Importance sampling for noise-induced amplitude and timing jitter in soliton transmission systems

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We discuss the application of importance-sampling techniques to the numerical simulation of transmission impairments induced by amplified spontaneous emission noise in soliton-based optical transmission systems. The method allows one to concentrate numerical simulations on the noise realizations that are most likely to result in transmission errors, thus leading to increases in speed of several orders of magnitude over standard Monte Carlo methods. We demonstrate the technique by calculating the probability distribution function of amplitude and timing fluctuations.

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A common source of transmission impairments in light-wave systems is the pulse amplitude and (or) timing jitter induced by amplified spontaneous emission noise.[1,2] Since amplified spontaneous emission (ASE) noise is a stochastic phenomenon, Monte Carlo simulations can be used to determine its effects on a system. The direct calculation of bit-error rates through standard Monte Carlo simulations is impossible, however, because error rates are required to be very small, e.g., one error per 1×10^9 bits or lower. As a result, one would need an exceedingly large number of realizations to observe even a single transmission error, and one would require even more to obtain reliable error estimates. To overcome this limitation, a common approximation is to numerically calculate the relevant variances and then extrapolate the results into the tails by assuming a Gaussian probability distribution function (pdf). It is clear, however, that this procedure is inadequate, since nonlinearity[3–5] and pulse interactions both contribute to make the resulting distributions non-Gaussian.[5–8]

Recently a technique known as importance sampling[9] (IS) was successfully applied to the direct simulation of transmission effects caused by polarization-mode dispersion.[10] In general, IS works by concentrating Monte Carlo trials on those configurations that are most likely to lead to transmission errors, thus significantly speeding up the simulations. Here we show how IS can be applied to numerical simulations of ASE-induced transmission impairments. For simplicity, we consider the case of a simple soliton-based transmission system (where, in the absence of noise, the pulse shape remains fixed), but it is anticipated that the technique can be extended to more-realistic systems and more-general transmission formats. The advantages of the method are substantial, allowing an increase in speed of several orders of magnitude over standard Monte Carlo simulations.

After we average out the periodic power variations due to fiber loss and amplification, the slowly varying envelope of an optical pulse propagates between amplifiers according to the nonlinear Schrödinger equation[3] (NLS)

$$i\,\frac{\partial u}{\partial z} + \frac{1}{2}\,\frac{\partial^2 u}{\partial t^2} + |u|^2 u = 0\,, \qquad (1)$$

where $z = \hat{z}/L$ and $t = \hat{t}_{\mathrm{ret}}/T_0$ are dimensionless distance and time, $\hat{z}$ and $\hat{t}_{\mathrm{ret}}$ are dimensional distance and retarded time, $T_0 = T_{\mathrm{FWHM}}/1.76$, and $L = T^2/|k''|$. Here T_{FWHM} is the pulse full width at half-maximum and k'' is the fiber dispersion coefficient [$\mathrm{ps}^2/\mathrm{km}$]. Also, $|u|^2 = L\gamma|q|^2$, where $|q|^2$ is the electric field intensity in watts and γ is the fiber nonlinear coefficient [$\mathrm{km}^{-1}\ \mathrm{W}^{-1}$]. Equation (1) admits of the well-known soliton solution

$$u_s(z,t) = A\,\mathrm{sech}[A(t - T - \Omega z)] \times \exp i[\Omega t + (A^2 - \Omega^2)z/2 + \Phi]\,, \qquad (2)$$

where the soliton parameters A, Ω, T, and Φ are constant. When the pulse reaches an amplifier at $z = nz_a$ (where z_a is the dimensionless amplifier spacing and $n = 1, 2, \ldots, N$, with N being the total number of amplifiers in the transmission line), a small amount of noise $\Delta u_n(t)$ is added to u: $u(n{z_a}^+, t) = u(n{z_a}^-, t) + \Delta u_n(t)$. (More realistically, fiber loss would also be compensated for by gain. Here we concentrate only on the effect of the noise.) Part of the noise is incorporated into the soliton, where it produces small changes of the soliton parameters.[3,4] The rest of the noise propagates along with the perturbed soliton. This process is repeated at each amplifier, resulting in a random walk of A, Ω, T, and Φ.[2]

The ASE noise term $\Delta u_n(t)$ can be modeled as classical zero-mean white noise: $\langle \Delta u_n(t) \rangle = 0$ and $\langle \Delta u_m(t) \Delta {u_n}^+(t') \rangle = \sigma^2 \delta_{m,n} \delta(t - t')$, where $\sigma^2 = [(G-1)^2/G \ln G](\gamma \hbar \omega_0 T/|k''|)\eta_{\mathrm{sp}}$. Here G is the amplifier gain, $\hbar$ is Planck's constant, ω_0 is the electric field's carrier frequency (for a wavelength of 1.55 μm, $\omega_0 = 1.2 \times 10^3\ \mathrm{ps}^{-1}$), and η_{sp} is the excess spontaneous-emission factor.[4] For typical system configurations, the noise amplitude at each amplifier is small, and thus the noise-induced changes of the soliton parameters at each individual amplifier are also usually small. In rare cases, however, these individual noise contributions combine to produce large deviations, thus resulting in potential transmission errors. It is precisely because these errors are rare that estimating their probability is difficult.

The main idea behind IS is to bias the pdfs used to generate the random Monte Carlo trials so that errors occur more frequently than would be the case otherwise.[9] It is well known, however, that it is not possible to increase the probability of generating errors merely by increasing the overall noise variance[9] in a high-dimensional system. The crucial step in applying IS is therefore to determine how to bias the distributions in the ways that are most likely to generate errors. In this case, this means determining the particular noise instantiations at each amplifier that produce large pulse amplitude or position variations at the fiber output. Fortunately, much is known about the dependence of a NLS soliton on its parameters A, Ω, T, and Φ, and this knowledge can be used to guide the biasing process. Since any value of these parameters provides an exact soliton of the NLS equation, no resistance is encountered if the noise at an amplifier changes one of them. This lack of resistance allows large variations to build up after many amplifiers. Furthermore, frequency fluctuations change the group velocity of the pulse, and subsequent propagation integrates this velocity shift to produce a large position shift. The goal is to use this knowledge to selectively bias the noise at each amplifier to induce larger-than-normal changes of the soliton amplitude and position at the fiber output.

To make these ideas definite, suppose that we are numerically solving a discretized version of NLS equation (1), using a split-step or other technique. At each amplifier, we add random noise Δu_n. If J is the total number of discrete time points in the computational domain (or, equivalently, the total number of Fourier modes), we can represent the noise by a vector $\mathbf{x}_n = (x_1, \ldots, x_{2J})^T$ giving the real and imaginary noise components at each discretized time point. In the unbiased case, the x_j are independent, identically distributed normal random variables with mean zero and variance $\sigma_a{}^2 = \sigma^2/(2\Delta t)$; explicitly, $p_{\mathbf{x}}(\mathbf{x}) = \exp(-\mathbf{x}^T\mathbf{x}/2\sigma_a{}^2)/(2\pi\sigma_a{}^2)^J$. Let $X = (\mathbf{x}_1; \ldots; \mathbf{x}_N)$ be the $2J \times N$ matrix that denotes all the noise components at all the amplifiers and suppose that we are interested in calculating the probability P that a variable $y(X)$ (e.g., amplitude A or position T) falls in a given range. Using IS, we can write an importance-sampled Monte Carlo estimate for P as[9]

$$P = \frac{1}{M} \sum_{m=1}^{M} I[y_m(X_m{}^*)]r(X_m{}^*)\,, \tag{3}$$

where M is the total number of samples, $X_m{}^*$ is the mth sample, and $I(y)$ is an indicator function that is equal to 1 if y falls in the prescribed range and is 0 otherwise. The quantity $r(X)$ is the IS likelihood ratio,[9] $r(X) = p_X(X)/p_X{}^*(X)$, where $p_X(X)$ is the unbiased pdf and $p_X{}^*(X)$ is the biased pdf that is actually used to draw the samples $X_m{}^*$. [If no biasing is used, $p_X{}^* \equiv p_X(X)$, $r(X) \equiv 1$, and Eq. (3) becomes a standard Monte Carlo estimator.] In our case, $r(X) = \prod_{n=1}^{N} r_x(\mathbf{x}_n)$, where $r_x(\mathbf{x}) = p_{\mathbf{x}}(\mathbf{x})/p_{\mathbf{x}}{}^*(\mathbf{x})$; i.e., $r(X)$ is the product of the individual likelihood ratios at each amplifier.

To bias the simulations, we need to determine the noise realizations that are most likely to result in variations of the soliton's final amplitude and position. To this end, we first decompose $u(z,t)$ just before each amplifier into its soliton and radiative components, $u(nz_a{}^-, t) = u_s(nz_a, t) + \delta u_n(nz_a, t)$. We determine the soliton component either by solving the Zakharov–Shabat eigenvalue problem[3,11] or by using the moment integrals for the soliton parameters A, Ω, T, and Φ.[4] Next, we note that over each span of fiber the radiation component plus the noise added by the amplifier will evolve according to the linearized NLS equation

$$w_z = \mathcal{L}\, w := (i/2)w_{tt} + 2i|u_s|^2 w + iu_s{}^2 w^*, \tag{4}$$

where $w(nz_a{}^+, t) = \delta u_n(nz_a, t) + \Delta u_n(t)$. The noise $\Delta u_n(t)$ produces changes both in the soliton and in the radiation component, but these can be separated by using the eigenfunctions of the adjoint $\mathcal{L}^+$ of the linearized NLS operator. These adjoint eigenfunctions are known analytically, of course, once the soliton parameters are determined. There are four adjoint eigenfunctions $u_p(z,t)$ $(P = A, \Omega, T, \Phi)$ associated with the changes in the four soliton parameters[3,4]; specifically, $\Delta P_n = \mathrm{Re} \int \underline{u}_p{}^*(z,t)\Delta u_n(t)\,\mathrm{d}t$. By definition, the radiation component is orthogonal to these adjoint modes.

To obtain the optimal biasing direction and amount, we examine the unbiased case and find the particular additive noise realizations at an amplifier most likely to produce a desired change, e.g., an amplitude shift ΔA. Since the noise pdf is Gaussian, maximizing the probability is the same as minimizing $\mathbf{x}^T\mathbf{x}$, which in a continuum approximation is proportional to $\int |\Delta u_n|^2\,\mathrm{d}t$. We therefore minimize $\int |\Delta u_n|^2\,\mathrm{d}t$ under the constraint $\mathrm{Re} \int \underline{u}_A{}^* \Delta u_n\,\mathrm{d}t = \Delta A$. This is a simple Lagrange multiplier problem with the solution $\Delta u_n = (\Delta A / \int |u_A|^2\,\mathrm{d}t) u_A$. The choice of ΔA at each amplifier involves a similar maximization of probability under the constraint that the values of ΔA_n sum to a desired final value ΔA_{tot}. For moderate values of ΔA_{tot}, the result of this calculation is well approximated by $\Delta A_n = \Delta A_{\mathrm{tot}}/N$. The procedure for biasing to produce position shifts in the soliton is the same in principle but somewhat more complicated in detail, requiring that a combination of the adjoint frequency and timing modes be used, since changes in the soliton's frequency affect its final position.

In the discretized version of the problem, each of the deterministic biasing directions (e.g., for amplitude or position) will be a vector $\mathbf{b}_n$. Once this biasing direction has been chosen, the actual biased Monte Carlo simulations are straightforward: An unbiased noise realization $\mathbf{x}_n$ is generated, and the biased noise realization $\mathbf{x}_n{}^*$ is obtained simply by addition of $\mathbf{b}_n$ to $\mathbf{x}_n$. Using $\mathbf{x}_n{}^* = \mathbf{x}_n + \mathbf{b}_n$, this then yields a likelihood ratio of $\mathbf{r}_{\mathbf{x}}(\mathbf{x}_n{}^*) = p_{\mathbf{x}}(\mathbf{x}_n + \mathbf{b}_n).p_{\mathbf{x}}(\mathbf{x}_n)$.

We performed importance-sampled Monte Carlo simulations to calculate the pdf of amplitude and timing jitter in a soliton-based transmission system. In the simulations we used a pulse FWHM of 17.6 ps, an amplifier spacing of 50 km, a fiber loss of 0.2 dB/km, a spontaneous-emission factor η_{sp} of 1.5, a fiber dispersion $k'' = -0.2\ \mathrm{ps}^2/\mathrm{km}$, and a total transmission

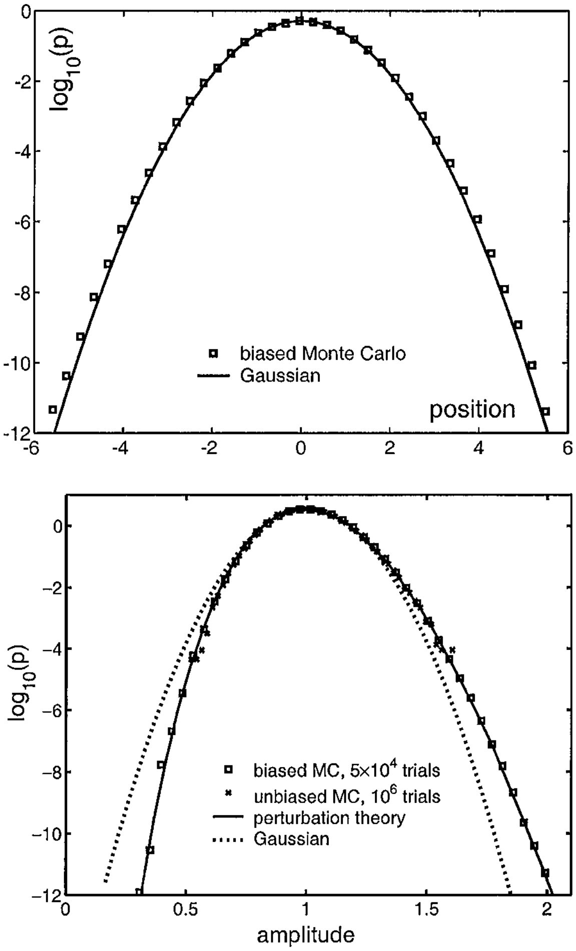

Fig. 1. pdf of (a) timing and (b) amplitude jitter in a soliton-based transmission system, reconstructed from 50,000 importance-sampled Monte-Carlo simulations. A simple model from soliton perturbation theory provides the expected result in both cases. We provide a considerably larger unbiased Monte Carlo simulation and a Gaussian fit in the case of amplitude jitter to demonstrate the efficiency and accuracy of the biased simulation.

distance of 10,000 km. In dimensionless units, these produce an amplifier spacing of 0.1, a power gain $G = 10$, and a total propagation distance of 20. The nonlinear coefficient of the fiber was taken to be 2.0 $\mathrm{km}^{-1}\ \mathrm{W}^{-1}$, which yields a dimensionless noise variance of $\sigma^2 = 6.9 \times 10^{-5}$.

Figure 1 shows the results of 50,000 importance-sampled Monte-Carlo simulations, subdivided into five biasing targets with 10,000 samples per target. Different choices of biasing generate different regions of the pdfs, and the results from each Monte Carlo realization are combined by use of a weighting scheme known as the balance heuristic.[12] Even with a relatively small number of Monte Carlo trials, the method produces values of amplitude and timing jitter that are far down into the tails of the probability distributions. These results agree with unbiased Monte Carlo simulations in the main portion of the pdf (the only region that can be reconstructed with unbiased simulations). The distribution of timing jitter also agrees with a Gaussian curve whose variance is determined by the noise amplitude and the norm of the linear modes. For the amplitude jitter, however, the deviation from Gaussian is evident. A simple model of amplitude fluctuations can be obtained via soliton perturbation theory[4]: $A_{n+1} = A_n + \sqrt{A_n}\, s_n$, where the s_n are independent, identically distributed normal random variables with mean zero and variance σ^2. For comparison, in Fig. 1 we show the results of importance-sampled numerical simulations of this reduced model: The agreement with the importance-sampled simulations of full NLS equation (1) is excellent throughout the range of amplitude values considered.

In conclusion, we have discussed the application of importance sampling to numerical simulations of rare events caused by amplifier noise, and we have demonstrated the method by calculating the pdf of amplitude and timing jitter in a soliton-based system. These results show that IS can be an effective tool in assessing the effect of amplifier noise in light-wave systems. Furthermore, the method assumes only small parameter changes at each amplifier and is therefore well suited to calculating the probability of large accumulated excursions of the soliton parameters.

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