Polarization-Mode Dispersion Emulation With Maxwellian Lengths and Importance Sampling

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***Abstract*—We describe an efficient polarization-mode dispersion (PMD) emulation method that combines a concatenation of birefringent sections having Maxwellian-distributed lengths with importance sampling. The method generates PMD statistics with accurate tail distributions using relatively few sections and reasonable numbers of samples, and allows the targeting of specific, arbitrarily large, first- and second-order PMD values.**

***Index Terms*—Importance sampling (IS), Monte Carlo (MC) simulation, optical fiber communications, polarization-mode dispersion (PMD).**

I. Introduction

IN RECENT years, it has become clear that, because of the random nature of polarization-mode dispersion (PMD), efficient tools for the controlled numerical/experimental production of PMD are necessary to test its effects [1], [2]. When implemented in hardware, such testing devices are known as PMD emulators or sources [3], [4]. The two most commonly used models (both in software and in hardware) employ a concatenation of fixed birefringent sections, with a choice of equal or nonequal section lengths [3]–[5]. Recently, importance sampling (IS) [6], [7] and multicanonical Monte Carlo (MC) [8] methods have been developed for these emulators. Compared with standard MC methods, these methods make it possible to efficiently estimate low probability events with a much smaller number of samples, thus, enabling the calculation of penalties at realistic probabilities [9], [10].

A drawback of fixed-length emulators, however, is that a large number of sections is required to accurately reproduce the statistical distributions of first- and second-order PMD of real fiber down to low probability values [6], [11]. This difficulty is intrinsic: When computing probabilities of quantities such as the PMD-induced penalty, one cannot use a small number of sections and simply rescale by the ratio between the probability associated with the real fiber's first-order or first- and second-order PMD and the values associated with the emulator. In general, the penalty depends upon more than just one or two moments. Thus, enough sections must be used to accurately reproduce all potentially contributory PMD states [6].

An alternative PMD emulation method has recently been proposed [12]. This method consists of a concatenation of sections connected by polarization controllers, where the section lengths are variable and Maxwellian-distributed. The model can be implemented both in software and in hardware [12], [13]. Compared with other PMD emulation schemes, it offers significant advantages: It generates the *exact* differential group delay (DGD) distribution with any number of sections.

Here, we describe how to apply IS to PMD emulation where polarization controllers and Maxwellian-distributed section lengths are used. Use of IS demonstrates that this emulation method requires a much smaller number of sections to produce accurate second-order PMD (SOPMD) statistics down to low probabilities. Furthermore, the application of IS to PMD emulation with Maxwellian-distributed section lengths allows the targeting of specific, arbitrarily large values of both first-order PMD and SOPMD. This is similar in spirit to the brownian bridge method [14], but more general because large SOPMD can also be targeted.

With this method, MC samples can be placed where desired when simulations are performed. Because of these benefits, the combined use of IS and Maxwellian-section lengths, therefore, results in a very efficient method for the calculation of realistic PMD-induced impairments.

II. IS for Emulators With Maxwellian Lengths

The growth of first-order PMD and SOPMD is described by the PMD concatenation equations [7], [15]. For linearly birefringent sections, and in the presence of polarization scramblers, it is possible to write these equations as $\boldsymbol{\tau}^{(n+1)} = \mathbf{R}_{n+1}(\boldsymbol{\tau}^{(n)} + \Delta\boldsymbol{\tau}^{(n+1)})$ and $\boldsymbol{\tau}_\omega^{(n+1)} = \mathbf{R}_{n+1}(\boldsymbol{\tau}_\omega^{(n+1)} + \Delta\boldsymbol{\tau}^{(n+1)} \times \boldsymbol{\tau}^{(n)})$, where $\boldsymbol{\tau}^{(n)}$ is the total PMD vector for the nth section, $\boldsymbol{\tau}_\omega^{(n)}$ is its frequency derivative (i.e., SOPMD), $\Delta\boldsymbol{\tau}^{(n+1)}$ is the frequency-independent individual contribution of the $(n+1)$th section, and $\mathbf{R}_{n+1}$ is the $(n+1)$th section's Müller rotation matrix. Note that if one is only interested in the statistics of these PMD vectors themselves, then $\mathbf{R}_{n+1}$ can be omitted, since the lengths of the vectors and their dot product are independent of it.

In emulators with fixed-length sections, the magnitude of $\Delta\boldsymbol{\tau}^{(n+1)}$ is a fixed predetermined value (which might be different for each section), and only its orientation varies. In the present method, however [12], each component of $\Delta\boldsymbol{\tau}^{(n)}$ is an independent Gaussian-distributed random variable (RV) with zero mean and variance σ_n^2, where $\langle \mathrm{DGD}^2 \rangle = 3\sum_{n=1}^N \sigma_n^2$ is the mean square DGD and N is the total number of sections. The joint probability density function (pdf) of the emulator's N sections is then given by $p(\Delta\boldsymbol{\tau}^{(1)}, \ldots, \Delta\boldsymbol{\tau}^{(N)}) = \exp[-\sum_{n=1}^N |\Delta\boldsymbol{\tau}^{(n)}|^2/2\sigma_n^2]/[(2\pi)^{N/2}\prod_{n=1}^N \sigma_n]^3$.

The first step when applying IS is to determine the most probable device configurations that produce a given value of

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first-order PMD and/or SOPMD. These configurations represent the best choice for biasing the MC simulations (the idea is to generate random samples around these most probable configurations). We first look at the simpler case of the DGD; then, we treat SOPMD and any linear combination of the two.

We express the contribution of each section with respect to a rotating reference frame [7] $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where $\mathbf{u}_1$ is aligned with $\boldsymbol{\tau}^{(n)}$, $\mathbf{u}_2$ with $\boldsymbol{\tau}_{\omega,\perp}^{(n)}$ (the depolarization [15]), and $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$. To maximize the probability of obtaining a given total $\mathrm{DGD} = |\sum_{n=1}^{N} \Delta\boldsymbol{\tau}^{(n)}|$, we need to minimize $\sum_{n=1}^{N} |\Delta\boldsymbol{\tau}^{(n)}|^2$ (the exponent of the joint pdf) with $\mathrm{DGD} = \mathrm{const}$. The solution to this problem is $\Delta\boldsymbol{\tau}^{(n+1)} = \mathbf{b}^{(n)}$, where $\mathbf{b}^{(n)} = \Delta b\, \mathbf{u}_1$ and $\Delta b = \mathrm{DGD}/N$. Thus, the configuration that maximizes the probability of obtaining a given DGD is that of equal-length sections, each aligned with the previous PMD vector. This is essentially the solution found in [6]; in this context, however, it means that: 1) the most probable configuration, and thus, the best biasing choice, is that of equal-length sections and 2) Δb is a parameter that is determined by the total DGD being sought.

The case of SOPMD is more complicated. The continuum limit of the first two concatenation equations is [7]

$$\frac{d\tau}{dz} = b_1, \quad \frac{d\tau_{\omega,\perp}}{dz} = b_3\tau - b_2\frac{\tau_{\omega,\|}}{\tau}, \quad \frac{d\tau_{\omega,\|}}{dz} = b_2\frac{\tau_{\omega,\perp}}{\tau} \tag{1}$$

where $\mathbf{b}(z) = \lim_{\Delta z \to 0} \Delta\boldsymbol{\tau}/\Delta z$ and (b_1,b_2,b_3) are its components with respect to U. These equations have the exact solution

$$\tau(L) = \int_0^L b_1(z)dz \tag{2a}$$

$$\tau_{\omega,\perp}(L) = \int_0^L b_3(z)\tau(z)\cos\xi(z,L)dz \tag{2b}$$

$$\tau_{\omega,\|}(L) = \int_0^L b_3(z)\tau(z)\sin\xi(z,L)dz \tag{2c}$$

where $\xi(z,L) = \int_z^L b_2(z')/\tau(z')dz'$. As in [7], the most likely way to realize a given SOPMD is to consider the largest of the two components, i.e., the depolarization $\tau_{\omega,\perp}$. Most probable configurations are now obtained by minimizing $\int_0^L b^2(z)dz$ subject to $\tau_{\omega,\perp}(L) = \mathrm{const}$, where $b^2(z) = |\mathbf{b}(z)|^2 = b_1^2(z) + b_2^2(z) + b_3^2(z)$, which is the continuum version of $\sum_{n=1}^{N} |\Delta\boldsymbol{\tau}^{(n)}|^2$. Using calculus of variations, the problem can be converted to a system of differential equations

$$\frac{db_1}{dz} = \frac{b_2^2 - b_3^2}{\tau}, \quad \frac{db_2}{dz} = -\frac{b_1 b_2}{\tau} - \frac{b_3}{\lambda}\sin\xi \tag{3}$$

with $b_3(z) = (\tau(z)/\lambda)\cos\xi(z,L)$, and where λ is a Lagrange multiplier. Equation (3) has one integral of the motion $b^2(z) = \mathrm{const}$. Solving (3) yields $\lambda = 2L/\pi$ and

$$\mathbf{b}(z) = b\left(\cos\left(\frac{\beta z}{L}\right)\mathbf{u}_1 + \sin\left(\frac{\beta z}{L}\right)\mathbf{u}_3\right) \tag{4}$$

with $\beta = \pi/2$. This is essentially the solution found in [7], except that: 1) the present calculation was not restricted in advance to in-plane contributions ($b_2 = 0$); 2) we have shown equal-length sections to also be the best biasing choice for SOPMD; and 3) b is also a parameter, determined by the total SOPMD being sought: $\mathrm{SOPMD} = N^2\Delta b^2/\pi$, where $\Delta b = b\Delta z$ is the DGD per section.

By choosing increasing values of Δb, larger values of DGD and/or SOPMD can be targeted. In addition, a more general calculus of variations analysis shows that by continuously varying β from 0 to π, any linear combination of first-order PMD and SOPMD can be obtained. By selectively choosing values for both Δb and β, it is, therefore, possible to obtain a full coverage of the DGD-SOPMD plane. This kind of coverage is often necessary when calculating outage probabilities [10]. Note that, unlike the case of fixed-length sections, any value of DGD and SOPMD is now allowed, and arbitrarily large values can be easily targeted. In addition, a simple relation exists between the biasing parameters Δb and β and the target values of first-order PMD and SOPMD. In fact, using (4) in (2) one finds

$$\mathrm{DGD} = N\Delta b \,\mathrm{sinc}\,\beta, \quad \mathrm{SOPMD} = (N\Delta b)^2\frac{1 - \mathrm{sinc}2\beta}{2\beta} \tag{5}$$

where $\mathrm{sinc}\,x = \sin x/x$.

Once the most probable configurations $\mathbf{b}^{(n)} = \mathbf{b}(n\Delta z)$ have been found, the last task is to use this information to bias MC PMD simulations. The goal is to choose biasing distributions that, for each section, randomly select values of $\Delta\boldsymbol{\tau}^{(n+1)}$ near $\mathbf{b}^{(n)}$. This is done by taking the components of $\Delta\boldsymbol{\tau}^{(n+1)}$ to be independent identically distributed Gaussian RVs with variance σ_n^2 and means equal to the components of $\mathbf{b}^{(n)}$. (Whenever the coordinate frame U is completely or partially undetermined (e.g., as when $n = 0, 1$), any indeterminate directions are taken to be randomly oriented.) The IS likelihood ratio [6], [7], which corrects computed probabilities for the biasing, is then $p(\Delta\boldsymbol{\tau}^{(1)}, \ldots, \Delta\boldsymbol{\tau}^{(N)})/p^*(\Delta\boldsymbol{\tau}^{(1)}, \ldots, \Delta\boldsymbol{\tau}^{(N)}) = \exp[-\sum_{n=1}^{N}(|\Delta\boldsymbol{\tau}^{(n)}|^2 - |\Delta\boldsymbol{\tau}^{(n)} - \mathbf{b}^{(n-1)}|^2/2\sigma_n^2)]$.

Fig. 1 illustrates the method by showing a few targets and how the MC samples cluster around these targets. For some of the regions, we also show the contour levels of the numerically reconstructed joint pdf of DGD and SOPMD, demonstrating how the method can be used to obtain statistical results in a limited region of the (DGD, SOPMD) plane.

III. Emulator Comparisons and Discussion

Using the IS techniques presented both here and in [6] and [7], we compared emulators with Maxwellian-length sections to fixed-length emulators, either with polarization controllers (hereafter, the "scramblers" model) or without (hereafter, the "waveplates" model). For each model, we numerically reconstructed the probability distributions of the DGD and SOPMD and compared them to their known counterparts for real fiber [16]. Fig. 2 shows, for the SOPMD, the value of the real fiber's pdf when the discrepancy between it and the pdf of each type of emulator first reaches 3 dB, as a function of the number of sections. The advantage of the emulator with Maxwellian length sections is evident.

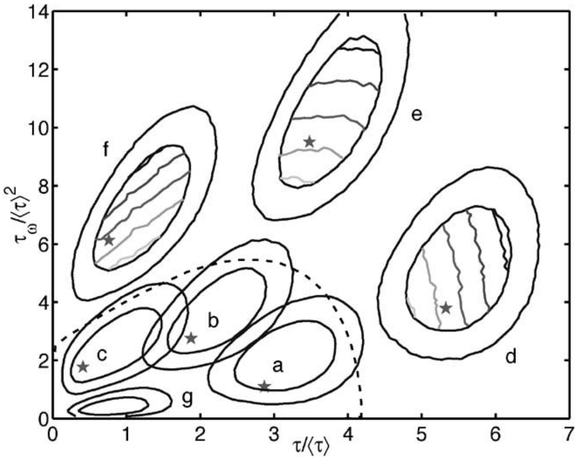

Fig. 1. Target spots (denoted by stars) for an emulator with 15 equal-mean Maxwellian-length sections, corresponding to $\beta = \pi/8$ (a, b), $\beta = \pi/2$ (c, d), or $\beta = 7\pi/8$ (e, f), and to $\Delta b/\sigma = 0.7$ (a, c, e) or $\Delta b/\sigma = 1.3$ (b, d, f). For each target, the contour lines indicate regions within which the density of MC samples is greater than 50% and 20% of the maximum. Region (g) shows the same curves for unbiased simulations, and the dotted line shows the accessible region of an emulator with 15 fixed-length birefringent sections and the same rms DGD. For targets (b), (d), and (f), the computed joint probability contours of DGD and SOPMD are also shown. Contours are at 10^{-n} with $n = 12, 14, 16, 18$ and 20, and 2×10^6 MC samples were used.

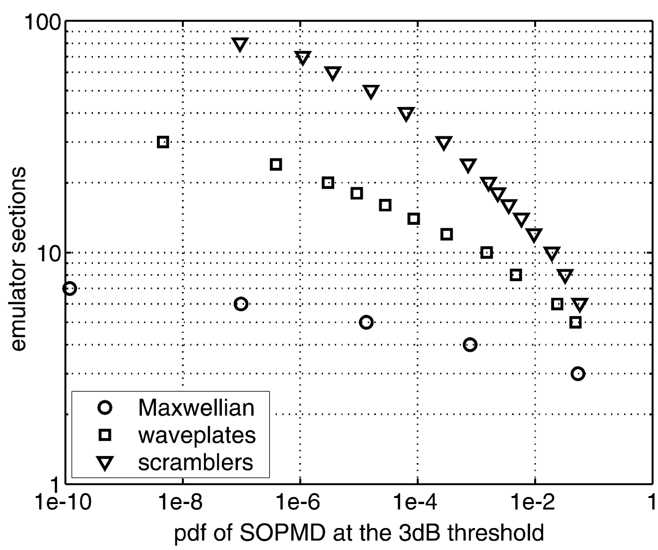

Fig. 2. Value of the real fiber SOPMD distribution at the point where the discrepancy with the emulator distribution first reaches 3 dB, as a function of the number of sections. The mean DGD is kept fixed at 1 ps, and equal sections are used. For all emulators, the data is obtained from importance-sampled simulations. A similar comparison holds for the DGD: To have a discrepancy of less than 3 dB at 10^{-6}, for example, more than 10 sections are needed with waveplates, and more than 80 with scramblers, while Maxwellian-length emulators yield the exact pdf with any number of sections.

Another advantage is seen by considering emulators with the same root-mean square DGD, i.e., $[\sum_{n=1}^{N} |\Delta\boldsymbol{\tau}^{(n)}|^2]^{1/2}$. For all such emulators, the central part of the DGD's pdf will be close to the same Maxwellian. The maximum total DGD $|\sum_{n=1}^{N} \Delta\boldsymbol{\tau}^{(n)}|$ will differ, however, and emulators with the largest maximum DGD will agree best with the Maxwellian in the *tails* of the pdf. As explained in Section II, the largest DGD is found for the case of equal lengths. Emulators with fixed equal-length sections are to be avoided, however, because of the artificial periodicities that they produce in the PMD autocorrelation function (ACF) [1], [17]. For fixed-length emulators, a tradeoff, therefore, exists between the need to suppress the ACF periodicities (which requires using nonequal lengths) and the need to properly reproduce the tails of the pdfs of DGD and SOPMD (which are best with equal-length sections). With Maxwellian-distributed lengths, however, the actual section lengths are random, and thus, the periodicities of the ACF are avoided even if the mean lengths are the same. Thus, with Maxwellian-distributed lengths, it is possible to avoid periodicities in the ACF without sacrificing accuracy in the tails of the pdfs of DGD and SOPMD.

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References

[1] R. Khosravani *et al.*, "Time and frequency domain characteristics of polarization-mode dispersion emulators," *IEEE Photon Technol. Lett.*, vol. 13, pp. 127–129, Feb. 2001.

[2] J. N. Damask *et al.*, "Method to measure and estimate total outage probability for PMD-impaired systems," *IEEE Photon. Technol. Lett.*, vol. 15, pp. 48–50, Jan. 2003.

[3] R. Chipman and R. Kinera, "High-order polarization mode dispersion emulator," *Opt. Eng.*, vol. 41, pp. 932–937, 2002.

[4] J. N. Damask *et al.*, "Demonstration of a programmable PMD source," *IEEE Photon. Technol. Lett.*, vol. 15, pp. 296–298, Feb. 2003.

[5] I. T. Lima *et al.*, "Comparison of polarization mode dispersion emulators," *J. Lightwave Technol.*, vol. 19, pp. 1872–1881, Dec. 2001.

[6] G. Biondini, W. L. Kath, and C. R. Menyuk, "Importance sampling for polarization-mode dispersion," *IEEE Photon. Technol. Lett.*, vol. 14, pp. 310–312, Mar. 2002.

[7] S. L. Fogal, G. Biondini, and W. L. Kath, "Multiple importance sampling for first- and second-order polarization-mode dispersion," *IEEE Photon. Technol. Lett.*, vol. 14, pp. 1273–1275, Sept. 2002.

[8] D. Yevick, "Multicanonical communication system modeling-application to PMD statistics," *IEEE Photon. Technol. Lett.*, vol. 14, pp. 1512–1514, 2002.

[9] I. T. Lima *et al.*, "Analysis of PMD compensators with fixed DGD using importance sampling," *IEEE Photon. Technol. Lett.*, vol. 14, pp. 627–629, May 2002.

[10] A. O. Lima *et al.*, "Statistical analysis of the performance of PMD compensators with multiple importance sampling," *IEEE Photon. Technol. Lett.*, vol. 15, pp. 1716–1718, Dec. 2003.

[11] M. Karlsson, "Probability density functions of the differential group delay in optical fiber communication systems," *J. Lightwave Technol.*, vol. 19, pp. 324–331, Mar. 2001.

[12] J. H. Lee, M. S. Kim, and Y. C. Chung, "Statistical PMD emulator using variable DGD elements," *IEEE Photon. Technol. Lett.*, vol. 15, pp. 54–56, Jan. 2003.

[13] L. Yan *et al.*, "Measurement of Q degradation due to polarization mode dispersion using multiple importance sampling," *Electron. Lett.*, vol. 39, pp. 974–975, 2003.

[14] M. Shtaif, "The Brownian-bridge method for simulating polarization mode dispersion in optical communications systems," *IEEE Photon. Technol. Lett.*, vol. 15, pp. 51–53, Jan. 2003.

[15] J. P. Gordon and H. Kogelnik, "PMD fundamentals: polarization-mode dispersion in optical fibers," *Proc. Nat. Acad. Sci.*, vol. 97, pp. 4541–4550, 2000.

[16] G. J. Foschini and C. D. Poole, "Statistical theory of polarization dispersion in single mode fibers," *J. Lightwave Technol.*, vol. 9, pp. 1439–1456, Nov. 1991.

[17] B. S. Marks, I. T. Lima, and C. R. Menyuk, "Autocorrelation function for polarization-mode dispersion emulators with rotators," *Opt. Lett.*, vol. 27, pp. 1150–1152, 2002.