ELSEVIER

23 June 1997

PHYSICS LETTERS A

Physics Letters A 230 (1997) 319–323

On the well-posedness of the Eckhaus equation

M.J. Ablowitz [a], G. Biondini [b], S. De Lillo [b]

[a] *Department of Applied Mathematics, University of Colorado, Campus Box 526, Boulder, CO 80309-0526, USA*

[b] *Dipartimento di Fisica, Università di Perugia and Istituto Nazionale di Fisica Nucleare, Sezione di Perugia, V. Pascoli 1, 06112 Perugia, Italy*

Received 20 November 1996; revised manuscript received 18 March 1997; accepted for publication 19 March 1997
Communicated by A.P. Fordy

Abstract

The Eckhaus equation, which is a nonlinear Schrödinger type equation that can be linearized to the free linear Schrödinger equation, is considered. A linearized analysis of the nonlinear problem indicates that the periodic boundary value problem is ill-posed. The exact solution also demonstrates the ill-posedness, even though the L^2 norm of the solution is a constant of the motion. The ill-posedness disappears on the infinite line with square integrable data, but the solitonic solutions, which are not square integrable, are seriously unstable. © 1997 Published by Elsevier Science B.V.

There are relatively few nonlinear equations which can be linearized and, in some cases, explicitly solved. The best known of these equations is the Burgers equation, which by a change of variables can be reduced to the linear heat equation. Another equation of interest, first examined in Ref. [1], and often referred to as the Eckhaus equation, is written as

$$i\psi_t + \psi_{xx} + 2(|\psi|^2)_x\psi + |\psi|^4\psi = 0,$$
$$\psi : \mathbb{R}^2 \to \mathbb{C}. \tag{1}$$

This equation is of nonlinear Schrödinger type, and admits a linearization to the free time dependent linear Schrödinger equation. Eq. (1) was found in Ref. [1] as an asymptotic multiscale reduction of certain classes of nonlinear partial differential equations. The Eckhaus equation was recently obtained [2] in the study of the instabilities of plane solitons associated with the Kadomtsev–Petviashvili equation. In Ref. [3] many of the properties of the Eckhaus equation were investigated, including the linearization, soliton solutions etc. The equation is also linearizable when the initial value problem on semi-infinite and finite domains is considered [4] or when an external parametric forcing is applied [5].

In this Letter we study the periodic boundary value problem for Eq. (1). We show that for square integrable initial data, the periodic solutions are ill-posed – i.e. the solution suffers from physically unacceptable instabilities, even though the L^2 norm of the solution is a constant of the motion. Interestingly, as the period L tends to infinity, the instabilities become less pronounced and eventually disappear in the infinite line case and the problem, for square integrable initial data, is well posed. However, for the "solitons", which tend to nonzero constants as $x \to \pm\infty$, the ill-posedness and instabilities persist.

We begin by considering the periodic case. It can be seen immediately that the null solution $\psi(x,t) \equiv 0$ is linearly stable, since the stability analysis reduces, in this special case, to the free linear time dependent Schrödinger equation for a small perturbation $\epsilon(x,t)$.

0375-9601/97/$17.00 © 1997 Published by Elsevier Science B.V. All rights reserved.
PII S0375-9601(97)00240-5