Universal Nature of the Nonlinear Stage of Modulational Instability

Gino Biondini[1,2,*] and Dionyssios Mantzavinos[2]
[1]*Department of Physics, State University of New York at Buffalo, Buffalo, New York 14260, USA*
[2]*Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14260, USA*
(Received 9 September 2015; published 27 January 2016)

We characterize the nonlinear stage of modulational instability (MI) by studying the longtime asymptotics of the focusing nonlinear Schrödinger (NLS) equation on the infinite line with initial conditions tending to constant values at infinity. Asymptotically in time, the spatial domain divides into three regions: a far left and a far right field, in which the solution is approximately equal to its initial value, and a central region in which the solution has oscillatory behavior described by slow modulations of the periodic traveling wave solutions of the focusing NLS equation. These results demonstrate that the asymptotic stage of MI is universal since the behavior of a large class of perturbations characterized by a continuous spectrum is described by the same asymptotic state.

DOI: 10.1103/PhysRevLett.116.043902

Introduction.—Modulational instability (MI)—i.e., the instability of a constant background to long wavelength perturbations—is one of the most ubiquitous phenomena in nonlinear science (e.g., see Ref. [1] and the references therein). The effect, which is known as Benjamin-Feir instability in the context of deep water waves [2], has been known since the 1960s, but it has received renewed attention in recent years and was also linked to the formation of rogue waves in optical media [3,4] and in the open sea [5].

The dynamics of many systems affected by MI is governed by the one-dimensional focusing nonlinear Schrödinger (NLS) equation, which models the evolution of weakly nonlinear dispersive wave packets in such diverse fields as water waves, plasmas, optics, and Bose-Einstein condensates. One can therefore study the initial (i.e., linear) stage of MI by linearizing the NLS equation around the constant background. One easily sees that all Fourier modes below a certain threshold are unstable, and the corresponding perturbations grow exponentially. However, the linearization ceases to be valid as soon as perturbations become comparable with the background. A natural question, then, is what happens at this point, which is referred to as the *nonlinear stage of MI*. Surprisingly, a precise characterization of the nonlinear stage of MI for generic, finite-energy perturbations has remained, by and large, an open problem for the last 50 years.

The NLS equation is a completely integrable system [6], and it admits an infinite number of conservation laws and exact N-soliton solutions for arbitrary N's, describing the elastic interaction of solitons [6]. By analogy with the case of localized initial conditions, a natural conjecture was that MI is therefore mediated by solitons [7,8]. The initial value problem (IVP) for the NLS equation can be solved via the inverse scattering transform (IST). In particular, the IST for the focusing NLS equation with zero boundary conditions (ZBC) at infinity (i.e., localized disturbances) was done in Ref. [6], and the IST for the defocusing NLS equation with nonzero boundary conditions (NZBC, i.e., solutions that tend to finite nonzero values at infinity) was done in Ref. [9]. However, only partial results [10–12] were available for the focusing NLS equation with NZBC until recently when, in Ref. [13], we developed a complete IST for this case. (Recall that the IST for systems with NZBC is notoriously more challenging, and the IVP for the vector NLS with NZBC was also only solved recently [14,15]). In Ref. [16] we then used the IST to study MI by computing the spectrum of the scattering problem for simple classes of perturbations of a constant background. In particular, we showed that there are classes of perturbations for which no solitons are present. Thus, since all generic perturbations of the constant background are linearly unstable, solitons cannot be the mechanism that mediates the MI, contradicting a recent conjecture [7]. Instead, in Ref. [16] we identified the instability mechanism within the context of the IST by showing that the instability comes from the continuous spectrum of the scattering problem associated with the NLS equation (see below for further details).

In this Letter we use the framework developed in Ref. [13] to characterize the nonlinear stage of MI. We do so by studying the longtime asymptotic behavior of localized perturbations of the constant background. We show that, generically, the longtime asymptotics of modulationally unstable fields on the whole line displays universal behavior and decomposes the xt plane into two plane wave regions—in which the solution is approximately equal to the background up to a phase—separated by a central region in which the leading-order behavior is described by a slowly modulated traveling wave.

The NLS equation and MI.—We write the focusing NLS equation as

$$iq_t + q_{xx} + 2(|q|^2 - q_o^2)q = 0, \qquad (1)$$

where $q(x,t)$ represents the complex envelope of a quasimonochromatic, weakly nonlinear dispersive wave packet, and the physical meaning of the variables x and t depends on the physical context. (For example, in optics, t represents propagation distance, while x is a retarded time.) Here, $q_o = |q_\pm| > 0$ is the background amplitude, and the NZBC satisfied by the field are

$$q_\pm = \lim_{x\to\pm\infty} q(x,t). \tag{2}$$

The term $-2q_o^2 q$ has been added to Eq. (1) so that $q_\pm$ are independent of time, and they can be removed by a trivial gauge transformation.

The constant background solution is simply $q_s(x,t) = q_o$. Linearizing Eq. (1) around this solution, one finds that all Fourier modes with $|\zeta| < 2q_o$ (where ζ is the Fourier variable) are unstable, and that the growth rate is $\gamma(\zeta) = |\zeta|\sqrt{4q_o^2 - \zeta^2}$. Below, we will use the IST for Eq. (1) with the NZBC (2), which was developed in Ref. [13], slightly reformulated in a way that is more convenient for the present purposes.

Recall that the NLS equation (1) is the zero-curvature condition $X_t - T_x + [X,T] = 0$ of the matrix Lax pair $\phi_x = X\phi$ and $\phi_t = T\phi$, with $X = ik\sigma_3 + Q$ and $T = -i(2k^2 + q_o^2 - |q|^2 - Q_x)\sigma_3 - 2kQ$, where $\sigma_3 = \mathrm{diag}(1,-1)$ is the third Pauli matrix, and

$$Q(x,t) = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}. \tag{3}$$

As usual, the first half of the Lax pair is referred to as the scattering problem and $q(x,t)$ as the potential, and the direct problem in the IST consists of determining the scattering data (i.e., the reflection coefficient, discrete eigenvalues, and norming constants) from the initial condition. This is done through the Jost eigenfunctions $\phi_\pm(x,t,k)$, which are the simultaneous matrix solutions of both parts of the Lax pair which reduce to plane waves, namely, $\phi_\pm(x,t,k) = E_\pm(k)e^{i\theta(x,t,k)\sigma_3} + o(1)$ as $x \to \pm\infty$, where $\pm i\lambda$ and $E_\pm(k) = I + i/(k+\lambda)\sigma_3 Q_\pm$ are, respectively, the eigenvalues and corresponding eigenvector matrices of $X_\pm = \lim_{x\to\pm\infty} X$, with $\lambda(k) = (k^2 + q_o)^{1/2}$ and $\theta(x,t,k) = \lambda x - \omega t$, with $\omega(k) = 2k\lambda$. These Jost eigenfunctions, which are the nonlinearization of the Fourier modes, are defined for all values of $k \in \mathbb{C}$ such that $\lambda(k) \in \mathbb{R}$, which comprise the continuous spectrum $\Sigma = \mathbb{R} \cup i[-q_o, q_o]$; see Fig. 1 (left panel). The scattering relation $\phi_-(x,t,k) = \phi_+(x,t,k)A(k)$ defines the scattering matrix $A(k)$ for $k \in \Sigma$, and the corresponding reflection coefficient is $r(k) = -a_{21}/a_{22}$. The zeros of $a_{11}(k)$ and $a_{22}(k)$ define the discrete spectrum of the problem, which leads to solitons. As usual, time evolution within IST is trivial. In particular, with the above normalization of the Jost eigenfunctions, all the scattering data are independent of time.

The focusing NLS equation (1) with the NZBC (2) possesses a rich family of soliton solutions [10,17–19], classified according to the possible placements of the discrete eigenvalue [13]. In particular, the so-called Akhmediev breathers provide a good representation for the growth of seeded perturbations [20,21]. Importantly, however, Akhmediev breathers are periodic in space, and they therefore possess infinite energy. Hence, they cannot describe the asymptotic state of localized (i.e., finite-energy) perturbations of the constant background. Moreover, as mentioned earlier, there exist generic perturbations of the constant background for which no discrete spectrum (and thus no solitons) is present. Instead, the key to describing the asymptotic stage of MI lies in the continuous spectrum. Indeed, as we showed in Ref. [16], $\omega(k)$ is purely imaginary for $k \in i[-q_o, q_o]$, and the Jost solutions for $k \in i[-q_o, q_o]$ are precisely the nonlinearization of the unstable Fourier modes. In fact, even their growth rate is the same, modulo the usual rescaling.

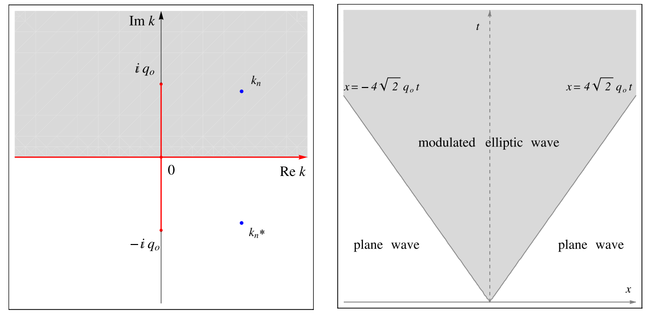

FIG. 1. (Left panel) The spectral k plane, showing the continuous spectrum Σ (the red lines), the regions where $\mathrm{Im}\lambda > 0$ (gray) and $\mathrm{Im}\lambda < 0$ (white), and a discrete eigenvalue k_n together with its symmetric counterpart. (Right panel) The asymptotic regime for the xt plane, showing the decomposition into two plane wave regions (white) and the modulated elliptic wave region (gray).

The inverse problem in the IST consists of reconstructing the solution $q(x,t)$ of the NLS equation from the scattering data and is formulated in terms of a Riemann-Hilbert problem, namely, the problem of reconstructing the meromorphic matrix $M(x,t,k)$, defined as $M(x,t,k) = (\phi_{+,1}/a_{22}, \phi_{-,2})e^{-i\theta\sigma_3}$ for $k \in \mathbb{C}^+ \backslash i[0, q_o]$ and $M(x,t,k) = (\phi_{-,1}, \phi_{+,2}/a_{11})e^{-i\theta\sigma_3}$ for $k \in \mathbb{C}^- \backslash i[-q_o, 0]$, where $\mathbb{C}^\pm = \{k \in \mathbb{C}\colon \mathrm{Im}k \gtrless 0\}$ and $\phi_{\pm,j}$ for $j = 1,2$ denote the columns of $\phi_\pm$. This is done by using the scattering relation and symmetries to obtain a jump condition $M^+(x,t,k) = M^-(x,t,k)V(x,t,k)$ for $k \in \Sigma$, where superscripts $\pm$ denote projection from the left or right of the contour Σ (oriented rightward along the real k axis and upward along the segment $i[-q_o, q_o]$). Explicitly,

$$V(x,t,k) = \begin{cases} \begin{pmatrix} 1+|r|^2 & r^* e^{2i\theta} \\ r e^{-2i\theta} & 1 \end{pmatrix}, & k \in \mathbb{R}, \\ \frac{iq_o}{k-\lambda}\begin{pmatrix} -r^* e^{2i\theta} & 1 \\ 1+|r|^2 & -r e^{-2i\theta} \end{pmatrix}, & k \in i[0, q_o], \end{cases}$$

plus a symmetric expression for $k \in i[-q_o, 0]$. Note that $\det M(x,t,k) = 1$ for $k \in \mathbb{C} \backslash \Sigma$ and $M(x,t,k) \to I$ as $k \to \infty$. The solution of the NLS equation is recovered via the usual reconstruction formula $q(x,t) = -2i \lim_{k\to\infty} k M_{12}$. The signature of MI in the inverse problem is the exponentially growing entries of $V(x,t,k)$ for $k \in i[-q_o, q_o]$ through the time dependence of $\theta(x,t,k)$.

Longtime asymptotics of finite-energy perturbations.—We now study the asymptotic state of MI for generic, finite-energy perturbations of a constant background. As mentioned earlier, we do so by computing the longtime asymptotics of the solutions of the focusing NLS equation with NZBC. As a concrete example we consider boxlike perturbations with $q(x,0) = q_o$ for $|x| > L$ and $q(x,0) = be^{i\beta}$ for $|x| < L$, in which case $r(k) = e^{2i\lambda L}[(b\cos\beta - q_o)k - ib\lambda\sin\beta]/[\lambda\mu\cot(2L\mu) - i(k^2 + q_o b\cos\beta)]$, with $\mu = \sqrt{k^2 + b^2}$. We emphasize, however, that the results described below are not limited to this example, and they apply to all localized perturbations such that the corresponding reflection coefficient has a small region of analyticity around the continuous spectrum and such that no discrete spectrum is present.

Recall that for linear evolution equations one computes the asymptotics of the solution as $t \to \infty$ via stationary phase or steepest descent by looking along lines $x = \xi t$, with ξ fixed [22]. In this far-field approximation, the solution essentially becomes the Fourier transform of the initial condition, modulated by the similarity variable ξ and evaluated at the critical points of the problem [23]. In the nonlinear case, instead, one must use the IST. The longtime asymptotics of solutions of the NLS equation with ZBC was computed through various approaches in Refs. [24,25]. Those results, however, do not apply in our case. Here, we used the more general nonlinearization of the steepest descent method, namely, the Deift-Zhou method for oscillatory Riemann-Hilbert problems [26].

Asymptotic stage of MI.—Since the implementation of the Deift-Zhou method is complicated, the details are reported elsewhere. On the other hand, the main results are straightforward. The key piece of information is the sign structure of $\mathrm{Im}\theta = \mathrm{Im}[\lambda(\xi - 2k)]t$ as a function of k for ξ fixed. Let $\xi_* = 4\sqrt{2}q_o$. For $|x| > \xi_* t$, there are two real stationary points in the complex k plane. This situation corresponds to the first and fourth panels of Fig. 2. For $|x| < \xi_* t$, there are two complex conjugate stationary points in the complex k plane. This situation corresponds to the second and third panels of Fig. 2.

Each of the four cases in Fig. 2 requires a different deformation of the Riemann-Hilbert problem. Correspondingly, the xt plane divides into three regions, as illustrated in the bifurcation diagram in Fig. 1 (right panel) [27]. Specifically, we note the following. (i) The range $x < -\xi_* t < 0$ is the left far field, plane wave region. Here, $|q(x,t)| = q_o + O(1/t^{1/2})$ as $t \to \infty$. Apart from a nonlinear contribution to the phase, the behavior is similar to the linear case.

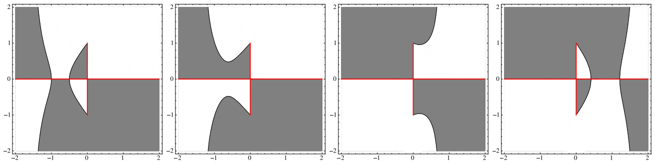

FIG. 2. The sign of $\mathrm{Im}\theta$ in the complex k plane for various values of the similarity variable $\xi = x/t$ and $q_o = 1$: (i) $\xi = -6$, corresponding to $x < -\xi_* t$; (ii) $\xi = -5.2$, corresponding to $-\xi_* t < x < 0$; (iii) $\xi = 3$, corresponding to $0 < x < \xi_* t$; and (iv) $\xi = 6.5$, corresponding to $x > \xi_* t$. Gray, $\mathrm{Im}\theta > 0$; white, $\mathrm{Im}\theta < 0$.

(ii) The range $-\xi_* t < x < \xi_* t$ is an oscillation region. Here, $q(x,t) = q_{\mathrm{asymp}}(x,t) + O(1/t^{1/2})$, with the asymptotic solution being a modulated traveling wave (elliptic) solution. This is the most interesting region, and it is described in some detail below. (iii) The range $x > \xi_* t > 0$ is a right far field, plane wave region. Here, $|q(x,t)| = q_o + O(1/t^{1/2})$ as $t \to \infty$, similar to region (i).

The kind of results described above are not unprecedented. Indeed, bifurcation diagrams dividing the longtime asymptotic behavior of solutions of the focusing NLS equation into regions of different genuses were obtained in different contexts in Refs. [28,29]. What is different here, however, is the physical setting, the specific results, and their physical interpretation.

The modulated traveling wave region.—We focus on the range $0 < x < \xi_* t$. (The solution in the range $-\xi_* t < x < 0$ is similar.) The leading-order solution in this region is expressed in terms of Jacobi elliptic functions and represents a slow modulation of the traveling wave (periodic) solutions of the focusing NLS equation [30]. In particular,

$$|q_{\mathrm{asymp}}(x,t)|^2 = (q_o + \alpha_{\mathrm{im}})^2 - 4q_o\alpha_{\mathrm{im}}\mathrm{sn}^2[C(x - 2\alpha_{\mathrm{re}}t - X); m], \qquad (4)$$

where $m = 4q_o\alpha_{\mathrm{im}}/C^2$ is the elliptic parameter, $C = \sqrt{\alpha_{\mathrm{re}}^2 + (q_o + \alpha_{\mathrm{im}})^2}$, and the slowly varying offset X is explicitly determined by the reflection coefficient. The four points $\pm iq_o$ and $\alpha_\pm = \alpha_{\mathrm{re}} \pm i\alpha_{\mathrm{im}}$ are the branch points associated with the elliptic solutions of the focusing NLS equation [31,32]; $\alpha_\pm$ are slowly varying functions of ξ, determined via a single, implicit equation that can be easily solved numerically. The slowly varying wave number, velocity, and period are, respectively, α_{re}, $2\alpha_{\mathrm{re}}$, and $2K(m)/C$ [33]. In particular, $\alpha \to 1/\sqrt{2}$ as $x \to \xi_* t$ and $\alpha \to iq_o$ as $x \to 0$. The first limit corresponds to the boundary between the genus-1 region and the plane wave region, in which case $m \to 0$ and the solution reduces to a constant. In the second limit, $m \to 1$, corresponding to the solitonic limit of the elliptic solution.

The universal profile of the solution amplitude in the oscillation region (neglecting for simplicity the ξ-dependent effect of the reflection coefficient) is shown in Fig. 3 at two different values of time. The envelope of $|q_{\mathrm{asymp}}|$ (the dashed lines), given by $q_o \pm \alpha_{\mathrm{im}}$, is time independent and depends only on ξ. Conversely, the oscillating structure is slowly varying in the xt frame. The boundary between the oscillation region and the plane wave regions can be understood within the context of Whitham modulation theory [31,32].

Discussion.—We have computed the longtime asymptotic behavior of a large class of perturbations of a constant background in a modulationally unstable medium for which no discrete spectrum is present. Recall that the linear stage of MI is characterized by exponential growth. As we showed [16], "linearizing" the IST (i.e., looking for solutions that are a small deviation from the constant background) yields exactly the same result as directly linearizing the NLS equation around the background. For longer times, however, the growth saturates and one obtains the asymptotic state described in our Letter. More precisely, in this Letter we showed that all such perturbations evolve towards an asymptotic state described by slow modulations of the traveling wave solutions of the focusing NLS equation. We emphasize the broad nature of our results. The initial conditions of the problem only determine a slowly varying offset for the elliptic solution via the reflection coefficient, whereas the structure of the solution as a modulated elliptic wave is independent of it. In this sense, the asymptotic stage of MI is universal.

Since the NLS equation has a wide range of applicability, from nonlinear optics to deep water waves, acoustics, plasmas, and Bose-Einstein condensates, we expect that the results of this Letter apply to all of the above physical contexts. In particular, our results provide explicit predictions about the behavior of laser pulses in optical fibers and gravity waves in one-dimensional deep water channels. The results also have potential connections to the phenomena of rogue waves [3,4] and integrable turbulence [34].

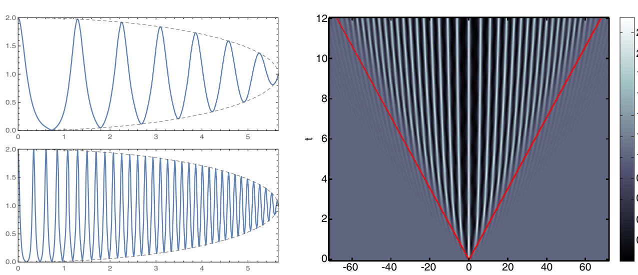

FIG. 3. (Left panels) The asymptotic solution $|q(x,t)|$ (the vertical axis) in the oscillation region as a function of $\xi = x/t$ (the horizontal axis) for $q_o = 1$. Top panel, $t = 4$; bottom panel, $t = 20$. Also shown (the dashed lines) is the time-independent envelope of the solution. (Right panel) Density plot from numerical simulations of Eq. (1) with a small Gaussian perturbation of the constant background. The red lines show the analytically predicted boundaries $x = \pm 4\sqrt{2} q_o t$.

MI is often studied in the framework of sideband perturbations of a constant background. The results of this Letter can therefore be compared to those in the case of periodic boundary conditions. There, the instability is ascribed to the presence of homoclinic solutions [35]. The initial stage of MI in that scenario was studied in Ref. [36] with a three-mode model. However, the IST machinery used to study the periodic case (namely, the theory of finite-genus solutions [37,38]) is very different from the one in the IVP with NZBC, used here [39]. Most importantly, the physics in the two cases is different. For example, (i) in the periodic case there is an amplitude threshold below which no instability occurs, whereas no such threshold exists on the infinite line, and (ii) in the periodic case, radiation cannot escape to infinity, and therefore it is doubtful that a longtime asymptotic state even exists. Also, sinusoidal excitations are a special case of perturbations with several Fourier components, each contributing with its own amplitude and phase. Such generic perturbations are characterized by their Fourier transform (or, equivalently, spectral data), and this is precisely the situation studied in this Letter.

The above results can also be compared to the semiclassical limit of the focusing NLS equation with ZBC [40]. The study of that scenario requires more sophisticated analysis, and the results are also more complicated. Moreover, numerical simulations of the semiclassical case become more and more sensitive to roundoff error as $\hbar \to 0$ [35]. In contrast, the present case does not appear to be as sensitive. The robustness of our analytical predictions is confirmed in Fig. 3, which shows a numerical simulation of Eq. (1) with a small Gaussian perturbation of the constant background. The numerical results show that there is an intermediate time range for which one sees the asymptotic behavior but no catastrophic roundoff. As a result, there appear to be no fundamental obstacles to the possibility of observing experimentally the behavior described in this Letter.

Semiclassical limits and longtime asymptotics problems are often studied using Whitham theory [23]. However, the Whitham equations for the focusing NLS equation are elliptic, and therefore the corresponding IVP is ill posed. This is well known in the case of ZBC (e.g., see Ref. [40]), and it remains true in the case of NZBC. While special solutions to the Whitham equations also exist in the focusing case [31,32], it should be clear that the IST-related methods used here are the only way to study the nonlinear stage of MI for generic perturbations of the constant background. Indeed, we see no obstacles to generalizing the present calculations to include the presence of discrete eigenvalues, which will allow for the first time a study of the interactions between solitons and radiation in modulationally unstable media.

This work was partially supported by the National Science Foundation under Grant No. DMS-1311847. D. M. also acknowledges an AMS-Simons Travel Grant.

*biondini@buffalo.edu

[1] V. E. Zakharov and L. A. Ostrovsky, Modulational instability: The beginning, Physica (Amsterdam) **238D**, 540 (2009).
[2] T. B. Benjamin and J. E. Feir, The disintegration of wave trains on deep water, J. Fluid Mech. **27**, 417 (1967).
[3] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev, and J. M. Dudley, The Peregrine soliton in nonlinear fibre optics, Nat. Phys. **6**, 790 (2010).
[4] D. R. Solli, C. Ropers, P. Koonath, and B. Jalali, Optical rogue waves, Nature (London) **450**, 1054 (2007).
[5] M. Onorato, A. R. Osborne, and M. Serio, Modulational Instability in Crossing Sea States: A Possible Mechanism for the Formation of Freak Waves, Phys. Rev. Lett. **96**, 014503 (2006).
[6] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP **34**, 62 (1972).
[7] V. E. Zakharov and A. A. Gelash, Nonlinear Stage of Modulation Instability, Phys. Rev. Lett. **111**, 054101 (2013).
[8] A. A. Gelash and V. E. Zakharov, Superregular solitonic solutions: A novel scenario for the nonlinear stage of modulation instability, Nonlinearity **27**, R1 (2014).
[9] V. E. Zakharov and A. B. Shabat, Interaction between solitons in a stable medium, Sov. Phys. JETP **37**, 823 (1973).
[10] E. A. Kuznetsov, Solitons in a parametrically unstable plasma, Sov. Phys. Dokl. **22**, 507 (1977).
[11] Y.-C. Ma, The perturbed plane-wave solutions of the cubic Schrödinger equation, Stud. Appl. Math. **60**, 43 (1979).
[12] J. Garnier and K. Kalimeris, Inverse scattering perturbation theory for the nonlinear Schrödinger equation with non-vanishing background, J. Phys. A **45**, 035202 (2012).
[13] G. Biondini and G. Kovačič, Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions, J. Math. Phys. (N.Y.) **55**, 031506 (2014).
[14] B. Prinari, M. J. Ablowitz, and G. Biondini, Inverse scattering transform for the vector nonlinear Schrödinger equation with non-vanishing boundary conditions, J. Math. Phys. (N.Y.) **47**, 063508 (2006).
[15] D. K. Kraus, G. Biondini, and G. Kovačič, The focusing Manakov system with nonzero boundary conditions, Nonlinearity **28**, 3101 (2015).
[16] G. Biondini and E. R. Fagerstrom, Integrable nature of modulational instability, SIAM J. Appl. Math. **75**, 136 (2015).
[17] D. H. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, J. Aust. Math. Soc. Series B, Appl. Math. **25**, 16 (1983).
[18] N. N. Akhmediev and V. I. Korneev, Modulational instability and periodic solutions of the nonlinear Schrödinger equation, Theor. Math. Phys. **69**, 1089 (1986).
[19] M. Tajiri and Y. Watanabe, Breather solutions to the focusing nonlinear Schrödinger equation, Phys. Rev. E **57**, 3510 (1998).
[20] M. J. Ablowitz and B. M. Herbst, On homoclinic structure and numerically induced chaos for the nonlinear Schrödinger equation, SIAM J. Appl. Math. **50**, 339 (1990).
[21] K. L. Henderson, D. H. Peregrine, and J. W. Dold, Unsteady water wave modulations: Fully nonlinear solutions and comparison with the nonlinear Schrödinger equation, Wave Motion **29**, 341 (1999).
[22] The point $x = 0$ appears to be special because, in the far-field approximation of the dynamics arising from localized perturbations, everything seems to arise from the origin, just like in the far-field asymptotics for linear problems [23].
[23] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[24] H. Segur and M. J. Ablowitz, Asymptotic solutions and conservation laws for the nonlinear Schrödinger equation, J. Math. Phys. (N.Y.) **17**, 710 (1976).
[25] V. E. Zakharov and S. V. Manakov, Asymptotic behavior of nonlinear waves systems integrated by the inverse scattering method, Sov. Phys. JETP **44**, 106 (1976).
[26] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the mKdV equation, Ann. Math. **137**, 295 (1993).
[27] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.116.043902 for a detailed expression of the solution in each region.
[28] R. Buckingham and S. Venakides, Long-time asymptotics of the nonlinear Schrödinger equation shock problem, Commun. Pure Appl. Math. **60**, 1349 (2007).
[29] A. Boutet de Monvel, V. P. Kotlyarov, and D. Shepelsky, Focusing NLS equation: Long-time dynamics of step-like initial data, Int. Math. Res. Not. **2011**, 1613 (2011).
[30] A special case of this solution was studied in Refs. [31,32] in the context of Whitham theory, but neither work studied the evolution of generic initial conditions.
[31] G. A. El', A. V. Gurevich, V. V. Khodorovskiĭ, and A. L. Krylov, Modulation instability and formation of a nonlinear oscillatory structure in a focusing medium, Phys. Lett. A **177**, 357 (1993).
[32] A. M. Kamchatnov, *Nonlinear Periodic Waves and Their Modulations* (World Scientific, Singapore, 2000).
[33] Here, $K(m)$ is the complete elliptic integral of the first kind. Since the wave is nonlinear, the wave number and the period are not related by a simple proportionality relation as for harmonic waves.
[34] V. E. Zakharov, Turbulence in integrable systems, Stud. Appl. Math. **122**, 219 (2009).
[35] M. J. Ablowitz, C. M. Schober, and B. M. Herbst, Numerical Chaos, Roundoff Errors and Homoclinic Manifolds, Phys. Rev. Lett. **71**, 2683 (1993).
[36] S. Trillo and S. Wabnitz, Dynamics of the nonlinear modulational instability in optical fibers, Opt. Lett. **16**, 986 (1991).
[37] A. R. Its and V. P. Kotlyarov, Explicit formulas for solutions of the NLS equation, Dokl. Akad. Nauk SSSR **11**, 965 (1976).
[38] Y.-C. Ma and M. J. Ablowitz, The periodic cubic NLS equation, Stud. Appl. Math. **65**, 113 (1981).
[39] In fact, the limiting process from the periodic case to the infinite line is highly nontrivial and is not yet properly understood.
[40] S. Kamvissis, K. D. McLaughlin, and P. D. Miller, *Semiclassical Soliton Ensembles for the Focusing Nonlinear Schrödinger Equation* (Princeton University Press, Princeton, NJ, 2003).