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Imaginary eigenvalues of Zakharov–Shabat problems with non-zero background

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ABSTRACT

The focusing Zakharov–Shabat scattering problem on the infinite line with non-zero boundary conditions for the potential is studied, and sufficient conditions on the potential are identified to ensure that the problem admits only purely imaginary discrete eigenvalues. The results, which generalize previous work by Klaus and Shaw, are applicable to the study of solutions of the focusing nonlinear Schrödinger equation with non-zero background.

1. Introduction

The nonlinear Schrödinger (NLS) equation is a universal model that describes the evolution of weakly nonlinear and quasi-monochromatic wave trains in media with cubic nonlinearities. As such, it arises in many disparate physical settings such as water waves, optics, acoustics, Bose–Einstein condensation, etc. It is also known since the pioneering work of Zakharov and Shabat in 1972 [30] that the NLS equation in one spatial dimension is a completely integrable system, and as such it can be written as the compatibility condition of an overdetermined pair of linear ordinary differential equations, which are called the Lax pair. Zakharov and Shabat also showed that the initial-value problem for the NLS equation could be solved by the inverse scattering transform. Accordingly, the first half of the Lax pair for the NLS equation is referred to as the Zakharov–Shabat scattering problem, and the solution of the NLS equation plays the role of a potential there. Therefore, the study of Zakharov–Shabat scattering problems has been an ongoing area of research (e.g., see [5,17,20,27]).

Recall that the NLS equation is the compatibility condition of the matrix Lax pair

$$\mathbf{v}_x = (-i\zeta\,\sigma_3 + Q\,(x,t))\,\mathbf{v}\,, \tag{1a}$$

$$\mathbf{v}_t = (2i\zeta^2\sigma_3 + 2kQ - iQ_x\sigma_3 - iQ^2\sigma_3)\,\mathbf{v}\,, \tag{1b}$$

where $\mathbf{v}(x,t,\zeta) = (v_1, v_2)^T$, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q\,(x,t) = i\begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix} \tag{2}$$

(with σ_1 to be used later). That is, the requirement $\mathbf{v}_{xt} = \mathbf{v}_{tx}$, together with the constraint $r = \nu q^*$, yields the NLS equation,

$$iq_t + q_{xx} - 2\nu|q|^2 q = 0\,, \tag{3}$$

where $q : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, subscripts denote partial derivatives and as usual $\nu = \mp 1$ denote the focusing and defocusing cases, respectively. Equation (1a) is referred to as the Ablowitz–Kaup–Newell–Segur scattering problem [3]. The Zakharov–Shabat scattering problem is the special case of (1) when $r(x,t) = \nu q^*(x,t)$ [with the asterisk denoting complex conjugation], in which case the compatibility condition of (1) yields precisely the NLS equation (3).

Equation (1a) can equivalently be written as the eigenvalue problem $\mathcal{L}\mathbf{v} = \zeta\mathbf{v}$ for the Dirac operator $\mathcal{L} = i\sigma_3(\partial_x - Q)$. The spectrum of the scattering problem is the set of all values of $\zeta \in \mathbb{C}$ such that nontrivial bounded eigenfunctions $\mathbf{v}(x,t,k)$ exist, and such values of ζ are referred to as the eigenvalues of the scattering problem. In particular, values $\zeta \in \mathbb{C}$ such that $\mathbf{v}(x,t,k) \in L^2(\mathbb{R})$ are referred to as the discrete eigenvalues of the problem. (As we discuss below, the above definition differs slightly from the one typically used in the development of the inverse scattering transform (IST), in which discrete eigenvalues are defined as the zeros of the analytic scattering coefficients.) The structure of the Lax pair implies that, when the potential evolves in time according to the

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NLS equation, the spectrum of $\mathcal{L}$ is independent of time. For this reason, we will drop the time dependence throughout this work.

In the defocusing case the Dirac operator $\mathcal{L}$ is self-adjoint [31], and therefore all eigenvalues are real. In the focusing case ($\nu = -1$), however, $\mathcal{L}$ is non-self adjoint. One can show that the reduction $r = \nu q^*$ implies that the spectrum of $\mathcal{L}$ is symmetric with respect to the real ζ-axis. It is also well known that, if the potential is even, the spectrum is also symmetric with respect to the imaginary ζ-axis. A natural question, however, is whether there exist special classes of potentials for which the spectrum possesses additional symmetries.

The above question was studied in 2002 by Klaus and Shaw [21]. Specifically, Klaus and Shaw considered a class of potentials $q(x)$ that are non-negative, smooth, L^1 functions on the real line, and such that $q(x)$ is nondecreasing for $x < 0$ and nonincreasing for $x > 0$. They were then able to show that any discrete eigenvalues ζ of (1a) are purely imaginary.

The study of nonlinear wave equations with non-zero boundary conditions (NZBC) also has a long history [17,31], and has received renewed attention in recent years (e.g., see [4,7–9,11,14,15,23] and references therein), due also in part to connections with various physical effects such as rogue waves [6,29], modulational instability [7,12,13], the dynamics of dispersive shock waves [1,2] and polarization shifts [10]. A limitation of Klaus and Shaw's result, however, is that it only applies to decaying potentials.

The properties of scattering operators with NZBC can be quite different from those of the same operators with ZBC. For example, it is well known that an "area theorem" exists for the Zakharov–Shabat operator with ZBC: no discrete eigenvalues can exist if the L^1 norm of the potential is less than $\pi/2$ [22]. (This bound, which improves the original one [3], is sharp.) However, it was recently shown that no generalization of the area theorem is possible for the same operators with NZBC, either in the focusing [7] or in the defocusing [14] case. In other words, the situation for the Zakharov–Shabat operator with NZBC is dissimilar to that for the same operators with ZBC, and is more similar instead to that for the Schrödinger operator $\mathcal{L} = -\partial^2 + q(x)$, which defines the scattering problem for the Korteweg–deVries equation [18]. On the other hand, in this work we show that the results of [21] do admit a straightforward generalization to potentials with NZBC.

To do this, we generalize the notion of "single-lobe" potentials to the case of NZBC. Specifically, we will call a single lobe potential with NZBC a function $q(x)$ which is: (i) smooth on real line, (ii) nondecreasing for $x < 0$ and nonincreasing for $x > 0$, (iii) limiting to $q(x) \to q_0$ as $x \to \pm\infty$, where $q_0 > 0$ is a constant, and (iv) $q(x) - q_0 \in L^1(\mathbb{R})$. This definition allows us to obtain the main result of this work, which is the following

Theorem 1.1. *Let $q(x)$ be a smooth, real-valued function on the real line such that*

$$q(x) \to q_0 \quad as \quad x \to \pm\infty, \quad q(x) > q_0 \quad for\, x \in \mathbb{R},$$

where $q_0 > 0$ is a constant. Moreover, let $q(x) - q_0 \in L^1(\mathbb{R})$. If $q(x)$ is nondecreasing for $x < 0$ and nonincreasing for $x > 0$, any discrete eigenvalue ζ of the scattering problem (1a) is purely imaginary, and $|\zeta| > q_0$.

In section 2 we give the proof of Theorem 1.1, and in section 3 we discuss a few examples to illustrate that both of the hypotheses of the theorem (namely, constant-phase and single-lobe conditions) are indeed necessary. Section 4 ends this work with a few concluding remarks.

2. Proof of Theorem 1.1

The strategy of the proof follows that in [21], but the implementation is somewhat different due to the NZBC. First we derive some upper bounds regarding the behavior of the Jost eigenfunctions corresponding to a discrete eigenvalue. Then we derive a constraint that relates discrete eigenvalues to certain integrals of the corresponding eigenfunctions. Finally we use the bounds to establish that the real part of the discrete eigenvalue must vanish identically.

2.1. Jost eigenfunctions and upper bound estimates

Recall that in the IST for the focusing NLS equation with NZBC [8] one defines the Jost eigenfunctions as the solutions of (1a) which tend to plane wave behavior either as $x \to \infty$ or as $x \to -\infty$. In particular, for our purposes it is sufficient to introduce the columns $\phi(x,\zeta)$ and $\psi(x,\zeta)$ as

$$\phi(x,\zeta) = \begin{pmatrix} \lambda + \zeta \\ -iq_0 \end{pmatrix} e^{-i\lambda x}(1 + o(1)), \quad x \to -\infty, \tag{4a}$$

$$\psi(x,\zeta) = \begin{pmatrix} -iq_0 \\ \lambda + \zeta \end{pmatrix} e^{i\lambda x}(1 + o(1)), \quad x \to +\infty, \tag{4b}$$

where $\lambda(\zeta)$ is defined by the equation $\lambda^2 = \zeta^2 + q_0^2$. The set of values $\zeta \in \mathbb{C}$ such that $\lambda(\zeta) \in \mathbb{R}$ comprises the discrete spectrum of the scattering problem. In our case, this is the set $\Sigma = \mathbb{R} \cup i[-q_0, q_0]$. Without loss of generality, one can define $\lambda(\zeta)$ for all $\zeta \in \mathbb{C}$ through the analytic continuation of the principal branch of the real square root off the positive real ζ-axis with a square-root sign discontinuity across the branch cut $[-iq_o, iq_o]$. It is easy to show that, with this definition, the sign of the imaginary part of $\lambda(\zeta)$ is the same as that of ζ away from the branch cut.

The Zakharov–Shabat scattering problem possesses the usual reflection symmetry such that for every eigenvalue ζ in the upper-half plane there is a corresponding eigenvalue ζ^* in the lower-half plane [8]. Thus, without loss of generality we can restrict ourself to studying the discrete eigenvalues in the upper-half plane.

The Jost eigenfunctions (4) are rigorously defined as the solutions of suitable linear integral equations [8]. For example,

$$\phi(x,\zeta) = \begin{pmatrix} \lambda + \zeta \\ -iq_0 \end{pmatrix} e^{-i\lambda x} + \int_{-\infty}^{x} G_-(x-y,\zeta)(Q(y) - Q_0)e^{i\lambda(y-x)}\phi(y,\zeta)\,dy, \tag{5}$$

where

$$G_-(x-y,\zeta) = \frac{1}{2\lambda}[(1 + e^{2i\lambda(x-y)})\lambda I - i(e^{2i\lambda(x-y)} - 1)(i\zeta\sigma_3 + Q_0)], \tag{6}$$

with

$$Q_0 = \begin{pmatrix} 0 & q_0 \\ -q_0 & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

and a similar equation for $\psi(x,\zeta)$. Using these integral equations, it was shown in [8] that, as is usually the case in the IST, both $\phi(x,\zeta)$ and $\psi(x,\zeta)$ admit analytic continuation to the upper half of the complex ζ plane.

Suppose now that $\lambda(\zeta) = \alpha + i\beta$ is a discrete eigenvalue corresponding to a certain value of ζ in the closure of the upper-half plane. It was shown in [8] that, as is usually the case in the IST, the associated Jost eigenfunctions $\phi(x,\zeta)$ and $\psi(x,\zeta)$ at this specific value of ζ are proportional each other, and that any of the $L^2(\mathbb{R})$ eigenfunctions $v(x,\zeta)$ associated to ζ are proportional to both of them. Because our definition of discrete eigenvalues require the corresponding eigenfunctions to be in $L^2(\mathbb{R})$ (as opposed to simply being bounded), it follows that $\beta = \operatorname{Im}\lambda$ must be strictly

positive, and therefore ζ is not allowed to be on $[0,\infty)\cup[0,iq_o]$. In other words, our definition of discrete eigenvalues excludes the case of improper, or embedded, eigenvalues, i.e., zeros of the analytic scattering coefficients on the continuous spectrum Σ.

Splitting (5) into components, and writing $\phi(x,\zeta)=(\phi_1,\phi_2)^T$, we immediately have the following bounds:

$$|\phi_1(x,\zeta)|\,e^{-\beta x}\le|\lambda+\zeta| + \frac{1}{2|\lambda|}\int_{-\infty}^{x}(q(y)-q_0)e^{-\beta y}\Big[q_0|\phi_1(y,\zeta)|(e^{-2\beta(x-y)}+1) + |\phi_2(y,\zeta)|(|\lambda-\zeta|e^{-2\beta(x-y)}+|\lambda+\zeta|)\Big]dy\,, \tag{8a}$$

$$|\phi_2(x,\zeta)|\,e^{-\beta x}\le q_0 + \frac{1}{2|\lambda|}\int_{-\infty}^{x}(q(y)-q_0)e^{-\beta y}\big[q_0|\phi_2(y,\zeta)|(e^{-2\beta(x-y)}+1) + |\phi_1(y,\zeta)|(|\lambda+\zeta|e^{-2\beta(x-y)}+|\lambda-\zeta|)\big]dy\,. \tag{8b}$$

Denoting $f(x,\zeta)=(|\phi_1(x,\zeta)|+|\phi_2(x,\zeta)|)e^{-\beta x}$, and taking the sum of (8), we have

$$f(x,\zeta)\le|\lambda+\zeta|+q_0 + \frac{1}{2|\lambda|}\int_{-\infty}^{x}(q(y)-q_0)\Big[q_0(e^{-2\beta(x-y)}+1) + (|\lambda+\zeta|+|\lambda-\zeta|)(e^{-2\beta(x-y)}+1)\Big]f(y,\zeta)\,dy\,.$$

Using Grönwall's inequality [19], we then obtain

$$f(x,\zeta)\le(|\lambda+\zeta|+q_0)\exp[J(x,\zeta)]\,, \tag{9a}$$

where

$$J(x,\zeta)=\frac{1}{2|\lambda|}\int_{-\infty}^{x}(q(y)-q_0)\Big[q_0(e^{-2\beta(x-y)}+1) + (|\lambda+\zeta|+|\lambda-\zeta|)(e^{-2\beta(x-y)}+1)\Big]dy\,. \tag{9b}$$

The exponent $J(x,\zeta)$ satisfies the following uniform bound as a function of x:

$$J(x,\zeta)\le\frac{1}{|\lambda|}(q_0+|\lambda+\zeta|+|\lambda-\zeta|)\int_{-\infty}^{x}(q(y)-q_0)dy\le C_o, \tag{10}$$

for all $x\in\mathbb{R}$, where C_o is a constant, since $q-q_o\in L_1(\mathbb{R})$. Hence we obtain

$$|\phi_1(x,\zeta)|+|\phi_2(x,\zeta)|\le C\,e^{\beta x}, \tag{11}$$

where $C=e^{C_o}$. Using similar methods, one can obtain a similar bound for the other Jost eigenfunction:

$$|\psi_1(x,\zeta)|+|\psi_2(x,\zeta)|\le C\,e^{-\beta x}. \tag{12}$$

Equations (11) and (12) imply $\lim_{x\to-\infty}(|\phi_1(x,\zeta)|+|\phi_2(x,\zeta)|)=\lim_{x\to\infty}(|\psi_1(x,\zeta)|+|\psi_2(x,\zeta)|)=0$. Since $v(x,\zeta)$ is proportional to both $\phi(x,\zeta)$ and $\psi(x,\zeta)$, we therefore have

$$\left[\frac{|v_1(x)|^2}{q(x)}\right]_0^{+\infty}-\left[\frac{|v_2(x)|^2}{q(x)}\right]_{-\infty}^{0}\le 0. \tag{13}$$

2.2. Relation between discrete eigenvalues and integrals of the eigenfunctions

We now derive a constraint that relates the discrete eigenvalue to certain integrals of the associated eigenfunction. It is convenient to write the scattering problem (1a) in component form as

$$iv_1'-iq(x)v_2=\zeta v_1, \tag{14a}$$

$$iv_2'+iq(x)v_1=-\zeta v_2, \tag{14b}$$

where $\mathbf{v}(x,\zeta)=(v_1,v_2)^T$ and primes denote differentiation with respect to x. Now suppose that ζ is a discrete eigenvalue and $\mathbf{v}(x,\zeta)$ the associated eigenfunction. We multiply the first equation of (14) by v_2^*, the second one by v_1^* and subtract, to get

$$i(v_1'v_2^*-v_2'v_1^*)-iq(x)(|v_1|^2+|v_2|^2)=\zeta(v_1v_2^*+v_2v_1^*),$$

from which, integrating, we obtain

$$i\int_{-\infty}^{\infty}(v_1'v_2^*-v_2'v_1^*)dx-i\int_{-\infty}^{\infty}q(x)(|v_1|^2+|v_2|^2)\,dx = \zeta\int_{-\infty}^{\infty}(v_1v_2^*+v_2v_1^*)\,dx\,. \tag{15}$$

The constraint (15) is the key that will allow us to prove that the discrete eigenvalues are purely imaginary. Note that all the integrals in (15) are convergent because $\mathbf{v}(x,\zeta)\in L^2(\mathbb{R})$. Also, with regard to the first integral in (15), note that, using integration by parts, we have

$$\int_{-\infty}^{\infty}(v_1'v_2^*-v_2'v_1^*)dx = [v_2^*v_1-v_1^*v_2]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}[v_2(v_1^*)'-v_1(v_2^*)']\,dx\,. \tag{16}$$

The integral in the right-hand side of this expression is the complex conjugate of that in the left-hand side. Since v_1 and v_2 tend to zero as $x\to\pm\infty$, we therefore obtain that both of these integrals are real-valued. Moreover, it should be obvious that the second and third integrals in (15) are also real-valued. Hence, if

$$\int_{-\infty}^{\infty}(v_1v_2^*+v_2v_1^*)\,dx\ne 0, \tag{17}$$

it follows from (15) that ζ must be purely imaginary. Below we show that indeed the relation (17) holds for single-lobe potentials with NZBC.

2.3. Single-lobe potentials

We now show that the relation (17) holds for all single-lobe potentials. We do so by applying the bounds derived in Section 2.1. From (14a) we have

$$v_2v_1^*=(v_1'v_1^*+i\zeta|v_1|^2)/q(x)\,. \tag{18}$$

Adding (18) and its complex conjugate, and integrating over $[0,\infty)$, we get

$$\int_0^\infty (v_2 v_1^* + v_1 v_2^*)\,dx = \int_0^\infty \frac{(|v_1|^2)'}{q(x)}dx + i(\zeta - \zeta^*)\int_0^\infty \frac{|v_1|^2}{q(x)}dx. \tag{19}$$

Now recall that $\operatorname{Im}\zeta > 0$ when $\operatorname{Im}\lambda > 0$, and that $\lambda = \alpha + i\beta$ with $\beta > 0$. Thus, the last term in (19) is strictly negative. That is, $i(\zeta - \zeta^*)\int_0^\infty (|v_1|^2/q)\,dx < 0$. Moreover, integration by part yields

$$\int_0^\infty \frac{(|v_1|^2)'}{q(x)}dx = \left[\frac{|v_1|^2}{q(x)}\right]_0^\infty + \int_0^\infty \frac{|v_1|^2 q'(x)}{q^2(x)}dx.$$

Hence we obtain

$$\int_0^\infty (v_2 v_1^* + v_1 v_2^*)dx < \left[\frac{|v_1|^2}{q(x)}\right]_0^\infty + \int_0^\infty \frac{|v_1|^2 q'(x)}{q^2(x)}dx. \tag{20a}$$

Next we turn our attention to the interval $(-\infty, 0]$. Multiplying (14b) by v_2^* and proceeding in a similar way as above, we obtain

$$\int_{-\infty}^0 (v_2 v_1^* + v_1 v_2^*)dx < \left[-\frac{|v_2|^2}{q(x)}\right]_{-\infty}^0 - \int_{-\infty}^0 \frac{|v_2|^2 q'(x)}{q^2(x)}dx. \tag{20b}$$

Since $q(x)$ is nondecreasing for $x < 0$ and nonincreasing for $x > 0$, combining (20a) and (20b) we deduce

$$\int_{-\infty}^\infty (v_2 v_1^* + v_1 v_2^*)dx < 0. \tag{21}$$

Thus, recalling the constraint (15) and the discussion in section 2.2, we conclude that ζ must be purely imaginary.

Finally, recall that $\operatorname{Im}\lambda = 0$ for $\zeta \in \mathbb{R} \cup i[-q_0, q_0]$. However, as we had previously observed, one must have $\operatorname{Im}\lambda > 0$ to have a discrete eigenvalue. Therefore, we must have not only that ζ is purely imaginary, but also that $|\zeta| > q_0$, which completes the proof.

3. Examples

We now review the various hypotheses of Theorem 1.1 by discussing a few concrete potentials. Without loss of generality we set the background amplitude q_o to 1 thanks to the scaling invariance of the NLS equation.

Our first example is the following box-like potential with NZBC:

$$q(x) = \begin{cases} 1\,, & |x| > L, \\ b\,e^{i\alpha}, & |x| \le L, \end{cases} \tag{22}$$

with $L > 0$, $b > 0$ and $\alpha \in \mathbb{R}$. This potential was studied in [7]. In particular, when $b > 1$ and simultaneously $\alpha = 2n\pi$, the function $q(x)$ satisfies the hypotheses of Theorem 1.1, and is the generalization of a potential barrier to NZBC. In [7] it was shown that indeed in this case all discrete eigenvalues are purely imaginary.

We now demonstrate that both the constant-phase and the single-lobe hypotheses in Theorem 1.1 are necessary. We begin with the constant-phase hypothesis. Consider first the case of the potential (22) with $b > 1$ but $\alpha \neq 2n\pi$. In this case $q(x)$ is not real-valued nor constant-phase, and indeed in [7] it was shown that, for certain values of α, some discrete eigenvalues are complex-valued.

Next we turn to the single-lobe hypothesis. Consider the piecewise-constant potential

$$q(x) = \begin{cases} 1, & |x| \ge L, \\ b, & 1 < |x| < L, \\ 0, & |x| < 1, \end{cases} \tag{23}$$

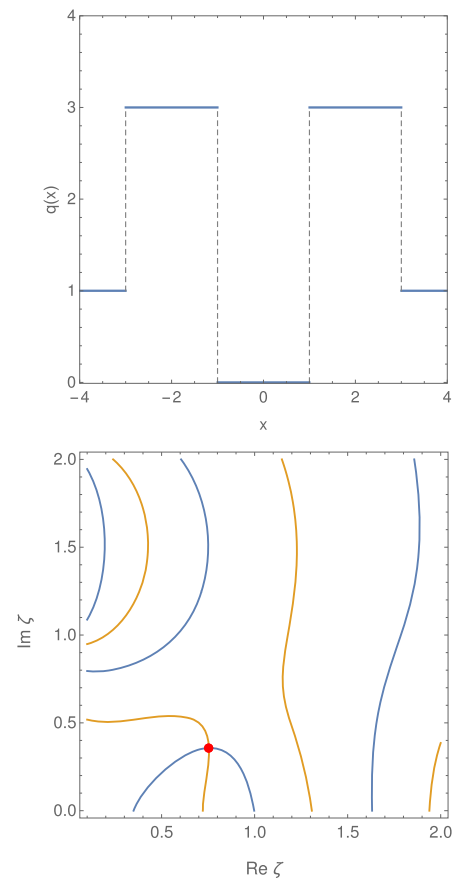

Fig. 1. Top: the potential (23) with $L = b = 3$. Bottom: The lines $\operatorname{Re} a(\zeta) = 0$ (blue) and $\operatorname{Im} a(\zeta) = 0$ (orange) in the complex ζ-plane for the potential (23) with $L = 3$ and $b = 10$. Their intersection (red dot) shows the discrete eigenvalue off the imaginary axis, whose approximate location is $\zeta_o = 0.963 + 0.232i$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

with $L > 1$. An example of such a potential is shown in the top panel of Fig. 1. While this potential is real-valued, it does not satisfy the single-lobe condition. Since like (22) it is piecewise constant, however, one can use similar techniques as in [7] to compute the scattering matrix analytically. More precisely, one first writes an explicit expression for a fundamental matrix solution of the scattering problem in each subdomains (namely, for $x < -L$, $-L < x < -1$, $-1 < x < 1$, $1 < x < L$ and $x > L$). Then, by expressing one of the two fundamental matrix solutions at each of the transition points (namely, $x = -L$, -1, 0, 1 and L) in terms of the other solution times a connection matrix, one can obtain the scattering matrix as a suitable product of these connection matrices.

Explicitly, one can define $\Phi_\pm(x,\zeta)$, $T_\pm(x,\zeta)$ and $T_0(x,\zeta)$ as the matrix solutions of (1a) which have the following representations:

$$\Phi_-(x,\zeta) = [(\lambda - k)I + i\sigma_1]\,e^{i\lambda x\sigma_3}\,, \quad x < -L\,, \tag{24a}$$

$$T_-(x,\zeta) = [(\mu - k)I + i\sigma_1]\,e^{i\mu x\sigma_3}\,, \quad -L < x < -1\,, \tag{24b}$$

$$T_0(x,\zeta) = e^{-i\zeta x\sigma_3}\,, \quad -1 < x < 1\,, \tag{24c}$$

$$T_+(x,\zeta) = [(\mu - k)I + i\sigma_1]\,e^{i\mu x\sigma_3}\,, \quad 1 < x < L\,, \tag{24d}$$

$$\Phi_+(x,\zeta) = [(\zeta + \lambda)I + i\sigma_1]\,e^{i\lambda x\sigma_3}\,, \quad x > L\,, \tag{24e}$$

where $\mu(\zeta) = (\zeta^2 + b^2)^{1/2}$, with the complex square root defined similarly as $\lambda(\zeta)$ with a branch cut on $[0, ib]$, and I is the 2×2 identity matrix as before. Then there exist connection matrices $A_1, \ldots, A_4$ such that, for all $x \in \mathbb{R}$,

$$\Phi_-(x,\zeta) = T_-(x,\zeta)\,A_1(\zeta)\,, \tag{25a}$$

$$T_-(x,\zeta) = T_0(x,\zeta)\, A_2(\zeta)\,, \tag{25b}$$

$$T_0(x,\zeta) = T_+(x,\zeta)\, A_3(\zeta)\,, \tag{25c}$$

$$T_+(x,\zeta) = \Phi_+(x,\zeta)\, A_4(\zeta)\,. \tag{25d}$$

Each of the matrices $A_1, \dots, A_4$ can be evaluated at the common point of the two adjecent subdomains (i.e., the transition points), to yield

$$A_1(\zeta) = T_-^{-1}(-L,\zeta)\Phi(-L,\zeta)\,, \tag{26a}$$

$$A_2(\zeta) = T_0^{-1}(-1,\zeta)T_-(-1,\zeta)\,, \tag{26b}$$

$$A_3(\zeta) = T_+^{-1}(1,\zeta)T_0(1,\zeta)\,, \tag{26c}$$

$$A_4(\zeta) = \Phi_+^{-1}(L,\zeta)T_+(L,\zeta)\,. \tag{26d}$$

Now note that $\Phi_\pm(x,\zeta)$ are just the Jost solutions of the scattering problem. The above relations then immediately yield

$$\Phi_-(x,\zeta) = \Phi_+(x,\zeta)\, A\,, \tag{27}$$

where the scattering matrix is

$$A(\zeta) = A_4 A_3 A_2 A_1\,. \tag{28}$$

Let us denote by $a(\zeta)$ as usual the entry of the scattering matrix that is analytic in the upper half plane. Since the potential (23) is even, $a(-\zeta^*) = 0$ whenever $a(\zeta) = 0$, implying that for each discrete eigenvalue ζ_o in the second quadrant of the complex ζ-plane there is a correspond discrete eigenvalue $-\zeta_o^*$ in the first quadrant. Thus, we can limit ourselves to studying the first quadrant of the complex plane. The bottom panel of Fig. 1 shows the contour lines $\mathrm{Re}\, a(\zeta) = 0$ (blue) and $\mathrm{Im}\, a(\zeta) = 0$ (orange) in the first quadrant for $L = 3$ and $b = 10$. Their intersection, denoted with a red dot, indicates the presence of a discrete eigenvalue approximately located at $\zeta_o = 0.963 + 0.232i$. The fact that ζ_o is off the imaginary axis confirms that, as in the case of zero boundary conditions, the single-lobe condition is a necessary requirement in order for discrete eigenvalues to be restricted to the imaginary axis.

4. Final remarks

In conclusion, we studied the Zakharov–Shabat scattering problem for the focusing nonlinear Schrodinger equation with nonzero boundary conditions. We identified sufficient conditions under which the scattering problem admits purely imaginary eigenvalues, and we illustrated the results by discussing several examples.

Note that, importantly, the condition that the potential be real-valued can be relaxed thanks to the phase invariance of the NLS equation, which means that Theorem 1.1 can be trivially extended to constant-phase potentials whose amplitude satisfies the above single-lobe condition.

Also note that, when $q(x)$ is real-valued, the change of dependent variable $v(x,\zeta) = v_1 + v_2$ and $w(x,\zeta) = v_1 - v_2$, transforms the Zakharov–Shabat scattering problem (1a) into the second-order ordinary differential equation

$$v'' + (\lambda + q^2(x) - iq'(x))\, v = 0\,, \tag{29}$$

with $\lambda = \zeta^2$. Equation (29) defines a non-self-adjoint Sturm–Liouville problem with the complex potential $V(x) = q^2(x) - iq'(x)$. Thus, the class of potentials of the Zakharov–Shabat scattering problem that admit purely imaginary eigenvalues is equivalent to that of potentials for which the complex Sturm–Liouville problem (29) admits purely real eigenvalues. Note also that, if $q(x)$ is even, the potential $V(x)$ possesses the so-called PT-symmetry (i.e., they are invariant under the combined action of space reflection and conjugation) [16]. As is well known, however, PT-symmetry is neither a necessary nor a sufficient condition for an eigenvalue problem in order to have purely real eigenvalues [24–26]. In the case of NZBC, this fact is further confirmed by the examples discussed in section 3. Indeed, the potential $q(x - x_o)$, with $q(x)$ given by (22), $\alpha = 0$ and $x_o \neq 0$, is not PT-symmetric (because it is not even), but nonetheless the corresponding SLP has purely real eigenvalues. Conversely, the potential $q(x)$ in (23) is PT-symmetric, but has complex eigenvalues. (Note that in [28] it was already pointed out that single-lobe potentials lead to a class of non PT-symmetric potentials for the Schrödinger equation (29) with only real eigenvalues.)

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