On the Nonlinear Schrödinger Equation on the Half Line with Homogeneous Robin Boundary Conditions

By G. Biondini and A. Bui

Boundary value problems for the nonlinear Schrödinger equation on the half line with homogeneous Robin boundary conditions are revisited using Bäcklund transformations. In particular: relations are obtained among the norming constants associated with symmetric eigenvalues; a linearizing transformation is derived for the Bäcklund transformation; the reflection-induced soliton position shift is evaluated and the solution behavior is discussed. The results are illustrated by discussing several exact soliton solutions, which describe the soliton reflection at the boundary with or without the presence of self-symmetric eigenvalues.

1. Introduction

The nonlinear Schrödinger (NLS) equation is a universal model for the behavior of weakly nonlinear, quasi-monochromatic wave packets, and they arise in a variety of physical settings. In many situations, the natural formulation of the problem gives rise to boundary value problems (BVPs). In particular, the NLS equation

$$iq_t + q_{xx} - 2\nu|q|^2q = 0 \tag{1}$$

on the half line $0 < x < \infty$ has been studied by various authors using several approaches [1–7]. Here, $q = q(x,t)$, and the values $\nu = \pm 1$ denote respectively the defocusing and focusing cases. It is well known that, in general, BVPs

Address for correspondence: Gino Biondini, State University of New York, Buffalo, NY, USA; e-mail: biondini@buffalo.edu

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for integrable nonlinear evolution equations can only be linearized for special kinds of boundary conditions (BCs), which are called linearizable. For the NLS equation, the linearizable BCs are the homogeneous Robin BCs [8]

$$q_x(0,t) + \alpha\, q(0,t) = 0\,, \tag{2}$$

where $\alpha \in \mathbb{R}$ is an arbitrary constant. Limiting cases are Dirichlet and Neumann BCs, which are obtained from (2) respectively as $\alpha \to \infty$ and $\alpha = 0$. In [3], the BVP for (1) with BCs (2) was considered using the extension of the potential to the whole real line introduced in [1, 5], and it was shown, that the discrete eigenvalues of the scattering problem appear in symmetric quartets—as opposed to pairs in the initial value problem (IVP). Moreover, symmetries of the discrete spectrum, norming constants, reflection coefficients and scattering data were obtained, and the reflection experienced by the solitons at the boundary was explained as a special form of soliton interaction. Similar results were obtained in [9] for the Ablowitz-Ladik system with certain linearizable BCs.

Several issues have not been adequately addressed in [3] nor in any of the previous works, however. On one hand, the method used in [3, 5] is not completely satisfactory: (i) For $\alpha \neq 0, \infty$, it is nontrivial to prove that the Lax pair of the NLS equation remains consistent at $x = 0$. (ii) The k-dependent extended potential has a pole in the complex k-plane, which complicates the inverse scattering transform (IST): the scattering eigenfunctions and analytic scattering coefficients develop poles in the complex k-plane. (iii) The extension cannot deal with self-symmetric discrete eigenvalues, namely eigenvalues on the imaginary axis (corresponding to stationary solitons), in the sense that in some of these cases it yields inconsistent symmetries for the scattering data. Here we avoid all of these complications by revisiting the problem using a symmetric Bäcklund transformation, introduced in [2, 6], which allows us to provide cleaner proofs of several statements and generalize several results of [3] to include the presence of self-symmetric eigenvalues. Since the Bäcklund transformation between the original potential and the transformed one is a system of nonlinear ordinary differential equations (ODEs), however, in order for this method to be effective one must be able to obtain the new potential only through linear transformations. Here we provide the explicit linearizing transformation of the system of ODEs. Finally, we revisit the calculation of the reflection-induced soliton position shift and we further discuss the behavior of solutions.

2. IST for the NLS equation

The study of the BVP in Section 3 will employ the IST for the IVP on the whole line. Here we therefore briefly define the quantities that will be used in the following sections. We refer the reader to Refs. [10, 11, 3, 12, 13, 14] for details.

The Lax pair for the NLS equation is

$$\Phi_x - ik\sigma_3\Phi = Q\,\Phi, \qquad \Phi_t + 2ik^2\sigma_3\Phi = H\,\Phi, \tag{3}$$

where $\Phi = \Phi(x,t,k)$ is a 2×2 matrix,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x,t) = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix},$$

$H(x,t,k) = i\sigma_3(Q_x - Q^2) - 2kQ$ and $r(x,t) = \nu q^*(x,t)$. Hereafter, the asterisk denotes complex conjugation, the superscript T matrix transpose, a prime signifies differentiation with respect to k and $[A,B] = AB - BA$ is the matrix commutator.

The modified eigenfunctions with constant limits as $x \to \pm\infty$ are $\mu(x,t,k) = \Phi(x,t,k)\,\mathrm{e}^{-i\theta(x,t,k)\sigma_3}$, where $\theta(x,t,k) = kx - 2k^2t$; they satisfy the modified Lax pair

$$\mu_x - ik[\sigma_3,\mu] = Q\mu\,, \quad \mu_t + 2ik^2[\sigma_3,\mu] = H\mu.$$

If $q(x,t)$ vanishes sufficiently fast as $x \to \pm\infty$, one can write Volterra integral equations that define the Jost solutions $\forall k \in \mathbb{R}$ as the solutions of the above system that reduce to the identity as $x \to \mp\infty$:

$$\mu_-(x,t,k) = I + \int_{-\infty}^{x} \mathrm{e}^{ik(x-y)\sigma_3} Q(y,t)\mu_-(y,t,k)\,\mathrm{e}^{-ik(x-y)\sigma_3}\mathrm{d}y, \tag{4a}$$

$$\mu_+(x,t,k) = I - \int_{x}^{\infty} \mathrm{e}^{ik(x-y)\sigma_3} Q(y,t)\mu_+(y,t,k)\,\mathrm{e}^{-ik(x-y)\sigma_3}\mathrm{d}y. \tag{4b}$$

We use the notation $A = (A^{(1)}, A^{(2)})$ to identify the columns of a 2×2 matrix A. The columns of $\mu_\pm$ can be analytically extended into the complex k-plane: $\mu_-^{(1)}$ and $\mu_+^{(2)}$ in the lower-half plane (LHP), $\mu_-^{(2)}$ and $\mu_+^{(1)}$ in the upper-half plane (UHP). The analyticity properties of $\Phi_\pm = \mu_\pm\,\mathrm{e}^{i\theta\sigma_3}$ follow accordingly.

Since $\det\Phi_\pm = 1$ for all $x \in \mathbb{R}$, Φ_- and Φ_+ are both fundamental matrix solutions of both parts of the Lax pair (3), and one can define the time-independent scattering matrix $A(k)$ as:

$$\Phi_-(x,t,k) = \Phi_+(x,t,k)\,A(k)\,, \qquad k \in \mathbb{R}. \tag{5}$$

The entries of the scattering matrix can be written in terms of Wronskians. In particular, $a_{11}(k) = \mathrm{Wr}(\Phi_-^{(1)}, \Phi_+^{(2)})$ and $a_{22}(k) = -\mathrm{Wr}(\Phi_-^{(2)}, \Phi_+^{(1)})$, which show that $a_{11}(k)$ and $a_{22}(k)$ can be analytically continued, respectively on $\mathrm{Im}\,k < 0$ and $\mathrm{Im}\,k > 0$ (but $a_{12}(k)$ and $a_{21}(k)$ are in general nowhere analytic).

The eigenfunctions and scattering data satisfy the symmetry relations

$$\Phi_\pm^{(1)}(x,t,k) = \sigma_\nu\big(\Phi_\pm^{(2)}(x,t,k^*)\big)^*, \qquad \Phi_\pm^{(2)}(x,t,k) = \nu\sigma_\nu\big(\Phi_\pm^{(1)}(x,t,k^*)\big)^*, \tag{6}$$

with

$$\sigma_\nu = \begin{pmatrix} 0 & 1 \\ \nu & 0 \end{pmatrix}, \tag{7}$$

implying

$$a_{22}(k) = a_{11}^*(k^*)\,, \qquad a_{21}(k) = \nu a_{12}^*(k)\,. \tag{8}$$

The asymptotic behavior of the eigenfunctions and scattering data as $k \to \infty$ is

$$\mu_- = I - \frac{1}{2ik}\sigma_3 Q + \frac{1}{2ik}\sigma_3 \int_{-\infty}^{x} q(y,t)r(y,t)\,\mathrm{d}y + O(1/k^2)\,, \tag{9a}$$

$$\mu_+ = I - \frac{1}{2ik}\sigma_3 Q - \frac{1}{2ik}\sigma_3 \int_{x}^{\infty} q(y,t)r(y,t)\,\mathrm{d}y + O(1/k^2)\,, \tag{9b}$$

and $A(k) = I + O(1/k)$.

Discrete eigenvalues are values of $k \in \mathbb{C}$ such that $a_{22}(k) = 0$ or $a_{11}(k) = 0$. In the defocusing case, there are no such values. In the focusing case, we assume that there exist a finite number of such zeros, and that they are simple. Note that $a_{22}(k_j) = 0$ if and only if $a_{11}(k_j^*) = 0$. Let k_j and $\bar{k}_j = k_j^*$, for $j = 1, \ldots, J$, be the zeros of a_{22} and a_{11}, respectively, with $\operatorname{Im} k_j > 0$. The Wronskian representations of the analytic scattering coefficients then yield

$$\Phi_-^{(2)}(x,t,k_j) = b_j\, \Phi_+^{(1)}(x,t,k_j)\,, \quad \Phi_-^{(1)}(x,t,\bar{k}_j) = \bar{b}_j\, \Phi_+^{(2)}(x,t,\bar{k}_j)\,. \tag{10}$$

In turn, these provide the residue relations that are needed in the inverse problem:

$$\operatorname*{Res}_{k=k_j}\left[\frac{\mu_-^{(2)}}{a_{22}}\right] = C_j\, e^{2i\theta(x,t,k_j)} \mu_+^{(1)}(x,t,k_j),$$

$$\operatorname*{Res}_{k=\bar{k}_j}\left[\frac{\mu_-^{(1)}}{a_{11}}\right] = \bar{C}_j\, e^{-2i\theta(x,t,\bar{k}_j)} \mu_+^{(2)}(x,t,\bar{k}_j)\,,$$

where the norming constants are $C_j = b_j/a_{22}'(k_j)$ and $\bar{C}_j = \bar{b}_j/a_{11}'(\bar{k}_j)$, and satisfy the symmetry relations $\bar{b}_j = -b_j^*$ and $\bar{C}_j = -C_j^*$.

The inverse problem [i.e., recovering the potential $q(x,t)$ from the scattering data] is formulated in terms of the matrix Riemann-Hilbert problem (RHP) defined by (5):

$$M^-(x,t,k) = M^+(x,t,k)(I - R(x,t,k))\,, \qquad \forall k \in \mathbb{R}, \tag{11}$$

where the sectionally meromorphic functions are

$$M^+ = \big(\mu_+^{(1)}\,,\ \mu_-^{(2)}/a_{22}\big), \quad M^- = \big(\mu_-^{(1)}/a_{11}\,,\ \mu_+^{(2)}\big),$$

the jump matrix is

$$R(x,t,k) = \begin{pmatrix} \nu|\rho(k)|^2 & e^{2i\theta(x,t,k)}\rho(k) \\ -\nu e^{-2i\theta(x,t,k)}\rho^*(k) & 0 \end{pmatrix},$$

and the reflection coefficient is $\rho(k) = a_{12}(k)/a_{22}(k)$. The matrices $M^{\pm}(x, t, k) - I$ are $O(1/k)$ as $k \to \infty$. After regularization, the RHP (11) can be solved via Cauchy projectors, and the asymptotics of $M^{\pm}$ as $k \to \infty$ yields the reconstruction formula

$$q(x,t) = -2i \sum_{j=1}^{J} C_j \mathrm{e}^{2i\theta(x,t,k_j)} \mu_{+,11}(x,t,k_j) + \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{2i\theta(x,t,k)} \rho(k) \mu_{+,11}(x,t,k)\,\mathrm{d}k.$$

In the reflectionless case [$\rho(k) = 0 \forall k \in \mathbb{R}$] with $\nu = -1$, the RHP reduces to an algebraic system, whose solution yields the pure J-soliton solution of the focusing NLS equation as

$$q(x,t) = 2i\,\frac{\det G^c}{\det G},$$

where $G = (G_{j,j'})$,

$$G^c = \begin{pmatrix} 0 & \mathbf{y}^{\mathrm{T}} \\ \mathbf{1} & G \end{pmatrix}, \qquad y_j = C_j\, e^{2i\theta(x,t,k_j)},$$

$$G_{j,j'} = \delta_{j,j'} + \mathrm{e}^{2i\theta(x,t,k_j')} C_{j'} \sum_{p=1}^{J} \frac{C_p^* \mathrm{e}^{-2i\theta(x,t,k_p^*)}}{(k_j - k_p^*)(k_p^* - k_{j'})},$$

$\mathbf{y} = (y_1, \ldots, y_J)^{\mathrm{T}}$ and $\mathbf{1} = (1, \cdots, 1)^{\mathrm{T}}$. In particular, the one-soliton solution is $q(x,t) = q_{1s}(x,t; A, V, \xi, \varphi)$, with

$$q_{1s}(x,t; A, V, \xi, \varphi) = A\,\mathrm{e}^{i[Vx + (A^2 - V^2)t + \varphi]}\,\mathrm{sech}[A(x - 2Vt - \xi)], \quad (12)$$

where $k_1 = (V + iA)/2$ and $C_1 = A\, e^{A\xi + i(\varphi + \pi/2)}$.

3. Bäcklund transformation and BVP

Suppose $q(x,t)$ and $\tilde{q}(x,t)$ both solve the NLS equation and the corresponding eigenfunctions $\Phi(x,t,k)$ and $\tilde{\Phi}(x,t,k)$ are related by the dressing transformation

$$\tilde{\Phi}(x,t,k) = B(x,t,k)\,\Phi(x,t,k). \quad (13)$$

Then, as we show next, q and $\tilde{q}$ are related by a Bäcklund transformation (or rather an auto-Bäcklund transformation, or a Darboux transformation [15]). If

$\Phi(x,t,k)$ is invertible, a necessary and sufficient condition for (13) is

$$B_x = ik[\sigma_3, B] + \tilde{Q}B - BQ\,. \tag{14}$$

If $B(x,t,k)$ is linear in k, one can write a solution (14) in the form

$$B(x,t,k) = 2ikI + b\,\sigma_3 + (\tilde{Q} - Q)\sigma_3, \tag{15}$$

$$b(x,t) = \alpha + \int_0^x (\tilde{q}(y,t)\tilde{r}(y,t) - q(y,t)r(y,t))\,\mathrm{d}y, \tag{16}$$

where α is an arbitrary constant. In particular, the off-diagonal entries of (14) yield the desired Bäcklund transformation between the original potential and the new one:

$$\tilde{q}_x - q_x = (\tilde{q} + q)\,b\,. \tag{17}$$

A corresponding relation exists between the Jost solution of the original Lax pair and those of the Bäcklund-transformed counterpart:

$$\tilde{\Phi}_\pm(x,t,k) = B(x,t,k)\,\Phi_\pm(x,t,k)\,B_{\pm\infty}^{-1}(k)\,, \tag{18a}$$

where $B_{\pm\infty}(k) = \lim_{x\to\pm\infty} B(x,t,k) = 2ikI + b_{\pm\infty}\sigma_3$. In turn, (18a) implies the following relation for the corresponding scattering matrices:

$$\tilde{A}(k) = B_{+\infty}(k)\,A(k)\,B_{-\infty}^{-1}(k)\,, \qquad k \in \mathbb{R}\,. \tag{18b}$$

Now let us impose a *mirror* symmetry. That is, suppose that the Bäcklund-transformed potential is the mirror image of the original one:

$$\tilde{q}(x,t) = q(-x,t)\,. \tag{19}$$

Then (17), evaluated at $x = 0$, yields the Robin BC (2). Note also that (19) implies $\lim_{x\to\infty} b(x,t) = \lim_{x\to-\infty} b(x,t) =: b_\infty$, and hence $\lim_{x\to\infty} B(x,t) = \lim_{x\to-\infty} B(x,t) =: B_\infty$.

In light of these results, one can use the Bäcklund transformation to solve the BVP for the NLS equation on the half line with the BCs (2). The strategy is as follows: (i) Given $q(x,0)$ for $x > 0$, the ODE (17) plus the initial condition $\tilde{q}(0,0) = q(0,0)$ define $\tilde{q}(x,0)$ for $x \geq 0$. (ii) In turn, (19) allows one to extend the original potential to all $x < 0$. (iii) One can now use the IST machinery of the IVP with the extended potential to solve the BVP. Note that, since the BVP is well-posed, the solution obtained in this way is the unique solution of the problem.

The mirror symmetry induces the following additional symmetries: for the eigenfunctions and the scattering matrix: for all $x, t, k \in \mathbb{R}$,

$$\tilde{\Phi}_\pm(x,t,k) = \sigma_3\Phi_\mp(-x,t,-k)\sigma_3\,, \qquad \tilde{A}(k) = \sigma_3 A^{-1}(-k)\sigma_3\,.$$

Combining this with (18a) we then have, for all $x, t, k \in \mathbb{R}$,

$$\Phi_\pm(x,t,k) = B^{-1}(x,t,k)\sigma_3\,\Phi_\mp(-x,t,-k)\,\sigma_3 B_\infty(k). \tag{20}$$

In turn, the scattering matrix $A(k)$ satisfies:

$$A(-k) = \sigma_3 B_\infty(k) A^{-1}(k) B_\infty^{-1}(k) \sigma_3 \,, \qquad k \in \mathbb{R}. \tag{21}$$

Or, in component form:

$$a_{22}(k) = a_{22}^*(-k^*) \,, \qquad \operatorname{Im} k \geq 0, \tag{22a}$$

$$a_{12}(k) = f(k) a_{12}(-k) \,, \qquad k \in \mathbb{R}, \tag{22b}$$

where

$$f(k) = \frac{2ik - b_\infty}{2ik + b_\infty} \tag{23}$$

and where as usual the symmetry for the analytic scattering coefficient was extended via the Schwartz reflection principle. In the case of Neumann BCs, obtained for $\alpha = 0$, the Bäcklund transformation (17) reduces to the even extension of the potential and $f(k) = 1 \ \forall k \in \mathbb{R}$. In the case of Dirichlet BCs, obtained in the limit $\alpha \to \infty$, the Bäcklund transformation becomes singular. Using the odd extension of the potential, however, one still obtains (22), with $f(k) = -1 \ \forall k \in \mathbb{R}$.

These results apply both in the defocusing and in the focusing case. In the latter case, (22) also induces an extra symmetry for the discrete spectrum (in addition to the up–down $k \mapsto k^*$ symmetry of the IVP). Namely, for each discrete eigenvalue k_n there exist a symmetric eigenvalue

$$k_{n'} = -k_n^*. \tag{24}$$

If $k_{n'} = k_n$ we say that the eigenvalue k_n is *self-symmetric*. Obviously, an eigenvalue is self-symmetric if and only if it is purely imaginary; that is, $k_{s'} = k_s$ if $k_s = iA_s$. Let us denote by S the number of self-symmetric eigenvalues in the UHP, which is of course the number of discrete eigenvalues lying on the positive imaginary semi-axis. Let us also denote the number of non-self-symmetric eigenvalues in the UHP as N, so that the total number of discrete eigenvalues on the UHP is $J = S + 2N$.

The last remaining task is to determine the value of b_∞. For $\alpha \neq 0$, the matrix $B(0, t, 0)$ is invertible, so (20) implies $\Phi^{(1)}(0, t, 0) = (b_\infty/\alpha)\, \Phi^{(2)}(0, t, 0)$. Also, evaluating (21) at $k = 0$ yields $A(0) = A^{-1}(0)$, which in turn implies $A(0) = a_{22}(0)\, I$ since $\det A(k) = 1 \forall k \in \mathbb{R}$. Thus $b_\infty = a_{22}(0)\, \alpha$. Now recall that the analytic scattering coefficients are determined completely in terms of the discrete spectrum and the reflection coefficient via the trace formulae [10, 11]. In particular:

$$a_{22}(k) = \mathrm{e}^{-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 - \nu |\rho(\zeta)|^2)}{\zeta - k} \mathrm{d}\zeta} \prod_{j=1}^{J} \frac{k - k_j}{k - k_j^*} \,, \qquad \operatorname{Im} k > 0. \tag{25}$$

Evaluating (25) at $k = 0$, using the fact that the integral vanishes there, and taking into account of the symmetry eigenvalues, one has $a_{22}(0) = (-1)^S$. Thus,

$$b_\infty = a_{22}(0)\,\alpha = (-1)^S\alpha. \tag{26}$$

Equation (26) also holds when $\alpha = 0$, since in this case the Bäcklund transformation reduces to the even extension.

4. Linearization of the Bäcklund transformation

As mentioned earlier, a key step in the method to solve the BVP is to generate the Bäcklund-transformed potential $\tilde{q}(x,t)$ via (17) given the original potential $q(x,t)$ for $x > 0$, thus obtaining the extension of $q(x,t)$ to the negative real axis via the mirror symmetry (19). On the other hand, (17) contains $b(x,t)$, which in turn depends on the unknowns $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ via (16). Therefore, obtaining $\tilde{q}(x,t)$ requires solving a nonlinear system of integro-differential equations. But since the NLS equation is integrable and the homogeneous Robin BCs (2) are "linearizable," one should be able to reduce the solution of the BVP to linear operations. Thus, for the Bäcklund transformation to provide an effective method to solve the BVP, one must be able to solve the above system of ODEs. We next show how this can be done.

We begin by differentiating (16), thus trivially rewriting the problem into a standard nonlinear system of ODEs:

$$\tilde{q}' = q' + (\tilde{q} + q)\,b, \qquad \tilde{r}' = r' + (\tilde{r} + r)\,b, \qquad b' = \tilde{q}\tilde{r} - qr \tag{27}$$

(in this section we will use prime to indicate partial differentiation with respect to x, for brevity), together with the *initial* conditions (ICs)

$$\tilde{q}(0,t) = q(0,t)\,, \qquad \tilde{r}(0,t) = r(0,t)\,, \qquad b(0,t) = \alpha. \tag{28}$$

One can verify by direct calculation that $[b^2 - (\tilde{q} - q)(\tilde{r} - r)]' = 0$, which, taking into account the ICs (28), yields

$$b^2 = (\tilde{q} - q)(\tilde{r} - r) + \alpha^2. \tag{29}$$

It is convenient to introduce the changes of dependent variable

$$u(x,t) = \tilde{q}(x,t) + q(x,t)\,, \qquad v(x,t) = \tilde{r}(x,t) + r(x,t)$$

and use (29) to rewrite the system (27) as

$$u' = bu + 2q', \qquad v' = bv + 2r', \qquad b' = b^2 + (ru + qv) - (4qr + \alpha^2). \tag{30}$$

We now perform the further change of variable

$$Y(x,t) = \exp\left[-\int_0^x b(y,t)\,\mathrm{d}y\right], \tag{31}$$

which is inverted by the Cole-Hopf transformation

$$b(x,t) = -Y'(x,t)/Y(x,t)\,,$$

and we introduce the following quadratic transformations

$$Y_1(x,t) = Y(x,t) \qquad Y_2(x,t) = Y'(x,t)\,, \tag{32a}$$

$$Y_3(x,t) = Y(x,t)\,u(x,t)\,, \qquad Y_4(x,t) = Y(x,t)\,v(x,t)\,. \tag{32b}$$

With these substitutions, (30) becomes the linear system of ODEs

$$Y_1' = Y_2\,, \qquad Y_2' = (4qr + \alpha^2)Y_1 - r\,Y_3 - q\,Y_4\,, \tag{33a}$$

$$Y_3' = 2q'Y_1\,, \qquad Y_4' = 2r'Y_1\,, \tag{33b}$$

and the ICs (28) become

$$Y_1(0,t) = 1\,, \quad Y_2(0,t) = -\alpha\,, \quad Y_3(0,t) = 2q(0,t)\,, \quad Y_4(0,t) = 2r(0,t). \tag{34}$$

Standard ODE results then guarantee that, under fairly mild regularity conditions on the potential, a global solution of the above system exists $\forall x \in \mathbb{R}$. Once the solution of (33) has been obtained, one can then back-substitute to recover the Bäcklund-transformed potentials as

$$\tilde{q}(x,t) = -q(x,t) + Y_3(x,t)/Y(x,t)\,, \tag{35a}$$

$$\tilde{r}(x,t) = -r(x,t) + Y_4(x,t)/Y(x,t)\,. \tag{35b}$$

The Bäcklund transformation—and with it the whole BVP—have therefore been completely linearized.

Note that $Y(x,t) \neq 0$ for all $x, t \in \mathbb{R}$; hence the right-hand side of (35) is always non-singular. Note also that the boundary data $q(0,t)$ and $r(0,t)$ appearing in the ICs (34) are not known in general when $\alpha \neq 0$. In practice, however, one only needs to solve the system (33) at $t = 0$: Once the extended potential $q(x,0)$ has been obtained for all $x \in \mathbb{R}$, one can compute the scattering data and therefore completely determine the solution of the BVP.

5. Norming constants and self-symmetric eigenvalues

We now obtain the relations between the norming constants of symmetric eigenvalues for the focusing NLS equation, including situations when self-symmetric eigenvalues are present. Explicitly, we show that, if b_n and $b_{n'}$ (or equivalently C_n and $C_{n'}$) are the norming constants associated respectively with a discrete eigenvalue k_n and its symmetric counterpart $k_{n'} = -k_n^*$, it is:

$$b_n b_{n'}^* = f(k_n)\,, \qquad C_n C_{n'}^* = -\frac{f(k_n)}{\big(a_{22}'(k_n)\big)^2}. \tag{36}$$

To derive these relations, we first separate the two columns of (20), obtaining

$$\Phi_\mp^{(1)}(x,t,k) = (2ik + b_\infty)\, B^{-1}(x,t,k)\sigma_3 \Phi_\pm^{(1)}(-x,t,-k), \tag{37a}$$

$$\Phi_\mp^{(2)}(x,t,k) = -(2ik - b_\infty)\, B^{-1}(x,t,k)\sigma_3 \Phi_\pm^{(2)}(-x,t,-k), \tag{37b}$$

for all $x, t, k \in \mathbb{R}$. Each of these relations can now also be extended to the appropriate half of the complex k-plane depending on the analyticity region of the eigenfunctions involved. Starting from the first of (10), one now uses successively (37a) to replace $\Phi_+^{(1)}(x,t,k_j)$, the second of (10) to replace $\Phi_-^{(1)}(-x,t,-k_j)$ (noting that $-k_j = k_{j'}^*$) and (37b) to replace $\Phi_+^{(2)}(-x,t,k_{j'}^*)$, finally obtaining

$$\left(1 - b_j b_{j'}^* \frac{2ik + b_\infty}{2ik - b_\infty}\right) \Phi_+^{(2)}(x,t,k_j) = 0\,.$$

Evaluating this relation in the limit $x \to \infty$ one then obtains the first of (36). The second of (36) follows trivially from the definition of C_j.

In particular, (36) implies that a self-symmetric eigenvalue $k_s = iA_s/2$ is allowed if and only if $f(k_s) > 0$, i.e., if and only if

$$A_s > |\alpha|. \tag{38}$$

Opposite limiting cases are Neumann BCs, when any self-symmetric eigenvalue is allowed, and Dirichlet BCs, when no self-symmetric eigenvalues can exist.

To effectively characterize the BVP, one should be able to uniquely determine the norming constants associated with the symmetric eigenvalues in terms of those associated with the original eigenvalues. We next show that this is indeed possible. Writing the norming constants as $C_j = A_j \mathrm{e}^{A_j \xi_j + i(\varphi_j + \pi/2)}$ for $j = 1, \ldots, J$ as before, the second of (36) implies

$$\xi_n + \xi_{n'} = \frac{1}{A_n}\big[\log|f(k_n)| - 2\log|a_{22}'(k_n)] - 2\log A_n\big], \tag{39a}$$

$$\varphi_n - \varphi_{n'} = \arg(f(k_n)) - 2\arg a_{22}'(k_n) + \pi. \tag{39b}$$

Recalling the trace formula (25), we then have, for all $j = 1, \ldots, J$,

$$a_{22}'(k_j) = \mathrm{e}^{-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(1+|\rho(\zeta)|^2)}{\zeta - k_j}\mathrm{d}\zeta} \prod_{m=1}^{J}{}' (k_j - k_m) \Big/ \prod_{m=1}^{J} (k_j - k_m^*)\,, \tag{40}$$

where the prime in the right-hand side indicates that the term with $m = j$ is omitted from the product. One can now use (40) in (39) to obtain $C_{n'}$ in terms of C_n.

Let us look at the relations (39) in more detail. For simplicity, we restrict ourselves to reflectionless solutions, in which case the integral in (40) is absent. We write the discrete eigenvalues as $k_j = (V_j + iA_j)/2$ for $j = 1, \ldots, J$, as before. Without loss of generality, we arrange the discrete eigenvalues in order of non-decreasing velocities, so that $k_{n'} = k_{2N+S-n+1}$ for all $n = 1, \ldots, N$ and $k_{(s+N)'} = k_{s+N}$ for all $s = 1, \ldots, S$. For the non-self-symmetric eigenvalues, (40) then yields

$$a'_{22}(k_n) = \frac{V_n}{iA_n(V_n + iA_n)} \prod_{p=N+1}^{N+S} \frac{V_n + i(A_n - A_p)}{V_n + i(A_n + A_p)} \prod_{m=1}^{N}{}' \frac{[(V_n - V_m) + i(A_n - A_m)]}{[(V_n + V_m) + i(A_n + A_m)]} \frac{[(V_n + V_m) + i(A_n - A_m)]}{[(V_n - V_m) + i(A_n + A_m)]}, \tag{41}$$

for $n = 1, \ldots, N$, while for the symmetric eigenvalues

$$a'_{22}(k_s) = \frac{1}{iA_s} \prod_{p=N+1}^{N+S}{}' \frac{A_s - A_p}{A_s + A_p} \prod_{m=1}^{N} \frac{(A_s - A_m)^2 + V_m^2}{(A_s + A_m)^2 + V_m^2},$$

for $s = N+1, \ldots, N+S$. Substituting these expressions in (39) we then obtain

$$\begin{aligned} \xi_n + \xi_{n'} = \frac{1}{A_n}\Bigg[& \log\left(1 + \frac{A_n^2}{V_n^2}\right) \\ & + \frac{1}{2}\log\frac{V_n^2 + (A_n + (-1)^S\alpha)^2}{V_n^2 + (A_n - (-1)^S\alpha)^2} + \sum_{p=N+1}^{N+S} \log\frac{V_n^2 + (A_n + A_p)^2}{V_n^2 + (A_n - A_p)^2} \\ & + \sum_{m=1}^{N}{}' \log\left(\frac{(V_n + V_m)^2 + (A_n + A_m)^2}{(V_n - V_m)^2 + (A_n - A_m)^2} \frac{(V_n - V_m)^2 + (A_n + A_m)^2}{(V_n + V_m)^2 + (A_n - A_m)^2}\right)\Bigg], \end{aligned}$$

$$\begin{aligned} \varphi_n - \varphi_{n'} = \pi & - 2\arg(A_n + iV_n) \\ & - \arg\left(\frac{V_n + i(A_n - (-1)^S\alpha)}{V_n + i(A_n + (-1)^S\alpha)}\right) - 2\sum_{p=N+1}^{N+S} \arg\left(\frac{V_n + i(A_n - A_p)}{V_n + i(A_n + A_p)}\right) \\ & - 2\sum_{m=1}^{N}{}' \arg\left(\frac{V_n - V_m + i(A_n - A_m)}{V_n + V_m + i(A_n + A_m)} \frac{V_n + V_m + i(A_n - A_m)}{V_n - V_m + i(A_n + A_m)}\right), \end{aligned}$$

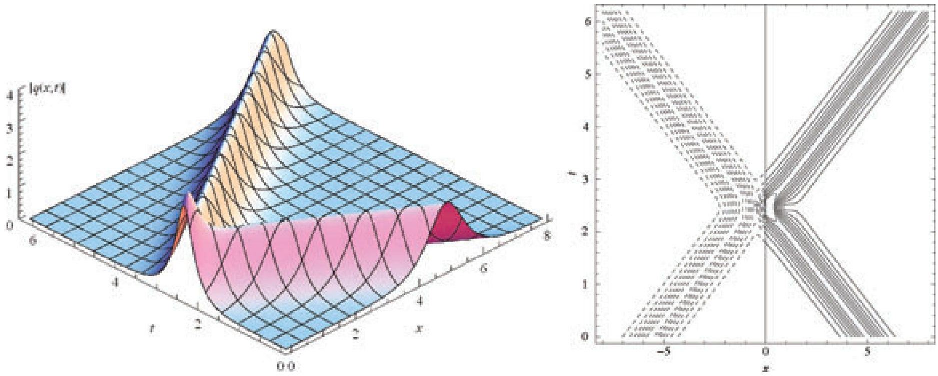

Figure 1. Soliton reflection at the boundary in the BVP for the NLS equation on the half line with Robin BCs at the origin, with $\alpha = -\sqrt{3}$ and no self-symmetric eigenvalues: $A_1 = 2$, $V_1 = -1$, $\xi_1 = 5$ and $\varphi_1 = 0$. Left-hand side: Three-dimensional plot. Right-hand side: Contour plot of the extended potential, including the mirror soliton.

for $n = 1, \dots, N$, together with

$$\xi_s = \frac{1}{A_s}\left[\frac{1}{2}\log\frac{A_s+(-1)^S\alpha}{A_s-(-1)^S\alpha} + \sum_{p=N+1}^{N+S}{}' \log\frac{A_s+A_p}{|A_s-A_p|} + \sum_{m=1}^{N}\log\frac{V_m^2+(A_s+A_m)^2}{V_m^2+(A_s-A_m)^2}\right],$$

for $s = N+1, \dots, N+S$.

Note that, even though (36) involves only one eigenvalue and its symmetric counterpart, the explicit relations between the norming constants are affected by *all* eigenvalues via the derivative of the scattering coefficient. Note also that no prescription is obtained for the phase of the norming constants associated with self-symmetric eigenvalues. Thus, changes in this phase do not affect the BCs.

6. Examples

We now illustrate the results by presenting several explicit soliton solutions of the BVP for the focusing NLS equation with Robin BCs. As in [3], we refer to solitons located in the physical domain as physical solitons and to their symmetric counterparts as mirror solitons.

Figure 1 shows a solution generated by one non-self-symmetric eigenvalue with $\alpha = -\sqrt{3}$. Here and below, the left portion of the figure shows a three-dimensional plot of the solution in the physical domain $x > 0$, while the right portion of the figure shows a contour plot of the extended solution over

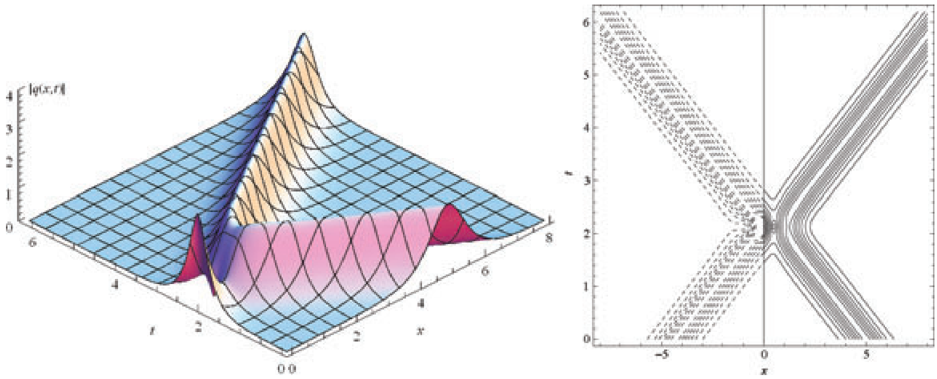

Figure 2. Same as Figure 1, but with $\alpha = \sqrt{3}$. Note the different position of the soliton after the reflection compared to Figure 1.

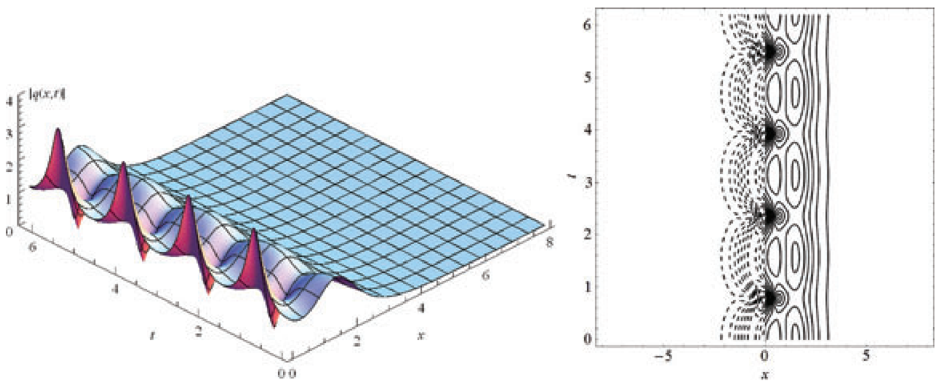

Figure 3. A self-symmetric bound state for the BVP with Robin BCs and $\alpha = 1$: $A_1 = 3/2$, $A_2 = 5/2$, and $\varphi_1 = \varphi_2 = 0$.

the whole real axis, including the mirror soliton. As discussed in [3], the apparent reflection experienced by the soliton at the boundary is nothing else but a role swap between the physical soliton and its symmetric counterpart. Figure 2 shows a solution generated by the same discrete eigenvalue and norming constant, but with $\alpha = \sqrt{3}$. Note the difference in the final position of the soliton after reflection. Figure 3 shows a bound state generated by two self-symmetric eigenvalues. Both solitons remains localized near the origin at all times, as in the IVP, and one can observe the usual beating between the two solitons. Nonetheless, the relations between the norming constants ensure that the solution satisfies the Robin BCs. Figure 4 shows a solution corresponding to one self-symmetric and one non-self-symmetric eigenvalue with $\alpha = -\sqrt{3}$. The soliton corresponding to the self-symmetric eigenvalue remains localized near the origin at all times, as before, while the soliton

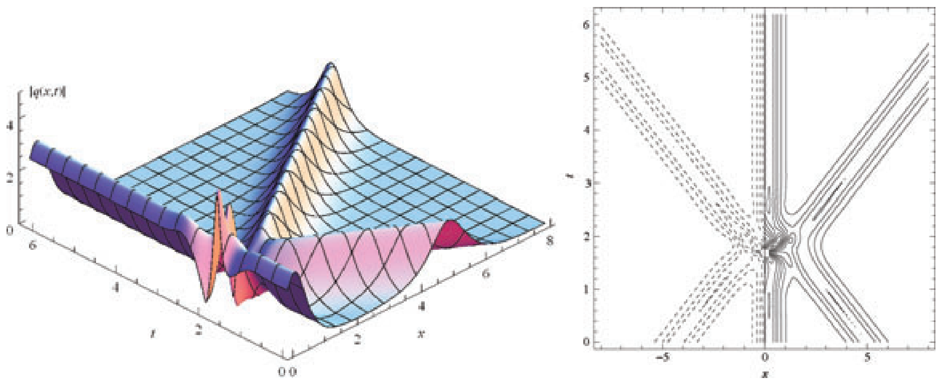

Figure 4. Soliton reflection for a BVP with $\alpha = -\sqrt{3}$ and one self-symmetric eigenvalue: $A_1 = 3$, $A_2 = 2$, $V_1 = 0$, $V_2 = -3/2$, $\xi_2 = 5$ and $\varphi_1 = \varphi_2 = 0$. Left-hand side: 3D plot. Right-hand side: contour plot showing both the physical solitons and the mirror soliton.

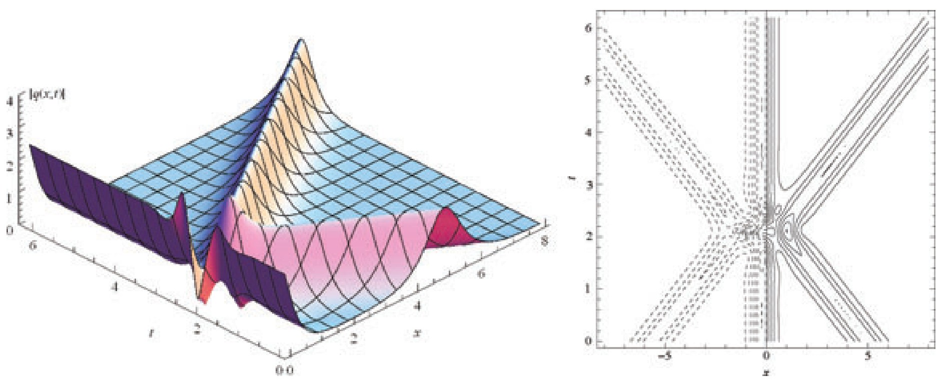

Figure 5. Same as Figure 4, but with $\alpha = \sqrt{3}$. Again, note the different position of the soliton after the reflection compared to Figure 4.

corresponding to the non-self-symmetric eigenvalue approaches the origin, interchanges its role with the mirror soliton upon collision, and then escapes to infinity. Figure 5 shows the solution generated by the same eigenvalues and norming constants but with $\alpha = \sqrt{3}$. Again, note the differences between the solutions, and in particular the different position of the self-symmetric soliton with respect to the origin. Finally, Figure 6 shows a solution generated by two self-symmetric eigenvalues and one non-self-symmetric eigenvalue, which combines the behavior in the previous examples.

Note that, even though all the above examples are, for simplicity, reflectionless solutions of the focusing case, the results apply also in the defocusing case, and also when the reflectionless coefficient is not identically zero.

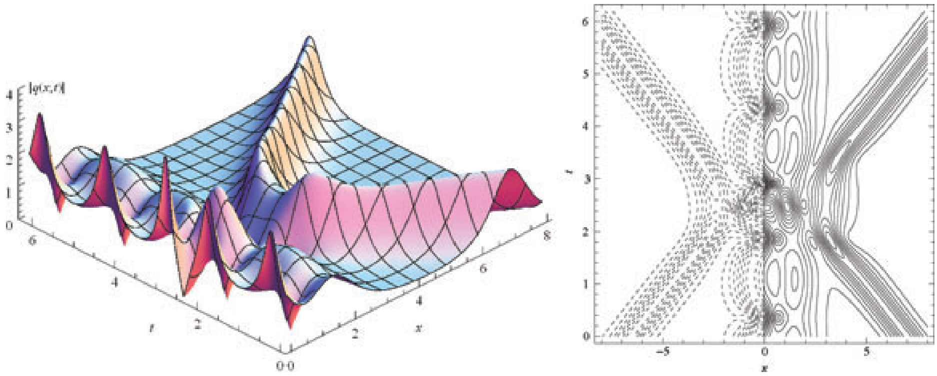

Figure 6. A solution of the BVP with Robin BCs with $\alpha = 1$ showing a traveling soliton and a self-symmetric bound-state at the origin: $A_1 = 3/2$, $A_2 = 5/2$, $A_3 = 2$, $V_1 = 0$, $V_2 = 0$, $V_3 = -1$, $\xi_3 = 7$ and $\varphi_1 = \varphi_2 = \varphi_3 = 0$.

7. Reflection-induced soliton position shift

As discussed in [3], the reflection experienced by a soliton at the boundary of the physical domain is accompanied by a corresponding position shift, which includes two components: one from the soliton interaction and one from the role swap between the physical soliton and its mirror. Here we generalize the calculations in [3] to cases with self-symmetric solitons (correcting minor typos in the process).

We label the discrete eigenvalues in order of nondecreasing velocity, as before, so that as $t \to -\infty$ the physical solitons correspond to the discrete eigenvalues $k_1, \dots, k_N$ and as $t \to \infty$ to $k_{S+N+1}, \dots, k_{S+2N}$. We also suppose for simplicity that all the non-zero velocities are non-degenerate, i.e., $V_{n+1} \neq V_n$ for all $n = 1, \dots, N-1$. As $t \to \pm\infty$, the solution of the NLS equation is

$$q(x,t) = \sum_{n=1}^{N} \left[q_{1s}(x,t;A_n,V_n,\xi_n^{\pm},\varphi_n^{\pm}) \right.$$
$$\left. + q_{1s}(x,t;A_{n'},V_{n'},\xi_{n'}^{\pm},\varphi_{n'}^{\pm}) \right] + q_{ss}(x,t) + o(1)\,,$$

where $q_{1s}(x,t;A,V,\xi,\varphi)$ is as in (12) and $q_{ss}(x,t)$ collects the contributions from all the self-symmetric solitons. The constants $\xi_n^{\pm}$ are obtained by explicitly computing the long-time asymptotics of the solution. In particular, for reflectionless solutions one has (e.g., see [10])

$$\xi_n^{\pm} = \xi_n + \frac{1}{A_n}\left[\log A_n + \log |a_{22}'(k_n)| \pm {\sum_{j=1}^{J}}' \sigma_{jn} \log \left| \frac{k_n - k_j}{k_n - k_j^*} \right| \right], \quad (42)$$

for all $n = 1, \ldots, N$ and all $n = S{+}N{+}1, \ldots, S{+}2N$, where σ_{jn} equals 1 for $j > n$ and -1 for $j < n$.

Consider a physical soliton identified by the discrete eigenvalue k_n as $t \to -\infty$. As $t \to -\infty$, its center tends to $x_n^-(t) = -2|V_n|t + \xi_n^-$ (note $V_n < 0$ since $n < N$). This trajectory reaches the boundary at $t = \xi_n^-/(2|V_n|) =: t_*$. If there were no soliton interactions and the reflection were purely linear, as $t \to \infty$ the center of the soliton would therefore be $x_n^{\rm lin} = 2|V_n|(t - t_*) = 2|V_n|t - \xi_n^-$. On the other hand, after the reflection the physical soliton is identified by the n'th eigenvalue, whose position as $t \to \infty$ is $x_{n'}^+(t) = 2V_{n'}t + \xi_{n'}^+$. It then follows that the total reflection-induced position shift is $\Delta\xi_n = x_{n'}^+ - x_n^{\rm lin} = \xi_{n'}^+ + \xi_n^-$. Or, equivalently,

$$\Delta\xi_n = \xi_n^- + \xi_{n'}^- + \delta\xi_{n'}\,, \tag{43}$$

where $\delta\xi_m$ is the total interaction-induced position shift:

$$\begin{aligned}\delta\xi_m = \xi_m^+ - \xi_m^- &= \frac{2}{A_m}\sum_{j=1}^{J}{}' \sigma_{jm}\log\left|\frac{k_m - k_j}{k_m - k_j^*}\right| \\ &= \frac{2}{A_m}\sum_{j=1}^{J}{}' \sigma_{jm}\,\log\frac{(V_m - V_j)^2 + (A_m - A_j)^2}{(V_m - V_j)^2 + (A_m + A_j)^2}\,.\end{aligned} \tag{44}$$

for all $m = 1, \ldots, N$ and all $m = S{+}N{+}1, \ldots, S{+}2N$. Moreover, (42) implies

$$\begin{aligned}\xi_n^- + \xi_{n'}^- = \xi_n + \xi_{n'} + \frac{1}{A_n}\bigg[&2\log A_n + 2\log|a_{22}'(k_n)| \\ &- \sum_{j=1}^{J}{}' \sigma_{jn}\log\left|\frac{k_n - k_j}{k_n - k_j^*}\right| - \sum_{j=1}^{J}{}' \sigma_{jn'}\log\left|\frac{k_{n'} - k_j}{k_{n'} - k_j^*}\right|\bigg],\end{aligned} \tag{45}$$

where we used that $|a_{22}'(k_n)| = |a_{22}'(k_{n'})|$ thanks to (22a). Now note that $\sigma_{jn'} = -\sigma_{j'n}$. By rearranging terms in one of the summations, it is then straightforward to see that the two sums completely cancel each other. Further using (39a) then yields

$$\xi_n^- + \xi_{n'}^- = \frac{1}{A_n}\log|f(k_n)| = \frac{1}{2A_n}\log\frac{V_n^2 + (A_n + (-1)^S\alpha)^2}{V_n^2 + (A_n - (-1)^S\alpha)^2}. \tag{46}$$

Inserting (44) and (46) in (43) then finally yields the reflection-induced position shift.

8. Comparison with linear behavior

It is instructive to compare the behavior of the solutions of the BVP for the NLS equation to that of the solution of the same BVP for the time-dependent linear Schrödinger (LS) equation,

$$iq_t + q_{xx} = 0. \tag{47}$$

Note first that, if $q(x,t)$ is obtained from an inverse Fourier transform (FT), defined as usual as

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ikx}\hat{q}(k,t)\,\mathrm{d}k, \tag{48}$$

and $\hat{q}(k,t)$ satisfies the symmetry

$$(k - i\alpha)\hat{q}(k,t) = (k + i\alpha)\,\hat{q}(-k,t), \tag{49}$$

$q(x,t)$ satisfies the homogeneous Robin BC (2) at $x = 0$, as one can verify by direct calculation.

Several methods can be used to show that the solution of the BVP for the LS (47) with BCs (2) is

$$\begin{aligned} q(x,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\theta(x,t,k)}\hat{q}(k,0)\,\mathrm{d}k + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\theta(x,t,k)} f_\alpha(k)\hat{q}(-k,0)\,\mathrm{d}k \\ & + 2\alpha\Theta(\alpha)\,\mathrm{e}^{i\theta(x,t,i\alpha)}\,\hat{q}(-i\alpha,0)\,, \end{aligned} \tag{50}$$

where $\hat{q}(k,t)$ is the one-sided FT,

$$\hat{q}(k,t) = \int_0^{\infty} \mathrm{e}^{-ikx} q(x,t)\,\mathrm{d}x, \tag{51}$$

$\theta(x,t,k) = kx - k^2 t$ for the LS equation, $f_\alpha(k) = (k + i\alpha)/(k - i\alpha)$, and $\Theta(\alpha)$ is the Heaviside step function, defined as $\Theta(\alpha) = 0$ for $\alpha < 0$ and $\Theta(\alpha) = 1$ for $\alpha > 0$. As usual, for Dirichlet and Neumann BCs (namely, $\alpha = 0$ and $\alpha \to \infty$) $f_\alpha(k)$ reduces to its limiting values of -1 and 1, respectively, and the last term in the right-hand side of (50) is absent in either case. Denoting by $q_1(x,t), \ldots, q_3(x,t)$ the individual terms in the right-hand side of (50), note that: each term separately solves the LS equation; $q_1(x,t)$ satisfies the IC, while $q_2(x,t)$ and $q_3(x,t)$ cancel each other exactly at $t = 0$; finally, $q_1(x,t)$ and $q_2(x,t)$ together satisfy the BCs, while $q_3(x,t)$ does so by itself. [Note also that $\mathrm{e}^{i\theta(i\alpha)} = \mathrm{e}^{-\alpha x + i\alpha^2 t}$, and $q_3(x,t)$ can be written via (48) with $\hat{q}_3(x,t) = -2i\alpha/(k - i\alpha)\,\Theta(\alpha)\,\mathrm{e}^{i\alpha^2 t}\hat{q}(-i\alpha,0)$.]

Let us now use the representation (50) to study the behavior of the solution. In the limiting cases of Dirichlet and Neumann BCs we have $q_2(x,t) = \mp q_1(-x,t)$, respectively. Noting that $q_1(x,0) = 0$ for $x < 0$, we therefore see that the solution describes the clean reflection of an incoming wave packet (accompanied with a π phase shift in the case of Dirichlet BCs). For $\alpha \neq 0$, however, $f_\alpha(k)$ describes a k-dependent phase: $f_\alpha(k) = \mathrm{e}^{2i \arctan(k/\alpha)+i\pi}$. As a result, $q_2(x,t)$ is not simply a mirror image of $q_1(x,t)$ as in the case of Dirichlet and Neumann BCs. Thus, for $\alpha \neq 0$, $q_2(x,t)$—and $q_3(x,t)$ when present—describe an unclean reflection.

Recall that the sum of $q_2(x,t)$ and $q_3(x,t)$ is exactly zero at $t=0$. On the other hand, this contribution becomes nonzero as soon as $t \neq 0$, which is a result of the parabolic nature of the LS equation. In particular, $q_2(x,t) \to 0$ in the L^∞ norm as $t \to \infty$. So, when $\alpha > 0$, the result of the reflection in the long-time asymptotic behavior of the solution is a standing wave localized near the boundary, a phenomenon that is reminescent of edge waves [16, 17].

9. Remarks

In summary, we have revisited the BVP for the nonlinear Schrödinger equation on the half line with homogeneous Robin BCs at the origin, using a symmetric Bäcklund transformation and clarifying several issues.

The present approach is obviously related the dressing method [13], but the Bäcklund transformation in this case is linearized via a different procedure from that used to generate the multi-soliton solutions [15, 13]. We also reiterate that the direct k-dependent extension proposed in [5] and used in [3] is perfectly adequate if no self-symmetric eigenvalues are present, but yields incorrect symmetries for the scattering eigenfunctions if an odd number of self-symmetric eigenvalues is present. Moreover, it is not clear whether a similar explicit extension could be written to solve BVPs for the Ablowitz-Ladik system or for the NLS equation with nonzero potential at infinity.

Of course Fokas' method or the method of Degasperis, Santini, and Manakov can also be used to solve these kinds of BVPs. Indeed, Fokas' method was used in [18] for the NLS equation and in [19] for the Ablowitz-Ladik system, and the method of Degasperis, Santini and Manakov was used for the NLS equation in [4]. In fact, both of these methods are more general, since they can be used with any kinds of BCs. The present method only applies to BVPs with linearizable BCs. For those BCs, however, the present method allows an effective characterization of the discrete spectrum (including cases with self-symmetric eigenvalues) and an explicit description of the long-time asymptotics of the solution of the BVP both as $t \to \infty$ and as $t \to -\infty$. At present, it is unclear how one can obtain the same kind of information with either Fokas' method or the method of Degasperis, Santini, and Manakov. In particular, note that ICs with self-symmetric solitons also produce purely imaginary eigenvalues in Fokas' formalism. The study of such solutions using Fokas' method therefore involves singularities on the integration contour of the RHP, and such situations were explicitly excluded in [18].

An important open question is whether the extended potential remains bounded as $x \to -\infty$. As we show in Appendix, for the LS equation, $\tilde{q}(x,t)$ can grow exponentially as $x \to \infty$ when $\alpha > 0$. When this happens, the extended potential does not admit a Fourier transform, and is outside the class of functions for which the IST applicable. In other words, for the LS equation with Robin BCs, one can obtain the solution by extension only if $\alpha < 0$. The situation is different for the NLS equation, since in this case the homogeneous term in the ODE for the mirror potential becomes nonlinear and contains $b(x,t)$, which can change sign. Therefore, determining the asymptotic behavior of $\tilde{q}(x,t)$ as $x \to \infty$ is non-trivial. Moreover, for the pure soliton solutions the extended potential is bounded for all x, independently of the sign of α and the sign of b_∞. (Indeed, one could interpret the condition (38) as a sufficient condition for the potential to guarantee decay of the mirror image as $x \to \infty$.) It remains to be addressed whether the same is true for solutions of the NLS equation with non-zero reflection coefficient.

On a more physical note, we find it remarkable that, while for $\alpha \neq 0$ the solutions of the LS equation experience an unclean reflection, the solitons of the NLS equation always reflect cleanly. Since the IST reduces to the Fourier transform in the linear limit, one would expect that the radiative parts of the solution (i.e., the portion originating from the reflection coefficient) will also experience an unclean reflection. Whether this is indeed the case is another issue that remains to be investigated.

Note added in proofs: We recently learned that the BVP for the NLS equation with homogeneous Robin BCs has also been studied in [20, 21], where some of the issues mentioned earlier were addressed. Also, BVPs for the vector NLS equation were recently studied in [22, 23].

Acknowledgments

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Appendix: Bäcklund transformation for the linear Schrödinger equation

Here, we show how BVPs for the LS equation on the half line $0 < x < \infty$ with homogeneous Robin BCs at $x = 0$ can be solved using a similar Bäcklund transformation as for the NLS equation. The key to do so is to use the 2×2 multiplicative Lax pair (3) instead of the simpler, scalar, and additive Lax pair that is available for linear evolution equations [24].

The LS equation is equivalent to the NLS equation with $\nu = 0$, and is therefore the compatibility condition of the Lax pair (3) with $r(x, t) = 0$. Thus, similarly to Section 3, if $q(x, t)$ and $\tilde{q}(x, t)$ both solve the LS equation and if their corresponding eigenfunctions $\Phi(x, t, k)$ and $\tilde{\Phi}(x, t, k)$ are related by (13), we say that $q(x, t)$ and $\tilde{q}(x, t)$ are related by a Bäcklund transformation. Looking for $B(x, t, k)$ linear in k, one again obtains (15), where now $b(x, t) \equiv \alpha$ $\forall x \in \mathbb{R}$, and the potentials satisfy the linear version of (17):

$$\tilde{q}_x - q_x = \alpha\,(q + \tilde{q}). \tag{A.1}$$

The approach to solve the BVP for the LS equation is then identical to that used in Section 3 for the NLS equation. Since $Q(x, t)$ has only one nonzero entry when $\nu = 0$, the Jost solutions can be found in closed form:

$$\Phi_-(x, t, k) = \mathrm{e}^{i\theta(x,t,k)\sigma_3} + \mathrm{e}^{i\theta(x,t,k)} \int_{-\infty}^{x} \mathrm{e}^{-2i\theta(y,t,k)} Q(y, t)\,\mathrm{d}y, \tag{A.2a}$$

$$\Phi_+(x, t, k) = \mathrm{e}^{i\theta(x,t,k)\sigma_3} - \mathrm{e}^{i\theta(x,t,k)} \int_{x}^{\infty} \mathrm{e}^{-2i\theta(y,t,k)} Q(y, t)\,\mathrm{d}y. \tag{A.2b}$$

We can now use the results from Section 3. In particular, (22b) yields

$$a_{12}(k) = \frac{2ik - \alpha}{2ik + \alpha}\, a_{12}(-k). \tag{A.3}$$

Now recall that the scattering matrix admits an integral representation, obtained from (5) via $A(k) = \lim_{x\to\infty} \mathrm{e}^{-i\theta(x,t,k)\sigma_3} \mu_-(x, t, k)\, \mathrm{e}^{i\theta(x,t,k)\sigma_3}$. In particular,

$$a_{12}(k) = \int_{-\infty}^{\infty} q(x, t)\mu_{-,22}(x, t, k)\, \mathrm{e}^{-2i\theta(x,t,k)}\, \mathrm{d}x\,,$$

But in our case $\Phi_{-,22}(x, t, k) = \mathrm{e}^{-i\theta(x,t,k)}$ $\forall x \in \mathbb{R}$. Hence it is simply

$$a_{12}(k) = \mathrm{e}^{4ik^2 t} \hat{q}(2k, t)\,,$$

where $\hat{q}(k, t)$ is the usual two-sided FT:

$$\hat{q}(k, t) = \int_{-\infty}^{\infty} \mathrm{e}^{-ikx} q(x, t)\, \mathrm{d}x. \tag{A.4}$$

Hence the symmetry (A.3) yields (49) upon a trivial rescaling $k \to 2k$.

Note that, even though we used a multiplicative Lax pair, the Jost eigenfunctions are still obtained by quadratures via (A.2), and no integral equations are needed. Next we show that the inverse problem can also be reduced to quadratures. Explicitly, from (A.2) we have

$$\Phi_-(x, t, k) - \Phi_+(x, t, k) = \mathrm{e}^{i\theta(x,t,k)} \int_{-\infty}^{\infty} \mathrm{e}^{-2i\theta(y,t,k)} Q(y, t)\, \mathrm{d}y. \tag{A.5}$$

Moreover, since $\Phi_{\pm,11}(x,t,k) = \mathrm{e}^{i\theta(x,t,k)}$ $\forall x \in \mathbb{R}$, we can express (A.5) as the scattering relation (5), where $A(k) = I + \mathrm{e}^{ik^2t}\hat{Q}(2k,t)$. Since $A(k)$ is time-independent, however, we can also evaluate the RHS at $t = 0$, obtaining

$$A(k) = I + \hat{Q}(2k, 0).$$

Taking the 1,2 component of the scattering relation (5), we then get

$$\mu_{-,12}(x,t,k) - \mu_{+,12}(x,t,k) = \mathrm{e}^{2i\theta(x,t,k)}\hat{q}(2k,0)\,,$$

which provides a scalar, additive RHP that can be solved explicitly using Cauchy projectors, yielding

$$\mu_{12}(x,t,k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\hat{q}(2\zeta,0)}{\zeta-k}\,\mathrm{e}^{2i\theta(x,t,\zeta)}\,\mathrm{d}\zeta.$$

(Recall that $\mu_{-,12}$ and $\mu_{+,12}$ are analytic in the UHP and LHP, respectively.) The asymptotic formula (9) then yields (upon a trivial rescaling) the reconstruction formula for the solution of the BVP in the form of the inverse Fourier transform (48), where as usual $\hat{q}(k,t) = \mathrm{e}^{-ik^2t}\hat{q}(k,0)$.

Finally, we note that, once the Bäcklund transformation has been established, it is also possible to bypass the 2×2 Lax pair (3) completely and use the simpler Lax pair

$$\mu_x - ik\mu = q, \qquad \mu_t + ik^2\mu = iq_x - kq \tag{A.6}$$

for the LS equation, where $\mu(x,t,k)$ is now a scalar function. This is because, unlike the case of the Bäcklund transformation for the NLS equation, (A.1) can be solved explicitly for $\tilde{q}(x,t)$ as

$$\tilde{q}(x,t) = q(x,t) + 2\alpha\int_0^x \mathrm{e}^{\alpha(x-y)}q(y,t)\mathrm{d}y, \qquad x > 0. \tag{A.7}$$

It is trivial to see that the extended potential thus obtained indeed satisfies the homogeneous Robin BC (2) at $x = 0$. One can now use this potential and the simpler IST for the Lax pair (A.6) on $-\infty < x < \infty$ to solve the BVP. It is also straightforward (albeit tedious) to verify by direct calculation that, when the extended potential is defined according to (A.7), its FT $\hat{q}(k,t)$, defined by (A.4), satisfies the corresponding symmetry (49). Indeed, one can show that

$$\hat{q}^{\,\mathrm{ext}}(k,t) = \hat{q}_o(k,t) + f_\alpha(k)\hat{q}_o(-k,t), \tag{A.8}$$

where to avoid confusion we denoted with $\hat{q}^{\,\mathrm{ext}}(k,t)$ the two-sided FT of the extended potential [given by (A.4)] and with $\hat{q}_o(k,t)$ the one-sided FT of the original one [given by (51)]. Note however that (A.7) implies that in general $\tilde{q}(x,t)$ grows exponentially as $x \to \infty$. Thus, for this method to be applicable, one needs to assume that $\alpha < 0$ to ensure that the extended potential vanishes as $x \to -\infty$. (Correspondingly, insertion of (A.8) in the inverse FT formula

(48) yields an incorrect expression for $q(x, t)$ when $\alpha > 0$, since it misses the third term in (50).)

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State University of New York

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