On the Spectrum of the Dirac Operator and the Existence of Discrete Eigenvalues for the Defocusing Nonlinear Schrödinger Equation

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We revisit the scattering problem for the defocusing nonlinear Schrödinger equation with constant, nonzero boundary conditions at infinity, i.e., the eigenvalue problem for the Dirac operator with nonzero rest mass. By considering a specific kind of piecewise constant potentials we address and clarify two issues, concerning: (i) the (non)existence of an area theorem relating the presence/absence of discrete eigenvalues to an appropriate measure of the initial condition; and (ii) the existence of a contribution to the asymptotic phase difference of the potential from the continuous spectrum.

1. Introduction

The nonlinear Schrödinger (NLS) equation is a universal model for the behavior of weakly nonlinear, quasi-monochromatic wave packets, and it arises in a variety of physical settings, such as nonlinear optics, plasmas, water waves, Bose–Einstein condensation, etc. In particular, the so-called defocusing NLS equation,

$$iq_t + q_{xx} - 2|q|^2 q = 0 \tag{1}$$

[where $q = q(x,t)$ and subscripts x and t denote partial differentiation throughout], describes the stable propagation of an electromagnetic beam in (cubic) nonlinear media with normal dispersion, and has been the subject

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of renewed applicative interest in the framework of recent experimental observations in Bose–Einstein condensates [1,2] and dispersive shock waves in optical fibers [3].

The defocusing NLS equation admits soliton solutions with nonzero boundary conditions (NZBCs) at infinity, so-called dark/gray solitons. The simplest, one-soliton solution has the form

$$q_s(x,t) = \mathrm{e}^{-2iq_o^2 t} q_o[\cos\alpha + i(\sin\alpha)\tanh[q_o(\sin\alpha)(x - x_o + 2q_o\, t\cos\alpha)]] \tag{2}$$

with $q_o > 0$, and α and x_o arbitrary real parameters. Dark solitons tend to a constant background, $q(x,t) \to q_\pm$ as $x \to \pm\infty$, with $|q_\pm| = q_o$, and appear as localized dips of intensity $q_o^2 \sin^2\alpha$. As in the case of zero boundary conditions (BCs), each dark soliton corresponds to a discrete eigenvalue of the scattering problem associated with the NLS equation (1) (see Section 2 for details).

Recall that the defocusing NLS equation (1) is the compatibility condition of the Lax pair

$$\Phi_x = -ik\sigma_3\Phi + Q\,\Phi, \quad \Phi_t = -2ik^2\sigma_3\Phi + T\,\Phi, \tag{3}$$

where $\Phi = \Phi(x,t,k)$ is a 2×2 matrix,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(x,t) = \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix},$$

and

$$T(x,t,k) = i\sigma_3(Q_x - Q^2) + 2kQ.$$

The first half of (3) is of course the Zakharov–Shabat scattering problem. As usual, the solution $q(x,t)$ of the NLS equation is referred to as the scattering potential and the eigenvalue k as the scattering parameter. To include dark solitons, one must study (1) with the BCs

$$q(x,t) \to q_\pm(t) = q_o\,\mathrm{e}^{-2iq_o^2 t + i\theta_\pm} \quad \text{as} \quad x \to \pm\infty,$$

with $\theta_\pm$ arbitrary real constants. The solution of the initial-value problem for the defocusing NLS equation with the above BCs by inverse scattering transform (IST) was first discussed almost 40 years ago [4], and the problem has been subsequently studied by several authors [5–12]. Nonetheless, many important issues still remain to be clarified for the IST with NZBCs, in spite of the experimental relevance of the problem. The main reason for this is that the IST is significantly more involved than in the case of decaying potentials (for which classic references exist, see for instance [13–15]), in particular as far as the analyticity of the eigenfunctions of the associated scattering problem is concerned. As a matter of fact, (the analogue of) Schwartz class is usually assumed for the potential (cf., for instance, [8,9]), which is clearly unnecessarily restrictive. In Refs. [10,11] the issue of establishing the analyticity of the eigenfunctions was addressed by reformulating the scattering problem in terms of a so-called energy dependent potential, but the drawback of that approach is a very complicated dependence of eigenfunctions and data on the scattering parameter.

A step forward in the direction of a rigorous IST was recently presented in Ref. [16], where, among other things, it was shown that the direct scattering problem is well defined for potentials q such that $q - q_\pm \in L^{1,2}(\mathbb{R}^\pm)$, where $L^{1,s}(\mathbb{R})$ is the complex Banach space of all measurable functions $f(x)$ for which $(1 + |x|)^s f(x)$ is integrable on the whole real line, and analyticity of eigenfunctions and scattering data was proved rigorously.

The purpose of this work is to address two further issues in the spectral theory of the scattering problem associated with the defocusing NLS equation with NZBCs. The first issue is whether an area theorem can be established, to relate the existence and location of discrete eigenvalues of the scattering problem to the area of the initial profile of the solution, suitably defined to take into account the NZBCs. For the focusing NLS equation with vanishing BCs, it was shown [17, 18] that there are no discrete eigenvalues and no spectral singularities if the L^1-norm of the potential is smaller that $\pi/2$. Conversely, it is well-known that no such result holds for the Korteweg-deVries equation, whose scattering problem [which is the time-independent Schrödinger equation] with positive initial datum can have discrete eigenvalues even for initial profiles with arbitrarily small area (see for instance [19]). Whether an area theorem can be proven for the existence of discrete eigenvalues for the defocusing NLS equation with NZBCs was still an open issue, however. The first result of this work is to show that no area theorem can be proved. We do so by providing explicit examples of box-type initial conditions (ICs) where at least one discrete eigenvalue exists, no matter how small the difference between the initial profile and the background field is.

The second issue addressed in this work is whether the radiative part of the spectrum can yield a nontrivial contribution to the asymptotic phase difference of the potential. It is well-known (see, for instance, [8]) that the trace formula for the NLS equation determines the asymptotic phase difference, $\arg(q_+/q_-)$, in terms of a contribution from the discrete spectrum and one from the continuous spectrum via the reflection coefficient (see Section 2 for details). In this work we show that the radiative component of the solution can indeed provide a nonzero contribution to the asymptotic phase difference of the potential. Again, we do so by explicitly providing examples of piecewise constant ICs corresponding to a nonzero asymptotic phase difference in the potential for which no discrete eigenvalues are present.

The relevance of the results presented in this work is twofold. From a theoretical point of view, recall that, for the defocusing NLS equation with NZBCs at infinity, the scattering problem is equivalent to the eigenvalue problem for the one-dimensional Dirac operator with nonzero rest mass. [That is, the first of (3) is equivalent to the eigenvalue problem $i\sigma_3(\partial_x - Q)\,\Phi = k\,\Phi$.] Therefore, the results in this work are also statements on the spectral properties of said operator.

Conversely, from an applied point of view, these results are relevant in the context of recent theoretical studies and experimental observations of defocusing NLS equation in the framework of dispersive shock waves in optical fibers (see, for instance, [3] regarding the appearance and evolution of dispersive shock waves when an input (reflectionless) pulse containing a large number of dark or gray solitons is injected in the fiber).

2. IST for the defocusing NLS equation with NZBCs

In this work we use a different, more convenient normalization for the eigenfunctions than the one commonly adopted in the literature; namely, one which allows the reduction $q_o \to 0$ to be taken continuously. To establish our notation, we therefore briefly review the IST for the defocusing NLS equation (1) with NZBCs. We refer the reader to [4, 8, 16] for further details.

It is also convenient to take both asymptotic values $q_\pm$ to be constant in time. To do so in a way that is compatible with the time evolution prescribed by (1), one can perform a trivial rescaling to remove the constant rotation due to the background field, obtaining a modified NLS equation in the form

$$iq_t + q_{xx} + 2\left(q_o^2 - |q|^2\right) q = 0. \tag{4}$$

The corresponding BCs then become

$$q(x,t) \to q_\pm = q_o\,\mathrm{e}^{i\theta_\pm} \quad \text{as} \quad x \to \pm\infty, \tag{5}$$

where $q_\pm$ are now independent of time. The Lax pair for the new NLS equation (4) is still given by (3), but where now

$$T(x,t,k) = i\sigma_3(Q_x - Q^2) + 2kQ + iq_o^2\sigma_3.$$

Hereafter, the asterisk denotes complex conjugation.

With the BCs (5), the asymptotic scattering problems [obtained by replacing Q with $Q_\pm = \lim_{x\to\pm\infty} Q(x,t)$ in (3)] have eigenvalues $\pm i\lambda$ where $\lambda^2 = k^2 - q_o^2$. It is then convenient to think of the variable k as belonging to a Riemann surface $\mathbb{K}$ consisting of a sheet $\mathbb{K}^+$ and a sheet $\mathbb{K}^-$ each coinciding with the complex plane cut along the semilines $\Sigma = (-\infty, -q_o] \cup [q_o, \infty)$, with its edges glued in such a way that $\lambda(k)$ is continuous through the cut. Without loss of generality, we label these sheets such that $\operatorname{Im}\lambda(k) > 0$ on $\mathbb{K}^+$ and $\operatorname{Im}\lambda(k) < 0$ on $\mathbb{K}^-$.

The continuous spectrum of the scattering problem is the set of all values of k for which $\lambda(k) \in \mathbb{R}$, namely $k \in \Sigma$. For all $k \in \Sigma$, the Jost solutions are defined as the simultaneous solutions of both parts of the Lax pair identified by the BCs

$$\Phi_\pm(x,t,k) \equiv Y_\pm(k)\,\mathrm{e}^{-i\Omega(x,t,k)\sigma_3} + o(1) \quad \text{as } x \to \pm\infty, \tag{6}$$

where

$$Y_\pm(k) = I - \frac{i}{k+\lambda}\sigma_3 Q_\pm, \tag{7}$$

$\Omega(x,t,k) = \lambda(x-2kt)$ and where the columns of $Y_\pm(k)$ are the corresponding eigenvectors of the asymptotic scattering problems.

Modified eigenfunctions with constant limits as $x \to \pm\infty$ can then be defined as $M_\pm(x,t,k) = \Phi_\pm(x,t,k)\,\mathrm{e}^{i\Omega(x,t,k)\sigma_3}$. If $q(x,t) - q_\pm$ vanishes sufficiently fast as $x \to \pm\infty$, one can write Volterra integral equations that define the modified eigenfunctions $\forall k \in \Sigma$. We use the notation $A = (A_1, A_2)$ to identify the columns of a 2×2 matrix A. The columns of $M_\pm$ can then be analytically extended into the complex $\mathbb{K}^\pm$ planes: $M_{-,1}$ and $M_{+,2}$ in $\mathbb{K}^+$, $M_{-,2}$ and $M_{+,1}$ in $\mathbb{K}^-$. The analyticity properties of $\Phi_\pm$ follow accordingly.

Because $\det\Phi_\pm = 2\lambda/(\lambda+k)$ $\forall (x,t) \in \mathbb{R}^2$ and $\forall k \in \Sigma \setminus \{\pm q_o\}$, Φ_- and Φ_+ are both fundamental matrix solutions of both parts of the Lax pair, and one can introduce the time-independent scattering matrix $S(k)$ as

$$\Phi_-(x,t,k) = \Phi_+(x,t,k)\,S(k), \quad k \in \Sigma \setminus \{\pm q_o\}. \tag{8}$$

We denote the entries of $S(k)$ as

$$S(k) = \begin{pmatrix} a(k) & \bar{b}(k) \\ b(k) & \bar{a}(k) \end{pmatrix}. \tag{9}$$

Since $\det\Phi_+ = \det\Phi_-$, we have $\det S(k) = 1$ for all $k \in \Sigma \setminus \{\pm q_o\}$. Moreover, the entries of the scattering matrix can be written in terms of Wronskians of the Jost solutions. In particular,

$$a(k) = \frac{\lambda+k}{2\lambda}\mathrm{Wr}(\Phi_{-,1},\ \Phi_{+,2}), \quad \bar{a}(k) = -\frac{\lambda+k}{2\lambda}\mathrm{Wr}(\Phi_{-,2},\ \Phi_{+,1}), \tag{10}$$

which show that $a(k)$ and $\bar{a}(k)$ can be analytically continued respectively on $\mathbb{K}^+$ and $\mathbb{K}^-$ (and may have poles at the branch points $k = \pm q_o$). On the other hand, $b(k)$ and $\bar{b}(k)$ are continuous for $k \in \Sigma$ but in general nowhere analytic.

As eigenfunctions and scattering data depend on λ, they are only single-valued for $k \in \mathbb{K}$ [as opposed to $k \in \mathbb{C}$]. To identify the appropriate sheet of $\mathbb{K}$, we will therefore label the eigenfunctions by making the λ dependence explicit when necessary. For instance, the Jost eigenfunctions satisfy the following symmetry relations with respect to the involution $(k,\lambda) \to (k^*,\lambda^*)$:

$$\Phi_{\pm,1}(x,t,k,\lambda) = \sigma_1(\Phi_{\pm,2}(x,t,k^*,\lambda^*))^*, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{11}$$

implying

$$a(k,\lambda) = \bar{a}^*(k^*,\lambda^*), \quad b(k,\lambda) = \bar{b}^*(k,\lambda^*). \tag{12}$$

Then $\det S(k) = 1$ yields

$$|a(k)|^2 = 1 + |b(k)|^2, \quad k \in \Sigma \setminus \{\pm q_o\}. \tag{13}$$

The scattering problem also admits a second involution: $(k,\lambda) \to (k^*, -\lambda^*)$. The corresponding symmetries for the eigenfunctions are given by:

$$(\Phi_{\pm,1}(x,t,k^*,-\lambda^*))^* = \frac{iq_\pm}{\lambda - k}\sigma_1 \Phi_{\pm,1}(x,t,k,\lambda), \tag{14}$$

$$(\Phi_{\pm,2}(x,t,k^*,-\lambda^*))^* = \frac{-iq_\pm^*}{\lambda - k}\sigma_1 \Phi_{\pm,2}(x,t,k,\lambda). \tag{15}$$

In turn, these symmetries imply the following relations for the scattering data:

$$a^*(k^*,-\lambda^*) = \frac{q_-}{q_+}a(k,\lambda), \quad \bar{a}^*(k^*,-\lambda^*) = \frac{q_+}{q_-}\bar{a}(k,\lambda), \tag{16}$$

$$b^*(k,-\lambda) = -\frac{q_+}{q_-^*}b(k,\lambda), \quad \bar{b}^*(k,-\lambda) = -\frac{q_+^*}{q_-}\bar{b}(k,\lambda). \tag{17}$$

In view of the inverse problem, is convenient to introduce a uniformization variable z defined by the conformal mapping (cf. [8])

$$z = k + \lambda(k),$$

which is inverted by

$$k = \frac{1}{2}\big(z + q_o^2/z\big), \quad \lambda = \frac{1}{2}\big(z - q_o^2/z\big).$$

Then: (i) the two sheets $\mathbb{K}^+$ and $\mathbb{K}^-$ are mapped onto the upper and lower half-planes $\mathbb{C}^\pm$ of the complex z-plane, respectively; (ii) the cut Σ on the Riemann surface is mapped onto the real z-axis; (iii) the segments $-q_o \leqslant k \leqslant q_o$ on $\mathbb{K}^+$ and $\mathbb{K}^-$ are mapped onto the upper and lower semicircles of radius q_o and center at the origin of the z-plane; (iv) the involution $(k,\lambda) \mapsto (k^*,\lambda^*)$ corresponds to $z \to z^*$, and $(k,\lambda) \mapsto (k^*,-\lambda^*)$ to $z \to q_o^2/z^*$. (Note also $k - \lambda = q_o^2/z$.)

The discrete spectrum is the set of all values of $k \in \mathbb{K}$ such that $a(k) = 0$ or $\bar{a}(k) = 0$. It is well known that for the scalar defocusing NLS equation such zeros are real and simple [8]. If $q - q_\pm \in L^{1,4}(\mathbb{R}^\pm)$, then one can prove there is a finite number of zeros, all of which belonging to the spectral gap $k \in (-q_o, q_o)$ [16], corresponding to $z \in C_o := \{z \in \mathbb{C} : |z| = q_o\}$. Moreover, the symmetries of the scattering data imply that $a(z) = 0$ if and only if $\bar{a}(z^*) = 0$. Let then $\{\zeta_j\}_{j=1}^N$ denote the zeros of $a(z)$ on the upper half of

C_o, and $\{\zeta_j^*\}_{j=1}^N$ the corresponding zeros of $\bar{a}(z)$ in the lower half. For all $j = 1, \ldots, N$, the Wronskian representations (10) yield

$$\Phi_{-,1}(x,t,\zeta_j) = b_j\, \Phi_{+,2}(x,t,\zeta_j), \quad \Phi_{-,2}(x,t,\zeta_j^*) = \bar{b}_j\, \Phi_{+,1}(x,t,\zeta_j^*), \quad (18)$$

for some complex constants b_j and $\bar{b}_j$, with $b_j^* = \bar{b}_j$ due to the symmetries (11). Moreover, writing the symmetries (14) and (15) in terms of the uniformization variable, one can easily verify that b_j and $\bar{b}_j$ also satisfy

$$b_j^* = -\frac{q_-}{q_+^*} b_j, \quad \bar{b}_j^* = -\frac{q_+^*}{q_-} \bar{b}_j.$$

In turn, these provide the residue relations

$$\operatorname*{Res}_{z=\zeta_j}\left[\frac{M_{-,1}}{a}\right] = C_j\, \mathrm{e}^{-2i\Omega(x,t,\zeta_j)} M_{+,2}(x,t,\zeta_j),$$
$$\operatorname*{Res}_{z=\zeta_j^*}\left[\frac{M_{-,2}}{\bar{a}}\right] = \bar{C}_j\, \mathrm{e}^{2i\Omega(x,t,\zeta_j)} M_{+,1}(x,t,\zeta_j^*)$$

(which will be needed in the inverse problem), where the norming constants are $C_j = b_j/a'(\zeta_j)$ and $\bar{C}_j = \bar{b}_j/\bar{a}'(\zeta_j^*)$, and satisfy the symmetry relations:

$$\bar{C}_j = C_j^*, \quad C_j^* = \frac{q_+}{q_+^*} C_j.$$

Note that the Jost solutions are continuous at the branch points $\pm q_o$, while the scattering coefficients generically have simple poles when $z = \pm q_o$, unless the columns of $\Phi_\pm(x,t,z)$ become linearly dependent at either $z = q_o$ or $z = -q_o$, or both. In this case, $a(z)$ and $\bar{a}(z)$ are nonsingular near the corresponding branch point. When this happens, in scattering theory the point $z = -q_o$ or $z = q_o$ is called a virtual level [8].

The asymptotic behavior of the eigenfunctions and scattering data as $z \to \infty$ and as $z \to 0$ is given by

$$M_\pm(x,t,z) = \begin{pmatrix} 1 & -iq(x,t)/z \\ iq^*(x,t)/z & 1 \end{pmatrix} + h.o.t. \quad \text{as } z \to \infty \quad (19a)$$

$$M_\pm(x,t,z) = \begin{pmatrix} q(x,t)/q_\pm & -iq_\pm/z \\ iq_\pm^*/z & q(x,t)/q_\pm^* \end{pmatrix} + h.o.t. \quad \text{as } z \to 0 \quad (19b)$$

implying $S(z) = I + O(1/z)$ as $z \to \infty$ and $S(z) = \operatorname{diag}(q_+/q_-, q_-/q_+) + O(z)$ as $z \to 0$.

The inverse problem [i.e., the problem of recovering the potential $q(x,t)$ from the scattering data] can be formulated as a matrix Riemann–Hilbert problem (RHP) in terms of the uniformization variable. Indeed, from (8) one obtains:

$$\mu^-(x,t,z) = \mu^+(x,t,z)(I - R(x,t,z)), \quad \forall z \in \mathbb{R}, \quad (20)$$

where the sectionally meromorphic functions are

$$\mu^+(x,t,z) = \big(M_{+,2},\ M_{-,1}/a\big), \quad \mu^-(x,t,z) = \big(M_{-,2}/\bar{a},\ M_{+,1}\big),$$

the jump matrix is

$$R(x,t,z) = \begin{pmatrix} |\rho(z)|^2 & \mathrm{e}^{-2i\Omega(x,t,z)}\rho(z) \\ -\mathrm{e}^{2i\Omega(x,t,z)}\rho^*(z) & 0 \end{pmatrix},$$

and the reflection coefficient is $\rho(z) = b(z)/a(z)$. The matrices $\mu^\pm(x,t,z) - I$ are $O(1/z)$ as $z \to \infty$. After regularization, to account for the poles at $z = 0$ (cf., (19)) and at the discrete eigenvalues $\{\zeta_j, \zeta_j^*\}_{j=1}^N$, the RHP (20) can be solved via Cauchy projectors, and the asymptotics of $\mu^\pm$ as $z \to 0$ yields the reconstruction formula

$$q(x,t) = q_+ \left[1 - \sum_{j=1}^N \frac{C_j}{\zeta_j} M_{+,2}(x,t,\zeta_j)\, \mathrm{e}^{-2i\Omega(x,t,\zeta_j)} \right.$$
$$\left. - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\rho(\zeta)}{\zeta} M_{+,2}(x,t,\zeta)\, \mathrm{e}^{-2i\Omega(x,t,\zeta)} d\zeta \right]. \tag{21}$$

Taking into account the analyticity properties of $a(z)$ in the UHP, the location of its zeros, as well as the symmetries, one can obtain for $\operatorname{Im} z > 0$ the following representation (trace formula):

$$a(z) = \prod_{j=1}^N \left(\frac{z - \zeta_j}{z - \zeta_j^*} \right) \exp\left[-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 - |\rho(\zeta)|^2)}{\zeta - z} d\zeta \right]. \tag{22}$$

Recalling that $a(z) \to q_+/q_-$ as $z \to 0$, we conclude that the potential satisfies

$$\frac{q_+}{q_-} = \prod_{j=1}^N \frac{\zeta_j}{\zeta_j^*} \exp\left[-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 - |\rho(z)|^2)}{z} dz \right], \tag{23}$$

sometimes referred to as the θ-condition [8].

3. Box-type piecewise-constant ICs

Consider a piecewise-constant IC of the type

$$q(x,0) = \begin{cases} q_- = q_o\, e^{i\theta_-} & x < -L, \\ q_c = h\, e^{i\alpha} & -L < x < L, \\ q_+ = q_o\, e^{i\theta_+} & x > L, \end{cases} \tag{24}$$

where h, q_o, and L are arbitrary nonnegative parameters, and α and $\theta_\pm$ are arbitrary phases. The above IC models a potential well or a potential "box" (both on a nonzero background) when $h < q_o$ and when $h > q_o$, respectively.

As we are only concerned with the discrete spectrum of the scattering operator, which is time-independent, hereafter we will consider the scattering problem at $t = 0$ and omit the time dependence from the eigenfunctions. (Recall that all scattering coefficients are time-independent with our normalizations.) The scattering problem (3) in each of three regions at $t = 0$ then takes the form

$$\Phi_x = (-ik\sigma_3 + Q_j)\Phi, \quad j = c, \pm, \tag{25}$$

with constant potentials

$$Q_\pm = \begin{pmatrix} 0 & q_\pm \\ q_\pm^* & 0 \end{pmatrix}, \quad Q_c = \begin{pmatrix} 0 & q_c \\ q_c^* & 0 \end{pmatrix}. \tag{26}$$

The scattering problem can be solved explicitly. Introducing

$$\lambda^2 = k^2 - q_o^2, \quad \mu^2 = k^2 - h^2, \tag{27}$$

one can easily find explicit solutions to (25):

$$\varphi_l(x,k) = Y_-(k)\,\mathrm{e}^{-i\lambda x\sigma_3} \quad x \leqslant -L, \tag{28a}$$

$$\varphi_c(x,k) = Y_c(k)\,\mathrm{e}^{-i\mu x\sigma_3} \quad -L \leqslant x \leqslant L, \tag{28b}$$

$$\varphi_r(x,k) = Y_+(k)\,\mathrm{e}^{-i\lambda x\sigma_3} \quad x \geqslant L, \tag{28c}$$

where $Y_\pm(k)$ is as in (7), and

$$Y_c(k) = I - \frac{i}{k+\mu}\sigma_3 Q_c.$$

We then have an explicit, simple representation for the Jost solutions $\Phi_\pm(x,k)$ in their respective regions:

$$\Phi_-(x,k) \equiv \varphi_l(x,k) \quad x \leqslant -L, \tag{29a}$$

$$\Phi_+(x,k) \equiv \varphi_r(x,k) \quad x \geqslant L. \tag{29b}$$

The form of the Jost solutions beyond these domains can then be obtained by writing each Jost solution as an appropriate linear combination of the fundamental matrix solution of the scattering problem in each region and then imposing continuity at the boundary. On the other hand, because we are only concerned with the determination of the scattering data, we can follow a simpler procedure: At the boundary of each region one can express the fundamental solution on the left as a linear combination of the fundamental solution on the right, and vice versa. In particular, we can introduce scattering matrices $S_-(k)$ and $S_+(k)$ such that

$$\varphi_l(-L,k) = \varphi_c(-L,k)S_-(k), \quad \varphi_r(L,k) = \varphi_c(L,k)S_+(k). \tag{30}$$

As a consequence, taking into account (3), we can express the scattering matrix $S(k)$ in (9) relating the Jost solutions $\Phi_\pm(x,k)$ as

$$S(k) = S_+^{-1}(k)S_-(k) = \mathrm{e}^{i\lambda L\sigma_3}\,Y_+^{-1}(k)Y_c(k)\,\mathrm{e}^{-2i\mu L\sigma_3}\,Y_c^{-1}(k)Y_-(k)\,\mathrm{e}^{i\lambda L\sigma_3}. \tag{31}$$

In particular, after some straightforward algebra, we obtain

$$\begin{aligned}2\lambda\mu\, e^{-2i\lambda L}a(k) = \mu\cos(2\mu L)\left[\lambda + k + \frac{q_+}{q_-}(\lambda - k)\right] \\ + i\sin(2\mu L)\left[-k(\lambda+k) + \frac{q_+}{q_-}k(\lambda-k) + q_c q_-^* + q_c^* q_+\right],\end{aligned}$$

$$\begin{aligned}2\lambda\mu\, b(k) = \sin(2\mu L)\left[k(q_+^* - q_-^*) + q_c^*(\lambda+k) - q_c\frac{q_-^*}{q_+^*}(\lambda - k)\right] \\ + i\mu\cos(2\mu L)\,(q_-^* - q_+^*).\end{aligned}$$

The remaining two entries in the scattering matrix can be determined from the symmetries of the scattering data.

Without loss of generality owing to the phase invariance of the NLS equation, we now set $\theta_+ = -\theta_- = \theta$. Also, owing to the reflection symmetry $x \to -x$ of the NLS equation, without loss of generality we can limit ourselves to considering $\theta \in [0, \pi/2]$. Taking into account that by assumption $|q_\pm| = q_o$ and $q_c = h\,\mathrm{e}^{i\alpha}$, we then obtain the following expressions for the scattering coefficients:

$$\begin{aligned}\lambda\mu\,\mathrm{e}^{-2i\lambda L - i\theta}a(k) = \mu\cos(2\mu L)\,[\lambda\cos\theta - ik\sin\theta] \\ + i\sin(2\mu L)\,[hq_o\cos\alpha - k\,(k\cos\theta - i\lambda\sin\theta)],\end{aligned} \tag{32}$$

$$\begin{aligned}\lambda\mu b(k) = -\mu q_o\sin\theta\cos(2\mu L) \\ - i\sin(2\mu L)\,[kq_o\sin\theta + he^{i\theta}\,(\lambda\sin(\theta+\alpha) + ik\cos(\theta+\alpha))].\end{aligned} \tag{33}$$

Note that both $a(k)$ and $b(k)$ can have a pole at $\lambda = 0$, in general, while all terms have a finite limit as $\mu \to 0$.

In the special case in which the asymptotic phase θ is taken to be zero, the above expressions of the scattering coefficients simplify to:

$$\lambda\mu e^{-2i\lambda L}a(k) = \lambda\mu\cos(2\mu L) + i\sin(2\mu L)\,(q_o\,h\cos\alpha - k^2), \tag{34}$$

$$\lambda\mu b(k) = h\sin(2\mu L)\,(k\cos\alpha - i\lambda\sin\alpha). \tag{35}$$

In the following sections we use the above expressions to answer the two issues raised in the introduction.

4. Discrete eigenvalues

Recall that the discrete eigenvalues of the scattering problem are the zeros of $a(k)$. As the scattering problem is self-adjoint, any such eigenvalues must be real. Moreover, owing to (13), the zeros of $a(k)$ can only be located in the spectral gap $-q_o < k < q_o$. When $\theta = 0$, the potential is an even function of x, and thus discrete eigenvalues come in opposite pairs $\pm k_j$ [20]. Thus, when $\theta = 0$ we can restrict our search to the range $0 \leqslant k < q_o$. It is convenient to introduce $r = h/q_o$ and $\omega = q_o L$, and to rescale the scattering parameter as $k = q_o y$. We investigate various scenarios.

4.1. Potential well

In this case $h < q_o$, implying $0 \leqslant r < 1$. Here we take $\theta = 0$ and use the corresponding expression (34) for $a(k)$. When looking for discrete eigenvalues in the range $h \leqslant k < q_o$ [i.e., for zeros of $a(y)$ with $y \in [r, 1)$], one has $\lambda = i q_o \sqrt{1 - y^2}$ and $\mu = \pm q_o \sqrt{y^2 - r^2}$, implying

$$e^{2\omega\sqrt{1-y^2}} a(y) = \cos\left(2\omega\sqrt{y^2 - r^2}\right) - \frac{y^2 - r\cos\alpha}{\sqrt{1-y^2}} \frac{\sin\left(2\omega\sqrt{y^2 - r^2}\right)}{\sqrt{y^2 - r^2}}. \quad (36)$$

[Note that all results are independent of the choice of the sign for μ.] The limit of $a(y)$ as $y \to 1^-$ (corresponding to $\lambda \to 0$, i.e., $k \to q_o$) shows that generically $a(y)$ has a pole at the branch points, because $r\cos\alpha < 1$, unless $\omega = n\pi/(2\sqrt{1 - r^2})$ for some $n \in \mathbb{N}$. The sign of the function $a(y)$ in a left neighborhood of the branch point depends on the sign of $\sin(2\omega\sqrt{1 - r^2})$.

4.1.1. Nonexistence of an area theorem. Here we consider potentials satisfying the condition $r \leqslant \cos\alpha$, and we show that all such potentials admit at least one discrete eigenvalue, no matter how small the value of $\omega > 0$ is. For this purpose, for any fixed $0 \leqslant r < 1$, consider values of ω sufficiently small that $2\omega\sqrt{1 - r^2} < \pi/2$, so that $\sin(2\omega\sqrt{1 - r^2}) > 0$. As a consequence, $\lim_{y\to 1^-} a(y) = -\infty$, regardless of the choice for $\alpha \in \mathbb{R}$ and $0 \leqslant r < 1$. On the other hand, $a(y)$ is continuous in the limit $y \to r^+$, and

$$\lim_{y\to r^+} e^{2\omega\sqrt{1-y^2}} a(y) = 1 + \frac{2r\omega}{\sqrt{1 - r^2}}(\cos\alpha - r).$$

This means that, for any choice of $\alpha \in \mathbb{R}$ and $0 \leqslant r < 1$ such that $\cos\alpha \geqslant r$, the function $a(y)$ changes sign in the interval $(r, 1)$. Therefore, by continuity $a(y)$ must have at least one zero in that interval. Note the area of the box (namely, the size of the perturbation from the constant background) is $2\omega(1-r)$. Thus, the above results imply that *no area theorem is possible for the defocusing NLS equation with NZBCs, because there exists a class of potentials for which the scattering problem has at least one discrete eigenvalue, no matter how small the area of the IC is*.

Next we look at generic values of ω and α for any fixed value of $0 \leqslant r < 1$ and discuss the existence, number and precise location of discrete eigenvalues. In locating the zeros of $a(y)$, it will be convenient to study separately the cases $\cos\alpha < r$ and $r \leqslant \cos\alpha$, as well as the ranges $y \in [0, r)$ and $y \in [r, 1)$.

4.1.2. Location of discrete eigenvalues: $y \in [r, 1)$ and $r \leqslant \cos\alpha$. Introducing the variable $\xi = \sqrt{y^2 - r^2}$, we can obtain all the discrete eigenvalues in the range $[r, 1)$ [i.e., the zeros of $a(y)$ in (36)] as the solutions of the transcendental equation

$$\tan(2\omega\xi) = \frac{\xi\sqrt{1-r^2-\xi^2}}{r^2 - r\cos\alpha + \xi^2}, \tag{37}$$

with $0 < \xi < \sqrt{1-r^2} < 1$. [Note $\xi = 0$ (i.e., $y = r$) is always a solution of (37), but it is a zero of $a(y)$ iff $2r\omega(r - \cos\alpha) = \sqrt{1-r^2}$. In particular, $y = r$ can never be a zero of $a(y)$ if $\cos\alpha \geqslant r$.] Note also that: (i) the numerator in the right-hand side (RHS) of (37) is positive $\forall \xi \in (0, \sqrt{1-r^2})$ and zero at the endpoints; (ii) the denominator equals $1 - r\cos\alpha > 0$ at $\xi = \sqrt{1-r^2}$, hence it is always positive in a neighborhood of this point; (iii) the tangent in the left-hand side (LHS) is positive $\forall \xi \in (0, \sqrt{1-r^2})$ if $\omega \leqslant \pi/(4\sqrt{1-r^2})$, but has one or more vertical asymptotes, and therefore takes on all positive and negative real values, if $\omega > \pi/(4\sqrt{1-r^2})$.

As $\cos\alpha \geqslant r$, the denominator in the RHS is negative $\forall \xi \in (0, \xi_o)$, positive $\forall \xi \in (\xi_o, \sqrt{1-r^2})$, and zero for $\xi = \xi_o$, where $\xi_o = \sqrt{r\cos\alpha - r^2}$. Thus, the RHS of (37) has a pole at $\xi = \xi_o$, and takes on all real values when $\xi \in (0, \sqrt{1-r^2})$. As a result, for all $r \neq 0$ (37) has exactly one nontrivial root whenever $0 < \omega < \pi/(2\sqrt{1-r^2})$, and one more root appears, as ω increases, each time the value of the tangent at $\xi = \sqrt{1-r^2}$ goes through a zero, i.e., at $\omega = \omega_n$, with $\omega_n = n\pi/(2\sqrt{1-r^2})$, for all $n \in \mathbb{N}$. Similar arguments apply when $r = 0$, except that in this case $\xi \equiv y$ and $\xi_o = 0$; here the RHS of (37) has no pole, but tends to 1 as $\xi \to 0$.

4.1.3. Location of discrete eigenvalues: $y \in [r, 1)$ and $\cos\alpha < r$. Discrete eigenvalues are still given by the solutions of (37), with ξ defined as above. However in this case the situation is slightly more complicated.

The RHS of (37) is positive $\forall \xi \in (0, \sqrt{1-r^2})$ and vanishes at $\xi = 0$ and $\xi = \sqrt{1-r^2}$, while the LHS is an increasing function in each interval of continuity. Again, suppose $r \neq 0$ first. Comparing the slope of the LHS and that of the RHS at $\xi = 0$, one sees that for all $\omega \in (0, \omega_o)$, with $\omega_o = \sqrt{1-r^2}/[2r(r-\cos\alpha)]$ exactly one intersection exists. On the other hand, for all $\omega_o < \omega < \omega_1$ (with ω_1 as above) no intersections exist. Finally, discrete eigenvalues in the range $y \in [r, 1)$ are again present for all $\omega > \omega_1$, and one more discrete eigenvalue appears at $\omega = \omega_n$, as in the previous case.

Note that as $\omega \to \omega_o^-$, the intersection tends to $\xi = 0$, i.e., $y = r$. Note also that the exclusion zone (ω_o, ω_1) might be empty if r and α are such that $\omega_1 \leqslant \omega_o$. As before, similar considerations apply when $r = 0$. We are now left to look for zeros of $a(y)$ for $y \in [0, r)$, for both choices of the sign of $r - \cos\alpha$.

4.1.4. Location of discrete eigenvalues: $y \in [0, r)$ and $r \leqslant \cos\alpha$. In this case $\lambda = iq_o\sqrt{1-y^2}$ as before, but $\mu = \pm iq_o\sqrt{r^2-y^2}$. (Again, the result is independent of the sign choice.) Then (36) is replaced by

$$e^{2\omega\sqrt{1-y^2}}a(y) = \cosh\left(2\omega\sqrt{r^2-y^2}\right) + \frac{r\cos\alpha - y^2}{\sqrt{1-y^2}}\frac{\sinh\left(2\omega\sqrt{r^2-y^2}\right)}{\sqrt{r^2-y^2}}. \tag{38}$$

All terms in the RHS are nonnegative except $r\cos\alpha - y^2$. Thus, a necessary condition for $a(y)$ to vanish is that this term be negative. In turn, because here we are considering $y \in [0, r)$, a necessary condition for this to happen is that $\cos\alpha < r$. Therefore, we can conclude that when $\cos\alpha \geqslant r$ no zeros of $a(y)$ exist for $y \in [0, r)$.

4.1.5. Location of discrete eigenvalues: $y \in [0, r)$ and $\cos\alpha < r$. We distinguish two sub-cases: (i) If $\cos\alpha < 0$, one should look for zeros of $a(y)$ in the entire range $y \in [0, r)$. (ii) If $0 \leqslant \cos\alpha < r$, the search can be restricted to the range $y \in [\sqrt{r\cos\alpha}, r)$. In either case, the zeros of $a(y)$ are given by the solutions of the transcendental equation

$$\tanh(2\omega\xi) = \frac{\xi\sqrt{1-r^2+\xi^2}}{r^2 - r\cos\alpha - \xi^2}, \tag{39}$$

where now $\xi = \sqrt{r^2-y^2}$. [The corresponding ranges for ξ are $\xi \in (0, r]$ in case (i) and $\xi \in (0, \sqrt{r(r-\cos\alpha)})$ in case (ii).] Both the LHS and the RHS of (39) are monotonically increasing functions of ξ. Moreover: (i) If $\cos\alpha < 0$, both the LHS and the RHS are continuous and positive $\forall\xi \in (0, r]$. (ii) If $0 \leqslant \cos\alpha < r < 1$, the RHS has a pole at $\xi = \xi_o$, with $\xi_o = \sqrt{r(r-\cos\alpha)}$, and it is negative for $\xi \in (\xi_o, \sqrt{r(r-\cos\alpha)})$, while for $\xi \in (0, \xi_o)$ it takes on all positive values. In both cases, however, comparing the slope of the LHS and that of the RHS at $\xi = 0$ one arrives at the same conclusion: when $\cos\alpha < r$, the scattering coefficient $a(y)$ has exactly one zero in the range $y \in [0, r)$ iff $\omega > \omega_o$ (with ω_o as before), and no zero otherwise. Indeed, it is easy to see that as $\omega \to \omega_o^+$, the zero tends to $\xi = 0$, i.e., $y = r$. Thus, this discrete eigenvalue is nothing else but the continuation of the solution branch discussed earlier. Moreover, as $\omega \to \infty$, this unique discrete eigenvalue tends to $y_\infty = y(\xi_\infty)$, where ξ_∞ is the value of ξ for which the RHS of (39) equals 1, i.e., $\xi_\infty = r(r - \cos\alpha)/\sqrt{r^2 - 2r\cos\alpha + 1}$, implying

$$y_\infty = \frac{r|\sin\alpha|}{\sqrt{r^2 - 2r\cos\alpha + 1}}. \tag{40}$$

4.1.6. Summary and examples. We can summarize the above results for the potential well (i.e., $r = h/q_o < 1$) as follows. In all cases (i.e., independently of the relative values of r and $\cos\alpha$), there is always at least one discrete eigenvalue. Moreover, for all $0 < \omega < \omega_1$ there is exactly one eigenvalue, and, as ω increases, one more eigenvalue appears at each $\omega = \omega_n$, with $\omega_n = n\pi/(2\sqrt{1 - r^2})$, for all $n \in \mathbb{N}$. Finally: (i) If $r \leqslant \cos\alpha$, all such eigenvalues are located in the range $k \in (h, q_o)$, and each of them tends asymptotically to $k = h$ as $\omega \to \infty$. (ii) If $r > \cos\alpha$, the statements about the location of the eigenvalues and their limit as $\omega \to \infty$ apply to all eigenvalues except one. This exceptional eigenvalue, which is present for all values of ω, is located in the range (h, q_o) for all $\omega < \omega_o$, at $k = h$ for $\omega = \omega_o$, and in the range $(y_\infty q_o, h)$ for all $\omega > \omega_o$. In addition, this eigenvalue tends to $y_\infty q_o$ in the limit $\omega \to \infty$. In all these cases, all the eigenvalues are always accompanied by their symmetric counterparts in the range $k \in (-q_o, 0]$.

The various cases are illustrated by a few representative examples in Fig. 1, while Fig. 2 shows the dependence of the discrete spectrum on ω when $\cos\alpha \geqslant r$ (left) and $\cos\alpha < r$ (right). It is worthwhile to note that, in all these cases, discrete eigenvalues always bifurcate from the branch points. In other words, *generically speaking the discrete eigenvalues appear through the formation of virtual levels*.

4.2. Further considerations

4.2.1. Potential barrier. In this case $h > q_o$. Taking again $\theta = 0$ and using the notation $k = q_o y$ and $\omega = q_o L$ introduced above, we now have $r > 1$ and $\mu = i q_o\sqrt{r^2 - y^2}$. We still obtain (38), but now with $r > 1$, and we need to look for zeros of $a(y)$ for $y \in [0, 1)$. (Contrary to the previous case, here the whole range of values y is allowed in principle.) In this case, $a(y)$ has a pole at the branch points unless $r\cos\alpha = 1$.

If $r\cos\alpha \geqslant 1$, all terms in the above expression are positive, and therefore no zeros of $a(y)$ exist for $y \in [0, 1)$—i.e., the problem does not have any discrete eigenvalues. If $r\cos\alpha < 1$, on the other hand, zeros of $a(y)$ may exist in principle. Introducing $\xi = \sqrt{r^2 - y^2}$, we find that the discrete eigenvalues are given by the solutions of the transcendental equation (39) as before, but where now $\xi \in (\sqrt{r^2 - 1}, r]$. In order for (39) to admit any solutions, one obviously needs its RHS to be strictly less than 1. In turn, reverting back to y

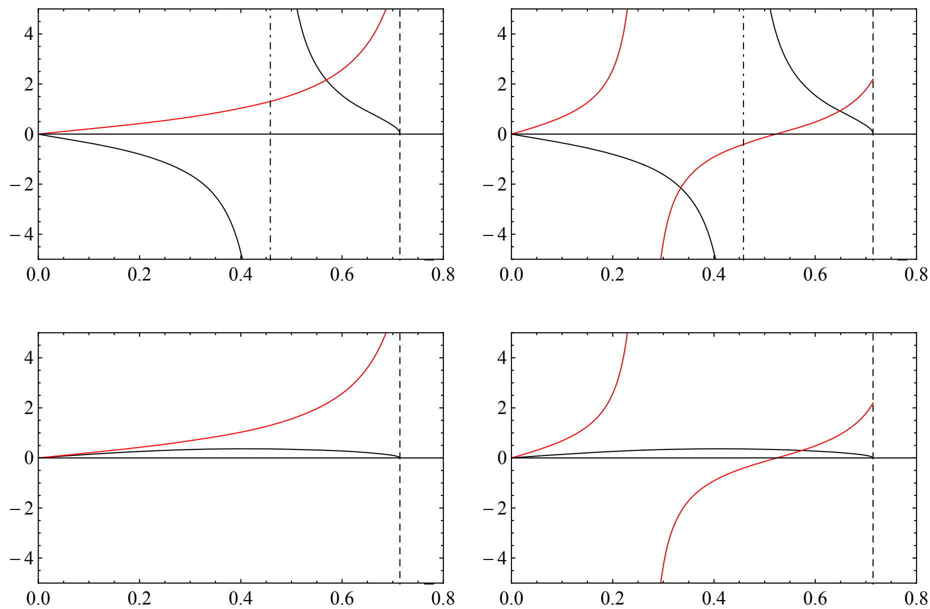

Figure 1. The LHS (red) and RHS (black) of (37) for a potential well with $r = 0.7$ as a function of ξ in $(0, \sqrt{1-r^2})$. Top left: $\alpha = 0$ ($\cos\alpha > r$) and $\omega = 1$, yielding one intersection. Top right: $\alpha = 0$ ($\cos\alpha > r$) and $\omega = 3$, yielding two intersections. Bottom left: $\alpha = \pi/2$ ($\cos\alpha < r$) and $\omega = 1$, yielding no intersections. Bottom right: $\alpha = \pi/2$ ($\cos\alpha < r$) and $\omega = 3$, yielding one intersection.

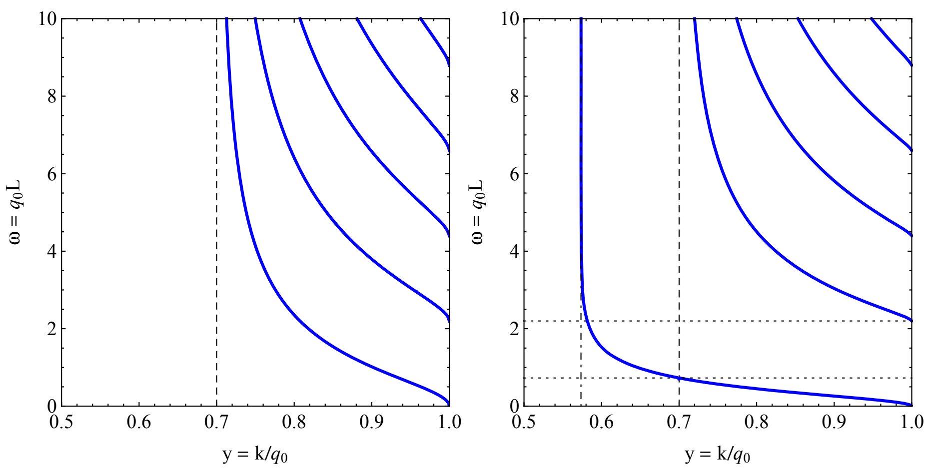

Figure 2. The discrete spectrum of the potential well as a function of ω for $r = 0.7$ and $\alpha = 0$ (left), $\alpha = \pi/2$ (right). The horizontal axis is $y = k/q_o$, the vertical axis is ω. The dashed vertical line identifies the value $y = r$. The dotted horizontal lines delimit the exclusion zone $\sqrt{1-r^2}/[2r(r-\cos\alpha)] \leqslant \omega \leqslant \pi/(2\sqrt{1-r^2})$. The dot–dashed vertical line shows the limiting value y_∞ given by (40).

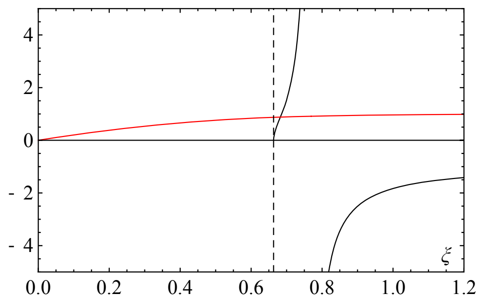

Figure 3. The LHS (red) and RHS (black) of (39) as a function of ξ in $(\sqrt{1-r^2}, r]$ for a potential barrier with $r = 1.2$, $\alpha = \pi/4$ and $\omega = 1$.

and taking into account that $y < 1$, this inequality implies

$$\frac{r^2 \sin^2 \alpha}{1 + r^2 - 2r \cos \alpha} < 1. \tag{41}$$

Conversely, whenever this condition is satisfied the right-hand side of (39) takes on all positive real values for $\xi \in (\sqrt{r^2 - 1}, r]$ exactly once, which ensures the existence of a discrete eigenvalue.

It is straightforward to check, however, that the inequality (41) is identically satisfied when $r \cos \alpha < 1$.

On the other hand, note that, unlike the case of a potential well, at most one discrete eigenvalue exists in this case (because the hyperbolic tangent is not an oscillating function). Indeed, it is easy to see that as $\omega \to \infty$, the unique discrete eigenvalue tends to the value of y for which the RHS of (39) equals 1, i.e., y_∞ given by (40) as before.

Summarizing, for a potential barrier (i.e., when $r > 1$) no discrete eigenvalues exist if $r \cos \alpha \geqslant 1$, and exactly one discrete eigenvalue (plus its symmetric counterpart) exists if $r \cos \alpha < 1$ (cf. Figure 3 for an example of the latter case).

Figure 4 shows the spectrum of the scattering problem as a function of ω for a few choices of α satisfying the condition $r \cos \alpha < 1$.

4.2.2. Black solitons. We now return to the general expression for $a(k)$ given in (32). Recall that the contribution to the solution obtained from a discrete eigenvalue at $k = 0$ is referred to as a black soliton. Also recall that the asymptotic phase difference of the potential is $\theta_+ - \theta_- = 2\theta$. It is straightforward to establish necessary and sufficient conditions to ensure that $k = 0$ is a discrete eigenvalue, i.e., a zero of $a(k)$. Indeed, when $k = 0$ both λ and μ are purely imaginary: $\lambda = iq_o$ and $\mu = ih$, and

$$e^{2q_o L - i\theta} a(0) = \cosh(2hL) \cos\theta + \sinh(2hL) \cos\alpha.$$

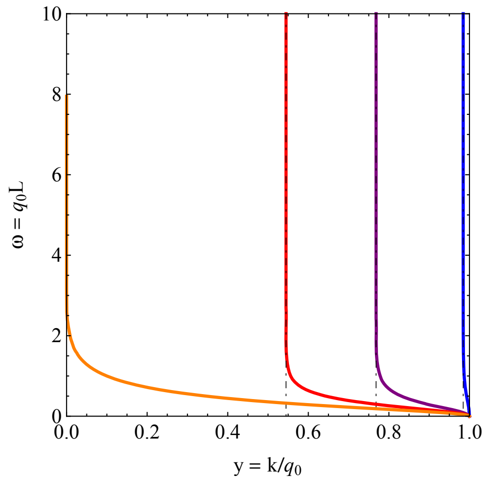

Figure 4. The discrete spectrum of the potential barrier as a function of ω for $r = 1.2$ and a few different values of α satisfying the condition $r\cos\alpha < 1$: $\alpha = \pi/4$ (blue), $\alpha = \pi/2$ (purple), $\alpha = 2\pi/3$ (red) and $\alpha = \pi$ (orange). As before, the horizontal axis is $y = k/q_o$, the vertical axis is ω. The dotted vertical lines show the limiting value y_∞ in each case.

Thus, $k = 0$ is a discrete eigenvalue if and only if either (i) $\cos\alpha = \cos\theta = 0$, for any choice of h, L, q_o; or (ii) $\tanh(2hL) = -\cos\theta\sec\alpha$. This second case obviously requires $\cos\theta\sec\alpha > -1$.

4.2.3. Step-like potentials. The reduction $L = 0$ yields the case in which the potential is a simple step, with the same constant amplitude q_o for $x \lessgtr 0$ and a phase jump of 2θ across $x = 0$. In this case, the scattering coefficient $a(k)$ in (32) reduces to

$$e^{-i\theta}a(k) = \cos\theta - i\frac{k}{\lambda}\sin\theta. \tag{42}$$

For $k \in [-q_o, q_o]$ one has $\lambda = i\sqrt{q_o^2 - k^2}$. The case $\theta = 0$ obviously corresponds to the trivial solution $q(x,t) \equiv q_o$. For any value $\theta \in (0, \pi/2]$ it is easy to see from (42) that there is exactly one discrete eigenvalue located at $k = q_o\cos\theta$. Note how, even in this case, the discrete eigenvalue appears from the branch point as soon as the potential deviates from the constant background.

5. Contribution of the continuous spectrum to the asymptotic phase difference

Recall that the condition (23) determines the asymptotic phase difference 2θ of the potential in terms of a contribution from the discrete spectrum and one from the continuous spectrum. The symmetries (16) imply that the reflection coefficient satisfies $|\rho(z)| = |\rho(q_o^2/z)|$ and the integral in (23) simplifies to

$$\int_{-\infty}^{\infty} \frac{\log(1-|\rho(z)|^2)}{z}dz = 2\int_{-\infty}^{-q_o} \frac{\log(1-|\rho(z)|^2)}{z}dz + 2\int_{q_o}^{\infty} \frac{\log(1-|\rho(z)|^2)}{z}dz.$$

The first term in the RHS is positive, while the second term is negative. (Recall $0 \leqslant 1-|\rho(z)|^2 < 1$.) Nonetheless, because no symmetry relates the two integrands, one should not expect an exact cancellation in general. This suggests that the radiative part of the spectrum can in principle produce a nontrivial contribution to the asymptotic phase difference of the potential. We next show that this is indeed the case by constructing an explicit example in which a nonzero asymptotic phase difference of the potential originates only from the contribution of the continuous spectrum.

The idea is to look for a similar result as for the potential barrier in section 4, where one can find parameter regimes for which the scattering problem has no discrete eigenvalues. The analysis for the potential barrier in Section 4 is limited to the case $\theta = 0$, for which there is no asymptotic phase difference. If one can obtain a similar result in the case $\theta \neq 0$, however, the asymptotic phase difference in the potential can only be due to radiation.

In light of the above considerations, we take $h > q_o$, implying $r = h/q_o > 1$, but now with $\alpha = 0$. As before we parameterize k as $k = q_o y$. As in this case the potential is not symmetric with respect to x, we need to consider both positive and negative values of y in $(-1, 1)$. Taking $\omega = q_o L$ as before, (32) becomes

$$\mathrm{e}^{-i\theta+2\omega\sqrt{1-y^2}}a(y) = \frac{1}{\sqrt{1-y^2}}\left[\frac{f_1(y)}{\sqrt{r^2-y^2}}\sinh\left(2\omega\sqrt{r^2-y^2}\right) + f_2(y)\cosh\left(2\omega\sqrt{r^2-y^2}\right)\right], \quad (43)$$

where

$$f_1(y) = r - y(y\cos\theta + \sqrt{1-y^2}\sin\theta), \quad (44a)$$

$$f_2(y) = \sqrt{1-y^2}\cos\theta - y\sin\theta. \quad (44b)$$

The hyperbolic sine and cosine in the RHS of (43), as well as the square roots in the denominator, are all positive $\forall y \in (-1, 1)$. Moreover, because $r > 1$, it is straightforward to show that $f_1(y) > 0$ $\forall\theta \in \mathbb{R}$ and $\forall y \in (-1, 1)$. The last term in the RHS of (43) is not sign-definite, however. Nonetheless,

(44) suggests that if θ is sufficiently small, $f_2(y)$ is also positive except in a neighborhood of $y = \pm 1$, where it becomes negative. Note, however, that $f_2(y) = O(\theta)$ as $\theta \to 0$, whereas $f_1(y)$ remains $O(1)$ as a function of θ for all $y \in [-1, 1]$, thereby keeping the RHS of (43) from becoming negative.

To prove that this is indeed the case, recall that we can take $\theta \in [0, \pi/2]$ without loss of generality, and note that for all $y \in [-1, 1]$ one has

$$f_1(y) \geqslant r - 1, \quad f_2(y) \geqslant -\sin\theta.$$

Therefore, a sufficient condition for the RHS of (43) to be strictly positive is that

$$\tanh(2\omega\sqrt{r^2 - y^2}) > \frac{\sqrt{r^2 - y^2}}{r - 1}\sin\theta \quad \forall y \in [-1, 1].$$

As the smallest value of the LHS in the above inequality (achieved at $y = \pm 1$) is $\tanh(2\omega\sqrt{r^2 - 1})$, while the largest value of the RHS (achieved at $y = 0$) is $[r/(r - 1)]\sin\theta$, a sufficient condition for the RHS of (43) to be strictly positive is that

$$\sin\theta < (1 - 1/r)\tanh(2\omega\sqrt{r^2 - 1}). \tag{45}$$

To summarize, *all potentials satisfying the condition* (45) *have a nonzero asymptotic phase difference but no discrete spectrum*. This result therefore demonstrates that the radiative part of the spectrum can indeed contribute to the asymptotic phase difference of the potential, provided the latter is sufficiently small.

Figure 5 shows, for a few values of θ, the boundary of the region in the $r\omega$-plane for which the condition (45) is satisfied. Because the RHS of (45) can be made arbitrarily close to 1 by taking sufficiently large values of r, for any given $\theta < \pi/2$ one can construct potentials in which the asymptotic phase difference is produced only by the continuous spectrum. On the other hand, because both factors in the RHS of (45) are strictly less than 1, an asymptotic phase difference $2\theta = \pi$ is not guaranteed. Note however that the condition (45) is sufficient but not necessary, as demonstrated in Fig. 6. Next we therefore look at the special case $\theta = \pi/2$.

When $r > 1$, $\alpha = 0$ and $\theta = \pi/2$, the discrete eigenvalues (if any) are given by the solutions of the transcendental equation

$$\tanh(2\omega\sqrt{r^2 - y^2}) = g(y), \quad g(y) = \frac{y\sqrt{r^2 - y^2}}{r - y\sqrt{1 - y^2}}.$$

The LHS and the denominator of $g(y)$ are strictly positive for $y \in [-1, 1]$, so solutions can only exist for $y \in (0, 1)$. There, the LHS is a decreasing function of y, taking on all values in the range $(\tanh(2\omega\sqrt{r^2 - 1}), \tanh(2\omega r))$.

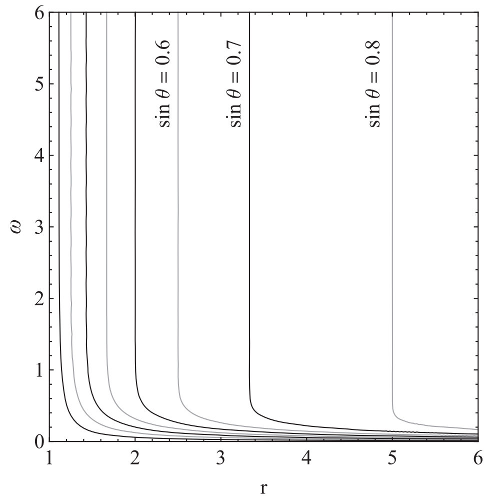

Figure 5. The boundaries of the ranges of r (horizontal axis) and ω (vertical axis) that guarantee that no discrete eigenvalues exist as a function of θ according to the condition (45).

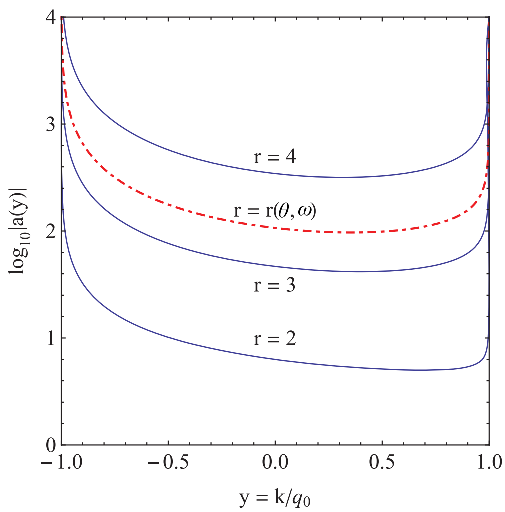

Figure 6. Plots of $\log_{10}|a(y)|$ (vertical axis) versus y (horizontal axis) for $\omega = 1$, $\theta = \pi/4$ and $r = 2$, 3 and 4 (solid blue curves). Also shown (dot–dashed red curve) is the curve obtained for the value of r that makes the two sides of (45) equal. Note how even for lower values of r no discrete eigenvalues exist.

On the other hand, it is straightforward to show that $g(y)$ has a local maximum at $y = y_r$, with $y_r = r/\sqrt{r^2+1}$ and $g(y_r) = 1$. Moreover, $g(0) = 0$ and $g(1) = \sqrt{1 - 1/r^2} < 1$. Therefore one is guaranteed at least one intersection for $y \in (0, 1)$, implying that for the specific class of potentials considered here, *when the asymptotic phase difference of the potential is π there always exists at least one discrete eigenvalue*.

6. Final remarks

Even though we have restricted our attention to potentials that could be characterized with elementary techniques, their study has nonetheless enabled us to draw general conclusions on the spectral properties of the scattering problem—namely, that no area theorem is possible and that the continuous spectrum can indeed provide a nonzero contribution to the asymptotic phase difference of the potential.

Also, because the initial datum $q(x, 0)$ coincides with its boundary values for all $|x| > L$, such ICs belong to the equivalent of compact support potentials in the case of zero BCs, and are therefore included in any functional class for which IST can conceivably be implemented, implying that the above conclusions hold in any such class.

On the other hand, other conclusions are specific to the examples considered here, and therefore open up the question of whether they remain true for generic potentials. One such question is whether an asymptotic phase difference of π always implies the existence of at least one discrete eigenvalue. Of course to answer this question in general one must investigate a broad class of ICs defined in an appropriate functional space. Such a study is outside the scope of this work. Nonetheless, the explicit examples discussed here should provide useful insight that will help the construction of a general theory.

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