Supplementary information to the article:

Experimental Observation and Theoretical Description of Multisoliton Fission in Shallow Water

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ON THE CHOICE OF THE REFERENCE LEVEL

Here we show through a numerical simulation of the KdV equation, that the choice of the reference level discussed in the paper, not only allows to calculate the soliton amplitudes, but also accounts for the right velocities of the emerging solitons. Specifically, we numerically solve the initial value problem for the T-KdV (Eqs. (2) of the paper), which we repeat here for convenience

$$u_\zeta - 6uu_\tau - \varepsilon^2 u_{\tau\tau\tau} = 0; \qquad u_0(\tau) = \cos(\tau)\,. \tag{1}$$

We recall that such equation describes in dimensionless units the dynamics in the tank, with τ being a retarded time introduced to remove the net drift due to the phase velocity c_0 (i.e., equivalent to shift from the lab frame to the frame of reference travelling at the natural velocity c_0 of the linear waves). We focus on the case of run B in table 1 of the paper, characterized by $\varepsilon^2 \simeq 0.029$, integrating Eqs. (1) up to $\zeta = 0.55$, which corresponds, in normalized units, to the last gauge in the tank (75 m). The result, reported in Fig. S1, shows the color level plot of the evolution (see Fig. S1(a)), and the output temporal profile (compared with the sinusoidal input, see Fig. S1(b)), respectively. On the other hand, the outcome of the integration of the direct periodic scattering problem (Eq. (3) in the paper) for the same problem is reported in Fig. 3(a) of the paper. This allows us to report in Fig. S1(b), the reference level $u_{ref} = -\lambda_{ref}$, where in the present case $\lambda_{ref} \equiv \lambda_{17} \simeq 0.73$ corresponds to the first band edge (for increasing

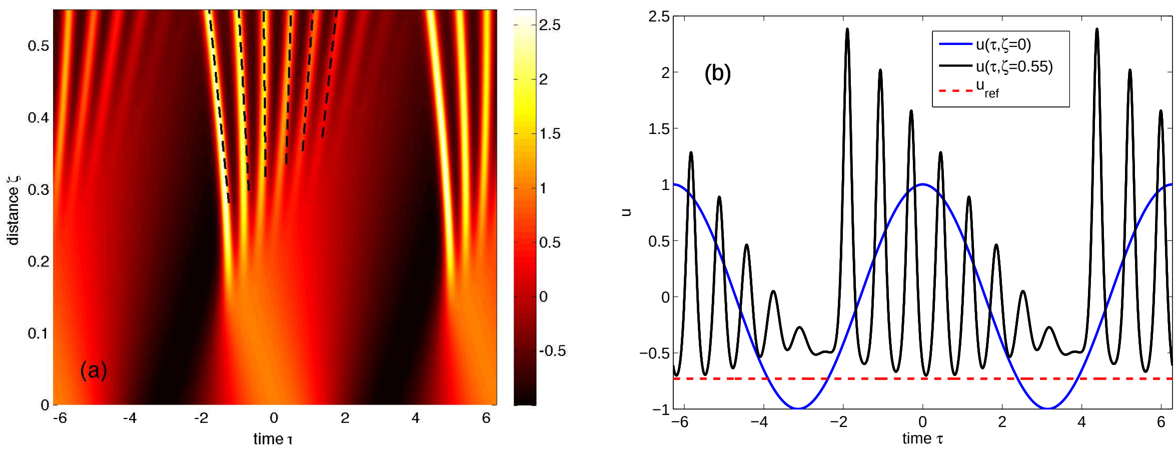

FIG. S1. Simulation based on the KdV equation (1) with $\varepsilon^2 = 0.029$, corresponding to the regime of run B in table 1 of the paper: (a) level colormap of the evolution; the superimposed dashed black lines stand for the velocities of the first six solitons calculated from the soliton amplitudes obtained via the scattering problem for the input $u_0(\tau) = \cos(\tau)$ reported in Fig. 3(b) of the paper (see also text below); (b) snapshot at $\zeta = 0.55$ ($z = 75$ m in real-world units); the dashed red line stands for the reference level $u_{ref} = -\lambda_{17} \simeq -0.73$, arising from the scattering data.

values of the spectral parameter λ), which does not fulfil the soliton constraint $W_n < \kappa = 0.01$. Such reference level (see dashed red line in Fig. S1(b)) accurately describes the background level over which the solitons progressively emerge, as one can clearly see in Fig. S1(b). The soliton peak corresponding to the n-th band is then calculated, as explained in the paper and illustrated in Fig. 3(b), according to the formula $A_n = 2(\lambda_{17} - \lambda_{2n})$, originally introduced by Osborne and Bergamasco [1]. This allows us to link with the expected velocities of the solitons. Indeed, it is well known that a soliton of the KdV has a definite velocity which depends on its amplitude as well as its background. In other words a finite non-vanishing background can change the soliton velocity, possibly even reversing its sign.

In the periodic case, the soliton-like excitations do not travel at strictly constant velocity, nor have they a strictly fixed amplitude. Nevertheless, as clearly visible in Fig. S1(a), once they separate, they travel with reasonably constant parameters (amplitude and velocity). Importantly, the solitons possess velocities ranging from negative to positive values (in the retarded frame τ), depending on their amplitudes. By knowing the amplitudes of the emerging solitons A_n, $n = 1, \ldots, 8$, we can calculate their expected velocities $v_n = d\tau/d\zeta$ according to the well known formula $v_n = -2A_n - 6u_{ref}$, which links the velocity of a fundamental soliton solution of Eq. (1) to its amplitude A_n and its background u_{ref}. The velocities calculated according to this formula for the first six solitons correspond to the dashed black lines with different slopes superimposed in Fig. S1(a). As shown, they describe with good accuracy the net velocity of the emerging solitons, thus further justifying the choice of the reference level discussed in the paper.

ADDITIONAL EXPERIMENTAL DATA

For the sake of completeness, we report in Fig. S2 below the traces relative to run A in Table 1 of the main paper, recorded for a still water depth $h = 40$ cm. Traces are displayed at $z = 5, 35, 55, 75$ m. At variance with Figs. 1-2 in the paper, here we report the traces with the same horizontal temporal scale in order to make visible the effect of the rigid translation of the whole wavetrain caused by the phase velocity c_0. In the regime illustrated in Fig. S2, the profile shown in Fig. S2(b) at 35 m is close to the breaking distance where the gradient catastrophe (in the dispersionless limit) would occur, which is estimated, in this case, to be $L_b = L_{nl} \simeq 38$m. The number of emerging soliton-like nonlinear waves remains quite limited even at 75 m, due to the relatively high value of $\varepsilon^2 \simeq 0.07$. This regime, characterized by few solitons and the virtual absence of soliton interactions between adjacent periods over the available propagation distance is essentially comparable to the regime of the experiment made by Zabusky and Galvin in the 70s, where they counted at most three main well developed solitons and a fourth less pronounced bump [2]. We point out that the comparison is made here only in terms of visible solitons since the experiments differ in many respect (Zabusky and Galvin used a ~ 30 m tank, measuring only at two gauges spaced few meters apart, and in the most nonlinear case they employed a water depth ~ 7 cm, consistently allowing for very shallow waves).

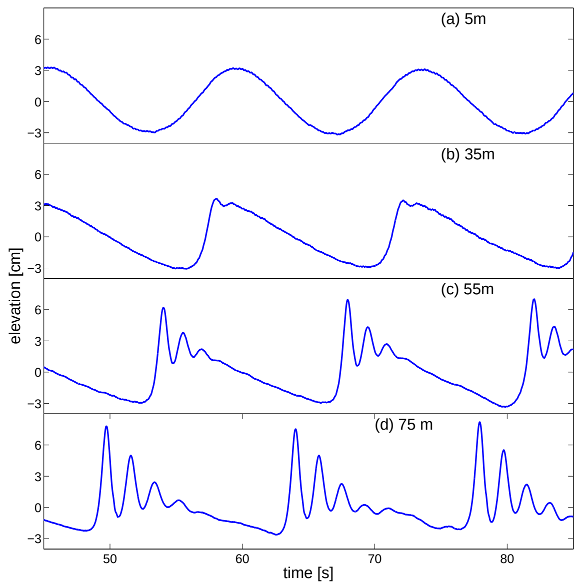

FIG. S2. Breaking of a harmonic wave in $h = 40$ cm depth (run A in table 1).

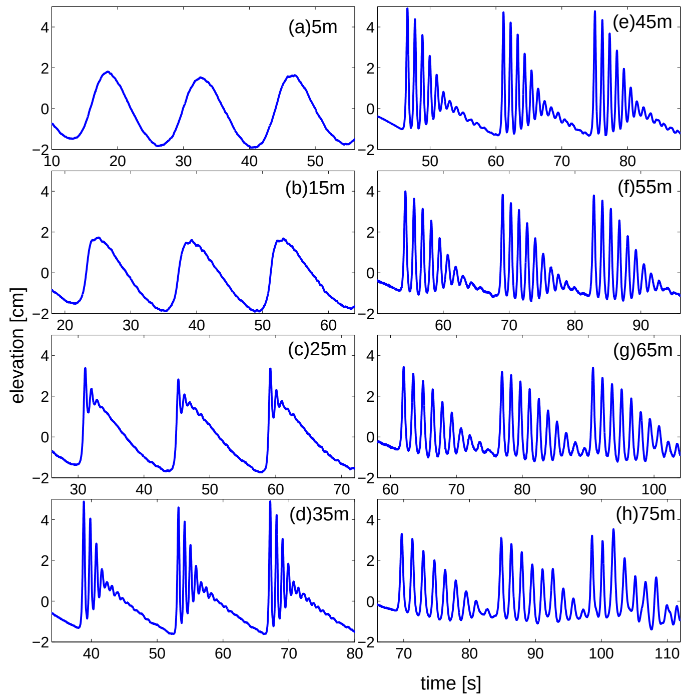

FIG. S3. Dispersive breaking over three periods of a harmonic wave in $h = 15$ cm depth (run C in table 1). Different temporal intervals are reported in order to follow the same periods, as in Figs. 1-2 of the paper.

For better clarity, we also report in Fig. S3, the full recorded data of a highly nonlinear run (run C in table 1), for which the number of fissioning solitons arising from the p-IST analysis of the experimental data fully agrees with the WKB prediction (see arrow labeled run C in Fig. 4 of the paper). The figure displays a temporal slot with the first three (out of total eight) periods. As shown measurements at the last two gauges start to become affected by the reflection of the forefront periods, more significantly at $z = 75$ m. It is worth emphasising that a maximum number of ten visible soliton-like bumps is reached at $z = 65$, while at $z = 75$ they are reduced to nine due to the interaction with the adjacent periods. Indeed tall solitons can hide more shallow ones even though the fission proceeds, especially when the evolution is probed at fixed distances as done in the experiment. This calls for a different approach, where solitons are counted via the p-IST analysis of the data, as explained in the paper. We find, in this particular case, eleven bands satisfying the soliton criterion $W_n \leq \kappa = 0.01$.

THE INVERSE SCATTERING THEORY AS A NONLINEAR FOURIER TRANSFORM

As we state in the introduction the periodic IST (p-IST) is fully equivalent to a nonlinear Fourier Transform. We do not discuss this point further, referring the reader to the numerous textbooks on IST. What is relevant in our analysis is the direct problem (equivalent to the direct Fourier Transform) that allows to associate the dominant nonlinear modes ("solitons") with the sufficiently narrow bands in the scattering problem [Eq. (3)], as illustrated in our Fig. 3a. We apply the numerical p-IST to our comprehensive set of experimental data in order to find such bands for different experimental conditions. Numerical p-IST was pioneered by Osborne and used by other authors [1, 3, 4]. It is worth pointing out few differences that the reader might find comparing with the work by other authors. The first difference concerns the relative band width W_n, which we impose to be such $W_n \leq \kappa = 0.01$ for the relative band to be classified as a soliton. An alternative formulation refers to the relative gap width $G_n = 1 - W_n$ (also known as soliton index [4]), in terms of which solitons are characterized as $G_n \geq (1 - \kappa) = 0.99$. For a cn-oidal wave this condition is equivalent to have a modulus of the Jacobian function $m_n \geq 0.99$ (a strict soliton would have $m_n = 1$). The second difference concerns the spectral parameter λ and the values $\lambda = \lambda_n$ which correspond to the band edges. Some authors assign to the spectral parameter the role of frequency to make more explicit the parallel with a linear Fourier analysis [3]. In our analysis, starting from the dimensionless spectral parameter λ, this would correspond to introduce a dimensional spectral parameter $\Omega^2 = \frac{\alpha\eta_0}{6\beta}\lambda$ (α, β being the coefficients of KdV, and η_0 the initial wave amplitude), which has the dimension of a square frequency. In this way our Fig. 3a could be redrawn against the frequency Ω, and soliton amplitudes (related to band locations) as well as the relative gap widths G_n could be reported against such renormalized spectral parameter as e.g. in [3]. This could be useful to graphically judge the threshold below which the bands can be classified as solitons. In our analysis, however, such condition is given by an analytical formula (reported in the text), whose outcome is shown, in the example of Fig. 3a, as a dashed vertical line labeled λ_s.

Finally we point out that the analysis in terms of p-IST or nonlinear Fourier Transform obviously implies to deal with the integrable conservative KdV, and hence to ignore the losses. These, however, can be non-negligible especially for the lower water levels [5], and the problem of determining their impact over the multi-soliton fission (along with other higher order effects such as deviation from a perfectly flat bottom or higher order dispersion) remains an open issue that needs further extended experimental and theoretical work.

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