Universal Nature of the Nonlinear Stage of Modulational Instability

Gino Biondini\(^1,2,*\) and Dionyssios Mantzavinos\(^2\)

\(^1\)Department of Physics, State University of New York at Buffalo, Buffalo, New York 14260, USA
\(^2\)Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14260, USA

(Received 9 September 2015; published 27 January 2016)

We characterize the nonlinear stage of modulational instability (MI) by studying the longtime asymptotics of the focusing nonlinear Schrödinger (NLS) equation on the infinite line with initial conditions tending to constant values at infinity. Asymptotically in time, the spatial domain divides into three regions: a far left and a far right field, in which the solution is approximately equal to its initial value, and a central region in which the solution has oscillatory behavior described by slow modulations of the periodic traveling wave solutions of the focusing NLS equation. These results demonstrate that the asymptotic stage of MI is universal since the behavior of a large class of perturbations characterized by a continuous spectrum is described by the same asymptotic state.

\(^*\)Department of Physics, State University of New York at Buffalo, Buffalo, New York 14260, USA

Introduction.—Modulational instability (MI)—i.e., the instability of a constant background to long wavelength perturbations—is one of the most ubiquitous phenomena in nonlinear science (e.g., see Ref. [1] and the references therein). The effect, which is known as Benjamin-Feir instability in the context of deep water waves [2], has been known since the 1960s, but it has received renewed attention in recent years and was also linked to the formation of rogue waves in optical media [3,4] and in the open sea [5].

The dynamics of many systems affected by MI is governed by the one-dimensional focusing nonlinear Schrödinger (NLS) equation, which models the evolution of weakly nonlinear dispersive wave packets in such diverse fields as water waves, plasmas, optics, and Bose-Einstein condensates. One can therefore study the initial (i.e., linear) stage of MI by linearizing the NLS equation around the constant background. One easily sees that all Fourier modes below a certain threshold are unstable, and the corresponding perturbations grow exponentially. However, the linearization ceases to be valid as soon as perturbations become comparable with the background. A natural question, then, is what happens at this point, which is referred to as the nonlinear stage of MI. Surprisingly, a precise characterization of the nonlinear stage of MI for generic, finite-energy perturbations has remained, by and large, an open problem for the last 50 years.

The NLS equation is a completely integrable system [6], and it admits an infinite number of conservation laws and exact \(N\)-soliton solutions for arbitrary \(N\)’s, describing the elastic interaction of solitons [6]. By analogy with the case of localized initial conditions, a natural conjecture was that MI is therefore mediated by solitons [7,8]. The initial value problem (IVP) for the NLS equation can be solved via the inverse scattering transform (IST). In particular, the IST for the focusing NLS equation with zero boundary conditions (ZBC) at infinity (i.e., localized disturbances) was done in Ref. [6], and the IST for the defocusing NLS equation with nonzero boundary conditions (NZBC, i.e., solutions that tend to finite nonzero values at infinity) was done in Ref. [9]. However, only partial results [10–12] were available for the focusing NLS equation with NZBC until recently when, in Ref. [13], we developed a complete IST for this case. (Recall that the IST for systems with NZBC is notoriously more challenging, and the IVP for the vector NLS with NZBC was also only solved recently [14,15]). In Ref. [16] we then used the IST to study MI by computing the spectrum of the scattering problem for simple classes of perturbations of a constant background. In particular, we showed that there are classes of perturbations for which no solitons are present. Thus, since all generic perturbations of the constant background are linearly unstable, solitons cannot be the mechanism that mediates the MI, contradicting a recent conjecture [7]. Instead, in Ref. [16] we identified the instability mechanism within the context of the IST by showing that the instability comes from the continuous spectrum of the scattering problem associated with the NLS equation (see below for further details).

In this Letter we use the framework developed in Ref. [13] to characterize the nonlinear stage of MI. We do so by studying the longtime asymptotic behavior of localized perturbations of the constant background. We show that, generically, the longtime asymptotics of modulationally unstable fields on the whole line displays universal behavior and decomposes the \(xt\) plane into two plane wave regions—in which the solution is approximately equal to the background up to a phase—separated by a central region in which the leading-order behavior is described by a slowly modulated traveling wave.

The NLS equation and MI.—We write the focusing NLS equation as

\[
iq_t + q_{xx} + 2(|q|^2 - q_0^2)q = 0, \quad (1)\]

\(iqt + q_{xx} + 2(|q|^2 - q_0^2)q = 0\)
where \( q(x, t) \) represents the complex envelope of a quasi-monochromatic, weakly nonlinear dispersive wave packet, and the physical meaning of the variables \( x \) and \( t \) depends on the physical context. (For example, in optics, \( t \) represents propagation distance, while \( x \) is a retarded time.) Here, \( q_o = |q_o| > 0 \) is the background amplitude, and the NZBC satisfied by the field are
\[
q_{\pm} = \lim_{x \to \pm \infty} q(x, t).
\]

The term \(-2q_o^2 q\) has been added to Eq. (1) so that \( q_{\pm}\) are independent of time, and they can be removed by a trivial gauge transformation.

The constant background solution is simply \( q_{\pm}(x, t) = q_o \). Linearizing Eq. (1) around this solution, one finds that all Fourier modes with \(|\xi| < 2q_o\) (where \( \xi \) is the Fourier variable) are unstable, and that the growth rate is \( \gamma(\xi) = |\xi| \sqrt{4q_o^2 - \xi^2} \). Below, we will use the IST for Eq. (1) with the NZBC (2), which was developed in Ref. [13], slightly reformulated in a way that is more convenient for the present purposes.

Recall that the NLS equation (1) is the zero-curvature condition \( X_T - T_X + [X, T] = 0 \) of the matrix Lax pair \( \phi_1 = X \phi \) and \( \phi_2 = T \phi \), with \( X = ik \sigma_3 + Q \) and \( T = -i(2k^2 + d_o^2 - |q|^2 - Q_0) \sigma_3 - 2kQ \), where \( \sigma_3 = \text{diag}(1, -1) \) is the third Pauli matrix, and
\[
Q(x, t) = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}.
\]

As usual, the first half of the Lax pair is referred to as the scattering problem and \( q(x, t) \) as the potential, and the direct problem in the IST consists of determining the scattering data (i.e., the reflection coefficient, discrete eigenvalues, and norming constants) from the initial condition. This is done through the Jost eigenfunctions \( \phi_{\pm}(x, t, k) \), which are the simultaneous matrix solutions of both parts of the Lax pair which reduce to plane waves, namely, \( \phi_{\pm}(x, t, k) = E_{\pm}(k)e^{i\theta_{\pm}(x, t, k)} + a(k)e^{-i\theta_{\pm}(x, t, k)} \) as \( x \to \pm \infty \), where \( \pm i \lambda, E_{\pm}(k) = I + i/(k + \lambda) \sigma_3 \), \( \theta_{\pm}(x, t, k) = \lambda x - \omega t \), with \( \omega(k) = (k^2 + q_o^2)^{1/2} \) and \( \theta(x, t, k) = \lambda x - \omega t \), with \( \omega(k) = 2k\lambda \). These Jost eigenfunctions, which are the nonlinearization of the Fourier modes, are defined for all values of \( k \in C \) such that \( \lambda(k) \in R \), which comprise the continuous spectrum \( \Sigma = R \cup [-q_o, q_o] \); see Fig. 1 (left panel). The scattering relation \( \phi_{-}(x, t, k) = \phi_{+}(x, t, k)A(k) \) defines the scattering matrix \( A(k) \) for \( k \in \Sigma \), and the corresponding reflection coefficient is \( r(k) = -a_{21}(k)/a_{22}(k) \). The zeros of \( a_{11}(k) \) and \( a_{22}(k) \) define the discrete spectrum of the problem, which leads to solitons. As usual, time evolution within IST is trivial. In particular, with the above normalization of the Jost eigenfunctions, all the scattering data are independent of time.

The focusing NLS equation (1) with the NZBC (2) possesses a rich family of soliton solutions [10,17–19], classified according to the possible placements of the discrete eigenvalue [13]. In particular, the so-called Akhmediev breathers provide a good representation for the growth of seeded perturbations [20,21]. Importantly, however, Akhmediev breathers are periodic in space, and they therefore possess infinite energy. Hence, they cannot describe the asymptotic state of localized (i.e., finite-energy) perturbations of the constant background. Moreover, as mentioned earlier, there exist generic perturbations of the constant background for which no discrete spectrum (and thus no solitons) is present. Instead, the key to describing the asymptotic stage of MI lies in the continuous spectrum. Indeed, as we showed in Ref. [16], \( \omega(k) \) is purely imaginary for \( k \in i[-q_o, q_o] \), and the Jost solutions for \( k \in i[-q_o, q_o] \) are precisely the nonlinearization of the unstable Fourier modes. In fact, even their growth rate is the same, modulo the usual rescaling.

![FIG. 1. (Left panel) The spectral k plane, showing the continuous spectrum \( \Sigma \) (the red lines), the regions where \( \text{Im} \lambda > 0 \) (gray) and \( \text{Im} \lambda < 0 \) (white), and a discrete eigenvalue \( k_n \) together with its symmetric counterpart. (Right panel) The asymptotic regime for the \( x_t \) plane, showing the decomposition into two plane wave regions (white) and the modulated elliptic wave region (gray).](https://example.com/fig1.png)
plus a symmetric expression for $k \in i[-q_\alpha,0]$. Note that 
\[ \det M(x,t,k) = 1 \quad \text{for} \quad k \in C \setminus \Sigma \quad \text{and} \quad M(x,t,k) \to I \quad \text{as} \quad k \to \infty. \] 
The solution of the NLS equation is recovered via the usual reconstruction formula $q(x,t) = -2\text{Im} k \to \infty k M_{12}$. 
The signature of MI in the inverse problem is the exponentially growing entries of $V(x,t,k)$ for $k \in i[-q_\alpha,q_\alpha]$ through the time dependence of $\theta(x,t,k)$.

Longtime asymptotics of finite-energy perturbations.---We now study the asymptotic state of MI for generic, finite-energy perturbations of a constant background. As mentioned earlier, we do so by computing the longtime asymptotics of the solutions of the focusing NLS equation with NZBC. As a concrete example we consider boxlike perturbations with $q(x,0) = q_o$ for $|x| > L$ and $q(x,0) = be^{ip}$ for $|x| < L$, in which case $r(k) = e^{2iL k_1 (b \cos \beta - q_o^2) - ib \sin \beta} / (i\mu \cot(2L\mu) - i(k^2 + q_o \cos \beta))$, with $\mu = \sqrt{k^2 + b^2}$. We emphasize, however, that the results described above are not limited to this example, and they apply to all localized perturbations such that the corresponding reflection coefficients have a small region of analyticity around the continuous spectrum and such that no discrete spectrum is present.

Recall that for linear evolution equations one computes the asymptotics of the solution as $t \to \infty$ via stationary phase or steepest descent by looking along lines asymptotics of the solutions of the focusing NLS equation. We now study the asymptotic state of MI for generic, finitenergy perturbations of a constant background. As mentioned earlier, we do so by computing the longtime asymptotics of the solutions of the focusing NLS equation with NZBC. As a concrete example we consider boxlike perturbations with $q(x,0) = q_o$ for $|x| > L$ and $q(x,0) = be^{ip}$ for $|x| < L$, in which case $r(k) = e^{2iL k_1 (b \cos \beta - q_o^2) - ib \sin \beta} / (i\mu \cot(2L\mu) - i(k^2 + q_o \cos \beta))$, with $\mu = \sqrt{k^2 + b^2}$. We emphasize, however, that the results described above are not limited to this example, and they apply to all localized perturbations such that the corresponding reflection coefficients have a small region of analyticity around the continuous spectrum and such that no discrete spectrum is present.

Recall that for linear evolution equations one computes the asymptotics of the solution as $t \to \infty$ via stationary phase or steepest descent by looking along lines asymptotics of the solutions of the focusing NLS equation. We now study the asymptotic state of MI for generic, finitenergy perturbations of a constant background. As mentioned earlier, we do so by computing the longtime asymptotics of the solutions of the focusing NLS equation with NZBC. As a concrete example we consider boxlike perturbations with $q(x,0) = q_o$ for $|x| > L$ and $q(x,0) = be^{ip}$ for $|x| < L$, in which case $r(k) = e^{2iL k_1 (b \cos \beta - q_o^2) - ib \sin \beta} / (i\mu \cot(2L\mu) - i(k^2 + q_o \cos \beta))$, with $\mu = \sqrt{k^2 + b^2}$. We emphasize, however, that the results described above are not limited to this example, and they apply to all localized perturbations such that the corresponding reflection coefficients have a small region of analyticity around the continuous spectrum and such that no discrete spectrum is present.

Recall that for linear evolution equations one computes the asymptotics of the solution as $t \to \infty$ via stationary phase or steepest descent by looking along lines asymptotics of the solutions of the focusing NLS equation. We now study the asymptotic state of MI for generic, finitenergy perturbations of a constant background. As mentioned earlier, we do so by computing the longtime asymptotics of the solutions of the focusing NLS equation with NZBC. As a concrete example we consider boxlike perturbations with $q(x,0) = q_o$ for $|x| > L$ and $q(x,0) = be^{ip}$ for $|x| < L$, in which case $r(k) = e^{2iL k_1 (b \cos \beta - q_o^2) - ib \sin \beta} / (i\mu \cot(2L\mu) - i(k^2 + q_o \cos \beta))$, with $\mu = \sqrt{k^2 + b^2}$. We emphasize, however, that the results described above are not limited to this example, and they apply to all localized perturbations such that the corresponding reflection coefficients have a small region of analyticity around the continuous spectrum and such that no discrete spectrum is present.

Recall that for linear evolution equations one computes the asymptotics of the solution as $t \to \infty$ via stationary phase or steepest descent by looking along lines asymptotics of the solutions of the focusing NLS equation. We now study the asymptotic state of MI for generic, finitenergy perturbations of a constant background. As mentioned earlier, we do so by computing the longtime asymptotics of the solutions of the focusing NLS equation with NZBC. As a concrete example we consider boxlike perturbations with $q(x,0) = q_o$ for $|x| > L$ and $q(x,0) = be^{ip}$ for $|x| < L$, in which case $r(k) = e^{2iL k_1 (b \cos \beta - q_o^2) - ib \sin \beta} / (i\mu \cot(2L\mu) - i(k^2 + q_o \cos \beta))$, with $\mu = \sqrt{k^2 + b^2}$. We emphasize, however, that the results described above are not limited to this example, and they apply to all localized perturbations such that the corresponding reflection coefficients have a small region of analyticity around the continuous spectrum and such that no discrete spectrum is present.

Recall that for linear evolution equations one computes the asymptotics of the solution as $t \to \infty$ via stationary phase or steepest descent by looking along lines asymptotics of the solutions of the focusing NLS equation. We now study the asymptotic state of MI for generic, finitenergy perturbations of a constant background. As mentioned earlier, we do so by computing the longtime asymptotics of the solutions of the focusing NLS equation with NZBC. As a concrete example we consider boxlike perturbations with $q(x,0) = q_o$ for $|x| > L$ and $q(x,0) = be^{ip}$ for $|x| < L$, in which case $r(k) = e^{2iL k_1 (b \cos \beta - q_o^2) - ib \sin \beta} / (i\mu \cot(2L\mu) - i(k^2 + q_o \cos \beta))$, with $\mu = \sqrt{k^2 + b^2}$. We emphasize, however, that the results described above are not limited to this example, and they apply to all localized perturbations such that the corresponding reflection coefficients have a small region of analyticity around the continuous spectrum and such that no discrete spectrum is present.

Recall that for linear evolution equations one computes the asymptotics of the solution as $t \to \infty$ via stationary phase or steepest descent by looking along lines asymptotics of the solutions of the focusing NLS equation. We now study the asymptotic state of MI for generic, finitenergy perturbations of a constant background. As mentioned earlier, we do so by computing the longtime asymptotics of the solutions of the focusing NLS equation with NZBC. As a concrete example we consider boxlike perturbations with $q(x,0) = q_o$ for $|x| > L$ and $q(x,0) = be^{ip}$ for $|x| < L$, in which case $r(k) = e^{2iL k_1 (b \cos \beta - q_o^2) - ib \sin \beta} / (i\mu \cot(2L\mu) - i(k^2 + q_o \cos \beta))$, with $\mu = \sqrt{k^2 + b^2}$. We emphasize, however, that the results described above are not limited to this example, and they apply to all localized perturbations such that the corresponding reflection coefficients have a small region of analyticity around the continuous spectrum and such that no discrete spectrum is present.

Recall that for linear evolution equations one computes the asymptotics of the solution as $t \to \infty$ via stationary phase or steepest descent by looking along lines asymptotics of the solutions of the focusing NLS equation. We now study the asymptotic state of MI for generic, finitenergy perturbations of a constant background. As mentioned earlier, we do so by computing the longtime asymptotics of the solutions of the focusing NLS equation with NZBC. As a concrete example we consider boxlike perturbations with $q(x,0) = q_o$ for $|x| > L$ and $q(x,0) = be^{ip}$ for $|x| < L$, in which case $r(k) = e^{2iL k_1 (b \cos \beta - q_o^2) - ib \sin \beta} / (i\mu \cot(2L\mu) - i(k^2 + q_o \cos \beta))$, with $\mu = \sqrt{k^2 + b^2}$. We emphasize, however, that the results described above are not limited to this example, and they apply to all localized perturbations such that the corresponding reflection coefficients have a small region of analyticity around the continuous spectrum and such that no discrete spectrum is present.

Recall that for linear evolution equations one computes the asymptotics of the solution as $t \to \infty$ via stationary phase or steepest descent by looking along lines asymptotics of the solutions of the focusing NLS equation. We now study the asymptotic state of MI for generic, finitenergy perturbations of a constant background. As mentioned earlier, we do so by computing the longtime asymptotics of the solutions of the focusing NLS equation with NZBC. As a concrete example we consider boxlike perturbations with $q(x,0) = q_o$ for $|x| > L$ and $q(x,0) = be^{ip}$ for $|x| < L$, in which case $r(k) = e^{2iL k_1 (b \cos \beta - q_o^2) - ib \sin \beta} / (i\mu \cot(2L\mu) - i(k^2 + q_o \cos \beta))$, with $\mu = \sqrt{k^2 + b^2}$. We emphasize, however, that the results described above are not limited to this example, and they apply to all localized perturbations such that the corresponding reflection coefficients have a small region of analyticity around the continuous spectrum and such that no discrete spectrum is present.
MI is often studied in the framework of sideband perturbations of a constant background. The results of this Letter can therefore be compared to those in the case of periodic boundary conditions. There, the instability is ascribed to the presence of homoclinic solutions [35]. The initial stage of MI in that scenario was studied in Ref. [36] with a three-mode model. However, the IST machinery used to study the periodic case (namely, the theory of finite-genus solutions [37,38]) is very different from the one in the IVP with NZBC, used here [39]. Most importantly, the physics in the two cases is different. For example, (i) in the periodic case there is an amplitude threshold below which no instability occurs, whereas no such threshold exists on the infinite line, and (ii) in the periodic case, radiation cannot escape to infinity, and therefore it is doubtful that a longtime asymptotic state even exists. Also, sinusoidal excitations are a special case of perturbations with several Fourier components, each contributing with its own amplitude and phase. Such generic perturbations are characterized by their Fourier transform (or, equivalently, spectral data), and this is precisely the situation studied in this Letter.

The above results can also be compared to the semiclassical limit of the focusing NLS equation with ZBC [40]. The study of that scenario requires more sophisticated analysis, and the results are also more complicated. Moreover, numerical simulations of the semiclassical case become more and more sensitive to roundoff error as $h \to 0$ [35]. In contrast, the present case does not appear to be as sensitive. The robustness of our analytical predictions is confirmed in Fig. 3, which shows a numerical simulation of Eq. (1) with a small Gaussian perturbation of the constant background. The numerical results show that there is an intermediate time range for which one sees the asymptotic behavior but no catastrophic roundoff. As a result, there appear to be no fundamental obstacles to the possibility of observing experimentally the behavior described in this Letter.

Semiclassical limits and longtime asymptotics problems are often studied using Whitham theory [23]. However, the Whitham equations for the focusing NLS equation are elliptic, and therefore the corresponding IVP is ill posed. This is well known in the case of ZBC (e.g., see Ref. [40]), and it remains true in the case of NZBC. While special solutions to the Whitham equations also exist in the focusing case [31,32], it should be clear that the IST-related methods used here are the only way to study the nonlinear stage of MI for generic perturbations of the constant background. Indeed, we see no obstacles to generalizing the present calculations to include the presence of discrete eigenvalues, which will allow for the first time a study of the interactions between solitons and radiation in modulationally unstable media.

This work was partially supported by the National Science Foundation under Grant No. DMS-1311847. D. M. also acknowledges an AMS-Simons Travel Grant.


[22] The point \( x = 0 \) appears to be special because, in the far-field approximation of the dynamics arising from localized perturbations, everything seems to arise from the origin, just like in the far-field asymptotics for linear problems [23].


[30] A special case of this solution was studied in Refs. [31,32] in the context of Whitham theory, but neither work studied the evolution of generic initial conditions.


[33] Here, \( K(m) \) is the complete elliptic integral of the first kind. Since the wave is nonlinear, the wave number and the period are not related by a simple proportionality relation as for harmonic waves.


[39] In fact, the limiting process from the periodic case to the infinite line is highly nontrivial and is not yet properly understood.