $s_{22}(k_o) = 0$ [respectively, $s_{11}(\overline{k_o}) = 0$]. Moreover, whenever such conditions are satisfied, the corresponding eigenfunctions decay exponentially at both spatial infinities.*

Corollary 15. *The set $K \cup \overline{K}$ of discrete eigenvalues (with $K \subset \mathbb{C}^+ \setminus \Sigma$) is comprised of a possibly infinite set of isolated points in $\mathbb{C} \setminus \Sigma$.*

Note that the set of discrete eigenvalues could possibly have one or more accumulation points in $\Sigma = \mathbb{R} \cup \Sigma_+ \cup \Sigma_-$ since, generically, $s_{11}(k)$ and $s_{22}(k)$ are not analytic there. Indeed, it is well known that such situations occur for the focusing NLS equation with zero BCs.[44] It is also possible that $s_{11}(k)$ could possess zeros along real k-axis even if the set of discrete eigenvalues $K \cup \overline{K}$ is finite. Indeed, while nongeneric, such situations are fairly common in the case of zero BCs.[4] Zeros of $s_{11}(k)$ and $s_{22}(k)$ along the continuous spectrum are referred to as spectral singularities.[45] In contrast, the scattering coefficients do not vanish on Σ_+^o or Σ_-^o:

Lemma 13. *Under the hypothesis of Theorem 1 and for $V \neq 0$,*

$$s_{11}(k) \neq 0, \quad r_{22}(k) \neq 0, \quad k \in \Sigma_{+2}^o \cup \Sigma_{-2}^o, \tag{83a}$$

$$s_{12}(k) \neq 0, \quad r_{12}(k) \neq 0, \quad k \in \Sigma_{+1}^o \cup \Sigma_{-2}^o, \tag{83b}$$

$$s_{21}(k) \neq 0, \quad r_{21}(k) \neq 0, \quad k \in \Sigma_{+2}^o \cup \Sigma_{-1}^o, \tag{83c}$$

$$s_{22}(k) \neq 0, \quad r_{11}(k) \neq 0, \quad k \in \Sigma_{+1}^o \cup \Sigma_{-1}^o. \tag{83d}$$

Note that the above statements do not hold in the case $V = 0$ (see Section 5). Indeed, when $V = 0$, the limiting case of a discrete eigenvalue on the branch cut gives rise to Akhmediev breathers.[29] Also, the coefficients can vanish on the boundary of Σ_+ or Σ_-, ie, at $k = p_\pm$ and $k = \overline{p_\pm}$. Indeed, in the case $V = 0$, such zeros lead to rational solutions such as the Peregrine breather and its generalizations.[29,46]

Recall that reflectionless potentials, ie, those for which $\rho(k) \equiv 0$, correspond to pure soliton solutions. As a consequence of Lemma 13, we see that there are no reflectionless potentials when $V \neq 0$, and all solutions must have a radiative component.

Recalling $\Phi(x, t, k)$ as given by (70), we see that the discrete eigenvalues correspond to singularities of $\Phi(x, t, k)$. It will be important to understand the residues of $\Phi(x, t, k)$ at the discrete eigenvalues for the inverse problem. Letting $\Phi_1(x, t, k)$ and $\Phi_2(x, t, k)$ denote the first and second columns of $\Phi(x, t, k)$, respectively, we have the following:

Lemma 14. *Under the hypothesis of Theorem 1, if $s_{22}(k)$ has a finite set of simple zeros, $K = \{k_1, \dots, k_N\} \subset \mathbb{C}^+ \setminus \Sigma$, there are norming constants $c_1, \dots, c_n \in \mathbb{C}$ such that*

$$\operatorname{Res}_{k=k_n} \Phi(x,t,k) = (\;0, \quad c_n \Phi_1(x,t,k_n)\,), \quad n = 1, \dots, N, \tag{84a}$$

$$\operatorname{Res}_{k=\overline{k_n}} \Phi(x,t,k) = \left(-\overline{c_n}\Phi_2(x,t,\overline{k_n}), 0\right), \quad n = 1, \dots, N. \tag{84b}$$

Higher-order zeros of the analytic scattering coefficients can be dealt with similarly, but we omit such cases for brevity.

3.5 | Scattering coefficients: Behavior at the branch points

Understanding the behavior of the scattering coefficients at the branch points will be necessary to properly formulate the inverse problem. Corollary 3 together with the Wronksian definitions (60) gives the behavior of the scattering coefficients at the branch points:

Corollary 16. *Under the hypothesis of Lemma 5 and for $V \neq 0$,*

$$s_{11}(k) = b^{(0)}_{\pm 11}(k - \overline{p_\pm})^{-1/4} + o(k - \overline{p_\pm})^{-1/4}, \quad k \to \overline{p_\pm}, \tag{85a}$$

$$s_{12}(k) = b^{(0)}_{+12}(k - p_+)^{-1/4} + o(k - p_+)^{-1/4}, \quad k \to p_+, \tag{85b}$$

$$s_{12}(k) = b^{(0)}_{-12}(k - \overline{p_-})^{-1/4} + o(k - \overline{p_-})^{-1/4}, \quad k \to \overline{p_-}, \tag{85c}$$

$$s_{21}(k) = b^{(0)}_{+21}(k - \overline{p_+})^{-1/4} + o(k - \overline{p_+})^{-1/4}, \quad k \to \overline{p_+}, \tag{85d}$$

$$s_{21}(k) = b^{(0)}_{-21}(k - p_-)^{-1/4} + o(k - p_-)^{-1/4}, \quad k \to p_-, \tag{85e}$$

$$s_{22}(k) = b^{(0)}_{\pm 22}(k - p_\pm)^{-1/4} + o(k - p_\pm)^{-1/4}, \quad k \to p_\pm \tag{85f}$$

for some constants $b^{(0)}_{\pm 11}$, $b^{(0)}_{\pm 12}$, $b^{(0)}_{\pm 21}$, $b^{(0)}_{\pm 22}$, where the limits must be taken from within the regions described by Corollary 5.

Analogous expansions can easily be given for the entries of $R(k)$, but we omit them for brevity.

Note that $b^{(0)}_{\pm 11} = 0$ exactly when $\mu_{+2}(x,t,k)$ and $\mu_{-1}(x,t,k)$ are linearly dependent at the branch points $\overline{p_\pm}$. Similarly, $b^{(0)}_{\pm 22} = 0$ exactly when the modified eigenfunctions $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ are linearly dependent at the branch points $p_\pm$. The symmetry (77a) shows that $s_{\pm 11} = 0$ exactly when $s_{\pm 22} = 0$. Moreover, note that $\mu_{+1}(x,t,k)$ and $\mu_{+2}(x,t,k)$ are always linearly dependent at the branch points p_+ and $\overline{p_+}$ (whenever they can be defined there) while $\mu_{-1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ are always linearly dependent at the branch points p_- and $\overline{p_-}$. As a result, $b^{(0)}_{\pm 11}$, $b^{(0)}_{\pm 12}$, $b^{(0)}_{\pm 21}$, and $b^{(0)}_{\pm 22}$ are either all zero or all nonzero depending on the linear dependence of $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ at $p_\pm$. We say that the "generic case" holds when $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ are linearly independent at both branch points so that $b^{(0)}_{\pm 11}$, $b^{(0)}_{\pm 12}$, $b^{(0)}_{\pm 21}$ and $b^{(0)}_{\pm 22}$ are all nonzero.

Corollary 17. *Under the hypothesis of Lemma 5 and for $V \neq 0$, in the generic case that $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ are linearly independent at the branch points $p_\pm$, we have*

$$\rho(k) = \rho_o + o(1), \quad k \to p_+, \tag{86a}$$

$$r(k) = r_o(k-p_-)^{1/2} + o(k-p_-)^{1/2}, \quad k \to p_- \tag{86b}$$

for some nonzero constants ρ_o, r_o, where the limits must be taken from within the regions described by (68).

There are numerous exceptional cases beyond the generic case discussed above. When necessary, higher-order expansions for the scattering coefficients about the branch points can be found using Corollary 4 together with the Wronskian representations (60):

Corollary 18. *Under the hypothesis of Lemma 6 and for $V \neq 0$,*

$$s_{11}(k) = b^{(0)}_{\pm 11}(k-\overline{p_\pm})^{-1/4} + b^{(1)}_{\pm 11}(k-\overline{p_\pm})^{1/4} + o(k-\overline{p_\pm})^{1/4}, \quad k \to \overline{p_\pm}, \tag{87a}$$

$$s_{12}(k) = b^{(0)}_{+12}(k-p_+)^{-1/4} + b^{(1)}_{+12}(k-p_+)^{1/4} + o(k-p_+)^{1/4}, \quad k \to p_+, \tag{87b}$$

$$s_{12}(k) = b^{(0)}_{-12}(k-\overline{p_-})^{-1/4} + b^{(1)}_{-12}(k-\overline{p_-})^{1/4} + o(k-\overline{p_-})^{1/4}, \quad k \to \overline{p_-}, \tag{87c}$$

$$s_{21}(k) = b^{(0)}_{+21}(k-\overline{p_+})^{-1/4} + b^{(1)}_{+21}(k-\overline{p_+})^{1/4} + o(k-\overline{p_+})^{1/4}, \quad k \to \overline{p_+}, \tag{87d}$$

$$s_{21}(k) = b^{(0)}_{-21}(k-p_-)^{-1/4} + b^{(1)}_{-21}(k-p_-)^{1/4} + o(k-p_-)^{1/4}, \quad k \to p_-, \tag{87e}$$

$$s_{22}(k) = b^{(0)}_{\pm 22}(k-p_\pm)^{-1/4} + b^{(1)}_{\pm 22}(k-p_\pm)^{1/4} + o(k-p_\pm)^{1/4}, \quad k \to p_\pm \tag{87f}$$

for some constants $b^{(0)}_{\pm 11}$, $b^{(0)}_{\pm 12}$, $b^{(0)}_{\pm 21}$, $b^{(0)}_{\pm 22}$, $b^{(1)}_{\pm 11}$, $b^{(1)}_{\pm 12}$, $b^{(1)}_{\pm 21}$, $b^{(1)}_{\pm 22}$, where the limits must be taken from within the regions described by Corollary 5.

We consider here only the exceptional case in which $b^{(0)}_{\pm 11}$, $b^{(0)}_{\pm 12}$, $b^{(0)}_{\pm 21}$, $b^{(0)}_{\pm 22}$ all zero and $b^{(1)}_{\pm 11}$, $b^{(1)}_{\pm 12}$, $b^{(1)}_{\pm 21}$, $b^{(1)}_{\pm 22}$ all nonzero. Other cases can be treated similarly.

Corollary 19. *Under the hypothesis of Lemma 6 and for $V \neq 0$, in the exceptional case that μ+1(x,t,k) and $\mu_{-2}(x,t,k)$ are linearly dependent at the branch points $p_\pm$ with $b^{(1)}_{\pm 11}$, $b^{(1)}_{\pm 12}$, $b^{(1)}_{\pm 21}$,*

$b_{\pm 22}^{(1)}$ *all nonzero, we have*

$$\rho(k) = \rho_o + o(1), \quad k \to p_+, \tag{88a}$$

$$r(k) = r_o(k - p_-)^{-1/2} + o(k - p_-)^{-1/2}, \quad k \to p_- \tag{88b}$$

for some nonzero constants ρ_o, r_o, where the limits must be taken from within the regions described by (68).

3.6 | Alternative solutions of the Lax pair

The Jost solutions $\phi_\pm(x,t,k)$ were defined to satisfy the asymptotic BCs (16). The preceding sections have explored the resulting analyticity properties. One could also seek solutions to an IVP for the Lax pair. We follow the recent work by Bilman and Miller (see Ref. 46 for details).

Lemma 15. *Suppose $q(x,t)$ is a bounded classical solution of (1) defined for $(x,t) \in \mathbb{R} \times [0,\infty)$. Then for each $k \in \mathbb{C}$, there exists a unique simultaneous fundamental solution $\Psi(x,t,k)$ of both parts of the Lax pair (3) together with the IC*

$$\Psi(0,0,k) = \mathrm{I}. \tag{89}$$

Moreover, $\Psi(x,t,k)$ is an entire function of k and $\det \Psi(x,t,k) \equiv 1$.

The proof proceeds by identifying $\Psi(x,t,k)$ as the value at $u = 1$ of the unique solution of the integral equation

$$\Psi(\xi(u),\tau(u),k) = \mathrm{I} + \int_0^u \left[X(\xi(s),\tau(s),k)x'(s) + T(\xi(s),\tau(s),k)t'(s)\right]\Psi(\xi(s),\tau(s),k)\,ds, \tag{90}$$

where $(\xi(\cdot),\tau(\cdot)) : [0,1] \to \mathbb{R} \times [0,\infty)$ is a smooth path from $(\xi(0),\tau(0)) = (0,0)$ to $(\xi(1),\tau(1)) = (x,t)$. One could choose any other base point (x_o,t_o) at which to normalize $\Psi(x,t,k)$ instead of (0,0). While the choice $x_o = 0$ is inconsequential, it is important to take $t_o = 0$, so that the relevant scattering data (see below) can be computed using the IC $q(x,0)$. Incidentally, the same fundamental matrix solution $\Psi(x,t,k)$ also proves to be useful when studying boundary value problems on the half line.[47,48]

In contrast to Theorem 1, Lemma 15 makes no requirement that $(q - q_\pm) \in L^1_x(\mathbb{R}^\pm)$. Moreover, whereas the regions of analyticity for $\phi_\pm(x,t,k)$ can only generically be shown to include the appropriate half-planes (minus the relevant branch cuts), $\Psi(x,t,k)$ is entire. These differences can be intuitively understood by recognizing that, for any fixed (x,t), the integral equations defining $\phi_\pm(x,t,k)$ integrate along infinitely long paths from the base points $(\pm\infty,t)$, whereas the integral equation for $\Psi(x,t,k)$ integrates along a finite path from the base point (0,0).

On the other hand, given any fundamental matrix solution to the Lax pair (3), one can readily obtain $\Psi(x,t,k)$. Indeed, we have the following:

Corollary 20. *Under the hypotheses of Theorem 1 and Lemma 15, we have*

$$\Psi(x,t,k) = \Phi(x,t,k)C^{-1}(k), \quad k \in \mathbb{C} \setminus (\Sigma \cup K \cup \overline{K}), \tag{91}$$

where $C(k) = \Phi(0,0,k)$.

The above corollary follows immediately from the uniqueness of the IVP in Lemma 15. Moreover, one could use (91) as a definition for $\Psi(x,t,k)$, with the exception of the removable singularities for $k \in \Sigma \cup K \cup \overline{K}$. Note that $C(k)$ is not an entire function of k. However, the discontinuities of $\Phi(x,t,k)$ and $C^{-1}(k)$ across Σ exactly cancel so that $\Psi(x,t,k)$ is entire. The symmetries (76) are then passed to $\Psi(x,t,k)$, though it follows directly from (74) and Lemma 15 that $\Psi(x,t,k)$ satisfies the following:

Corollary 21. *Under the hypotheses of Lemma 15, we have*

$$\overline{\Psi(x,t,\overline{k})} = -\sigma_* \Psi(x,t,k)\sigma_*, \quad k \in \mathbb{C}. \tag{92}$$

4 | INVERSE PROBLEM: RHP FORMULATION

We now turn our attention to the inverse problem (namely, recovering the solution of the NLS equation from its scattering data), which we will formulate as a matrix RHP. Proofs for all the results in this section are given in Section 7.2. We begin by introducing the sectionally meromorphic matrix function

$$M(x,t,k) = \Phi(x,t,k)\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}, \quad k \in \mathbb{C} \setminus \Sigma, \tag{93}$$

where $\Phi(x,t,k)$ is given by (70) and $\theta_o(x,t,k) = k(x - 2kt)$, as in (43). In light of Lemma 4 and Corollary 7, we see that $M(x,t,k) = \mathrm{I} + O(1/k)$ as $k \to \infty$. Note that $\det M(x,t,k) = 1$.

Since $\Phi(x,t,k)$ satisfies the Lax pair (3), we have the following:

Lemma 16 (Modified Lax pair). *The matrix $M(x,t,k)$ defined by (93) satisfies the modified Lax pair*

$$M_x(x,t,k) - ik[\sigma_3, M(x,t,k)] = Q(x,t)M(x,t,k), \tag{94a}$$

$$M_t(x,t,k) + 2ik^2[\sigma_3, M(x,t,k)] = \big(i\sigma_3\big(Q_x(x,t) - Q^2(x,t)\big) + 2kQ(x,t)\big)M(x,t,k). \tag{94b}$$

4.1 | Jump matrix and residue conditions

As before, we use the superscripts $\pm$ to denote the nontangential left/right limits toward the oriented contours, with Σ oriented as in Figure 2. We first express the discontinuity of $M(x,t,k)$ across Σ^o. We should note that in many previous works, $\theta_\pm(x,t,k)$ were used instead of $\theta_o(x,t,k)$ in the definition (93). The use of $\theta_o(x,t,k)$, however, results in a considerable simplification of the jumps across the branch cuts compared to $\theta_\pm(x,t,k)$, since, unlike $\theta_\pm(x,t,k)$, $\theta_o(x,t,k)$ is entire.

Lemma 17. *Under the hypothesis of Theorem 1 and for $V \neq 0$, the matrix $M(x,t,k)$ defined in (93) satisfies the jump condition*

$$M^+(x,t,k) = M^-(x,t,k)J(x,t,k), \quad k \in \Sigma^o, \tag{95}$$

where the jump matrix $J(x,t,k)$ is given by

$$J(x,t,k) = \mathrm{e}^{i\theta_o(x,t,k)\sigma_3} J_o(k)\,\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}, \tag{96}$$

with

$$J_o(k) = \begin{cases} \begin{pmatrix} 1 & \rho(k) \\ \overline{\rho(k)} & 1+\rho(k)\overline{\rho(k)} \end{pmatrix}, & k \in \mathbb{R}, \\ \begin{pmatrix} i\rho(k)\mathrm{e}^{-i\delta} & 0 \\ -i\mathrm{e}^{-i\delta} & -i\mathrm{e}^{i\delta}/\rho(k) \end{pmatrix}, & k \in \Sigma^o_{+1}, \\ \begin{pmatrix} i\mathrm{e}^{-i\delta}/\overline{\rho(\overline{k})} & -i\mathrm{e}^{i\delta} \\ 0 & \overline{-i\rho(\overline{k})\mathrm{e}^{i\delta}} \end{pmatrix}, & k \in \Sigma^o_{+2}, \\ \begin{pmatrix} 1 & -r(k) \\ 0 & 1 \end{pmatrix}, & k \in \Sigma^o_{-1}, \\ \begin{pmatrix} 1 & 0 \\ \overline{r(\overline{k})} & 1 \end{pmatrix}, & k \in \Sigma^o_{-2}, \end{cases} \tag{97}$$

where $\rho(k)$ and $r(k)$ are given in (68).

Special cases, as well as the reduction to $V = 0$, are discussed in Sections 5 and 6.

As a consequence of Lemma 13 we see that the entries of the jump matrix are nonsingular on $\Sigma^o_+ \cup \Sigma^o_-$, but potentially have unbounded growth toward the branch points. Corollaries 17 and 19 describe some of the possible behaviors of the jump matrix at these potential singularities. In particular, in the generic case we see that $J_o(k)$ is continuous at the branch points.

Note that we have broken the $x \mapsto -x$ symmetry in the definition of $M(x,t,k)$, which rescales the columns of $\phi_-(x,t,k)$ using the analytic scattering coefficients from the right, $s_{11}(k)$ and $s_{22}(k)$. One could instead define $M(x,t,k)$ using the analytic scattering coefficients from the left, $r_{11}(k)$ and $r_{22}(k)$ to rescale $\phi_+(x,t,k)$. Indeed, defining

$$\widehat{M}(x,t,k) = \begin{cases} \left(\phi_{-2}(x,t,k), \dfrac{\phi_{+1}(x,t,k)}{r_{11}(k)}\right)\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}, & k \in \mathbb{C}^+ \setminus \Sigma, \\ \left(\dfrac{\phi_{+2}(x,t,k)}{r_{22}(k)}, \phi_{-1}(x,t,k)\right)\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}, & k \in \mathbb{C}^- \setminus \Sigma, \end{cases} \tag{98}$$

we find that $\widehat{M}^+(x,t,k) = \widehat{M}^-(x,t,k)\widehat{J}(x,t,k)$, where

$$\widehat{J}(x,t,k) = J(x,t,k)[(\rho, r, \theta_o, \Sigma_+, \Sigma_-) \mapsto (\widehat{\rho}, \widehat{r}, -\theta_o, \Sigma_-, \Sigma_+)], \tag{99}$$

with $\widehat{\rho}(k) = r_{21}(k)/r_{11}(k)$ and $\widehat{r}(k) = 1/r_{11}(k)s_{12}(k)$.

As usual, when a nonempty discrete spectrum is present, the matrix $M(x,t,k)$ acquires pole singularities at the eigenvalues forming the discrete spectrum, and these singularities must be taken into account to complete the formulation of the RHP. Specifically, letting $M_1(x,t,k)$ and $M_2(x,t,k)$ denote the first and second columns of $M(x,t,k)$, respectively, from (84) we have the following residue conditions for $M(x,t,k)$:

Lemma 18. *Under the hypothesis of Theorem 1, if $s_{22}(k)$ has a finite set of simple zeros, $K = \{k_1, \dots, k_N\} \subset \mathbb{C}^+ \setminus \Sigma$, then $M(x,t,k)$ is analytic in $\mathbb{C} \setminus (\Sigma \cup K \cup \overline{K})$. Moreover, $M(x,t,k)$ has simple poles at each $k_n \in K$ and $\overline{k_n} \in \overline{K}$, and there are norming constants $c_1, \dots, c_n \in \mathbb{C}$ such that*

$$\mathrm{Res}_{k=k_n} M(x,t,k) = \left(\ 0, \quad c_n \mathrm{e}^{2i\theta_o(x,t,k_n)} M_1(x,t,k_n)\right), \quad n = 1, \dots, N, \tag{100a}$$

$$\mathrm{Res}_{k=\overline{k_n}} M(x,t,k) = \left(-\overline{c_n}\mathrm{e}^{-2i\theta_o(x,t,\overline{k_n}} M_2(x,t,\overline{k_n}),\ 0\ \right), \quad n = 1, \dots, N. \tag{100b}$$

4.2 | Growth conditions

In addition to the normalization, jump condition and residue conditions for $M(x,t,k)$, one must specify appropriate growth conditions near the branch points.[46]

Corollary 3 describes the behavior of the Jost solutions near the branch points. Specifically, if $(q - q_\pm) \in L^{1,1}_x(\mathbb{R}^\pm)$ then the eigenfunctions $\phi_{+1}(x,t,k)$ and $\phi_{-2}(x,t,k)$ have $-1/4$ power growth toward their respective branch points $p_\pm$. The behavior of $s_{11}(k)$ and $s_{22}(k)$ near the branch points then determines the growth conditions for the inverse problem. Recall that in the generic case (see Section 3.5), $s_{11}(k)$ and $s_{22}(k)$ have $-1/4$ power growth toward the branch points $\overline{p_\pm}$ and $p_\pm$, respectively. In such cases, Corollaries 3 and 16 give the following result:

Lemma 19. *Let $V \neq 0$ and the hypothesis of Lemma 5 be satisfied. In the generic case that $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ are linearly independent at the branch points $p_\pm$, we have*

$$M(x,t,k) = \begin{cases} \left(B^{(0)}_{p_+}(x,t) + o(1)\right)(k - p_+)^{-\sigma_3/4}, & k \to p_+, \\ \left(B^{(0)}_{\overline{p_+}}(x,t) + o(1)\right)(k - \overline{p_+})^{+\sigma_3/4}, & k \to \overline{p_+}, \\ B^{(0)}_{p_-}(x,t) + o(1), & k \to p_-, \\ B^{(0)}_{\overline{p_-}}(x,t) + o(1), & k \to \overline{p_-} \end{cases} \tag{101}$$

for some invertible matrices $B^{(0)}_{p_\pm}(x,t)$, $B^{(0)}_{\overline{p_\pm}}(x,t)$.

Note that the invertibility of the matrices $B^{(0)}_{p_\pm}(x,t)$, $B^{(0)}_{\overline{p_\pm}}(x,t)$ is an immediate consequence of the linear independence of $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ at the branch points. In particular, Lemma 19 shows that the following limits exist:

$$\lim_{k\to p_+} M(x,t,k)(k - p_+)^{+\sigma_3/4}, \qquad \lim_{k\to p_-} M(x,t,k)$$

$$\lim_{k\to\overline{p_+}} M(x,t,k)(k-\overline{p_+})^{-\sigma_3/4}, \qquad \lim_{k\to\overline{p_-}} M(x,t,k). \tag{102}$$

The requirement that these limits exist will serve as the growth conditions for the RHP in the generic case to guarantee uniqueness of solutions.

The asymptotic behavior changes in the exceptional case in which $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ are linearly dependent at one or both of the branch points $p_\pm$. Specifically, suppose we are in the case where $b^{(0)}_{\pm 11}$, $b^{(0)}_{\pm 22}$ are zero and $b^{(1)}_{\pm 11}$, $b^{(1)}_{\pm 22}$ are nonzero, where $b^{(0)}_{\pm 11}$, $b^{(0)}_{\pm 22}$, $b^{(1)}_{\pm 11}$, and $b^{(1)}_{\pm 22}$ are as in Corollary 18. In such cases, Corollaries 4 and 18 give the following result:

Lemma 20. *Let $V \neq 0$ and the hypothesis of Lemma 6 be satisfied. In the exceptional case that $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ are linearly dependent at the branch points $p_\pm$ with $b^{(1)}_{\pm 11}$ and $b^{(1)}_{\pm 22}$ (as given by Corollary 18) nonzero, we have*

$$M(x,t,k) = \begin{cases} \left(B^{(0)}_{p_+}(x,t) + B^{(1)}_{p_+}(x,t)(k-p_+)^{1/2} + o(k-p_+)^{1/2}\right)(k-p_+)^{-1/4}, & k\to p_+, \\ \left(B^{(0)}_{\overline{p_+}}(x,t) + B^{(1)}_{\overline{p_+}}(x,t)(k-\overline{p_+})^{1/2} + o(k-p_+)^{1/2}\right)(k-\overline{p_+})^{-1/4}, & k\to \overline{p_+}, \\ \left(B^{(0)}_{p_-}(x,t) + B^{(1)}_{p_-}(x,t)(k-p_-)^{1/2} + o(k-p_+)^{1/2}\right)(k-p_-)^{-1/4+\sigma_3/4}, & k\to p_-, \\ \left(B^{(0)}_{\overline{p_-}}(x,t) + B^{(1)}_{\overline{p_-}}(x,t)(k-\overline{p_-})^{1/2} + o(k-p_+)^{1/2}\right)(k-\overline{p_-})^{-1/4-\sigma_3/4}, & k\to \overline{p_-} \end{cases} \tag{103}$$

for some matrices $B^{(0)}_{p_\pm}(x,t)$, $B^{(0)}_{\overline{p_\pm}}(x,t)$, $B^{(1)}_{p_\pm}(x,t)$, $B^{(1)}_{\overline{p_\pm}}(x,t)$, with $\det B^{(0)}_{p_\pm}(x,t) = \det B^{(0)}_{\overline{p_\pm}}(x,t) = 0$.

Other exceptional cases can be treated similarly. Higher-order expansions for $M(x,t,k)$ about the branch points can be found by placing further restrictions on the potential.

Note that the asymmetry between the growth conditions at p_- and $\overline{p_-}$ on one hand and those at p_+ and $\overline{p_+}$ on the other is a result of the choice to rescale $\phi_-(x,t,k)$ using the analytic scattering coefficients from the right to define $M(x,t,k)$. If one took $\widehat{M}(x,t,k)$ as defined in (98) instead, the growth conditions would match exactly those for $M(x,t,k)$ but with p_+ and p_- interchanged.

4.3 | RHP, linear algebraic-integral equations, and reconstruction formula

Together, the results of Sections 4.1 and 4.2 describe the properties of the matrix $M(x,t,k)$ through its definition (93) in terms of the Jost eigenfunctions. Specifically, in the generic case in which the analytic Jost solutions are linearly independent at the branch points, we have:

Definition 1 (RHP)**.** Determine a matrix $M(x,t,k)$ satisfying the following conditions:

(i) $M(x,t,k)$ is analytic for $k \in \mathbb{C} \setminus (\Sigma \cup K \cup \overline{K})$, with $K \subset \mathbb{C}^+ \setminus \Sigma$ finite,
(ii) $M(x,t,k)$ satisfies the jump condition (95), with $J(x,t,k)$ given by (97),
(iii) $M(x,t,k) = \mathrm{I} + O(1/k),\ k \to \infty$,
(iv) $M(x,t,k)$ has simple poles at each $k_n \in K$ and $\overline{k_n} \in \overline{K}$ satisfying the residue conditions (100),
(v) $M(x,t,k)$ satisfies the growth conditions (102) at the branch points $p_\pm$ and $\overline{p_\pm}$ (ie, the limits exist).

Theorem 2. *For $V \neq 0$, if $(q - q_\pm) \in L_x^{1,1}(\mathbb{R}^\pm)$ and $(q - q_\pm)_x \in L_x^1(\mathbb{R}^\pm)$ for all $t \in \mathbb{R}$ with $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ linearly independent at both branch points $p_\pm$, then the matrix $M(x,t,k)$ defined by (93) satisfies RHP 1.*

We now invert the perspective and seek to recover $M(x,t,k)$ just from the five properties in Definition 1. That is, we show how the solution of the above RHP can be converted into that of a suitable set of linear algebraic-integral equations.

We first have:

Lemma 21. *If $M(x,t,k)$ is any solution of the RHP 1, then* $\det M(x,t,k) = 1$ *for all $k \in \mathbb{C} \setminus (\Sigma \cup K \cup \overline{K})$. Moreover, $M^\dagger(x,t,k)^{-1}$ also solves the RHP.*

The first part of Lemma 21 is easily proved by applying the determinant to the conditions in RHP 1 to arrive at a scalar RHP seeking an entire function which is $1 + O(1/k)$ as $k \to \infty$. The second part of Lemma 21 is verified through straightforward calculations.

For brevity, in what follows we suppress the dependence of M, J, and θ_o on x and t wherever this does not cause ambiguity. Again letting $M_1(k)$ and $M_2(k)$ denote the first and second columns of $M(k)$, respectively, we have the following:

Theorem 3. *If the RHP 1 admits a solution $M(k)$, it is given as a solution to the following system of linear algebraic-integral equations:*

$$M(k) = \mathrm{I} + \sum_{n=1}^{N} \left(\frac{-\overline{c_n} e^{-2i\theta_o(\overline{k_n})} M_2(\overline{k_n})}{k - \overline{k_n}}, \frac{c_n e^{2i\theta_o(k_n)} M_1(k_n)}{k - k_n} \right) + \frac{1}{2\pi i} \int_\Sigma \frac{M^-(\xi)(J(\xi) - I)}{\xi - k} d\xi, \quad k \in \mathbb{C} \setminus (\Sigma \cup K \cup \overline{K}), \tag{104a}$$

$$M_1(k_n) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{m=1}^{N} \frac{\overline{c_m} e^{-2i\theta_o(\overline{k_m})} M_2(\overline{k_m})}{k_n - \overline{k_m}} + \frac{1}{2\pi i} \int_\Sigma \frac{[M^-(\xi)(J(\xi) - \mathrm{I})]_1}{\xi - k_n} d\xi, \tag{104b}$$

$$M_2(\overline{k_n}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{m=1}^{N} \frac{c_m e^{2i\theta_o(k_m)} M_1(k_m)}{\overline{k_n} - k_m} + \frac{1}{2\pi i} \int_\Sigma \frac{[M^-(\xi)(J(\xi) - \mathrm{I})]_2}{\xi - \overline{k_n}} d\xi. \tag{104c}$$

The final step in the inverse problem is to reconstruct the solution of the NLS equation from that of the RHP. This is done without any appeal to the direct problem, but instead by using only those conditions on $M(x,t,k)$ imposed by the RHP 1.

Lemma 22. *Let $M(x,t,k)$ solve the RHP 1. Then $M(x,t,k)$ satisfies the modified Lax pair (94) with*

$$Q(x,t) := -i \lim_{k \to \infty} k[\sigma_3, M(x,t,k)]. \tag{105}$$

Corollary 22 (Reconstruction formula). *Let $M(x,t,k)$ solve the RHP 1. The corresponding solution of the NLS equation (1) is given by*

$$q(x,t) = -2i \sum_{n=1}^{n} c_n \mathrm{e}^{2i\theta_o(x,t,k_n)} M_{11}(x,t,k_n) - \frac{1}{\pi} \int_\Sigma [M^-(x,t,\xi)(J(x,t,\xi)-I)]_{12} d\xi. \tag{106}$$

4.4 | Existence and uniqueness of solutions of the RHP

The issue of the existence and uniqueness of a solution to the RHP 1 is nontrivial because of the singular behavior of $M(x,t,k)$ at the branch points (see Section 4.2). On the other hand, following recent work,[46] one can define an alternative matrix and RHP that is also regular at the branch points, as we show next.

To begin, we choose $R > 0$ large enough so that the ball B_R of radius R centered at the origin of the complex k-plane contains both branch cuts Σ_+, Σ_- and all zeros of the analytic scattering coefficients $s_{11}(k)$ and $s_{22}(k)$. Note that the large k behavior (61) of the scattering coefficients guarantees that such an R always exists. We then introduce a modified, sectionally analytic matrix,

$$\widetilde{M}(x,t,k) = \begin{cases} \Phi(x,t,k)\,\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}, & k \in \mathbb{C} \setminus ((-\infty,-R] \cup B_R \cup [R,\infty)), \\ \Psi(x,t,k)\,\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}, & k \in B_R, \end{cases} \tag{107}$$

with $\Phi(x,t,k)$ and $\Psi(x,t,k)$ given by (70) and (91), respectively. Corollary 20 and Lemma 17 immediately give the following:

Corollary 23. *Under the hypotheses of Theorem 1 and Lemma 15, the matrix $\widetilde{M}(x,t,k)$ defined in (107) satisfies the jump condition*

$$\widetilde{M}^+(x,t,k) = \widetilde{M}^-(x,t,k)\widetilde{J}(x,t,k), \quad k \in \widetilde{\Sigma}, \tag{108}$$

where the contour

$$\widetilde{\Sigma} = (-\infty,-R) \cup \partial B_R \cup (R,\infty) \tag{109}$$

is oriented as in Figure 3 and the jump matrix $\widetilde{J}(x,t,k)$ is given by

$$\widetilde{J}(x,t,k) = \mathrm{e}^{i\theta_o(x,t,k)\sigma_3} \widetilde{J}_o(k) \mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}, \tag{110}$$

with

$$\widetilde{J}_o(k) = \begin{cases} J_o(k), & k \in (-\infty,R) \cup (R,\infty), \\ C(k), & k \in \partial B_R \cap \mathbb{C}^+, \\ C(k)^{-1}, & k \in \partial B_R \cap \mathbb{C}^-, \end{cases} \tag{111}$$

where $J_o(k)$ is given in (97) and $C(k)$ is given by (91).

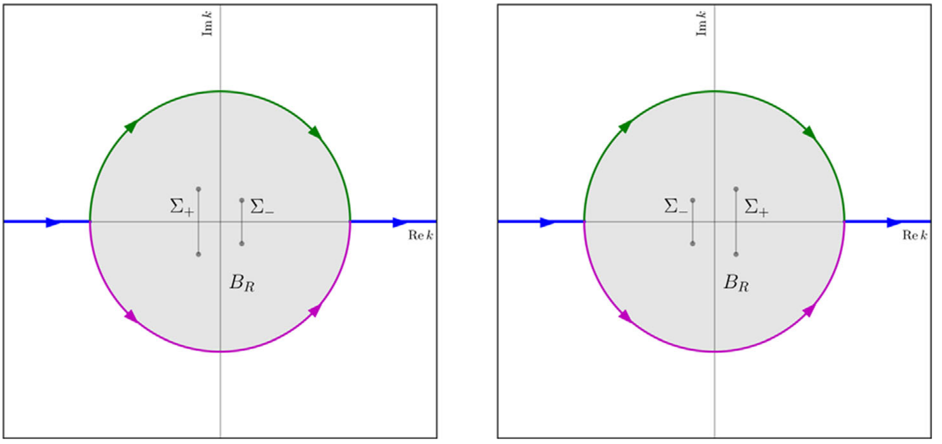

FIGURE 3 Orientation of the contour $\tilde{\Sigma} = (-\infty, -R) \cup \partial B_R \cup (R, \infty)$. The branch cuts $\Sigma_\pm$ are included for $V > 0$ (left) and $V < 0$ (right) with $A_- < A_+$. Note that $\widetilde{M}(x, t, k)$ is analytic on the branch cuts and along $(-R, R)$

Note that $\widetilde{M}(x, t, k)$ matches exactly $M(x, t, k)$ for large k, and as such has the same normalization at infinity. Importantly, not also that, since all the zeros of $s_{11}(k)$ and $s_{22}(k)$ are inside B_R where $\Psi(x, t, k)$ is analytic instead, $\widetilde{M}(x, t, k)$ is sectionally analytic (not just sectionally meromorphic like $M(x, t, k)$). This means that no residues conditions will be needed in the modified RHP. Moreover, $\widetilde{M}(x, t, k)$ is analytic on both branch cuts $\Sigma_\pm$, including at the branch points. The absence of jumps across the branch cuts is a departure from the formalism of Ref. 46, and results from the use of $\theta_o(x, t, k)$ instead of $\theta_\pm(x, t, k)$ to define $\widetilde{M}(x, t, k)$.

The definition (107) and Corollary 23 give rise to the following modified RHP:

Definition 2 (Modified RHP)**.** Determine a matrix $\widetilde{M}(x, t, k)$ satisfying the following conditions:

(i) $\widetilde{M}(x, t, k)$ is analytic for $k \in \mathbb{C} \setminus \tilde{\Sigma}$,
(ii) $\widetilde{M}(x, t, k)$ satisfies the jump condition (108), with $\tilde{J}(x, t, k)$ given by (111),
(iii) $\widetilde{M}(x, t, k) = \mathrm{I} + O(1/k)$, $k \to \infty$.

Theorem 4. *If $(q - q_\pm) \in L^1_x(\mathbb{R}^\pm)$ and $(q - q_\pm)_x \in L^1_x(\mathbb{R}^\pm)$ for all $t \in \mathbb{R}$, then the matrix $\widetilde{M}(x, t, k)$ defined by (107) satisfies the modified RHP 2.*

The important difference between the RHP 1 and the RHP 2 is that the latter has no singular behavior at the branch points. All the information about the behavior of $M(x, t, k)$ near the branch points, as well as all possible remnants of any discrete spectrum, are encoded into the jump matrix $\tilde{J}(x, t, k)$ along ∂B_R. This new RHP then falls under the framework developed by Zhou.[42] For convenience, we state Zhou's vanishing lemma explicitly:

Lemma 23. *(Zhou's vanishing lemma, theorem 9.3 in Ref. 42) Let $\tilde{\Sigma}$ be an oriented contour which is a finite union of simple smooth closed curves (possibly extending to infinity) with a finite number of self-intersections. Consider a generic RHP that consists of finding a matrix $\widetilde{M}(x, t, k)$ satisfying the following conditions:*

(i) $\widetilde{M}(x,t,k)$ *is analytic for* $k \in \mathbb{C} \setminus \widetilde{\Sigma}$,
(ii) $\widetilde{M}^{+}(x,t,k) = \widetilde{M}^{-}(x,t,k)\widetilde{J}(x,t,k),\ k \in \widetilde{\Sigma}$,
(iii) $\widetilde{M}(x,t,k) = \mathrm{I} + O(1/k),\ k \to \infty$.

Suppose the contour $\widetilde{\Sigma}$ *is Schwarz symmetric and if the jump matrix* $\widetilde{J}(x,t,k)$ *satisfies the following:*

(a) $\widetilde{J}(x,t,k)$ *is* $C^1(\widetilde{\Sigma})$,
(b) $\widetilde{J}(x,t,k) = \widetilde{J}^{\dagger}(x,t,k),\ k \in \widetilde{\Sigma} \setminus \mathbb{R}$, *where* $\dagger$ *denotes the Schwarz conjugate-transpose,*
(c) $\mathrm{Re}\,\widetilde{J}(x,t,k)$ *is positive definite for* $k \in \widetilde{\Sigma} \cap \mathbb{R}$.

Then the RHP admits a unique solution.

Lemma 23 does not apply to the original RHP 1 (since, for example, the contour is not closed), but it does apply to the modified RHP 2:

Theorem 5. *Suppose* $\rho(k) \in C^1(\mathbb{R} \setminus (-R,R))$ *and* $C(k) \in C^1(\partial B_R \cap \mathbb{C}^{\pm})$ *with* $C(k)C^{\dagger}(k) = \mathrm{I}$. *The modified RHP 2 admits a unique solution.*

Importantly, note that the above conditions on $\rho(k)$ and the matrix $C(k)$ are automatically satisfied when they are generated through the direct problem (see Section 3 for details).

Finally, we show how solutions of the modified RHP are related to those of the original RHP to establish the uniqueness of solutions of the original RHP 1. We do so by constructing an appropriate map between solutions of the two RHPs. Specifically, if $M_o(x,t,k)$ is any solution of the original RHP 1, let $C_o(k) = M_o(0,0,k)$. We know $C_o(k)$ has unit determinant by Lemma 21. With $C_o(k)$ fixed, we now define the following map F_o operating on matrix-valued functions $m(x,t,k)$:

$$F_o(m)(x,t,k) = \begin{cases} m(x,t,k), & k \in \mathbb{C} \setminus (\Sigma \cup B_R), \\ m(x,t,k)\,\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}\, C_o^{-1}(k)\,\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}, & k \in B_R \setminus \Sigma, \end{cases} \tag{112}$$

where the ball B_R of radius R is taken large enough to contain all singularities of the original RHP. We then show that, for *any* solution $M(x,t,k)$ of the original RHP, $\widetilde{M}(x,t,k) := F_o(M)(x,t,k)$ is indeed a solution of the modified RHP, with $C(k)$ replaced by $C_o(k)$. The map therefore allows one to establish the uniqueness of solutions to the original RHP 1:

Theorem 6. *If* $\rho(k) \in C^1(\mathbb{R} \setminus (-R,R))$ *and if there exists a solution* $C_o(k)$ *of the original RHP 1 at* $(x,t) = (0,0)$ *satisfying* $C_o(k)C_o^{\dagger}(k) = \mathrm{I}$, *then any solution of the original RHP is unique for all* $(x,t) \in \mathbb{R} \times \mathbb{R}^{+}$.

5 | REDUCTIONS: SYMMETRIC AMPLITUDES, ONE-SIDED BCs, AND ZERO VELOCITY

The general framework of the previous sections admits several distinguished reductions, as we discuss next.

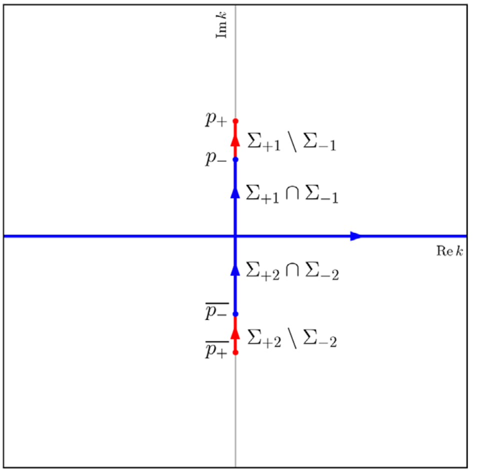

FIGURE 4 Continuous spectrum for $V = 0$ assuming $A_- < A_+$. The blue segments indicate where all four Jost solutions are defined generically, while the red segments indicate where only three of the solutions are defined generically

Symmetric amplitudes. The case of symmetric amplitudes is obtained when $A_- = A_+ = A$. This is a straightforward reduction of the general formalism of the previous sections, as all of the individual results go through without adjustment in this case. The only difference is simply that now the two branch cuts Σ_+ and Σ_- in the complex k-plane have the same height.

One-sided BCs. The general formalism also goes through in the case of one-sided BCs, namely, the case $A_- = 0$ (or, equivalently, $A_+ = 0$, due to the invariance of NLS under space reflection). Actually, due to the Galilean and phase invariance of NLS, in this case one can also take $V = 0$ and $\delta = 0$ without loss of generality. This particular reduction was studied in Refs. 34, 40. The only differences from the general formalism of the previous sections is that now $\lambda_- \equiv k$ for all $k \in \mathbb{C}$ and hence $\Sigma_- = \{0\}$.

5.1 | No counterpropagating flows

This reduction corresponds to the case of zero asymptotic velocity, ie, $V = 0$, and was studied in Refs. 29, 31 for equal amplitudes and in Ref. 33 for unequal amplitudes. The general approach presented in the previous sections can be successfully implemented in this case as well. However, special consideration is required because when $V = 0$ the segments Σ_+ and Σ_- are partially overlapping (cf. Figure 4) and, therefore, the domains of applicability of certain results change. In what follows, we take $A_- \leq A_+$ so that the overlapping portion of the branch cuts is given by Σ_- and the nonoverlapping portion is given by $\Sigma_+ \setminus \Sigma_-$. This is done without loss of generality thanks to the reflection symmetry of NLS, ie, the symmetry under the transformation $x \mapsto -x$. Specifically, Corollaries 5, 8, 10, and 11 are modified as follows:

Corollary 24 (Analog of Corollary 5). *Under the hypothesis of Theorem 1 and for $V = 0$, the scattering coefficients admit the following Wronskian representations:*

$$s_{11}(k) = \mathrm{Wr}[\phi_{-1}(x,t,k), \phi_{+2}(x,t,k)], \quad k \in \mathbb{R} \cup \mathbb{C}^- \cup \Sigma_{-1}^o \setminus \{\overline{p_+}, \overline{p_-}\}, \tag{113a}$$

$$s_{12}(k) = \mathrm{Wr}[\phi_{-2}(x,t,k), \phi_{+2}(x,t,k)], \quad k \in \mathbb{R} \cup \Sigma_{+1}^o \cup \Sigma_{-2}^o, \tag{113b}$$

$$s_{21}(k) = \mathrm{Wr}[\phi_{+1}(x,t,k), \phi_{-1}(x,t,k)], \quad k \in \mathbb{R} \cup \Sigma_{-1}^o \cup \Sigma_{+2}^o, \tag{113c}$$

$$s_{22}(k) = \mathrm{Wr}[\phi_{+1}(x,t,k), \phi_{-2}(x,t,k)], \quad k \in \mathbb{R} \cup \mathbb{C}^+ \cup \Sigma_{-2}^o \setminus \{p_+, p_-\}. \tag{113d}$$

Moreover, $s_{22}(k)$ and $s_{11}(k)$ are analytic in $\mathbb{C}^\pm \setminus \Sigma$, respectively.

Corollary 25 (Analog of Corollary 8). *Under the hypothesis of Theorem 1 and for $V = 0$, the left scattering coefficients can be extended through the Wrosnkian representations,*

$$r_{11}(k) = \mathrm{Wr}[\phi_{+1}(x,t,k), \phi_{-2}(x,t,k)], \quad k \in \mathbb{R} \cup \mathbb{C}^+ \cup \Sigma_{-2}^o \setminus \{p_+, p_-\}, \tag{114a}$$

$$r_{12}(k) = \mathrm{Wr}[\phi_{+2}(x,t,k), \phi_{-2}(x,t,k)], \quad k \in \mathbb{R} \cup \Sigma_{+1}^o \cup \Sigma_{-2}^o, \tag{114b}$$

$$r_{21}(k) = \mathrm{Wr}[\phi_{-1}(x,t,k), \phi_{+1}(x,t,k)], \quad k \in \mathbb{R} \cup \Sigma_{-1}^o \cup \Sigma_{+2}^o, \tag{114c}$$

$$r_{22}(k) = \mathrm{Wr}[\phi_{-1}(x,t,k), \phi_{+2}(x,t,k)], \quad k \in \mathbb{R} \cup \mathbb{C}^- \cup \Sigma_{-1}^o \setminus \{\overline{p_+}, \overline{p_-}\}. \tag{114d}$$

Moreover, $r_{11}(k)$ and $r_{22}(k)$ are analytic in $\mathbb{C}^\pm \setminus \Sigma$, respectively.

Corollary 26 (Analog of Corollary 10). *Under the hypothesis of Theorem 1 and for $V = 0$, (59a), (59b), and (63) can be extended to $k \in \Sigma_{-1}^o \cup \Sigma_{+2}^o$, $\Sigma_{+1}^o \cup \Sigma_{-2}^o$, and Σ_-^o, respectively. That is:*

$$\phi_{-1}(x,t,k) = s_{11}(k)\phi_{+1}(x,t,k) + s_{21}(k)\phi_{+2}(x,t,k), \quad k \in \mathbb{R} \cup \Sigma_{-1}^o \cup \Sigma_{+2}^o, \tag{115a}$$

$$\phi_{-2}(x,t,k) = s_{12}(k)\phi_{+1}(x,t,k) + s_{22}(k)\phi_{+2}(x,t,k), \quad k \in \mathbb{R} \cup \Sigma_{+1}^o \cup \Sigma_{-2}^o, \tag{115b}$$

$$\phi_{+1}(x,t,k) = r_{11}(k)\phi_{-1}(x,t,k) + r_{21}(k)\phi_{-2}(x,t,k), \quad k \in \mathbb{R} \cup \Sigma_-^o, \tag{115c}$$

$$\phi_{+2}(x,t,k) = r_{12}(k)\phi_{-1}(x,t,k) + r_{22}(k)\phi_{-2}(x,t,k), \quad k \in \mathbb{R} \cup \Sigma_-^o. \tag{115d}$$

Corollary 27 (Analog of Corollary 11). *Under the hypothesis of Theorem 1 and for $V = 0$,*

$$\phi_{+1}(x,t,k) \in L_x^\infty(\mathbb{R}), \quad k \in \mathbb{R} \cup \Sigma_-^o, \tag{116a}$$

$$\phi_{+2}(x,t,k) \in L_x^\infty(\mathbb{R}), \quad k \in \mathbb{R} \cup \Sigma_-^o, \tag{116b}$$

$$\phi_{-1}(x,t,k) \in L_x^\infty(\mathbb{R}), \quad k \in \mathbb{R} \cup \Sigma_{-1}^o \cup \Sigma_{+2}^o, \tag{116c}$$

$$\phi_{-2}(x,t,k) \in L_x^\infty(\mathbb{R}), \quad k \in \mathbb{R} \cup \Sigma_{+1}^o \cup \Sigma_{-2}^o. \tag{116d}$$

Recall that the condition $(q - q_\pm) \in L_x^1(\mathbb{R})$ assumed in Theorem 1 does not allow one to generically obtain solutions to the Lax pair at $k = \pm iA_-$ bounded as $x \to -\infty$. Consequently, these points cannot generically be included as part of the continuous spectrum. On the other hand, Lemma 5 allows one to define $\mu_-(x,t,k)$ at the branch points under more strict conditions on the potential. In such cases, the columns of $\mathrm{e}^{-if_-(x,t)\sigma_3}\mu_-(x,t,k)\mathrm{e}^{i\theta_-(x,t,k)\sigma_3}$ (although linearly independent) are solutions to the Lax pair at $k = \pm iA_-$ bounded as $x \to -\infty$. Since $\phi_+(x,t,\pm iA_-)$ remains a fundamental matrix solution bounded as $x \to \infty$ when $A_- < A_+$, relations analogous to (115) show that $\mathrm{e}^{-if_-(x,t)\sigma_3}\mu_-(x,t,k)\mathrm{e}^{i\theta_-(x,t,k)\sigma_3}$ is a solution to the Lax pair bounded for all $x \in \mathbb{R}$, and the branch points $\pm iA_-$ can be included in the continuous spectrum.

Next, Corollary 12, Lemmas 9, 11, and 13 are modified as follows:

Corollary 28 (Analog of Corollary 12). *Under the hypothesis of Theorem 1 and for $V = 0$, the continuous spectrum is given by $\Sigma^o \setminus \{\pm iA_-\}$. If, in addition, $(q - q_-) \in L_x^{1,1}(-\infty, a)$ for some $a \in \mathbb{R}$ and $A_- < A_+$, then the continuous spectrum is given by Σ^o.*

Lemma 24 (Analog of Lemma 9). *Under the hypothesis of Theorem 1 and for $V = 0$, we have the symmetries*

$$\overline{s_{22}(\bar{k})} = s_{11}(k), \quad k \in \mathbb{R} \cup \mathbb{C}^- \cup \Sigma_{-1}^o \setminus \{\overline{p_+}, \overline{p_-}\}, \tag{117a}$$

$$\overline{s_{12}(\bar{k})} = -s_{21}(k), \quad k \in \mathbb{R} \cup \Sigma_{+2}^o \cup \Sigma_{-1}^o, \tag{117b}$$

$$\overline{r_{11}(\bar{k})} = r_{22}(k), \quad k \in \mathbb{R} \cup \mathbb{C}^- \cup \Sigma_{-1}^o \setminus \{\overline{p_+}, \overline{p_-}\}, \tag{117c}$$

$$\overline{r_{21}(\bar{k})} = -r_{12}(k), \quad k \in \mathbb{R} \cup \Sigma_{+2}^o \cup \Sigma_{-1}^o. \tag{117d}$$

Lemma 25 (Analog of Lemma 11). *Under the hypothesis of Theorem 1 and for $V = 0$,*

$$s_{22}^+(k) = i\mathrm{e}^{-i\delta}s_{12}(k), \qquad r_{11}^+(k) = -i\mathrm{e}^{-i\delta}r_{12}(k), \qquad k \in \Sigma_{+1}^o \setminus \Sigma_{-1}, \tag{118a}$$

$$s_{22}^{+}(k) = e^{-2i\delta}s_{11}(k), \qquad r_{11}^{+}(k) = \mathrm{e}^{-2i\delta}r_{22}(k), \qquad k \in \Sigma_{+1}^{o} \cap \Sigma_{-1}^{o}, \tag{118b}$$

$$s_{11}^{+}(k) = i\mathrm{e}^{+i\delta}s_{21}(k), \qquad r_{22}^{+}(k) = -i\mathrm{e}^{+i\delta}r_{21}(k), \qquad k \in \Sigma_{+2}^{o} \setminus \Sigma_{-2}, \tag{118c}$$

$$s_{11}^{+}(k) = e^{+2i\delta}s_{22}(k), \qquad r_{22}^{+}(k) = \mathrm{e}^{+2i\delta}r_{11}(k), \qquad k \in \Sigma_{+2}^{o} \cap \Sigma_{-2}^{o}. \tag{118d}$$

Lemma 26 (Analog of Lemma 13)**.** *Under the hypothesis of Theorem 1 and for $V = 0$,*

$$s_{11}(k) \neq 0, \quad r_{22}(k) \neq 0, \quad k \in \Sigma_{+2}^{o} \setminus \Sigma_{-2}, \tag{119a}$$

$$s_{12}(k) \neq 0, \quad r_{12}(k) \neq 0, \quad k \in \Sigma_{+1}^{o} \setminus \Sigma_{-1}, \tag{119b}$$

$$s_{21}(k) \neq 0, \quad r_{21}(k) \neq 0, \quad k \in \Sigma_{+2}^{o} \setminus \Sigma_{-2}, \tag{119c}$$

$$s_{22}(k) \neq 0, \quad r_{11}(k) \neq 0, \quad k \in \Sigma_{+1}^{o} \setminus \Sigma_{-1}. \tag{119d}$$

Note, however, that no statement can be made about the possibility of zeros of the scattering coefficients on $\Sigma_+ \cap \Sigma_-$. The above results can be proved exactly as their counterparts for $V \neq 0$.

If $A_- \neq A_+$, then the behavior of the scattering coefficients follows exactly as in the case of $V \neq 0$ treated in Section 3.5. On the other hand, if $A_- = A_+$ then the branch points come together and Corollaries 16, 17, 18, and 19 must be adjusted accordingly.

Corollary 29 (Analog of Corollary 16 when $A_- = A_+$)**.** *Under the hypothesis of Lemma 5 and for $V = 0$ and $A_+ = A_- = A$,*

$$s_{11}(k) = b_{\pm 11}^{(0)}(k \mp iA)^{-1/2} + o(k \mp iA)^{-1/2}, \quad k \to \pm iA, \tag{120a}$$

$$s_{12}(k) = b_{\pm 12}^{(0)}(k \mp iA)^{-1/2} + o(k \mp iA)^{-1/2}, \quad k \to \pm iA, \tag{120b}$$

$$s_{21}(k) = b_{\pm 21}^{(0)}(k \mp iA)^{-1/2} + o(k \mp iA)^{-1/2}, \quad k \to \pm iA, \tag{120c}$$

$$s_{22}(k) = b_{\pm 22}^{(0)}(k \mp iA)^{-1/2} + o(k \mp iA)^{-1/2}, \quad k \to \pm iA \tag{120d}$$

for some constants $b_{\pm 11}^{(0)}$, $b_{\pm 12}^{(0)}$, $b_{\pm 21}^{(0)}$, $b_{\pm 22}^{(0)}$, where each limit must be taken from within the regions described by Corollary 24.

Corollary 30 (Analog of Corollary 17 when $A_- = A_+$)**.** *Under the hypothesis of Lemma 5 and for $V = 0$ and $A_- = A_+ = A$, in the generic case that $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ are linearly independent at the branch point iA, we have*

$$\rho(k) = \rho_o + o(1), \quad k \to iA, \tag{121a}$$

$$r(k) = r_o(k - iA) + o(k - iA), \quad k \to iA \tag{121b}$$

for some nonzero constants ρ_o, r_o where the limits must be taken from within the regions described by (68).

Corollary 31 (Analog of Corollary 18 when $A_- = A_+$)**.** *Under the hypothesis of Lemma 6 and for $V = 0$ and $A_- = A_+ = A$,*

$$s_{11}(k) = b^{(0)}_{\pm 11}(k \mp iA)^{-1/2} + b^{(1)}_{\pm 11} + o(1), \quad k \to \pm iA, \tag{122a}$$

$$s_{12}(k) = b^{(0)}_{\pm 12}(k \mp iA)^{-1/2} + b^{(1)}_{\pm 12} + o(1), \quad k \to \pm iA, \tag{122b}$$

$$s_{21}(k) = b^{(0)}_{\pm 21}(k \mp iA)^{-1/2} + b^{(1)}_{\pm 21} + o(1), \quad k \to \pm iA, \tag{122c}$$

$$s_{22}(k) = b^{(0)}_{\pm 22}(k \mp iA)^{-1/2} + b^{(1)}_{\pm 22} + o(1), \quad k \to \pm iA \tag{122d}$$

for some constants $b^{(0)}_{\pm 11}, b^{(0)}_{\pm 12}, b^{(0)}_{\pm 21}, b^{(0)}_{\pm 22}, b^{(1)}_{\pm 11}, b^{(1)}_{\pm 12}, b^{(1)}_{\pm 21}, b^{(1)}_{\pm 22}$, where each limit must be taken from within the regions described by Corollary 24.

Corollary 32 (Analog of Corollary 19 when $A_- = A_+$)**.** *Under the hypothesis of Lemma 5 and for $V = 0$ and $A_- = A_+ = A$, in the exceptional case that $\mu_{+1}(x, t, k)$ and $\mu_{-2}(x, t, k)$ are linearly dependent at the branch point iA with $b^{(1)}_{\pm 11}, b^{(1)}_{\pm 12}, b^{(1)}_{\pm 21}, b^{(1)}_{\pm 22}$ all nonzero, we have*

$$\rho(k) = \rho_o + o(1), \quad k \to iA, \tag{123a}$$

$$r(k) = r_o + o(1), \quad k \to iA \tag{123b}$$

for some nonzero constants ρ_o, r_o, where the limits must be taken from within the regions described by (68).

Lemma 17 is also adjusted as follows:

Lemma 27 (Analog of Lemma 17)**.** *Under the hypothesis of Theorem 1 and for $V = 0$, the matrix $M(x, t, k)$ defined by (93) satisfies the jump condition*

$$M^+(x, t, k) = M^-(x, t, k) J(x, t, k), \quad k \in \Sigma^o, \tag{124}$$

where $J(x,t,k)$ is still given by (96) but now with

$$J_o(k) = \begin{cases} \begin{pmatrix} 1 & \rho \\ \overline{\rho} & 1+\rho(k)\overline{\rho(\overline{k})} \end{pmatrix}, & k \in \mathbb{R}, \\ \begin{pmatrix} i\rho(k)\mathrm{e}^{-i\delta} & 0 \\ -i\mathrm{e}^{-i\delta} & -i\mathrm{e}^{i\delta}/\rho(k) \end{pmatrix}, & k \in \Sigma_{+1}^{o} \setminus \Sigma_{-1}, \\ \begin{pmatrix} i\mathrm{e}^{-i\delta}/\overline{\rho(\overline{k})} & -i\mathrm{e}^{i\delta} \\ 0 & -i\overline{\rho(\overline{k})}\mathrm{e}^{i\delta} \end{pmatrix}, & k \in \Sigma_{+2}^{o} \setminus \Sigma_{-2}, \\ \begin{pmatrix} i\rho(k)\mathrm{e}^{-i\delta} & -i(1+\rho(k)\overline{\rho(\overline{k})})\mathrm{e}^{i\delta} \\ -i\mathrm{e}^{-i\delta} & i\overline{\rho(\overline{k})}\mathrm{e}^{i\delta} \end{pmatrix}, & k \in \Sigma_{+1}^{o} \cap \Sigma_{-1}^{o}, \\ \begin{pmatrix} -i\rho(k)\mathrm{e}^{-i\delta} & -i\mathrm{e}^{i\delta} \\ -i(1+\rho(k)\overline{\rho(\overline{k})})\mathrm{e}^{-i\delta} & -i\overline{\rho(\overline{k})}\mathrm{e}^{i\delta} \end{pmatrix}, & k \in \Sigma_{+2}^{o} \cap \Sigma_{-2}^{o}. \end{cases} \tag{125}$$

The calculation of the jump matrix $J_o(k)$ in Lemma 27 is carried out in Section 7.2. As a consequence of Lemma 26, the entries of the jump matrix are nonsingular $\Sigma_+^o \setminus \Sigma_-^o$, potentially with unbounded growth toward the branch points. Corollaries 17 and 19 describe some of the possible behaviors of the jump matrix at these potential singularities when $A_- \neq A_+$, while Corollaries 30 and 32 treat the analogous scenarios for $A_- = A_+$. In particular, in the generic case we see that $J_o(k)$ is continuous at the branch points. Also notice that the jumps for $V = 0$ on $\mathbb{R}$, $\Sigma_{+1}^o \setminus \Sigma_{-1}$ and $\Sigma_{+2}^o \setminus \Sigma_{-2}$ as given in (125), respectively, match the jumps for $V \neq 0$ on $\mathbb{R}$, Σ_{+1}^o and Σ_{+2}^o as given in (97).

We have broken the $x \mapsto -x$ symmetry twice to arrive at the jump matrix $J_o(k)$: once in the definition of $M(x,t,k)$ by (93), which rescaled the columns of $\phi_-(x,t,k)$ using the analytic scattering coefficients from the right, $s_{11}(k)$ and $s_{22}(k)$, and a second time by taking $A_- \leq A_+$. If instead one takes $A_- \geq A_+$, then the overlapping portion of the branch cuts is given by Σ_+ while the nonoverlapping portion is given by $\Sigma_- \setminus \Sigma_+$. In that case, the jumps for $V = 0$ on $\mathbb{R}$, $\Sigma_{-1}^o \setminus \Sigma_{+1}$ and $\Sigma_{-2}^o \setminus \Sigma_{+2}$, respectively, match the jumps for $V = 0$ on $\mathbb{R}$, Σ_{-1}^o and Σ_{-2}^o in (97), while the jumps on the overlap $\Sigma_- \cap \Sigma_+$ match the corresponding jumps in (125). For $V = 0$, we make the choice $A_- \leq A_+$ so that all jumps of $M(x,t,k)$ as defined in (93) can be expressed in terms of $\rho(k)$.

If $A_- \neq A_+$, then the growth conditions follow exactly as in the case of $V \neq 0$ treated in Section 4.2 (cf. Lemmas 19 and 20). On the other hand, if $A_- = A_+$ then, as remarked above, the branch points come together and Lemmas 19 and 20 must also be adjusted accordingly:

Lemma 28 (Analog of Lemma 19 when $A_- = A_+$). *Under the hypothesis of Lemma 5 and for $V = 0$ and $A_- = A_+ = A$, in the generic case that $\mu_{+1}(x,t,k)$, $\mu_{-2}(x,t,k)$ are linearly independent at the branch point iA, we have*

$$M(x,t,k) = \left[B_{\pm iA}^{(0)}(x,t) + o(1)\right](k \mp iA)^{\mp\sigma_3/4}, \quad k \to \pm iA \tag{126}$$

for some invertible matrices $B_{\pm iA}^{(0)}(x,t)$.

Lemma 29 (Analog of Lemma 20 when $A_- = A_+$). *Under the hypothesis of Lemma 6 and for $V = 0$ and $A_- = A_+ = A$, in the exceptional case that $\mu_{+1}(x,t,k)$, $\mu_{-2}(x,t,k)$ are linearly dependent at $k = iA$ with $b^{(1)}_{-11}$ and $b^{(1)}_{+22}$ (as given by Corollary 31) nonzero, we have*

$$M(x,t,k) = \left[B^{(0)}_{\pm iA}(x,t) + B^{(1)}_{\pm iA}(x,t)(k \mp iA)^{1/2} + o(k \mp iA)^{1/2}\right](k \mp iA)^{-1/4}, \quad k \to \pm iA \quad (127)$$

for some matrices $B^{(0)}_{\pm iA}(x,t)$, $B^{(1)}_{\pm iA}(x,t)$ with $\det B^{(0)}_{\pm iA}(x,t) = 0$.

As a result of Lemma 28, we see that, in the generic case, the following limits exist:

$$\lim_{k \to \pm iA} M(x,t,k)(k \mp iA)^{\pm\sigma_3/4}. \quad (128)$$

Finally, Theorem 2 is adjusted as follows:

Theorem 7 (Analog of Theorem 2). *For $V = 0$ and $A_- < A_+$, if $(q - q_\pm) \in L^{1,1}_x(\mathbb{R})$ and $(q - q_\pm)_x \in L^1_x(\mathbb{R})$ for all $t \in \mathbb{R}$ with $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ linearly independent at both branch points $p_\pm$, then $M(x,t,k)$ as given by (93) satisfies the modified RHP 1 with $J(x,t,k)$ given by (125).*

Theorem 8 (Analog of Theorem 2). *For $V = 0$ and $A_- = A_+ = A$, if $(q - q_\pm) \in L^{1,1}_x(\mathbb{R})$ and $(q - q_\pm)_x \in L^1_x(\mathbb{R})$ for all $t \in \mathbb{R}$ with $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ linearly independent at the branch point iA, then $M(x,t,k)$ as given by (93) satisfies the modified RHP 1 with $J(x,t,k)$ given by (125) and growth conditions given by (128).*

The formal expressions for the system of algebraic-integral equations in Theorem 3 and reconstruction formula in Corollary 22 are unchanged for $V = 0$.

Also, the work in Section 4.4 is entirely unchanged.

6 | RIEMANN PROBLEMS

Recall that Riemann problems are IVPs with step-like ICs.[49–51] We now compute explicitly the scattering data and growth conditions for the various Riemann problems described by the general framework of this paper.

Proofs for all the results in this section are given in Section 7.4.

6.1 | Riemann problem for a pure two-sided step with counterpropagating flows

Consider the IC

$$q(x,0) = \begin{cases} A_+ e^{-iVx+i\delta}, & x \geq 0, \\ A_- e^{+iVx-i\delta}, & x < 0, \end{cases} \qquad 0 < A_- \leq A_+. \quad (129)$$

The special case $A_- = A_+$, $\delta = 0$, and $V > 0$ for this problem was considered in Ref. 38. Note, however, that the normalization for the Jost solutions and the sectionally meromorphic matrices is different in this work.

Explicitly, at $t = 0$ we have

$$\phi_+(x,0,k) = \frac{1}{d_+(k)} \begin{cases} \mathrm{e}^{-iVx\sigma_3/2} E_+(k) \mathrm{e}^{i\lambda_+ x\sigma_3}, & x \geq 0, \\ \mathrm{e}^{+iVx\sigma_3/2} E_-(k) \mathrm{e}^{i\lambda_- x\sigma_3} E_-^{-1}(k) E_+(k), & x < 0, \end{cases} \tag{130a}$$

and

$$\phi_-(x,0,k) = \frac{1}{d_-(k)} \begin{cases} \mathrm{e}^{-iVx\sigma_3/2} E_+(k) \mathrm{e}^{i\lambda_+ x\sigma_3} E_+^{-1}(k) E_-(k), & x \geq 0, \\ \mathrm{e}^{+iVx\sigma_3/2} E_-(k) \mathrm{e}^{i\lambda_- x\sigma_3}, & x < 0. \end{cases} \tag{130b}$$

Then, $\phi_-(x,0,k) = \phi_+(x,0,k) S(k)$ with $S(k) = (d_+(k)/d_-(k)) E_+^{-1}(k) E_-(k)$.

Hence, we find

$$\begin{aligned} s_{22}(k) &= \frac{1}{d_+(k) d_-(k) (\lambda_- + (k - V/2)) A_+ \mathrm{e}^{i\delta}} \left[(\lambda_+ - (k + V/2)) A_- \mathrm{e}^{-i\delta} + (\lambda_- + (k - V/2)) A_+ \mathrm{e}^{i\delta} \right], \\ \rho(k) &= \frac{i A_+ \mathrm{e}^{i\delta}}{\lambda_+ + (k + V/2)} \frac{(\lambda_+ + (k + V/2)) A_- \mathrm{e}^{-i\delta} - (\lambda_- + (k - V/2)) A_+ \mathrm{e}^{i\delta}}{(\lambda_+ - (k + V/2)) A_- \mathrm{e}^{-i\delta} + (\lambda_- + (k - V/2)) A_+ \mathrm{e}^{i\delta}}, \\ r(k) &= \frac{4i\lambda_+ \lambda_- \mathrm{e}^{i\delta}}{[(\lambda_+ + (k + V/2)) \mathrm{e}^{2i\delta} - (\lambda_+ - (k + V/2)) \mathrm{e}^{-2i\delta}] A_- - 2(k - V/2) A_+}. \end{aligned} \tag{131}$$

Lemma 30. *For the pure two-sided step IC with $V \neq 0$ and $0 < A_- \leq A_+$, there are no discrete eigenvalues. Furthermore, if $\delta = 0$ then*

(i) If $A_+ = A_-$ then $\rho(k)$ has no zeros and $r(k)$ has no poles.
(ii) If $A_+ \neq A_-$ then $\rho(k)$ has a zero and $r(k)$ has a pole only at

$$k = \frac{V}{2} \left(\frac{A_+ + A_-}{A_+ - A_-} \right). \tag{132}$$

In Section 7.4, we show that the modified eigenfunctions $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ are linearly independent at the branch points, so that the growth conditions are given by Lemma 19. The matrix $M(x,t,k)$ as defined in (93) then satisfies the RHP 1 with $K = \emptyset$ and with $\rho(k)$ and $r(k)$ given by (131).

6.2 | Riemann problem for a pure two-sided step without counterpropagating flows

Consider now the IC

$$q(x,0) = \begin{cases} A_+ \mathrm{e}^{+i\delta}, & x \geq 0, \\ A_- \mathrm{e}^{-i\delta}, & x < 0, \end{cases} \qquad 0 < A_- \leq A_+, \tag{133}$$

which corresponds to (129) with $V = 0$. Much of the work from the preceding section is valid with $V = 0$. In particular, the explicit Jost solutions $\phi_\pm(x,t,k)$ at $t = 0$ and the scattering coefficients are given again by expressions (130) and (131), now with $V = 0$. Specifically, the reflection coefficient $\rho(k)$ is given by

$$\rho(k) = \frac{iA_+ e^{i\delta}}{\lambda_+ + k} \frac{(\lambda_+ + k)A_- e^{-i\delta} - (\lambda_- + k)A_+ e^{i\delta}}{(\lambda_+ - k)A_- e^{-i\delta} + (\lambda_- + k)A_+ e^{i\delta}}. \tag{134}$$

Lemma 31. *For the pure two-sided step IC with $0 < A_- \le A_+$, $\delta = 0$ and $V = 0$, there are no discrete eigenvalues. Furthermore,*

(i) If $A_+ = A_-$ then $\rho(k) \equiv 0$.
(ii) If $A_+ \neq A_-$ then $\rho(k)$ has a zero only at $k = 0$.

In Section 7.4, we show that for $A_+ \neq A_-$ and $V = 0$ the modified eigenfunctions $\mu_{+1}(x,t,k)$ and $\mu_{-2}(x,t,k)$ are linearly independent at the branch points, so that the growth conditions are given by Lemma 19. Then, the matrix $M(x,t,k)$ defined by (93) satisfies a modified RHP 1 with $K = \emptyset$, $J(x,t,k)$ given by (125), and $\rho(k)$ and $r(k)$ given by (131).

On the other hand, if $A_+ = A_- = A$, $V = 0$ and $\delta = 0$, then the modified eigenfunctions are linearly dependent at the branch points and $s_{11}(k) \equiv s_{22}(k) \equiv 1$, so that the growth conditions are instead given by (127).

6.3 | Riemann problem for a pure one-sided step

Finally, consider the IC

$$q(x,0) = \begin{cases} A e^{-iVx + i\delta}, & x \ge 0, \\ 0, & x < 0, \end{cases} \qquad A > 0. \tag{135}$$

To fit the previous framework, we take $A_+ = A$ and $A_- = 0$. Due to the Galilean and phase invariances of the NLS, here we can actually assume $V = \delta = 0$. Thus, much of the work for the two-sided Riemann problem (133) can be reused after setting $A_+ = A$, $A_- = 0$, and $\delta = 0$. Note that with $A_- = 0$ and $V = 0$ we have $\lambda_- = k$. Explicitly, at $t = 0$ we have

$$\phi_+(x,0,k) = \frac{1}{d(k)} \begin{cases} E(k) e^{i\lambda x \sigma_3}, & x \ge 0, \\ e^{ikx\sigma_3} E(k), & x < 0, \end{cases} \tag{136a}$$

and

$$\phi_-(x,0,k) = \begin{cases} E(k) e^{i\lambda x \sigma_3} E^{-1}(k), & x \ge 0, \\ e^{ikx\sigma_3}, & x < 0, \end{cases} \tag{136b}$$

where $E(k) = E_+(k)$, $d(k) = d_+(k)$ with $V = \delta = 0$. Then $\phi_-(x, 0, k) = \phi_+(x, 0, k)S(k)$ with $S(k) = d(k)E^{-1}(k)$. We then find

$$s_{22}(k) = \frac{1}{d(k)}, \qquad \rho(k) = -\frac{iA}{\lambda + k}. \tag{137}$$

We see that there are no discrete eigenvalues, $\rho(k)$ has no singularities, and $\rho(k) \neq 0$ for any $k \in \mathbb{C}$. Moreover, we easily see that the growth conditions are given by Lemma 19 (ignoring those on p_- and $\overline{p_-}$). Then, $M(x, t, k)$ satisfies a modified version of RHP 1 with $K = \emptyset$ and $J(x, t, k)$ given by

$$J(x,t,k) = \begin{cases} \begin{pmatrix} 1 & \rho(k)\mathrm{e}^{2i\theta_o(x,t,k)} \\ -\rho(k)\mathrm{e}^{-2i\theta_o(x,t,k)} & -\frac{2\lambda}{iA}\rho \end{pmatrix}, & k \in \mathbb{R}, \\ \begin{pmatrix} i\rho(k) & 0 \\ -i\mathrm{e}^{-2i\theta_o(x,t,k)} & -i/\rho(k) \end{pmatrix}, & k \in \Sigma^o_{+1}, \\ \begin{pmatrix} -i/\rho(k) & -i\mathrm{e}^{2i\theta_o(x,t,k)} \\ 0 & i\rho(k) \end{pmatrix}, & k \in \Sigma^o_{+2}, \end{cases} \tag{138}$$

with $\rho(k)$ given by (137).

7 | Proofs

In this section, we include proofs and calculations for the various theorems, lemmas, and corollaries stated in the previous sections.

7.1 | Direct problem

Jost solutions for the exact potentials $q_\pm(x,t)$. We begin by obtaining solutions $\widetilde{\psi}_\pm(x, t, k)$ to the first part of the Lax pair (3), Writing

$$X_\pm(x, t, k) = \mathrm{e}^{-if_\pm(x,t)\sigma_3}\widehat{X}_\pm(k)\mathrm{e}^{if_\pm(x,t)\sigma_3}, \tag{139a}$$

with

$$\widehat{X}_\pm(k) = ik\sigma_3 + A_\pm\sigma_3\mathrm{e}^{\pm i\delta}\sigma_1, \tag{139b}$$

we can write the first part of the Lax pair equivalently as

$$(\mathrm{e}^{if_\pm(x,t)\sigma_3}\widetilde{\psi}_\pm(x, t, k))_x = \widehat{X}_\pm(k \pm V/2)\mathrm{e}^{if_\pm(x,t)\sigma_3}\widetilde{\psi}_\pm(x, t, k). \tag{140}$$

Now $\widehat{X}_\pm(k \pm V/2)$ has eigenvector and eigenvalue matrices $E_\pm(k)$ and $i\lambda(k;A_\pm)\sigma_3$, with $E_\pm(k)$ and $\lambda(k;A)$ as defined in (12b) and (19), respectively. Then

$$\widetilde{\psi}_\pm(x,t,k) = \mathrm{e}^{-if_\pm(x,t)\sigma_3}E_\pm(k)\mathrm{e}^{i\lambda_\pm(k)x\sigma_3} \tag{141}$$

is a fundamental matrix solution to the first part of the Lax pair (3). We now seek simultaneous solutions $\widetilde{\phi}_\pm(x,t,k)$ of both parts of (3). To this end, note that, since $\widetilde{\phi}_\pm(x,t,k)$ and $\widetilde{\psi}_\pm(x,t,k)$ are both solutions to the first part of (3), we have

$$\widetilde{\phi}_\pm(x,t,k) = \widetilde{\psi}_\pm(x,t,k)B_\pm(t,k) \tag{142}$$

for some matrix $B_\pm(t,k)$. Differentiating with respect to t, we find

$$\frac{dB_\pm(t,k)}{dt} = \widetilde{\psi}_\pm^{-1}(T_\pm\widetilde{\psi}_\pm - (\widetilde{\psi}_\pm)_t)B_\pm(t,k) = -2i\lambda_\pm(k)(k \mp V/2)\sigma_3 B_\pm(t,k),$$

(where we suppressed the dependence of $\widetilde{\psi}$ and $T_\pm$ for brevity), so that

$$B_\pm(t,k) = \mathrm{e}^{-2i\lambda_\pm(k)(k\mp V/2)t\sigma_3}B_\pm(0,k). \tag{143}$$

Taking $B_\pm(0,k) = \mathrm{I}$ gives the simultaneous fundamental matrix solutions

$$\widetilde{\phi}_\pm(x,t,k) = \mathrm{e}^{-if_\pm(x,t)\sigma_3}E_\pm(k)\mathrm{e}^{i\theta_\pm(x,t,k)\sigma_3}. \tag{144}$$

Proof of Lemma 1 (Branch cut for $\lambda_\pm(k)$). The analyticity properties of $\lambda(k;A)$ together with the facts that $\lambda(k;A) = k + O(1/k)$ as $k \to \infty$ and $\lambda(k;A) \in \mathbb{R}$ exactly when $k \in \mathbb{R} \cup i[-A,A]$ while $\lambda(k;A) \in i\mathbb{R}$ exactly when $k \in i\mathbb{R} \setminus i(-A,A)$ establish (22a) and (22b). Let $k - iA = r_1\mathrm{e}^{i\varphi_1}$ and $k + iA = r_2\mathrm{e}^{i\varphi_2}$, with $-\pi/2 \le \varphi_1, \varphi_2 < 3\pi/2$. Then $\overline{k} - iA = \overline{k+iA} = r_2\mathrm{e}^{i\widetilde{\varphi}_1}$ and $\overline{k} + iA = \overline{k - iA} = r_1\mathrm{e}^{i\widetilde{\varphi}_2}$, with

$$\widetilde{\varphi}_1 = \begin{cases} -\varphi_2, & -\pi/2 \le \varphi_2 \le \pi/2, \\ 2\pi - \varphi_2, & \pi/2 < \varphi_2 < 3\pi/2, \end{cases}$$

and

$$\widetilde{\varphi}_2 = \begin{cases} -\varphi_1, & -\pi/2 \le \varphi_1 \le \pi/2, \\ 2\pi - \varphi_1, & \pi/2 < \varphi_1 < 3\pi/2, \end{cases}$$

so that $-\pi/2 \le \widetilde{\varphi}_1, \widetilde{\varphi}_2 < 3\pi/2$. If $-\pi/2 \le \varphi_1, \varphi_2 \le \pi/2$, then

$$\lambda(\overline{k};A) = \sqrt{r_1r_2}\mathrm{e}^{i(-\varphi_2-\varphi_1)/2} = \overline{\lambda(k;A)}. \tag{145}$$

On the other hand, if $\pi/2 \le \varphi_1, \varphi_2 < 3\pi/2$, then

$$\lambda(\overline{k};A) = \sqrt{r_1r_2}\mathrm{e}^{i(-\varphi_2-\varphi_1+4\pi)/2} = \overline{\lambda(k;A)}. \tag{146}$$

The remaining two cases never occur, so (22c) is proved. Also, $(-k) - iA = -(k + iA) = r_2 e^{i\hat{\varphi}_1}$ and $(-k) + iA = -(k - iA) = r_1 e^{i\hat{\varphi}_2}$, with

$$\hat{\varphi}_1 = \begin{cases} \varphi_2 + \pi, & -\pi/2 \le \varphi_2 < \pi/2, \\ \varphi_2 - \pi, & \pi/2 \le \varphi_2 < 3\pi/2, \end{cases}$$

and

$$\hat{\varphi}_2 = \begin{cases} \varphi_1 + \pi, & -\pi/2 \le \varphi_1 < \pi/2, \\ \varphi_1 - \pi, & \pi/2 \le \varphi_1 < 3\pi/2, \end{cases}$$

so that $-\pi/2 \le \hat{\varphi}_1, \hat{\varphi}_2 < 3\pi/2$. If $-\pi/2 \le \varphi_1, \varphi_2 < \pi/2$, then

$$\lambda(-k; A) = \sqrt{r_1 r_2} e^{i(\varphi_1 + \varphi_2 + 2\pi)/2} = -\lambda(k; A). \tag{147}$$

On the other hand, if $\pi/2 \le \varphi_1, \varphi_2 < 3\pi/2$, then

$$\lambda(-k; A) = \sqrt{r_1, r_2} e^{i(\varphi_1 + \varphi_2 - 2\pi)/2} = -\lambda(k; A). \tag{148}$$

The case of $-\pi/2 \le \varphi_1 < \pi/2$ and $\pi/2 \le \varphi_2 < 3\pi/2$ can only happen when $\varphi_1 = -\pi/2$ and $\varphi_2 = \pi/2$ so that $k \in i(-A, A)$. In such case,

$$\lambda(-k; A) = \sqrt{r_1 r_2} e^{i(-\pi/2 + \pi + \pi/2 - \pi)/2} = \sqrt{r_1 r_2} e^{i(-\pi/2 + \pi/2)/2} = \lambda(k; A). \tag{149}$$

The remaining case cannot occur, so (22d) is proved. Combining (22c) and (22d) and recalling that $\lambda(k; A) \in \mathbb{R}$ for $k \in i[-A, A]$ gives (22e). ■

Proof of Lemma 2 (Properties of $d_\pm(k)$). Here, we compute the jump of $d_\pm(k)$ across the branch cut $\Sigma_\pm$. To simplify the argument, we first explicitly define $(D(k; A))^{1/2}$, where $D(k; A) = \frac{2\lambda(k;A)}{\lambda(k;A)+k}$ with $\lambda(k; A)$ as defined in (19). We claim that $\operatorname{Re} D(k; A) > 0$ except where $\lambda(k; A) = 0$, in which case $D(k; A) = 0$ (ie, at the branch points $k = \pm iA$). Indeed,

$$D(k; A) = \frac{2\lambda}{\lambda + k} = \frac{2\lambda}{|\lambda + k|^2}(\bar{\lambda} + \bar{k}),$$

so that

$$\operatorname{Re} D(k; A) = \frac{2}{|\lambda + k|^2}\left(|\lambda|^2 + \lambda_{\mathrm{re}} k_{\mathrm{re}} + \lambda_{\mathrm{im}} k_{\mathrm{im}}\right).$$

Lemma 1 shows that λ_{re} and k_{re} have the same sign, as do λ_{im} and k_{im}. This then proves the claim. Next, note that, similar to (29),

$$D^+(k; A) = \frac{4\lambda^2}{A^2} \frac{1}{D(k; A)}, \quad k \in i(-A, A).$$

Since $\lambda > 0$ on $i(-A, A)$, then using $\sqrt{\cdot}$ defined as the principal square root with branch cut along $\{0\} \cup \mathbb{R}^-$,

$$\sqrt{D^+(k;A)} = \frac{2\lambda}{A}\frac{1}{\sqrt{D(k;A)}}.$$

Since $\operatorname{Re} D(k+\epsilon; A) > 0$ for k in $i(-A, A)$ and any $\epsilon > 0$, then

$$\sqrt{D(k;A)}^{+} = \sqrt{D^+(k;A)} = \frac{2\lambda}{A}\frac{1}{\sqrt{D(k;A)}}, \qquad k \in i(-A, A).$$

Correspondingly, $d_\pm(k) := \sqrt{D(k \pm V/2; A_\pm)}$ satisfies the jump in Lemma 2.

The asymptotic behavior of $d_\pm(k)$ follows directly from that of $D_\pm(k)$. ■

Integral equations for $\mu_\pm(x,t,k)$. We now establish the integral equations (35) for $\mu_\pm(x,t,k)$, We first define

$$\psi_\pm(x,t,k) = \mathrm{e}^{-i\theta_\pm(x,t,k)\sigma_3} E_\pm^{-1}(k) \mathrm{e}^{if_\pm(x,t)\sigma_3}\phi_\pm(x,t,k),$$

so that $\psi_\pm(x,t,k) = \mathrm{I} + o(1)$ as $x \to \pm\infty$. Recalling that $\phi_\pm(x,t,k)$ satisfies the Lax pair, we have

$$\begin{aligned}
&\mathrm{e}^{-if_\pm(x,t)\sigma_3}\left(\mp i\frac{V}{2}\sigma_3 E_\pm(k) + E_\pm(k) i\lambda_\pm\sigma_3\right)\mathrm{e}^{i\theta_\pm(x,t,k)\sigma_3}\psi_\pm(x,t,k) + \mathrm{e}^{-if_\pm(x,t)\sigma_3}E_\pm(k)\mathrm{e}^{i\theta_\pm\sigma_3}(\psi_\pm)_x \\
&= (X_\pm + \Delta Q_\pm)\mathrm{e}^{-if_\pm(x,t)\sigma_3}E_\pm(k)\mathrm{e}^{i\theta_\pm(x,t,k)\sigma_3}\psi_\pm.
\end{aligned}$$

Now with $\widehat{X}_\pm(k)$ as defined in (139a), we have

$$\begin{aligned}
\mathrm{e}^{if_\pm(x,t)\sigma_3} X_\pm(x,t,k) \mathrm{e}^{-if_\pm(x,t)\sigma_3} E_\pm(k) &= \left(\widehat{X}_\pm(k \pm V/2) \mp i\frac{V}{2}\sigma_3\right)E_\pm(k) \\
&= E_\pm(k) i\lambda_\pm\sigma_3 \mp i\frac{V}{2}\sigma_3 E_\pm(k).
\end{aligned} \tag{150}$$

Thus,

$$\mathrm{e}^{-if_\pm(x,t)\sigma_3}E_\pm(k)\mathrm{e}^{i\theta_\pm(x,t,k)\sigma_3}(\psi_\pm)_x = \Delta Q_\pm(x,t)\mathrm{e}^{-if_\pm(x,t)\sigma_3}E_\pm(k)\mathrm{e}^{i\theta_\pm(x,t,k)\sigma_3}\psi_\pm.$$

Formally integrating, we arrive at the integral equations for $\psi_\pm(x,t,k)$,

$$\psi_\pm(x,t,k) = \mathrm{I} + \int_{\pm\infty}^{x} \mathrm{e}^{-i\theta_\pm(y,t,k)\sigma_3}E_\pm^{-1}(k)\mathrm{e}^{2if_\pm(y,t)\sigma_3}\Delta Q_\pm(y,t)E_\pm(k)\mathrm{e}^{i\theta_\pm(y,t,k)\sigma_3}\psi_\pm(y,t,k)dy.$$

Finally, recognizing that $\mu_\pm(x,t,k) = E_\pm(k)\mathrm{e}^{i\theta_\pm(x,t,k)\sigma_3}\psi_\pm(x,t,k)\mathrm{e}^{-i\theta_\pm(x,t,k)\sigma_3}$ gives the corresponding integral equations (35) for $\mu_\pm(x,t,k)$.

Proof of Theorem 1 (Analyticity of the Jost solutions). Here we use the integral equations for $\mu_\pm(x,t,k)$ to find and prove the regions of analyticity and continuity for the Jost solutions under the assumption that $(q-q_\pm)\in L^1_x(\mathbb{R})$ for all $t\in\mathbb{R}$.

The proof follows nearly identically to the proof in Ref. 29. Comparing the integral equations there and here, we have only trivial differences: (a) In Ref. 29, the eigenfunctions are expressed in terms of the uniformization variable $z=\lambda+k$. (b) Here, the definition of $\lambda_\pm(k)$ causes a shift of $\mp V/2$ compared to the $\lambda(k)$ that appears in Ref. 29. (c) Here, we have an extra factor $\mathrm{e}^{2if_\pm(y,t)\sigma_3}$. These differences cause no issue with the analysis of the Neumann iterates. We start by rewriting the integral equations (35) as

$$\mu_\pm(x,t,k)=E_\pm(k)\left[I+\int_{\pm\infty}^{x}\mathrm{e}^{i\lambda_\pm(x-y)\sigma_3}E_\pm^{-1}\Delta\widehat{Q}_\pm(y,t)\mu_\pm(y,t,k)\mathrm{e}^{-i\lambda_\pm(x-y)\sigma_3}dy\right],\tag{151}$$

where

$$\widehat{Q}_-(x,t)=\mathrm{e}^{2if_-(x,t)\sigma_3}Q(x,t),\tag{152a}$$

$$\Delta\widehat{Q}_\pm(x,t)=\mathrm{e}^{2if_\pm(x,t)\sigma_3}\Delta Q_\pm(x,t).\tag{152b}$$

Letting $w(x,t,k)$ be the first column of $W(x,t,k)=E_-^{-1}(k)\mu_-(x,t,k)$, we have

$$w(x,t,k)=\begin{pmatrix}1\\0\end{pmatrix}+\int_{-\infty}^{x}C(x,y,t,k)w(y,t,k)dy,\tag{153a}$$

where

$$C(x,y,t,k)=\mathrm{diag}\big(1,\mathrm{e}^{-2i\lambda_-(x-y)}\big)E_-^{-1}(k)\Delta\widehat{Q}_-(y,t)E_-(k).\tag{153b}$$

Note that the bounds of integration imply $x-y\geq 0$. Now we introduce a Neumann series for w,

$$w(x,t,k)=\sum_{n=0}^{\infty}w^{(n)}(x,t,k),\tag{154a}$$

with

$$w^{(0)}=\begin{pmatrix}1\\0\end{pmatrix},\quad w^{(n+1)}(x,t,k)=\int_{-\infty}^{x}C(x,y,t,k)w^{(n)}(y,t,k)dy.\tag{154b}$$

Introducing the L^1 vector norm $\|w\|:=|w_1|+|w_2|$ and the corresponding subordinate matrix norm $\|C\|$, we then have

$$\|w^{(n+1)}(x,t,k)\|\leq\int_{-\infty}^{x}\|C(x,y,t,k)\|\|w^{(n)}(y,t,k)\|dy.$$

Note that

$$\|E_-(k)\| = 1 + \frac{A_-}{|\lambda_- + (k - V/2)|}, \quad \|E_-^{-1}(k)\| = \frac{1}{|D_-(k)|}\left(1 + \frac{A_-}{|\lambda_- + (k - V/2)|}\right).$$

Thus,

$$\begin{aligned}\|C(x,y,t,k)\| &\le \|\operatorname{diag}\big(1, \mathrm{e}^{-2i\lambda_-(x-y)}\big)\| \|E_-^{-1}(k)\| \|\Delta\widehat{Q}_-(y,t)\| \|E_-(k)\| \\ &= c(k)\big(1 + \mathrm{e}^{2\,\mathrm{Im}\,\lambda_-(x-y)}\big)|q(y,t) - q_-(y,t)|,\end{aligned}$$

where

$$c(k) = \|E_-^{-1}(k)\| \|E_-(k)\| = \frac{1}{|D_-(k)|}\left(1 + \frac{A_-}{|\lambda_- + (k - V/2)|}\right)^2$$

is the condition number of $E_-(k)$. Recall that $\mathrm{Im}\,\lambda_- \le 0$ for $k \in \mathbb{R} \cup \mathbb{C}^- \cup \Sigma_{-1}$, and that $c(k) \to \infty$ as $k \to p_-, \overline{p_-}$. Thus, given $\epsilon > 0$, we restrict our attention to the domain

$$U_\epsilon = \mathbb{R} \cup \mathbb{C}^- \cup \Sigma_{-1}^o \setminus B_\epsilon(\overline{p_-}),$$

where $B_\epsilon(k_o) = \{k \in \mathbb{C} \;:\; |k - k_o| < \epsilon\}$. We next prove that, for all $k \in U_\epsilon$ and for all $n \in \mathbb{N}$,

$$\|w^{(n)}(x,t,k)\| \le \frac{M^n(x,t)}{n!}, \tag{155a}$$

where

$$M(x,t) = 2c_\epsilon \int_{-\infty}^x |q(y,t) - q_-(y,t)|dy, \tag{155b}$$

and $c_\epsilon := \max_{k\in U_\epsilon} c(k)$. The claim is trivially true for $n = 0$. Also note that for all $k \in U_\epsilon$ and for all $y \le x$ we have $1 + \mathrm{e}^{2\,\mathrm{Im}\,\lambda_-(x-y)} \le 2$. Thus if (155a) holds, then

$$\|w^{(n+1)}(x,t,k)\| \le \frac{2c_\epsilon}{n!}\int_{-\infty}^x |q(y,t) - q_-(y,t)|M^n(y,t)dy = \frac{1}{n!(n+1)}M^{n+1}(x,t)$$

proving the induction step. Thus, for all $\epsilon > 0$, if $q(x,t) - q_-(x,t) \in L^1(-\infty, a]$ for some $a \in \mathbb{R}$, the Neumann series converges absolutely and uniformly with respect to $k \in U_\epsilon$ for $x \in (-\infty, a)$. This demonstrates that $\mu_{-1}(x,t,k)$ and thus $\phi_{-1}(x,t,k)$ is defined for $k \in \mathbb{R} \cup \mathbb{C}^- \cup \Sigma_{-1}^o \setminus \{\overline{p_-}\}$, continuous from the right for $k \in \mathbb{R} \cup \Sigma_{-2}^o$, and analytic for $k \in \mathbb{C}^- \setminus \Sigma_{-2}$.

The arguments for the remaining eigenfunctions are similar. ∎

Proof of Lemma 3 (Asymptotics of $\phi_\pm(x,t,k)$ as $k \to \infty$). We now determine the asymptotic behavior for $\mu_\pm(x,t,k)$ as k goes to infinity under the assumption that $(q - q_\pm) \in L_x^1(\mathbb{R})$, and find an explicit asymptotic expansion up to $o(1/k)$ to reconstruct the potential $q(x,t)$.

Consider the formal expansion

$$\mu_-(x,t,k) = \sum_{n=0}^{\infty} \mu^{(n)}(x,t,k), \tag{156a}$$

with

$$\mu^{(0)}(x,t,k) = E_-(k), \tag{156b}$$

$$\mu^{(n+1)}(x,t,k) = \int_{-\infty}^{x} E_- \mathrm{e}^{i\lambda_-(x-y)\sigma_3} E_-^{-1} \Delta\hat{Q}_-(y,t)\mu^{(n)}(y,t,k)\mathrm{e}^{-i\lambda_-(x-y)\sigma_3} dy. \tag{156c}$$

Let B_d and B_o denote the diagonal and off-diagonal parts of a matrix B, respectively. The above expression gives an asymptotic expansion for the columns of $\mu_-(x,t,k)$ as $k \to \infty$ within the appropriate regions of the complex k-plane for each column (see Section 2.3). We now show that if the potential admits a continuous derivative with $(q-q_-)_x \in L^1_x(-\infty,a)$ for some $a \in \mathbb{R}$, then

$$\mu_d^{(2m)} = O\left(\frac{1}{k^m}\right), \quad \mu_o^{(2m)} = O\left(\frac{1}{k^{m+1}}\right), \quad \mu_d^{(2m+1)} = O\left(\frac{1}{k^{m+1}}\right), \quad \mu_o^{(2m+1)} = O\left(\frac{1}{k^{m+1}}\right), \tag{157}$$

within the appropriate region of the complex k-plane for each column. Explicitly, the first column is valid for $k \in \mathbb{C}^-$ while the second column is valid for $k \in \mathbb{C}^+$. To aid in the argument, we define

$$R^{(n)}(x,t,k) = \int_{-\infty}^{x} \mathrm{e}^{2i\lambda_-(x-y)\sigma_3} \left(\Delta\hat{Q}_-(y,t)\mu_d^{(n)}(y,t,k) \right.$$
$$\left. - \frac{iA_-}{\lambda_- + (k-V/2)} \sigma_1 \Delta\hat{Q}_-(y,t)\mu_o^{(n)}(y,t,k) \right) dy, \tag{158}$$

and simultaneously show that

$$R^{(2m)} = O\left(\frac{1}{k^{m+1}}\right), \quad R^{(2m-1)} = O\left(\frac{1}{k^{m+1}}\right). \tag{159}$$

To clarify the logic, in the induction step we will assume that (157) is true for $\mu^{(n-1)}$ and $\mu^{(n)}$, and (159) is true for $R^{(n-1)}$. We will then show that (159) holds for $R^{(n)}$, which will be used to show that (157) holds for $\mu^{(n+1)}$. Defining $\mu^{(-1)} = 0$ and $R^{(-1)} = 0$, and noting that the claim is clearly true for $\mu^{(0)}$ gives the necessary base cases. Next, integrating by parts we find

$$R^{(n)} = \frac{i}{2\lambda_-}\sigma_3 \left(\Delta\hat{Q}_-\mu_d^{(n)} - \frac{iA_-}{\lambda_- + (k-V/2)} \sigma_1 \Delta\hat{Q}_-\mu_o^{(n)} \right)$$
$$- \frac{i}{2\lambda_-}\sigma_3 \int_{-\infty}^{x} \mathrm{e}^{2i\lambda_-(x-y)\sigma_3} \left[(\hat{Q}_-)_x \mu_d^{(n)} - \frac{iA_-}{\lambda_- + (k-V/2)} \sigma_1 (\hat{Q}_-)_x \mu_o^{(n)} \right.$$
$$\left. + \Delta\hat{Q}_-(\mu_d^{(n)})_x - \frac{iA_-}{\lambda_- + (k-V/2)} \sigma_1 \Delta\hat{Q}_-(\mu_o^{(n)})_x \right] dy,$$

so that

$$R^{(n)} = O\left(\frac{\mu_d^{(n)}}{k}\right) + O\left(\frac{\mu_o^{(n)}}{k^2}\right) + O\left(\frac{(\mu_d^{(n)})_x}{k}\right) + O\left(\frac{(\mu_o^{(n)})_x}{k^2}\right). \tag{160}$$

Now

$$\begin{aligned} E_-^{-1}\Delta\widehat{Q}_-\mu^{(n)} = {}& \frac{1}{D_-}\left[\Delta\widehat{Q}_-\mu_o^{(n)} - \frac{iA_-}{\lambda_- + (k - V/2)}\sigma_1\Delta\widehat{Q}_-\mu_d^{(n)}\right] \\ & + \frac{1}{D_-}\left[\Delta\widehat{Q}_-\mu_d^{(n)} - \frac{iA_-}{\lambda_- + (k - V/2)}\sigma_1\Delta\widehat{Q}_-\mu_o^{(n)}\right], \end{aligned}$$

where the first term is a diagonal matrix and the second is off-diagonal. Then

$$\begin{aligned} e^{i\lambda_-(x-y)\sigma_3}E_-^{-1}\Delta\widehat{Q}_-\mu^{(n)}e^{-i\lambda(x-y)\sigma_3} = {}& \frac{1}{D_-}\left[\Delta\widehat{Q}_-\mu_o^{(n)} - \frac{iA_-}{\lambda_- + (k - V/2)}\sigma_1\Delta\widehat{Q}_-\mu_d^{(n)}\right] \\ & + \frac{1}{D_-}e^{2i\lambda_-(x-y)\sigma_3}\left[\Delta\widehat{Q}_-\mu_d^{(n)} - \frac{iA_-}{\lambda_- + (k - V/2)}\sigma_1\Delta\widehat{Q}_-\mu_o^{(n)}\right], \end{aligned}$$

and

$$\begin{aligned} E_-e^{i\lambda_-(x-y)\sigma_3}E_-^{-1}\Delta\widehat{Q}_-\mu^{(n)}e^{-i\lambda(x-y)\sigma_3} = {}& \frac{1}{D_-}\left[\Delta\widehat{Q}_-\mu_o^{(n)} - \frac{iA_-}{\lambda_- + (k - V/2)}\sigma_1\Delta\widehat{Q}_-\mu_d^{(n)}\right] \\ & + \frac{iA_-}{2\lambda_-}\sigma_1 e^{2i\lambda_-(x-y)\sigma_3}\left[\Delta\widehat{Q}_-\mu_d^{(n)} - \frac{iA_-}{\lambda_- + (k - V/2)}\sigma_1\Delta\widehat{Q}_-\mu_o^{(n)}\right] \\ & + \frac{1}{D_-}e^{2i\lambda_-(x-y)\sigma_3}\left[\Delta\widehat{Q}_-\mu_d^{(n)} - \frac{iA_-}{\lambda_- + (k - V/2)}\sigma_1\Delta\widehat{Q}_-\mu_o^{(n)}\right] \\ & + \frac{iA_-}{2\lambda_-}\sigma_1\left[\Delta\widehat{Q}_-\mu_o^{(n)} - \frac{iA_-}{\lambda_- + (k - V/2)}\sigma_1\Delta\widehat{Q}_-\mu_d^{(n)}\right], \end{aligned}$$

where the first two terms are diagonal and the last two are off-diagonal. Then, we have

$$\mu_d^{(n+1)} = \frac{1}{D_-}\int_{-\infty}^{x}\left(\Delta\widehat{Q}_-\mu_o^{(n)} - \frac{iA_-}{\lambda_- + (k - V/2)}\sigma_1\Delta\widehat{Q}_-\mu_d^{(n)}\right)dy + \frac{iA_-}{2\lambda_-}\sigma_1 R^{(n)}, \tag{161a}$$

$$\mu_o^{(n+1)} = \frac{iA_-}{2\lambda_-}\sigma_1\int_{-\infty}^{x}\left(\Delta\widehat{Q}_-\mu_o^{(n)} - \frac{iA_-}{\lambda_- + (k - V/2)}\sigma_1\Delta\widehat{Q}_-\mu_d^{(n)}\right)dy + \frac{1}{D_-}R^{(n)}. \tag{161b}$$

Differentiating and reindexing, we find

$$(\mu_d^{(n)})_x = \Delta\widehat{Q}_-\mu_o^{(n-1)} - A_-\sigma_1\sigma_3 R^{(n-1)}, \tag{162a}$$

$$(\mu_o^{(n)})_x = \Delta\widehat{Q}_-\mu_d^{(n-1)} + i\sigma_3 R^{(n-1)}, \tag{162b}$$

so that with (160) we have

$$R^{(n)} = O\left(\frac{\mu_d^{(n)}}{k}\right) + O\left(\frac{\mu_o^{(n)}}{k^2}\right) + O\left(\frac{\mu_d^{(n-1)}}{k^2}\right) + O\left(\frac{\mu_o^{(n-1)}}{k}\right) + O\left(\frac{R^{(n-1)}}{k}\right).$$

By induction, we see that if $n = 2m$, then $R^{(2m)} = O(1/k^{m+1})$, so that with (161) we have

$$\mu_d^{(2m+1)} = O\left(\mu_o^{(2m)}\right) + O\left(\frac{\mu_d^{(2m)}}{k}\right) + O\left(\frac{1}{k^{m+2}}\right) = O\left(\frac{1}{k^{m+1}}\right),$$

$$\mu_o^{(2m+1)} = O\left(\frac{\mu_o^{(2m)}}{k}\right) + O\left(\frac{\mu_d^{(2m)}}{k^2}\right) + O\left(\frac{1}{k^{m+1}}\right) = O\left(\frac{1}{k^{m+1}}\right),$$

while if $n = 2m-1$, then $R^{(2m-1)} = O(\frac{1}{k^{m+1}})$, so that with (161) we have

$$\mu_d^{(2m)} = O\left(\mu_o^{(2m-1)}\right) + O\left(\frac{\mu_d^{(2m-1)}}{k}\right) + O\left(\frac{1}{k^{m}}\right) = O\left(\frac{1}{k^{m}}\right),$$

$$\mu_o^{(2m)} = O\left(\frac{\mu_o^{(2m-1)}}{k}\right) + O\left(\frac{\mu_d^{(2m-1)}}{k^2}\right) + O\left(\frac{1}{k^{m+1}}\right) = O\left(\frac{1}{k^{m+1}}\right),$$

which completes the induction. Similar argument gives the corresponding asymptotics for μ_+.

From the above, we see that

$$\mu_- = \mu_d^{(0)} + \mu_o^{(0)} + \mu_d^{(1)} + \mu_o^{(1)} + \mu_d^{(2)} + o(1/k).$$

Computing these terms explicitly up to order $1/k$, we have

$$\mu_d^{(0)} = \mathrm{I}, \quad \mu_o^{(0)} = \frac{i}{2k} A_- \sigma_1,$$

$$\mu_d^{(1)} = \frac{i}{2k} \int_{-\infty}^{x} \left[\Delta\hat{Q}_-, A_-\sigma_1\right] dy + o(1/k), \quad \mu_o^{(1)} = \frac{i}{2k} \sigma_3 \Delta\hat{Q}_- + o(1/k),$$

$$\mu_d^{(2)} = \frac{i}{2k} \int_{-\infty}^{x} \Delta\hat{Q}_- \sigma_3 \Delta\hat{Q}_- dy + o(1/k),$$

where the last two make use of the Riemann-Lebesgue lemma. Then

$$\mu_- = \mathrm{I} + \frac{i}{2k} \sigma_3 \mathrm{e}^{2if_- \sigma_3} Q + \frac{i}{2k} \int_{-\infty}^{x} \left([\mathrm{e}^{2if_- \sigma_3} \Delta Q_-, A_-\sigma_1] + \Delta Q_- \sigma_3 \Delta Q_-\right) dy + o(1/k).$$

The 12-entry of this expression then gives (39). ∎

Proof of Corollary 2 (Real differentiability of $\phi_\pm(x,t,k)$). We now show that under the assumption $(q-q_\pm)\in L^1_x(\mathbb{R})$, the Jost solutions $\phi_+(x,t,k)$ and $\phi_-(x,t,k)$ are real differentiable for all $k\in\mathbb{R}$. Consider the first column of the integral equation (35) for $\mu_{+1}(x,t,k)$,

$$\mu_{+1}(x,t,k)=E_{+1}(k)+\int_\infty^x K(x-y,k)\mathrm{e}^{2if_+(y,t)\sigma_3}\Delta Q_+(y,t)\mu_{+1}(y,t,k)dy, \tag{163}$$

where

$$K(\xi,k)=E_+(k)\mathrm{diag}\big(1,\mathrm{e}^{-2i\lambda_+\xi}\big)E_+^{-1}(k).$$

Formally differentiating with respect to k, we have

$$\begin{aligned}\frac{\partial\mu_{+1}}{\partial k}(x,t,k)&=\frac{\partial E_{+1}}{\partial k}(k)+\int_\infty^x\frac{\partial K}{\partial k}(x-y,k)\mathrm{e}^{2if_+(y,t)\sigma_3}\Delta Q_+(y,t)\mu_{+1}(y,t,k)dy\\&\quad+\int_\infty^x K(x-y,k)\mathrm{e}^{2if_+(y,t)\sigma_3}\Delta Q_+(y,t)\frac{\partial\mu_{+1}}{\partial k}(y,t,k)dy.\end{aligned}\tag{164}$$

Note that the integral equation (164) has exactly the same kernel as (163), and just the integrated term is different. Analysis of the Neumann iterates for (164), similar to the proof of Theorem 1, then shows that if $(q-q_+)\in L^1_x(a,\infty)$ for some $a\in\mathbb{R}$, then $\partial\mu_{+1}/\partial k$ is well-defined and continuous for $k\in\mathbb{R}$, with continuity restricted to $\mathbb{R}$. Similar analysis for the remaining eigenfunctions gives the result. ■

Proof of Lemma 5 (Well-defined modified eigenfunctions at the branch points). Here, we show that under the assumption $(q-q_\pm)\in L^{1,1}_x(\mathbb{R})$, the modified eigenfunctions $\mu_\pm(x,t,k)$ can be extended to the branch points. Again consider the integral equation (163) for $\mu_{+1}(x,t,k)$.

Note that

$$\lim_{k\to -V/2\pm iA_+}K(\xi,k)=\mathrm{I}+A_+\xi(\sigma_3\mathrm{e}^{i\delta\sigma_3}\sigma_1\mp\sigma_3).$$

Analysis of the Neumann iterates similar to the proof of Theorem 1 shows that if $(q-q_+)\in L^{1,1}_x(a,\infty)$ for some $a\in\mathbb{R}$, then $\mu_{+1}(x,t,k)$ is well-defined and continuous at the branch points $k=p_+,\overline{p_+}$. Note that continuity at $k=\overline{p_+}$ is restricted to $k\in\Sigma_{+2}$. Similar argument for the remaining eigenfunctions at their respective branch points gives:

- If $(q-q_+)\in L^{1,1}_x(a,\infty)$ for some $a\in\mathbb{R}$, then
  - $\mu_{+1}(x,t,k)$ is continuous at the branch points $k=p_+,\overline{p_+}$, where continuity at $k=\overline{p_+}$ is restricted to $k\in\Sigma_{+2}$,
  - $\mu_{+2}(x,t,k)$ is continuous at the branch points $k=p_+,\overline{p_+}$, where continuity at $k=p_+$ is restricted to $k\in\Sigma_{+1}$.
- If $(q-q_-)\in L^{1,1}_x(-\infty,a)$ for some $a\in\mathbb{R}$, then
  - $\mu_{-1}(x,t,k)$ is continuous at the branch points $k=p_-,\overline{p_-}$, where continuity at $k=p_-$ is restricted to $k\in\Sigma_{-1}$,
  - $\mu_{-2}(x,t,k)$ is continuous at the branch points $k=p_-,\overline{p_-}$, where continuity at $k=\overline{p_-}$ is restricted to $k\in\Sigma_{-2}$.

Moreover, the normalizations $\mu_\pm(x,t,k) = E_\pm(k)(1+o(1))$ as $x \to \pm\infty$ imply that the modified eigenfunctions are nonzero at the branch points.
Lemma 5 then follows. ■

Proof of Lemma 6 (Expansion of modified eigenfunctions at the branch points). We now improve the expansions of $\mu_\pm(x,t,k)$ about the branch points from the previous lemma under the more strict assumption that $(q - q_\pm) \in L_x^{1,2}(\mathbb{R})$.

We again consider $\mu_{+1}(x,t,k)$. It is convenient to introduce the variable

$$z(k) = \lambda_+(k) + (k + V/2),$$

so that

$$k + V/2 = \frac{1}{2}(z - A_+^2/z), \quad \lambda_+ = \frac{1}{2}(z + A_+^2/z).$$

Note that $|z| \geq A_+$ for all $k \in \mathbb{C}$, with $|z| = A_+$ exactly when $k \in \Sigma_+$. Furthermore, the branch points $k = -V/2 \pm iA_+$ correspond to $z = \pm iA_+$. The reason for introducing the variable z is that, while the derivatives of the eigenfunctions with respect to k are not well-defined at the branch points, those with respect to z are, which will enable us to obtain asymptotic estimates near the branch points.

With some abuse of notation, we write all k-dependence as z-dependence, so that the integral equations (163) and (164) become

$$\mu_{+1}(x,t,z) = E_{+1}(z) + \int_\infty^x K(x-y,z)\mathrm{e}^{2if_+(y,t)\sigma_3}\Delta Q_+(y,t)\mu_{+1}(y,t,z)dy,$$

and

$$\frac{\partial \mu_{+1}}{\partial z}(x,t,z) = \frac{\partial E_{+1}}{\partial z}(z) + \int_\infty^x \frac{\partial K}{\partial z}(x-y,z)\mathrm{e}^{2if_+(y,t)\sigma_3}\Delta Q_+(y,t)\mu_{+1}(y,t,z)dy$$
$$+ \int_\infty^x K(x-y,z)\mathrm{e}^{2if_+(y,t)\sigma_3}\Delta Q_+(y,t)\frac{\partial \mu_{+1}}{\partial z}(y,t,z)dy, \tag{165}$$

respectively. Note that

$$\lim_{z\to\pm iA_+} K(\xi,z) = \mathrm{I} + A_+\xi(\sigma_3\mathrm{e}^{i\delta\sigma_3}\sigma_1 \mp \sigma_3),$$

$$\lim_{z\to\pm iA_+} \frac{\partial K}{\partial z}(\xi,z) = -i\xi\,\mathrm{I} - iA_+\xi^2(\sigma_3\mathrm{e}^{i\delta\sigma_3}\sigma_1 \mp \sigma_3).$$

Analysis of the Neumann iterates for (165), similar to the proof of Theorem 1, then shows that if $(q - q_+) \in L_x^{1,2}(a,\infty)$ for some $a \in \mathbb{R}$, then $\partial\mu_{+1}/\partial z$ is well-defined and continuous at $z = iA_+$, with continuity restricted to $|z| \geq A_+$. Then

$$\frac{\partial \mu_{+1}}{\partial z}(x,t,z) = \frac{\partial \mu_{+1}}{\partial z}(x,t,iA_+) + o(1), \quad z \to iA_+.$$

Since

$$\mu_{+1}(x,t,z)=\mu_{+1}(x,t,iA_+)+\int_{iA_+}^{z}\frac{\partial\mu_{+1}}{\partial z}(0,0,s)ds,$$

we have

$$\mu_{+1}(x,t,z)=\beta_{p_+}^{(0)}(x,t)+\widetilde{\beta}_{p_+}^{(1)}(x,t)(z-iA_+)+o(z-iA_+),\quad z\to iA_+,$$

where $\beta_{p_+}^{(0)}(x,t)=\mu_{+1}(x,t,z=iA_+)$ and $\widetilde{\beta}_{p_+}^{(1)}(x,t)=\frac{\partial\mu_{+1}}{\partial z}(x,t,z=iA_+)$. In terms of k, we have

$$\mu_{+1}(x,t,k)=\beta_{p_+}^{(0)}(x,t)+\widetilde{\beta}_{p_+}^{(1)}(x,t)(\lambda_++k-p_+)+o(\lambda_++k-p_+),\quad k\to p_+.$$

Note that

$$\lambda_+(k)=((k+V/2)^2+A_+^2)^{1/2}=(k-p_+)^{1/2}(k-\overline{p_+})^{1/2},$$

so that

$$\lambda_++k-p_+=(2iA_+)^{1/2}(k-p_+)^{1/2}+o(k-p_+)^{1/2},\quad k\to p_+.$$

Then

$$\mu_{+1}(x,t,k)=\beta_{p_+}^{(0)}(x,t)+\beta_{p_+}^{(1)}(x,t)(k-p_+)^{1/2}+o(k-p_+)^{1/2},\quad k\to p_+,$$

where $\beta_{p_+}^{(1)}(x,t)=(2iA_+)^{1/2}\,\widetilde{\beta}_{p_+}^{(1)}(x,t)$.

Using similar analysis for the remaining eigenfunctions (instead with $z(k)=\lambda_-(k)+(k-V/2)$ for $\mu_-(x,t,k)$), we see that Lemma 6 then follows. ■

Proof of Lemma 7 (Behavior of $d_\pm(k)$ at the branch points). At the branch points $p_\pm$, we have

$$\lambda_\pm(k)=(k-p_\pm)^{1/2}(k-\overline{p_\pm})^{1/2}=(2iA_\pm)^{1/2}(k-p_\pm)^{1/2}+o(1),\quad k\to p_\pm,$$

$$\lambda_\pm(k)+(k\pm V/2)=iA_\pm+o(1),\quad k\to p_\pm,$$

so that

$$D_\pm(k)=\left(\frac{8}{iA_\pm}\right)^{1/2}(k-p_\pm)^{1/2}+o(1),\quad k\to p_\pm.$$

Similar calculation gives the asymptotic behavior at the branch points $\overline{p_\pm}$. ■

Proof of Lemmas 8 and 9 (First symmetry, Jost solutions, and scattering coefficients). Here we show relations between the Jost solutions and their Schwarz conjugates, and the corresponding relations between the scattering coefficients.

Consider $\psi_\pm(x,t,k) = -\sigma_* \overline{\phi_\pm(x,t,\overline{k})}\sigma_*$, defined columnwise wherever $\phi_\pm(x,t,\overline{k})$ exists. Then

$$(\psi_\pm)_x = -\sigma_* \overline{(\phi_\pm)_x(x,t,\overline{k})}\sigma_* = \left(-\sigma_* \overline{X(x,t,\overline{k})}\sigma_*\right)\left(-\sigma_* \overline{\phi_\pm(x,t,\overline{k})}\sigma_*\right) = X(x,t,k)\psi_\pm.$$

Similar calculation shows that $(\psi_\pm)_t = T(x,t,k)\psi_\pm$. Then $-\sigma_* \overline{\phi_\pm(x,t,\overline{k})}\sigma_*$ satisfies the Lax pair (3). Comparing the asymptotic behavior as $x \to \pm\infty$, we see that

$$\overline{\phi_\pm(x,t,\overline{k})} = -\sigma_* \phi_\pm(x,t,k)\sigma_*, \tag{166}$$

which is to be understood columnwise wherever the appropriate columns are defined. Writing (166) in terms of the columns gives (76) and proves Lemma 8. In particular, (166) holds for all $k \in \mathbb{R}$, which implies that

$$\overline{S(\overline{k})} = -\sigma_* S(k)\sigma_*, \quad k \in \mathbb{R},$$

so that

$$\overline{s_{22}(\overline{k})} = s_{11}(k), \quad k \in \mathbb{R}, \tag{167a}$$

$$\overline{s_{12}(\overline{k})} = -s_{21}(k), \quad k \in \mathbb{R}. \tag{167b}$$

Moreover, using (76), we see that

$$\overline{s_{22}(\overline{k})} = \mathrm{Wr}[\sigma_* \phi_{+2}(x,t,k), -\sigma_* \phi_{-1}(x,t,k)] = \mathrm{Wr}[\phi_{-1}(x,t,k), \phi_{+2}(x,t,k)] = s_{11}(k) \tag{168}$$

for $k \in \mathbb{C}^- \setminus \{\overline{p_+}, \overline{p_-}\}$, so that (167a) can be extended. We can similarly extend (167b) to $k \in \mathbb{R} \cup \Sigma_{+2}^o \cup \Sigma_{-1}^o$. Recalling (67) then completes the proof. ■

Proof of Lemmas 10 and 11 (Second symmetry, Jost solutions, and scattering coefficients). We now determine the discontinuities of the Jost solutions across the appropriate portions of the branch cuts $\Sigma_\pm$, and the corresponding jumps for the scattering coefficients.

Consider the transformation $\lambda_+ \mapsto -\lambda_+$. Simple algebraic manipulations yield

$$E_\pm(\lambda_\pm \mapsto -\lambda_\pm) = \mathrm{I} + \frac{iA_\pm}{-\lambda_\pm + (k \pm V/2)} \mathrm{e}^{\pm i\delta\sigma_3}\sigma_1 = \left(\frac{2\lambda_\pm}{iA_\pm D_\pm}\right) E_\pm \mathrm{e}^{\pm i\delta\sigma_3}\sigma_1,$$

so that with Lemma 2 we have

$$\left(\frac{1}{d_\pm}\mathrm{e}^{-if_\pm\sigma_3}E_\pm \mathrm{e}^{i\theta_\pm\sigma_3}\right)(\lambda_\pm \mapsto -\lambda_\pm) = -i\frac{1}{d_\pm}\mathrm{e}^{-if_\pm\sigma_3}E_\pm \mathrm{e}^{\pm i\delta\sigma_3}\sigma_1 \mathrm{e}^{-i\theta_\pm\sigma_3}$$
$$= -i\frac{1}{d_\pm}\mathrm{e}^{-if_\pm\sigma_3}E_\pm \mathrm{e}^{i\theta_\pm\sigma_3}\mathrm{e}^{\pm i\delta\sigma_3}\sigma_1.$$

The asymptotics (16) then give

$$\phi_\pm(\lambda_\pm \mapsto -\lambda_\pm) = -i\phi_\pm \mathrm{e}^{\pm i\delta\sigma_3}\sigma_1(1+o(1)), \quad x \to \pm\infty,$$

which is to be understood columnwise. This, together with (27), proves Lemma 10.

The Wronskian representations (60) give

$$s_{22}^+(k) = \mathrm{Wr}\big[\phi_{+1}^+(x,t,k), \phi_{-2}^+(x,t,k)\big], \quad k \in \Sigma_{+1}^o \cup \Sigma_{-1}^o.$$

Noting that $\phi_{-1}(x,t,k)$ is analytic on Σ_{+1} while $\phi_{+2}(x,t,k)$ is analytic on Σ_{-1} for $V \neq 0$, we have

$$s_{22}^+(k) = -i\mathrm{e}^{-i\delta}\mathrm{Wr}[\phi_{+2}(x,t,k), \phi_{-2}(x,t,k)] = i\mathrm{e}^{-i\delta}s_{12}(k), \quad k \in \Sigma_{+1}^o,$$

$$s_{22}^+(k) = -i\mathrm{e}^{-i\delta}\mathrm{Wr}[\phi_{+1}(x,t,k), \phi_{-1}(x,t,k)] = -i\mathrm{e}^{-i\delta}s_{21}(k), \quad k \in \Sigma_{-1}^o.$$

The symmetries (77) give the corresponding jumps for $s_{11}(k)$. The jumps for $r_{11}(k)$ and $r_{22}(k)$ are then easily found using (67). ■

Proof of Lemma 12 (Discrete eigenvalues). Here we show the correspondence between zeros of the analytic scattering coefficients and bounded solutions to the Lax pair (3) away from the continuous spectrum and branch points.

Let $s_{22}(k_o) = 0$ for some $k_o \in \mathbb{C}^+ \setminus \Sigma$. Then from the Wronskian definition (60d), we see that $\phi_{-2}(x,t,k_o)$ and $\phi_{+1}(x,t,k_o)$ are linearly dependent so that both decay as $x \to \pm\infty$, establishing the existence of a bounded solution to the Lax pair (3) for $k = k_o$ which decays at both spatial infinities.

Conversely, let $v(x,t)$ be a nontrivial bounded solution to the Lax pair for $k = k_o \in \mathbb{C}^+ \setminus \Sigma$. Suppose $s_{22}(k_o) =: s_o \neq 0$, so that $\Phi(x,t) = (\phi_{+1}(x,t,k_o), \phi_{-2}(x,t,k_o))$ is a fundamental matrix solution and

$$v(x,t) = \Phi(x,t)c \tag{169}$$

for some constant vector c. Since $k_o \in \mathbb{C}^+ \setminus \Sigma$, we have $\operatorname{Im}\lambda_\pm(k_o) > 0$. Correspondingly, the asymptotic behavior (16) gives

$$\lim_{x\to\infty} \|\phi_{+1}(x,t,k_o)\| = 0, \qquad \lim_{x\to-\infty} \|\phi_{-2}(x,t,k_o)\| = 0. \tag{170}$$

If $\phi_{+1}(x,t,k_o)$ is bounded for all x, then

$$s_o = \lim_{x\to-\infty} \mathrm{Wr}[\phi_{+1}(x,t,k_o), \phi_{-2}(x,t,k_o)] = 0,$$

which is a contradiction. Arguing similarly for $\phi_{-2}(x,t,k_o)$, we see

$$\lim_{x\to-\infty} \|\phi_{+1}(x,t,k_o)\| = \infty, \qquad \lim_{x\to\infty} \|\phi_{-2}(x,t,k_o)\| = \infty.$$

On the other hand, (169) and (170) give

$$\lim_{x\to\infty}\|v(x,t)\| = \lim_{x\to\infty}\|c_2\phi_{-2}(x,t,k_o)\|, \quad \lim_{x\to-\infty}\|v(x,t)\| = \lim_{x\to-\infty}\|c_1\phi_{+1}(x,t,k_o)\|.$$

Since v is bounded for all x, we must have $c = 0$, which is a contradiction. Thus $s_{22}(k_o) = 0$.
The symmetries (76) and (77) give the corresponding statement for $k \in \mathbb{C}^- \setminus \Sigma$. ■

Proof of Lemma 13 (Nonvanishing scattering coefficients on branch cuts). We now show that the scattering coefficients are nonvanishing on the branch cuts.

We first show that if u, v are solutions to the scattering problem (3a), then

$$\frac{\partial}{\partial x}\left(u^\dagger(x,t,k)v(x,t,k)\right) = 0.$$

Indeed, the symmetry $X^\dagger(x,t,k) = -X(x,t,k)$ gives

$$\frac{\partial}{\partial x}\left(u^\dagger v\right) = u_x^\dagger v + u^\dagger v_x = u^\dagger X^\dagger v + u^\dagger X v = -u^\dagger X v + u^\dagger X v = 0.$$

For $k \in \Sigma_{+1}^o$, taking $u = v = \phi_{+1}$ or $u = v = \phi_{+2}$ gives

$$\frac{\partial}{\partial x}\left(\phi_{+j}^\dagger(x,t,k)\phi_{+j}(x,t,k)\right) = 0, \quad j = 1, 2.$$

Using the symmetry (76a) and taking the limit as $x \to \infty$, we see that

$$\phi_{+j}^\dagger(x,t,k)\phi_{+j}(x,t,k) = 1, \quad j = 1, 2.$$

If either $s_{12}(k_o) = 0$ or $s_{22}(k_o) = 0$ for some $k_o \in \Sigma_{+1}^o$, the Wronskians (60b) and (60d) give $\phi_{+2}(x,t,k_o) = c_o\phi_{-2}(x,t,k_o)$ or $\phi_{+1}(x,t,k_o) = c_o\phi_{-2}(x,t,k_o)$ for some $c_o \in \mathbb{C}$.
With $V \neq 0$ and $k \in \Sigma_{+1}$, we have $\phi_{-2}(x,t,k_o) \to 0$ as $x \to -\infty$ so that, for the appropriate j,

$$\phi_{+j}^\dagger(x,t,k)\phi_{+j}(x,t,k) = \lim_{x\to-\infty}\left(\overline{c_o}\phi_{-2}^\dagger(x,t,k)\right)(c_o\phi_{-2}(x,t,k)) = 0,$$

which is a contradiction. Thus, $s_{12}(k) \neq 0$ and $s_{22}(k) \neq 0$ for all $k \in \Sigma_{+1}^o$. The symmetries give $s_{21}(k) \neq 0$ and $s_{11}(k) \neq 0$ for $k \in \Sigma_{+2}^o$.
Similar argument shows that $s_{21}(k) \neq 0$ and $s_{22}(k) \neq 0$ for all $k \in \Sigma_{-1}^o$. The symmetries again give $s_{12}(k) \neq 0$ and $s_{11}(k) \neq 0$ for $k \in \Sigma_{-2}^o$. ■

Proof of Lemma 14 (Residues of Jost solutions). We now express the residues of $\Phi(x,t,k)$. From Corollary 5, we see that if $s_{22}(k_o) = 0$ for some discrete eigenvalue $k_o \in \mathbb{C}^+ \setminus \Sigma$, then

$$\mathrm{Wr}[\phi_{+1}(x,t,k_o), \phi_{-2}(x,t,k_o)] = 0, \quad \forall x, t \in \mathbb{R}.$$

Neither $\phi_{+1}(x,t,k_o)$ nor $\phi_{-2}(x,t,k_o)$ can be identically zero due to the normalizations in (16). Then,

$$\phi_{-2}(x,t,k_o) = C_n \phi_{+1}(x,t,k_o), \quad \forall x,t \in \mathbb{R}, \quad C_n \neq 0.$$

Now $s_{11}(\overline{k_o}) = 0$ by Lemma 9. Hence, $\phi_{-1}(x,t,\overline{k_o})$ and $\phi_{+2}(x,t,\overline{k_o})$ are also proportional, and $\overline{k_o}$ is a discrete eigenvalue as well. In particular, from Lemma 8 we have

$$-\sigma_* \overline{\phi_{-1}(x,t,\overline{k_o})} = C_n \sigma_* \overline{\phi_{+2}(x,t,\overline{k_o})}, \quad \forall x,t \in \mathbb{R},$$

so that

$$\phi_{-1}(x,t,\overline{k_o}) = -\overline{C_n} \phi_{+2}(x,t,\overline{k_o}), \quad \forall x,t \in \mathbb{R}.$$

If k_o is a simple root of $s_{22}(k)$ so that $s_{22}'(k_o) \neq 0$, then

$$\operatorname{Res}_{k=k_o}\left[\frac{\phi_{-2}(x,t,k)}{s_{22}(k)}\right] = c_n\, \phi_{+1}(x,t,k_o), \tag{171a}$$

$$\operatorname{Res}_{k=\overline{k_o}}\left[\frac{\phi_{-1}(x,t,k)}{s_{11}(k)}\right] = -\overline{c_n}\, \phi_{+2}(x,t,\overline{k_o}), \tag{171b}$$

with

$$c_n = \frac{C_n}{s_{22}'(k_o)}.$$

Writing the relations (169) in terms of $\Phi(x,t,k)$ then gives the result. ■

7.2 | Inverse problem

Proof of Lemma 17 (Calculation of the jump matrices). Here we compute the jump conditions satisfied by $M(x,t,k)$ as defined by (93).

For simplicity of calculation, we first compute the jump matrices for $\Phi(x,t,k)$ as given by (70). Noting that

$$\Phi(x,t,k) = M(x,t,k)\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}, \quad k \in \mathbb{C} \setminus \Sigma,$$

the jump condition (95) for $M(x,t,k)$ is equivalent to the jump condition

$$\Phi^+(x,t,k) = \Phi^-(x,t,k) J_o(k), \quad k \in \Sigma^o$$

for $\Phi(x,t,k)$, where $J(x,t,k)$ and $J_o(k)$ are related by (96). The symmetries in Lemma 8 imply that

$$J_o(k) = -\sigma_* \overline{J_o(\overline{k})} \sigma_*, \quad k \in \Sigma^o_{\pm}. \tag{172}$$

We now compute $J_o(k)$ for $V \neq 0$. ■

Jump for $k \in \mathbb{R}$. Rearranging (59), we have

$$\phi_{+1}(x,t,k) = \frac{\phi_{-1}(x,t,k)}{s_{11}(k)} - \frac{s_{21}(k)}{s_{11}(k)} \phi_{+2}(x,t,k), \qquad k \in \mathbb{R},$$

$$\frac{\phi_{-2}(x,t,k)}{s_{22}(k)} = \frac{s_{12}(k)}{s_{22}(k)} \frac{\phi_{-1}(x,t,k)}{s_{11}(k)} + \left(1 - \frac{s_{21}(k)s_{12}(k)}{s_{11}(k)s_{22}(k)}\right) \phi_{+2}(x,t,k), \qquad k \in \mathbb{R}.$$

Recalling $\rho(k)$ as defined in (68a) and the symmetry (79), we then have

$$\Phi^+(x,t,k) = \Phi^-(x,t,k) \begin{pmatrix} 1 & \rho(k) \\ \overline{\rho(k)} & 1 + \rho(k)\overline{\rho(k)} \end{pmatrix}, \quad k \in \mathbb{R}. \tag{173}$$

Jumps for $k \in \Sigma^o_+$. Recall that $\phi_{-2}(x,t,k)$ is analytic for $k \in \Sigma^o_{+1}$. Then

$$\left(\frac{\phi_{-2}(x,t,k)}{s_{22}(k)}\right)^+ = \frac{s^-_{22}(k)}{s^+_{22}(k)} \left(\frac{\phi_{-2}(x,t,k)}{s_{22}(k)}\right)^-, \quad k \in \Sigma^o_{+1}.$$

Lemma 11 then gives

$$\frac{s^-_{22}(k)}{s^+_{22}(k)} = -\frac{i\mathrm{e}^{i\delta}}{\rho(k)}, \quad k \in \Sigma^o_{+1}.$$

Recalling (59b) which was extended to Σ^o_{+1}, we have

$$\frac{\phi_{-2}(x,t,k)}{s_{22}(k)} = \rho(k)\phi_{+1}(x,t,k) + \phi_{+2}(x,t,k), \quad k \in \Sigma^o_{+1}.$$

Solving for $\phi_{+2}(x,t,k)$ and using (81a), we see that

$$\phi^+_{+1}(x,t,k) = i\rho(k)\mathrm{e}^{-i\delta}\phi_{+1}(x,t,k) - i\mathrm{e}^{-i\delta} \frac{\phi_{-2}(x,t,k)}{s_{22}(k)}, \quad k \in \Sigma^o_{+1}.$$

Together we have

$$\Phi^+(x,t,k) = \Phi^-(x,t,k) \begin{pmatrix} i\rho(k)\mathrm{e}^{-i\delta} & 0 \\ -i\mathrm{e}^{-i\delta} & -i\mathrm{e}^{i\delta}/\rho(k) \end{pmatrix}, \quad k \in \Sigma^o_{+1}. \tag{174}$$

The symmetry (172) gives

$$\Phi^+(x,t,k) = \Phi^-(x,t,k)\begin{pmatrix} ie^{-i\delta}/\overline{\rho(\overline{k})} & -ie^{i\delta} \\ 0 & \overline{-i\rho(\overline{k})e^{i\delta}} \end{pmatrix}, \quad k \in \Sigma^o_{+2}. \tag{175}$$

Jumps for $k \in \Sigma^o_-$. Recall that $\phi_{+1}(x,t,k)$ is analytic for $k \in \Sigma^o_{-1}$. Then

$$\phi^+_{+1}(x,t,k) = \phi^-_{+1}(x,t,k), \quad k \in \Sigma^o_{-1}.$$

Solving (71c) for $\phi_{-1}(x,t,k)$ we have

$$\phi_{-1}(x,t,k) = \frac{1}{r_{11}(k)}\phi_{+1}(x,t,k) - \frac{r_{21}(k)}{r_{11}(k)}\phi_{-2}(x,t,k), \quad k \in \Sigma^o_{-1}.$$

Lemma 11 then gives

$$\left(\frac{\phi_{-2}(x,t,k)}{s_{22}(k)}\right)^+ = \frac{1}{s_{21}(k)r_{11}(k)}\phi_{+1}(x,t,k) - \frac{r_{21}(k)s_{22}(k)}{s_{21}(k)r_{11}(k)}\left(\frac{\phi_{-2}(x,t,k)}{s_{22}(k)}\right), \quad k \in \Sigma^o_{-1}.$$

From (67) we see

$$s_{21}(k)r_{11}(k) = -r_{21}(k)s_{22}(k) = -1/r(k), \quad k \in \mathbb{R} \cup \Sigma^o_{-1},$$

with $r(k)$ as defined in (68b). Together we have

$$\Phi^+(x,t,k) = \Phi^-(x,t,k)\begin{pmatrix} 1 & -r(k) \\ 0 & 1 \end{pmatrix}, \quad k \in \Sigma^o_{-1}. \tag{176}$$

The symmetry (172) gives

$$\Phi^+(x,t,k) = \Phi^-(x,t,k)\begin{pmatrix} 1 & 0 \\ \overline{r(\overline{k})} & 1 \end{pmatrix}, \quad k \in \Sigma^o_{-2}. \tag{177}$$

Proof of Lemmas 19 and 20 (Growth conditions). The growth conditions follow immediately from the definition of $M(x,t,k)$ and Corollaries 3 and 16 in the generic case, or Corollaries 4 and 18 in the considered exceptional case. To see that the matrices $B^{(0)}_{p_\pm}$ and $B^{(0)}_{\overline{p_\pm}}$ are invertible in the generic case, taking determinants of both sides of (102) yields

$$\det M(x,t,k) = \begin{cases} \det\left[B^{(0)}_{p_\pm}(x,t) + o(1)\right], & k \to p_\pm, \\ \det\left[B^{(0)}_{\overline{p_\pm}}(x,t) + o(1)\right], & k \to \overline{p_\pm}. \end{cases}$$

Since $M(x,t,k)$ has unit determinant for all $k \in \mathbb{C} \setminus \Sigma$, the invertibility of $B^{(0)}_{p_\pm}$ and $B^{(0)}_{\overline{p_\pm}}$ follows.

On the other hand, for the considered exceptional case, taking determinants of both sides of (103) yields

$$\det M(x,t,k) = \begin{cases} \det\left[B^{(0)}_{p_\pm}(x,t) + O(k-p_\pm)^{1/2}\right](k-p_+)^{-1/2}, & k \to p_\pm, \\ \det\left[B^{(0)}_{\overline{p_\pm}}(x,t) + O(k-p_\pm)^{1/2}\right](k-\overline{p_\pm})^{-1/2}, & k \to \overline{p_+}, \end{cases}$$

from which we see that we must have $\det B^{(0)}_{p_\pm}(x,t) = \det B^{(0)}_{\overline{p_\pm}}(x,t) = 0$. ∎

Proof of Theorem 3 (Linear algebraic-integral equations). Here we convert the RHP 1 to a set of linear algebraic-integral equations.

Letting

$$\widehat{M}(k) = M(k) - \mathrm{I} - \sum_{n=1}^{N} \frac{\mathrm{Res}_{\xi=k_n}[M(\xi)]}{k-k_n} - \sum_{n=1}^{N} \frac{\mathrm{Res}_{\xi=\overline{k_n}}[M(\xi)]}{k-\overline{k_n}},$$

we see that $\widehat{M}(k)$ satisfies a modified RHP, similar to the RHP 1, but where the jump condition (95) is replaced by

$$\widehat{M}^+(k) = \widehat{M}^-(k) + M^-(k)(J(k)-\mathrm{I}), \quad k \in \Sigma^o,$$

and $\widehat{M}(k) = O(1/k)$ as $k \to \infty$. Introducing the Cauchy projector,

$$P[f](k) = \frac{1}{2\pi i}\int_\Sigma \frac{f(\xi)}{\xi - k} d\xi, \quad k \in \mathbb{C} \setminus \Sigma$$

for $f : \Sigma \to \mathbb{C}$ and applying P to $\widehat{M}^\pm(k)$, we see that

$$P[\widehat{M}^\pm](k) = \frac{1}{2\pi i}\left[\int_{\Sigma_{+2}\cup\Sigma_{-2}} \frac{\widehat{M}^+(\xi)}{\xi-k}d\xi + \int_{\Sigma_{+1}\cup\Sigma_{-1}} \frac{\widehat{M}^-(\xi)}{\xi-k}d\xi\right] + \begin{cases} \pm\widehat{M}(k), & k \in \mathbb{C}^\pm \setminus \Sigma, \\ 0, & k \in \mathbb{C}^\mp \setminus \Sigma. \end{cases}$$

The above expression is obtained using the analyticity of $\widehat{M}(k)$ in $\mathbb{C}\setminus\Sigma$ to close the contour in the appropriate half-plane. To do so, one must add and subtract the integral along the "opposite" side of Σ_+ and Σ_- in that half-plane.

Specifically, consider $P[\widehat{M}^+](k)$. Writing

$$\begin{aligned} P[\widehat{M}^+](k) &= \frac{1}{2\pi i}\left[\int_{\Sigma_{+2}\cup\Sigma_{-2}} \frac{\widehat{M}^+(\xi)}{\xi-k}d\xi + \int_{\Sigma_{+1}\cup\Sigma_{-1}} \frac{\widehat{M}^-(\xi)}{\xi-k}d\xi\right] \\ &\quad + \frac{1}{2\pi i}\left[\int_{\mathbb{R}\cup\Sigma_{+1}\cup\Sigma_{-1}} \frac{\widehat{M}^+(\xi)}{\xi-k}d\xi + \int_{-(\Sigma_{+1}\cup\Sigma_{-1})} \frac{\widehat{M}^-(\xi)}{\xi-k}d\xi\right], \end{aligned}$$

we see that the contour for the last two terms can be closed in the upper half-plane (see Figure 5). When k is in the upper half-plane, these last two terms then reduce to $\widehat{M}(k)$, while they reduce to 0 when k is in the lower half-plane. Similar argument gives the corresponding result for $P[\widehat{M}^-](k)$.

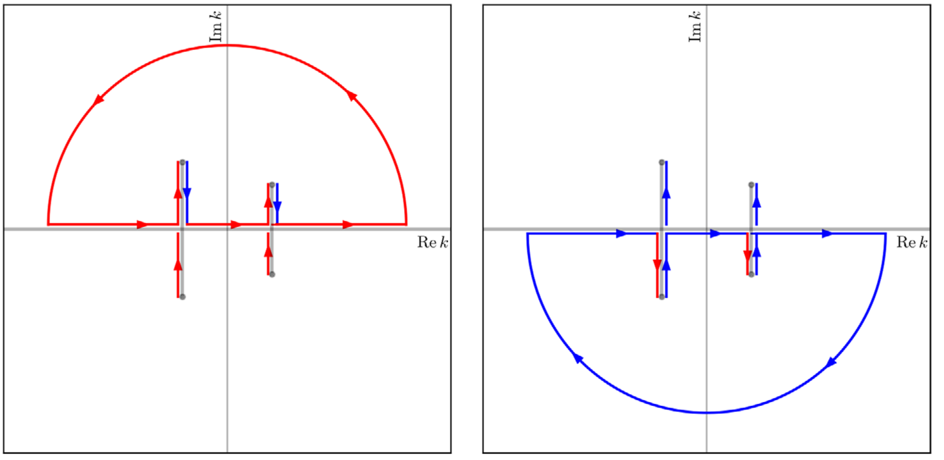

FIGURE 5 Closing the contours of integration for the Cauchy projector P applied to $\widehat{M}^+$ (left) and $\widehat{M}^-$ (right). Red and blue contours involve integrals with $\widehat{M}^+$ and $\widehat{M}^-$, respectively. The integrals over the blue (red) contours on the left (right) are added and subtracted to close the contour in the upper (lower) half-plane

Applying P to the jump condition for $\widehat{M}(k)$ then gives

$$\widetilde{M}(k) = \frac{1}{2\pi i}\int_{\Sigma}\frac{M^-(\xi)(J(\xi)-I)}{\xi - k}d\xi, \quad k \in \mathbb{C}\setminus\Sigma.$$

This then gives the algebraic-integral equation

$$M(k) = \mathrm{I} + \sum_{n=1}^{N}\frac{\mathrm{Res}_{\xi=k_n}[M(\xi)]}{k-k_n} + \sum_{n=1}^{N}\frac{\mathrm{Res}_{\xi=\overline{k_n}}[M(\xi)]}{k-\overline{k_n}} + \frac{1}{2\pi i}\int_{\Sigma}\frac{M^-(\xi)(J(\xi)-I)}{\xi-k}d\xi, \quad k\in\mathbb{C}\setminus\Sigma. \tag{178}$$

Evaluating the first and second columns of (176) at $k = k_n$ and $k = \overline{k_n}$, respectively, and using the residue conditions (84) then gives (104b) and (104c), which closes the system.

Recall that $M^-(k)$ has singularities at the branch points. As such, the convergence of the above improper integrals over Σ must be considered. The growth conditions (102) show that $M(k)$ has at worst fourth-root singularity at the branch points. On the other hand, Corollary 17 shows that $J(k)$ is continuous at the branch points. Thus we see that the integrands in each improper integral have at worst fourth-root singularity at the branch points, so that the integrals converge. ∎

Proof of Lemma 22 ($M(x,t,k)$ satisfies the modified Lax pair). We now show that if $M(x,t,k)$ solves the RHP 1 then it also satisfies the modified Lax pair (3) with $Q(x,t)$ given by (105), under the assumption of an appropriate vanishing lemma. Specifically, we assume that a modified RHP 1 with the normalization instead given as $M(x,t,k) = O(1/k)$, $k\to\infty$ has only the trivial solution.

We first define $\widetilde{X}(x,t,k)$ and $\widetilde{T}(x,t,k)$ by

$$\begin{aligned}\widetilde{X}(x,t,k)M(x,t,k) &= M_x(x,t,k) - ik[\sigma_3, M(x,t,k)] - Q(x,t)M(x,t,k),\\ \widetilde{T}(x,t,k)M(x,t,k) &= M_t(x,t,k) + 2ik^2[\sigma_3, M(x,t,k)]\\ &\quad - i\sigma_3\big(Q_x(x,t) - Q^2(x,t)\big)M(x,t,k) + 2kQ(x,t)M(x,t,k),\end{aligned}$$

where $Q(x,t)$ is given by (105). In particular, $Q(x,t) = -i[\sigma_3, M^{(1)}(x,t)]$, where $M^{(1)}(x,t)$ is defined by the asymptotic expansion

$$M(x,t,k) = \mathrm{I} + \frac{M^{(1)}(x,t)}{k} + \frac{M^{(2)}(x,t)}{k^2} + O(1/k^3). \tag{179}$$

We will show that $\widetilde{X}(x,t,k)M(x,t,k)$ and $\widetilde{T}(x,t,k)M(x,t,k)$ satisfy the same jump condition as $M(x,t,k)$ but are $O(1/k)$ as $k \to \infty$. Assumption the above stated vanishing lemma, we then conclude that

$$\widetilde{X}(x,t,k)M(x,t,k) = 0, \quad \widetilde{T}(x,t,k)M(x,t,k) = 0. \tag{180}$$

Indeed, using the asymptotic expansion (177), simple algebra yields $\widetilde{X}(x,t,k)M(x,t,k) = O(1/k)$ as $k \to \infty$. Moreover,

$$\begin{aligned}(\widetilde{X}(x,t,k)M(x,t,k))^+ &= \widetilde{X}(x,t,k)(M^-(x,t,k)J(x,t,k)) \\ &= M_x^-(x,t,k)J(x,t,k) + M^-(x,t,k)J_x(x,t,k) \\ &\quad - ik[\sigma_3, M^-(x,t,k)J(x,t,k)] - Q(x,t)M^-(x,t,k)J(x,t,k),\end{aligned}$$

where $J(x,t,k)$ is the jump matrix (97). Noting that $J_x(x,t,k) = ik[\sigma_3, J(x,t,k)]$ and more algebra gives

$$(\widetilde{X}(x,t,k)M(x,t,k))^+ = (\widetilde{X}(x,t,k)M(x,t,k))^- J(x,t,k), \quad k \in \Sigma,$$

and so we conclude that

$$\widetilde{X}(x,t,k)M(x,t,k) = 0. \tag{181}$$

To see that $\widetilde{T}(x,t,k)M(x,t,k)$ is $O(1/k)$, note that

$$Q^2(x,t) = i\sigma_3 Q(x,t)M^{(1)}(x,t) + iQ(x,t)M^{(1)}(x,t)\sigma_3,$$

$$Q_x(x,t) = 2\big[M^{(2)}\sigma_3, \sigma_3\big] - i\big[\sigma_3, Q(x,t)M^{(1)}(x,t)\big].$$

In the above, we have used the facts that σ_3 and $Q(x,t)$ anticommute and that (179) implies

$$M_x^{(1)}(x,t) = i\big[\sigma_3, M^{(2)}(x,t)\big] + Q(x,t)M^{(1)}(x,t).$$

From there, straightforward algebra shows that $\widetilde{T}(x,t,k)M(x,t,k) = O(1/k)$ as $k \to \infty$. Arguing as with the jump for $\widetilde{X}(x,t,k)M(x,t,k)$, instead noting that $J_t(x,t,k) = -2ik^2[\sigma_3, J(x,t,k)]$, shows that

$$(\widetilde{T}(x,t,k)M(x,t,k))^+ = (\widetilde{T}(x,t,k)M(x,t,k))^- J(x,t,k), \quad k \in \Sigma,$$

from which we conclude (178). Equivalently, $M(x,t,k)$ satisfies the modified Lax pair (94) with $Q(x,t)$ defined by (105). ■

Proof of Corollary 22 (Reconstruction formula). We now find the solution $q(x,t)$ corresponding to a solution $M(x,t,k)$ to the RHP 1, assuming the same vanishing lemma used in Lemma 22.

Lemma 22 shows that $M(x,t,k)\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}$ satisfies the Lax pair (3) with $Q(x,t)$ given by (105). Then defining

$$q(x,t) := Q_{12}(x,t) = -2i \lim_{k\to\infty} [kM_{12}(x,t,k)], \tag{182}$$

we see that $q(x,t)$ satisfies the NLS equation (1) as the compatibility condition of the Lax pair (3). Moreover, from (176) we see

$$M(k) = \mathrm{I} + \frac{1}{k}\left[\sum_{n=1}^{N} \mathrm{Res}_{\xi=k_n}[M(\xi)] + \sum_{n=1}^{N} \mathrm{Res}_{\xi=\overline{k_n}}[M(\xi)] + \frac{1}{2\pi i}\int_\Sigma M^-(\xi)(J(\xi)-I)d\xi\right]$$
$$+o(1/k), \quad k\to\infty. \tag{183}$$

Combining (100), (180), and (181) then gives (106). ■

Proof of Theorem 5 (Existence and uniqueness of solutions for the modified RHP 2). Here we establish the existence of a unique solution for the modified RHP 2. We do so by showing that the jump matrix $\widetilde{J}(x,t,k)$ satisfies the conditions of Lemma 23.

Condition (a) is satisfied by the assumptions that $\rho(k) \in C^1(\mathbb{R}\setminus(-R,R))$ and $C(k) \in C^1(\partial B_R \cap \mathbb{C}^\pm)$. Condition (b) is satisfied by the assumption $C(k)C^\dagger(k) = \mathrm{I}$. It is straightforward to verify that condition (c) is met due to the structure of $\widetilde{J}(x,t,k)$. Indeed, for $k \in \widetilde{\Sigma}\cap\mathbb{R}$ we have $\widetilde{J}^\dagger(x,t,k) = \widetilde{J}(x,t,k)$, so that $\widetilde{J}(x,t,k)$ is an invertible Hermitian matrix and is thus positive definite. We then conclude that $\mathrm{Re}\,\widetilde{J}(x,t,k)$ is positive definite. ■

Proof of Theorem 6 (Uniqueness of solutions for the original RHP 1). The proof relies on the existence of a solution of the original RHP at $(x,t) = (0,0)$ to construct the map F_o. We need to show that $\widetilde{M}(x,t,k) = F_o(M)(x,t,k)$ solves the modified RHP 2 (after filling all removable singularities) for any solution $M(x,t,k)$ of the original RHP 1. The key here is that, even though $C_o(k)$ is fixed by this given solution, when applied to *any other* solution $M(x,t,k)$ of the original RHP, the map $F_o(M)$ still produces a solution of the modified RHP. Essentially, this is because $\Phi(x,t,k) = M(x,t,k)\,\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}$ [cf. (93)] is a fundamental matrix solution of the Lax pair. However, given any fundamental matrix solution $\phi(x,t,k)$ of both parts of the Lax pair, $\phi(x,t,k)c(k)$ is also a fundamental matrix solution of both parts of the Lax pair for any invertible matrix $c(k)$.

It is immediately clear that the normalization at infinity is satisfied, and that the jump condition is satisfied on $(-\infty,R)\cup(R,\infty)$. It is straightforward to check that $\widetilde{M}(x,t,k)$ satisfies the jump condition on ∂B_R. Indeed, we have

$$\widetilde{M}^+(x,t,k) = M(x,t,k) = \widetilde{M}^-(x,t,k)\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}C_o(k)\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}, \quad k\in\partial B_R\cap\mathbb{C}^+,$$

which verifies the jump on $\partial B_R\cap\mathbb{C}^+$, with similar calculation for the jump on $\partial B_R\cap\mathbb{C}^-$. Continuing, it is easy to show that the jump on $\Sigma\cap B_R$ is removed. Indeed, for $k\in\Sigma\setminus B_R$ we have

$$\widetilde{M}^+(x,t,k) = M^+(x,t,k)\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}(C_o^+(k))^{-1}\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}$$
$$= M^-(x,t,k)J(x,t,k)\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}(C_o^-(k)J_o(k))^{-1}\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}$$

$$= M^-(x,t,k)J(x,t,k)\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}J_o^{-1}(k)\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}(C_o^-(k))^{-1}\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3}$$
$$= \widetilde{M}^-(x,t,k),$$

since $J(x,t,k)\mathrm{e}^{i\theta_o(x,t,k)\sigma_3}J_o^{-1}(k)\mathrm{e}^{-i\theta_o(x,t,k)\sigma_3} = \mathrm{I}$.

It remains to be shown that $\widetilde{M}(x,t,k)$ is analytic at the discrete eigenvalues and branch points. Straightforward algebra using the residue conditions for $M(x,t,k)$ and $C_o(k)$ shows that $\widetilde{M}(x,t,k)$ has no residues at the discrete eigenvalues. Finally, to see that $\widetilde{M}(x,t,k)$ is analytic at the branch points, the growth conditions (102) for $M(x,t,k)$ and $C_o(k)$ in the generic case show that $\widetilde{M}(x,t,k)$ has well-defined limit and is indeed analytic at the branch points (note that applying determinants to the growth conditions shows that the limiting matrix values must be invertible). We then see that $\widetilde{M}(x,t,k)$ satisfies the RHP 2 with $C_o(k)$ in place of $C(k)$. As in Theorem 5, the assumptions that $\rho(k) \in C^1(\mathbb{R} \setminus (-R,R))$ and $C_o(k) \in C^1(\partial B_R \cap \mathbb{C}^\pm)$ together with $C_o(k)C_o^\dagger(k) = \mathrm{I}$ show that this modified RHP admits a unique solution.

We now show that solutions of the original RHP are unique. Suppose that $M(x,t,k)$ and $M'(x,t,k)$ are two distinct solutions of the original RHP, and let $\widetilde{M}(x,t,k) = F_o(M)(x,t,k)$ and $\widetilde{M}'(x,t,k) = F_o(M')(x,t,k)$. Since we have already proved that solutions of the modified RHP are unique, we have that $\widetilde{M}(x,t,k) = \widetilde{M}'(x,t,k)$. However, since the map (112) is invertible, the equality of $\widetilde{M}(x,t,k)$ and $\widetilde{M}'(x,t,k)$ immediately implies the equality of $M(x,t,k)$ and $M'(x,t,k)$. ■

7.3 | Reductions

Proof of Lemma 25 (Second symmetry, scattering coefficients with $V = 0$). Here we recompute the jumps of the scattering coefficients across the branch cuts in the case that $V = 0$.

The Wronskian representations (113) give

$$s_{22}^+(k) = \mathrm{Wr}\left[\phi_{+1}^+(x,t,k), \phi_{-2}^+(x,t,k)\right], \quad k \in \Sigma_{+1}^o.$$

The calculation for $k \in \Sigma_{+1}^o \setminus \Sigma_{-1}$ follows exactly as in the proof of Lemma 11 for $k \in \Sigma_{+1}^o$. We proceed with the calculation for $k \in \Sigma_{+1}^o \cap \Sigma_{-1}^o$. From Lemma 10 and the Wronskian representation (60a) we see

$$s_{22}^+(k) = \mathrm{e}^{-2i\delta}s_{11}(k), \quad k \in \Sigma_{+1}^o \cap \Sigma_{-1}^o.$$

The symmetries (77) give the corresponding jump for $s_{11}(k)$. The jumps for $r_{11}(k)$ and $r_{22}(k)$ are then given by (67). ■

Proof of Lemma 27 (Calculation of the jump matrices for $V = 0$). We now compute the jumps in the special case $V = 0$. We need only compute the jumps for $k \in \Sigma_{+1}^o \cap \Sigma_{-1}^o$ and $k \in \Sigma_{+2}^o \cap \Sigma_{-2}^o$, as remaining jumps on $\mathbb{R}$, $\Sigma_{+1}^o \setminus \Sigma_{-1}$ and $\Sigma_{+2}^o \setminus \Sigma_{-2}$ will match the corresponding jumps on $\mathbb{R}$, Σ_{+1}^o and Σ_{+2}^o with $V \neq 0$. Note that (58) and (62) can now be extended to $(\Sigma_{+1}^o \cap \Sigma_{-1}^o) \cup (\Sigma_{+2}^o \cap \Sigma_{-2}^o)$.

We first consider the jump for $k \in \Sigma_{+1}^o \cap \Sigma_{-1}^o$. Unlike in the case of nonzero velocity, both $\phi_{+1}(x,t,k)$ and $\phi_{-2}(x,t,k)$ have jumps across the contour. However, we again find

$$\phi^+_{+1}(x,t,k) = i\rho(k)\mathrm{e}^{-i\delta}\phi_{+1}(x,t,k) - i\mathrm{e}^{-i\delta}\frac{\phi_{-2}(x,t,k)}{s_{22}(k)}, \quad k \in \Sigma^o_{+1} \cap \Sigma^o_{-1}.$$

From Lemma 26, we have

$$\frac{\phi_{-1}(x,t,k)}{s_{11}(k)} = (1+\rho(k)\overline{\rho(\overline{k})})\phi_{+1}(x,t,k) - \overline{\rho(\overline{k})}\frac{\phi_{-2}(x,t,k)}{s_{22}(k)}, \quad k \in \Sigma^o_{+1} \cap \Sigma^o_{-1}.$$

Combining with (81d) and Lemma 118 gives

$$\left(\frac{\phi_{-2}(x,t,k)}{s_{22}(k)}\right)^+ = -i(1+\rho(k)\overline{\rho(\overline{k})})\mathrm{e}^{i\delta}\phi_{+1}(x,t,k) + i\overline{\rho(\overline{k})}\mathrm{e}^{i\delta}\frac{\phi_{-2}(x,t,k)}{s_{22}(k)}, \quad k \in \Sigma^o_{+1} \cap \Sigma^o_{-1}.$$

Together, we have

$$\Phi^+(x,t,k) = \Phi^-(x,t,k)\begin{pmatrix} i\rho(k)\mathrm{e}^{-i\delta} & -i(1+\rho(k)\overline{\rho(\overline{k})})\mathrm{e}^{i\delta} \\ -i\mathrm{e}^{-i\delta} & i\overline{\rho(\overline{k})}\mathrm{e}^{i\delta} \end{pmatrix}, \quad k \in \Sigma^o_{+1} \cap \Sigma^o_{-1}. \tag{184}$$

The symmetry (172) gives

$$\Phi^+(x,t,k) = \Phi^-(x,t,k)\begin{pmatrix} -i\rho(k)\mathrm{e}^{-i\delta} & -i\mathrm{e}^{i\delta} \\ -i(1+\rho(k)\overline{\rho(\overline{k})})\mathrm{e}^{-i\delta} & -i\overline{\rho(\overline{k})}\mathrm{e}^{i\delta} \end{pmatrix}, \quad k \in \Sigma^o_{+2} \cap \Sigma^o_{-2}. \tag{185}$$

■

7.4 | Riemann problems

Proof of Lemma 30 (Pure two-sided step with counterpropagating flows). Here we consider the possible zeros and poles of $s_{22}(k)$, $\rho(k)$ and $r(k)$ for $V \neq 0$ and $0 < A_- \leq A_+$.

Suppose $s_{22}(k) = 0$. Then

$$(\lambda_- + (k - V/2))A_+\mathrm{e}^{i\delta} + (\lambda_+ - (k + V/2))A_-\mathrm{e}^{-i\delta} = 0$$

and so $|\lambda_- + (k - V/2)|A_+ = |\lambda_+ - (k + V/2)|A_-$. Note that

$$|\lambda_- + (k - V/2)| \geq A_-, \quad k \in \mathbb{C},$$

$$|\lambda_- + (k - V/2)| = A_- \Leftrightarrow k \in \Sigma_-,$$

$$|\lambda_+ - (k + V/2)| \leq A_+, \quad k \in \mathbb{C},$$

$$|\lambda_+ - (k + V/2)| = A_+ \Leftrightarrow k \in \Sigma_+.$$

Since Σ_+ and Σ_- are disjoint, at least one of the above inequalities is strict. Then the above equality implies $A_-A_+ < A_+A_-$. Thus, $s_{22}(k) \neq 0$ for all $k \in \mathbb{C}$, meaning that there are no discrete eigenvalues.

In addition, with $\delta = 0$, if $\rho(k) = 0$ then

$$(\lambda_+ + (k + V/2))A_- = (\lambda_- + (k - V/2))A_+.$$

Squaring, expressing $A_\pm^2$ in terms of $\lambda_\pm, k$, and canceling common factors gives

$$(\lambda_+ + (k + V/2))(\lambda_- - (k - V/2)) = (\lambda_- + (k - V/2))(\lambda_+ - (k + V/2)).$$

Expanding and simplifying gives $(k - V/2)\lambda_+ = (k + V/2)\lambda_-$, which after squaring again and simplifying gives

$$(A_+^2 - A_-^2)k^2 - V(A_+^2 + A_-^2)k + \frac{V^2}{4}(A_+^2 - A_-^2) = 0. \tag{186}$$

If $A_+ = A_-$ then $\rho(k)$ has a possible zero at $k = 0$. Plugging back into $\rho(k)$ shows that this is not an actual zero.

On the other hand, if $A_+ \neq A_-$, we get two possible zeros, at

$$k_\pm = \frac{V}{2}\left(\frac{A_+ \pm A_-}{A_+ \mp A_-}\right).$$

Plugging back into $\rho(k)$, we see that k_+ is indeed a zero of $\rho(k)$ for $A_+ \neq A_-$, while k_- is not.

If $A_+ = A_-$, then $r(k)$ has no singularities. On the other hand, if $A_+ \neq A_-$, then $r(k)$ has singularity at $k = k_+$, finishing the argument. ∎

Behavior at the branch points for the pure two-sided step with counterpropagating flows. Here we consider the linear dependence of the modified eigenfunctions $\mu_{+1}(x, t, k)$ and $\mu_{-2}(x, t, k)$ at the branch points for $V \neq 0$ and $0 < A_- \leq A_+$.

From (130), we see that

$$\mathrm{Wr}[\mu_{+1}(x, t, k), \mu_{-2}(x, t, k)] = 1 + \frac{A_+A_-}{(\lambda_+ + (k + V/2))(\lambda_- + (k - V/2))}\mathrm{e}^{-2i\delta}.$$

Suppose that $\mathrm{Wr}[\mu_{+1}(x, t, p_+), \mu_{-2}(x, t, p_+)] = 0$. Straightforward algebra shows that the requiring the right-hand side of the above expression to vanish implies $A_- \cos(2\delta) - A_+ = iV$. Since $V \neq 0$, this cannot be true. Similar calculation shows that

$\mathrm{Wr}[\mu_{+1}(x, t, p_-), \mu_{-2}(x, t, p_-)] \neq 0.$

Proof of Lemma 31 (Pure two-sided step without counterpropagating flows). Here we consider the possible zeros and poles of $s_{22}(k)$ and $\rho(k)$ for $V = 0$ and $0 < A_- \leq A_+$.

Suppose $s_{22}(k) = 0$ with $\delta = 0$. Following the same argument as for $V \neq 0$, we now must have $A_+(\lambda_+ + k) = -A_-(\lambda_+ - k)$, and $k \in \Sigma_+ \cap \Sigma_- = \Sigma_-$. Letting $k = is$ with $s \in [-A_-, A_-]$, we have

$$A_+\left(\sqrt{A_-^2 - s^2} + is\right) = -A_-\left(\sqrt{A_+^2 - s^2} - is\right).$$

If $A_+ = A_- = A$, then $s_{22}(k) = 0$ at $k = \pm iA$. In such case, $\rho(k) = 0$ identically and the jumps across $\Sigma_{+1} \setminus \Sigma_{-1}$ and $\Sigma_{+2} \setminus \Sigma_{-2}$ disappear. On the other hand, if $A_+ \neq A_-$, then $s_{22}(k) \neq 0$ for any $k \in \mathbb{C}$, and there are no discrete eigenvalues.

Note that the calculation done to arrive at (184) can again be used here, now with $V = 0$. Then if $\rho(k) = 0$, we have $(A_+^2 - A_-^2)k^2 = 0$. If $A_+ = A_-$, then as stated before $\rho(k)$ is identically zero. On the other hand, if $A_+ \neq A_- > 0$, then $\rho(k)$ has a possible zero at $k = 0$. Plugging back into $\rho(k)$ verifies that this is indeed a zero. ■

Behavior at the branch points for the pure two-sided step without counterpropagating flows. Here we consider the linear dependence of the modified eigenfunctions $\mu_{+1}(x, t, k)$ and $\mu_{-2}(x, t, k)$ at the branch points for $V = 0$ and $0 < A_- \leq A_+$.

From (136), we see that

$$\mathrm{Wr}[\mu_{+1}(x, t, k), \mu_{-2}(x, t, k)] = 1 + \frac{A_+ A_-}{(\lambda_+ + k)(\lambda_- + k)} \mathrm{e}^{-2i\delta}.$$

Straightforward algebra shows that requiring the right-hand side of the above expression to vanish implies $A_- \cos(2\delta) = A_+$. Since $A_- \leq A_+$, this can only happen when $A_- = A_+ = A$ and $\delta = n\pi$ (ie, no phase difference). In such case, we have $s_{22}(k) \equiv 1$.

8 | CONCLUSIONS

In summary, we presented the formulation of the IST for the focusing NLS with a general class of NZBCs at infinity consisting of counterpropagating waves. The spectrum of the scattering problem is characterized by the presence of four distinct branch points. Thus, even if one takes into account the multivaluedness of the asymptotic eigenvalues by introducing a suitable two-sheeted Riemann surface, the resulting curve has genus one and, therefore, it is not possible to introduce a uniformization variable to map it back to a single copy of the complex plane. Accordingly, we developed the formalism by explicitly taking into account the nonanalyticity of the asymptotic eigenvalues and by making a suitable choice of branch cuts. We also explicitly studied the limiting behavior of the Jost eigenfunctions and scattering coefficients at the branch points. We formulated the inverse problem as a matrix RHP with jumps along the real axis and the branch cuts, converted the problem to a set of linear algebraic-integral equations, and obtained a reconstruction formula for the potential. We discussed several exact reductions as special cases. One of them is the case when no counterpropagating flows are present, namely, $V = 0$, which had been studied in Ref. 33. Even in that case, however, our formalism is slightly different from that of Ref. 33 in a few respects (such as proof of analyticity, different sectionally meromorphic matrix, etc.). Finally, we considered various Riemann problems as specific examples.

The availability of the IST makes it possible to calculate the long-time asymptotic behavior of solutions with the given class of ICs. Similar problems were recently considered in Ref. 49 using the genus-one Whitham modulation equations, and it was shown that, in many cases, Whitham theory provides an effective asymptotic description for the behavior of solutions. However, it was also shown there that there are many cases in which the genus-one Whitham equations are not sufficient to fully characterize the behavior of solutions. To fully describe those cases, either higher-genus theory or the full power of the IST are needed. Moreover, even when effective, Whitham theory is only a formal perturbation theory, and does not provide rigorous estimates. The long-time asymptotics using the IST was computed in Refs. 30, 31 in the special case $A_+ = A_-$ and $V = 0$. Until recently, the case $V \neq 0$ was only studied in a special case (a Riemann problem with equal amplitudes and $V > 0$) in Ref. 38. Moreover, even in that case, the analysis only applies to the case of large V.

While in the process of finalizing the present manuscript, we learned that a similar problem was also independently considered in a recent preprint,[52] where the IST was concisely formulated and various scenarios for the long-time asymptotics were presented and discussed. The main differences between the formalism of the IST in Ref. 52 and the one in the present work are that a different normalization was used for the Jost eigenfunctions, that no discrete spectrum for the scattering problem, and consequently no poles in the RHP, were allowed in Ref. 52, and that the issue of existence and uniqueness of solutions of the RHP was not addressed in Ref. 52. As is well known, each discrete eigenvalue contributes a soliton to the solution of the NLS equation. On one hand, as shown in Ref. 53, discrete eigenvalues greatly complicate the long-time asymptotics; on the other hand, as shown in Ref. 32, the presence of discrete eigenvalues leads to very interesting interaction phenomena between solitons and radiation, including transmission, trapping, and the emergence of soliton-generated wakes.

As usual, the IST was developed under the assumption of existence of solution. However, one could use the results of the present work to prove well-posedness in appropriate function spaces. At the same time, as was discussed at length, the issue of existence and uniqueness of solutions of the RHP is nontrivial. This is because of the fact that the associated jumps occur along an open contour. Here, we addressed this problem by introducing a modified RHP using a similar approach as in Ref. 46. However, even in the case $V = 0$, it is not entirely clear what conditions must be included in the original RHP to ensure the existence of solutions. This question is left as a topic for future work. Another complication that is addressed by introduction of the modified RHP is the possible presence of zeros of the analytic scattering coefficients on the continuous spectrum Σ (cf. (28)). Such zeros lead to so-called spectral singularities in the RHP.[45] We have shown (cf. Lemma 13) that, when $V \neq 0$, no such singularities are possible on Σ_+^o and Σ_-^o. However, one could have singularities for $k \in \mathbb{R}$. Moreover, when $V = 0$ the scattering coefficients could vanish on the portion of Σ where all four Jost eigenfunctions are defined (cf. Lemma 26). The presence of spectral singularities not only complicates the analysis, but also leads to different long-time asymptotic behavior for the solutions (eg, see Refs. 4, 54 for the case of zero BCs).

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