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Two-dimensional reductions of the Whitham modulation system for the Kadomtsev–Petviashvili equation

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Abstract

Two-dimensional reductions of the Kadomtsev–Petviashvili(KP)–Whitham system, namely the overdetermined Whitham modulation system for five dependent variables that describe the periodic solutions of the KP equation, are studied and characterized. Three different reductions are considered corresponding to modulations that are independent of x, independent of y, and of t (i.e. stationary), respectively. Each of these reductions still describes dynamic, two-dimensional spatial configurations since the modulated cnoidal wave, generically, has a nonzero speed and a nonzero slope in the xy plane. In all three of these reductions, the integrability of the resulting systems of equations is proven, and various other properties are elucidated. Compatibility with conservation of waves yields a reduction in the number of dependent variables to two, three and four, respectively. As a byproduct of the stationary case, the Whitham modulation system for the classical Boussinesq equation is explicitly obtained.

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1. Introduction and background

The description of dispersive wave propagation has been a classical topic of study dating back to the works of Boussinesq, Stokes, Rayleigh, Korteweg and de Vries and others in the nineteenth century, and it continues to attract significant attention. A scenario of both theoretical and applicative interest is that in which dispersive effects are much smaller than nonlinear ones, a regime that often leads to the generation of dispersive shock waves. Indeed, a large number of works have been devoted to this subject (e.g. see [18] and references therein). The mathematical framework for the description of small dispersion problems in one spatial dimension and the formation of dispersive shock waves in that context have been well characterized, beginning with the seminal work of Whitham [41]. However, our understanding of dispersive wave propagation and dispersive shock waves in more than one spatial dimension is much less developed.

The purpose of this work is to study special approximate solutions of the Kadomtsev–Petviashvili (KP) equation [25],

$$\left(u_t + 6uu_x + \varepsilon^2 u_{xxx}\right)_x + \sigma u_{yy} = 0\,, \tag{1.1}$$

where $0 < \varepsilon \ll 1$, subscripts x, y and t denote partial differentiation and the values $\sigma = \mp 1$ distinguish between the KPI and KPII variants of the KP equation, respectively. The KP equation, which is a two-dimensional generalization of the celebrated Korteweg–de Vries (KdV) equation, similarly arises in such diverse fields as plasma physics [24, 25, 28], fluid dynamics [6, 26], nonlinear optics [7, 31], Bose–Einstein condensates [23, 38] and ferromagnetic media [39]. The KP equation is also, like the KdV equation, a completely integrable infinite-dimensional Hamiltonian system whose solutions possess a rich mathematical structure [5, 8, 22, 24, 26, 27, 29]. The initial-value problem for the KP equation is in principle amenable to exact solution via the inverse scattering transform (IST) [5, 27, 29]. Yet, even though considerable work has been devoted to the development of the IST for the KP equation throughout the last twenty years [12–15, 42], the IST has rarely been used to study the dynamical behavior of solutions of the KP equation [43]. Conversely, asymptotic methods such as Whitham modulation theory have recently been shown to be quite effective in this regard [4, 11, 32, 34].

In this work, we derive and characterize several asymptotic reductions of the KP equation, which we rewrite in evolution form as

$$u_t + 6uu_x + \varepsilon^2 u_{xxx} + \sigma v_y = 0\,, \qquad v_x = u_y\,. \tag{1.2}$$

The linear dispersion relation of (1.2), obtained by looking for small-amplitude plane-wave solutions $u(x,y,t) = u_o + A\,\mathrm{e}^{i\theta(x,y,t)}$ with $|A| \ll |u_o|$, $\theta(x,y,t) = (kx + ly - \omega t)/\varepsilon$, is $\omega = (6u_o + \sigma q^2)k - k^3$, with $q = l/k$. In addition, the KP equation admits nonlinear, exact traveling wave solutions in the form of 'cnoidal waves'

$$u\,(x,y,t) = r_1 - r_2 + r_3 + 2\,(r_2 - r_1)\,\mathrm{cn}^2\,(2\theta K_m; m)\,, \qquad v\,(x,y,t) = qu + p\,, \tag{1.3}$$

where $\mathrm{cn}(\cdot)$ denotes the Jacobian elliptic cosine [30], $\theta(x,y,t)$ is as above, $K_m = K(m)$ and $E_m = E(m)$ are the complete elliptic integrals of the first and second kind, respectively, and

$$m = \frac{r_2 - r_1}{r_3 - r_1} \tag{1.4}$$

is the elliptic parameter. The above solution is completely determined by five parameters: r_1, r_2, r_3, q and p. The local wavenumber k and l in the x and y directions and the frequency ω are then obtained as

$$k = \sqrt{r_3 - r_1}\,/2K_m\,, \tag{1.5a}$$
$$l = qk\,, \tag{1.5b}$$
$$\omega = \left(V + \sigma q^2\right) k\,, \tag{1.5c}$$

with parameter

$$V = \frac{\omega}{k} - \sigma q^2 = 2\left(r_1 + r_2 + r_3\right). \tag{1.6}$$

In [4], the method of multiple scales was used to derive the so-called KP–Whitham system (KPWS), i.e. a system of quasilinear first-order partial differential equations (PDEs) that describes the slow modulation of the above periodic solutions of the KP equation. One begins by seeking a solution of (1.2) in the form $u = u(\theta, x, y, t)$, with rapidly varying variable $\theta(x, y, t)$ defined through its derivatives:

$$\theta_x = k\,(x,y,t)/\varepsilon, \qquad \theta_y = l\,(x,y,t)/\varepsilon, \qquad \theta_t = -\omega\,(x,y,t)/\varepsilon\,. \tag{1.7}$$

Here, $k(x,y,t)$ and $l(x,y,t)$ are the local wave numbers in the x and y directions, respectively, and $\omega(x,y,t)$ is the wave's local frequency. Imposing the equality of the mixed second derivatives of θ results in the compatibility conditions

$$k_t + \omega_x = 0\,, \tag{1.8a}$$
$$l_t + \omega_y = 0\,, \tag{1.8b}$$
$$k_y - l_x = 0\,, \tag{1.8c}$$

called the 'conservation of waves' equations. One also introduces the dependent variable

$$q\,(x,y,t) = \frac{l}{k} \tag{1.9}$$

consistent with the above periodic solutions, along with the slowly varying independent variables x, y and t. It was then shown in [4] that to leading order one recovers the solution (1.3). When the parameters of the above periodic solution are slowly modulated with respect to x, y or t, they satisfy a system of Whitham modulation equations. When writing down these equations, it is convenient to define the 'convective derivative'

$$\frac{D}{Dy} = \frac{\partial}{\partial y} - q\frac{\partial}{\partial x}. \tag{1.10}$$

In component form, the KPWS is then comprised of the following PDEs

$$\frac{\partial r_i}{\partial t} + \left(V_j + \sigma q^2\right)\frac{\partial r_j}{\partial x} + 2\sigma q\frac{Dr_j}{Dy} + \sigma\nu_j\frac{Dq}{Dy} + \sigma\frac{Dp}{Dy} = 0\,, \qquad j = 1,2,3, \tag{1.11a}$$

$$\frac{\partial q}{\partial t} + \left(V_2 + \sigma q^2\right)\frac{\partial q}{\partial x} + 2\sigma q\frac{Dq}{Dy} + (4-\nu_4)\frac{Dr_1}{Dy} + (2+\nu_4)\frac{Dr_3}{Dy} = 0\,, \tag{1.11b}$$

$$\frac{\partial p}{\partial x} - (1-\alpha)\frac{Dr_1}{Dy} - \alpha\frac{Dr_3}{Dy} + \nu_5\frac{\partial q}{\partial x} = 0\,, \tag{1.11c}$$

$$b_1\frac{Dr_1}{Dy} + b_2\frac{Dr_2}{Dy} + b_3\frac{Dr_3}{Dy} + b_4\frac{\partial q}{\partial x} = 0\,. \tag{1.11d}$$

[Note that (1.8*a*) is a consequence of the three equations (1.11*a*), while (1.8*b*) and (1.8*c*) are equivalent to (1.11*b*) and (1.11*d*), respectively.] Here, $V_1,\dots,V_3$ are the characteristic speeds of the Whitham system for the KdV equation, namely

$$V_1 = V - 2b\frac{K_m}{K_m - E_m}\,, V_2 = V - 2b\frac{(1-m)\,K_m}{E_m - (1-m)\,K_m}\,, V_3 = V + 2b\frac{(1-m)\,K_m}{mE_m}\,, \tag{1.12}$$

and $b = 2(r_2 - r_1)$ is the amplitude of the cnoidal wave solution (1.3), while the remaining coefficients are

$$\nu_1 = \frac{V}{6} + \frac{b}{3m}\frac{(1+m)\,E_m - K_m}{K_m - E_m}\,, \quad \nu_2 = \frac{V}{6} + \frac{b}{3m}\frac{(1-m)^2\,K_m - (1-2m)\,E_m}{E_m - (1-m)\,K_m}\,, \tag{1.13a}$$

$$\nu_3 = \frac{V}{6} + \frac{b}{3m}\frac{(2-m)\,E_m - (1-m)\,K_m}{E_m}\,, \quad \nu_4 = \frac{2mE_m}{E_m - (1-m)\,K_m}\,, \tag{1.13b}$$

$$\nu_5 = r_1 - r_2 + r_3\,, \quad \alpha = \frac{E_m}{K_m}\,, \quad b_1 = (1-m)\,(K_m - E_m)\,, \tag{1.13c}$$

$$b_2 = E_m - (1-m)\,K_m, \quad b_3 = -mE_m, \quad b_4 = 2\,(r_2 - r_1)\,(1-m)\,K_m. \tag{1.13d}$$

Importantly, the modulation system (1.11) contains six PDEs for the five dependent variables r_1, r_2, r_3, q and p, and is therefore overdetermined in general. In [4], the initial value problem for the system (1.11) was shown to be compatible provided that (1.11*c*) and (1.11*d*) hold at $t = 0$, in which case it was shown that (1.11*c*) and (1.11*d*) remain satisfied for all $t > 0$. Consequently, in [4] a reduced system consisting of the five PDEs (1.11*a*)–(1.11*c*) was considered, which is a minimal set of equations for the five dependent variables $\mathbf{r} = (r_1, r_2, r_3, q, p)^T$ that can be written as

$$I_4\,\frac{\partial \mathbf{r}}{\partial t} + A_5\,\frac{\partial \mathbf{r}}{\partial x} + B_5\,\frac{\partial \mathbf{r}}{\partial y} = 0\,, \tag{1.14}$$

where $I_4 = \mathrm{diag}(1,1,1,1,0)$ and A_5 and B_5 are 5×5 matrices whose explicit form is given in (A.6). In [4] and [11] the term 'KPWS' was used to refer to the five equations (1.14). However, in this work we will show that, in order for the modulation system to inherit the integrability properties of the KP equation, it is crucial to consider all six equations (1.11) on an equal footing. Accordingly, we will henceforth refer to the five-component system (1.14) as the 'partial' KPWS, and we will refer to the six equations (1.11) as the 'full' KPWS.

Generally, all the dependent variables in (1.11) depend on two spatial dimensions (x and y) and one temporal dimension (t), so we refer to the KPWS (1.11) as $(2+1)$-dimensional, or equivalently 3-dimensional. Various asymptotic reductions of the system (1.11) and their properties were studied in [11], and the soliton limit of (1.11) was used in [32–34] to study various concrete physical problems. However, a number of important questions remain open. Among them is the issue of integrability. It is generally believed that asymptotic expansions of integrable systems preserve integrability at any order, and since the KPWS arises as the leading order of such an expansion for the KP equation (1.1), which is integrable, one would naturally expect that the modulation system is also integrable. The Haantjes tensor test is a necessary condition for the integrability of a system of strictly hyperbolic hydrodynamic type equations [20] (see also the appendix). On the other hand, as was already mentioned in [4], the partial KPWS (1.14) fails the Haantjes tensor test. The reason why this is the case is that,

in order for (1.14) to be compatible with the KP equation (1.2), the initial conditions for k and q must be related by the constraint (1.8c), i.e. $k_y = l_x$, or, equivalently, (1.11d). When they are not, solutions of the partial KPWS do not describe an asymptotic expansion of solutions of the KP equation, which explains why (1.14) alone is not integrable despite the integrability of the KP equation. Thus, one is faced with the conundrum that the partial KPWS (1.14) is compatible but not integrable, while the full KPWS (1.11) is overdetermined and therefore not compatible in general.

At the same time, it was shown in [11] that the harmonic and soliton limits of the system (1.11) are in fact integrable. An obvious question is then whether there are other integrable reductions of (1.14) and compatible reductions of (1.11), and if so how one can identify them. In this work, we begin to address this question by studying and characterizing the two-dimensional ($1+1$ and $2+0$) reductions of the full KPWS (1.11).

Specifically, in the following sections we prove the following:

Theorem. *If the dependent variable* $\mathbf{r}(x,y,t) = (r_1, r_2, r_3, q, p)^T$ *of the full KPWS* (1.11) *is independent of any one of the independent variables, the system admits an integral of motion, and* $\mathbf{r}$ *satisfies a reduced modulation system that is both compatible and integrable. In particular:*

- *If* $\partial\mathbf{r}/\partial x \equiv 0$*, the wavenumber* k *is constant and* $\mathbf{r}$ *satisfies* (2.2)*,* (2.5)*,* (2.16)*, and* (2.19)*.*
- *If* $\partial\mathbf{r}/\partial y \equiv 0$*, the wavenumber* l *is constant and* $\mathbf{r}$ *satisfies* (3.2) *and* (3.14)*.*
- *If* $\partial\mathbf{r}/\partial t \equiv 0$*, the frequency* ω *is constant and* $\mathbf{r}$ *satisfies* (4.7)*.*

The above theorem is proved in sections 2–4. Specifically, in section 2 we consider the situation in which all fields are independent of x, and in section 3 the situation in which all fields are independent of y. In section 4 we study the situation in which all fields are stationary, i.e. independent of t. In the course of proving the theorem, we will also show that the corresponding reductions of the partial KPWS (1.14) are integrable if and only if the third conservation of waves equation is added to them.

Several further remarks on integrability are worth mentioning. Whitham modulation equations are within the class of quasi-linear, first-order PDE known as equations of hydrodynamic type [17]. In [36, 37], Tsarev obtained necessary and sufficient conditions for the integrability of (1+1)-dimensional hydrodynamic type equations. Equations that are diagonalizable in Riemann invariants and satisfy the so-called semi-Hamiltonian property exhibit an infinite number of conservation laws and are locally solved using a generalization of the hodograph method. But obtaining Riemann invariants is, in general, an unsolved problem for systems of more than two equations. It turns out that, in the case of strictly hyperbolic, conservative hydrodynamic type equations in (1+1) or (2+0)-dimensions, the vanishing of the Haantjes tensor is not only necessary but also sufficient to prove diagonalizability and the semi-Hamiltonian property [19]. Our proof of the theorem rests on this fact so it is not necessary to diagonalize the modulation systems.

As noted earlier, the KPWS (1.11) was derived using the multiple scales method in [4]. In addition to the conservation of waves equations (1.8), three solvability conditions determine the full KPWS. An alternative method to derive the Whitham modulation equations is the method of averaged conservation laws, Whitham's original approach to modulation theory [40]. In the appendix, we average the KP conservation laws associated with mass, momentum (in the x direction), and irrotationality, then show that they are equivalent to the three solvability conditions obtained in [4]. Consequently, the KPWS and its dimensional reductions are conservative so that we can use the Haantjes tensor test to prove their integrability.

We emphasize that, as in [32–34], even when the solution of the KPWS is independent of one independent variable, the reduced systems of equations still generically describe two-dimensional, dynamical configurations of the KP equation, because a nonzero value of ω in (1.5*c*) implies propagation of the cnoidal wave, and q describes the orientation of the periodic wave in the xy plane. Variations of q with respect to x or y correspond to curved wave profiles.

To avoid possible confusion, we should note that in this work we are using the normalization of [4], not that of [11, 32–34]. In the latter works, the coefficient 6 in front of the term uu_x in (1.2) was absent and the cnoidal wave's period was normalized to 2π, whereas here it is normalized to unity. As a result, several formulas are adjusted accordingly.

2. The YT system

In this section we consider solutions of the KPWS in which all fields are independent of x. When solutions are independent of x, the partial KPWS (1.14) reduces to a system of five PDEs in the independent variables y and t, which we refer to as the 'YT system'. In vector form, this YT system is

$$I_4 \frac{\partial \mathbf{r}}{\partial t} + B_5 \frac{\partial \mathbf{r}}{\partial y} = 0\,, \tag{2.1}$$

where $\mathbf{r} = (r_1, r_2, r_3, q, p)^T$ and $I_4 = \mathrm{diag}(1,1,1,1,0)$ as before, and the coefficient matrix B_5 is given in (A.6*b*). The corresponding reduction of the full KPWS (1.11) is simply given by (2.1) augmented with (1.11*d*) (which again expresses the compatibility condition $k_y = l_x$). Below we show that, on the one hand, (2.1) is not integrable by itself. On the other hand, upon enforcing the compatibility condition $k_y = l_x$ required by the full KPWS (1.11), the x-independent reduction of the KPWS is both compatible and integrable.

2.1. Reduction of the YT system to a three-component system

The system (2.1) can be reduced through a suitable change of variables. Explicitly, the last row of (2.1) is

$$\frac{\partial r_1}{\partial y} + \alpha \frac{\partial s_2}{\partial y} = 0\,, \tag{2.2}$$

where we use the transformation

$$s_1 = r_3 - r_2\,, \quad s_2 = r_3 - r_1\,, \quad s_3 = r_2 - r_1\,. \tag{2.3}$$

We make the choice not to define these variables in cyclic fashion in order to preserve the property that $s_j \geqslant 0\ \forall j = 1,2,3$ when the Riemann-type variables $r_1, \ldots, r_3$ are well-ordered. The transformation (2.3) leads to the set of four equations

$$\frac{\partial s_j}{\partial t} + 2\sigma q \frac{\partial s_j}{\partial y} + \sigma \Delta\nu_j \frac{\partial q}{\partial y} = 0\,, \quad j = 1,2,3, \tag{2.4*a*}$$

$$\frac{\partial q}{\partial t} + 6\frac{\partial r_1}{\partial y} + (\nu_4 + 2)\frac{\partial s_2}{\partial y} + 2\sigma q \frac{\partial q}{\partial y} = 0\,, \tag{2.4*b*}$$

with $\Delta\nu_1 = \nu_3 - \nu_2$, $\Delta\nu_2 = \nu_3 - \nu_1$ and $\Delta\nu_3 = \nu_2 - \nu_1$. Note that the variable p does not appear in (2.4) nor in (1.11*d*). Therefore, its value can be determined by integrating (1.11*a*) for one $j \in \{1,2,3\}$, e.g.

$$\sigma\frac{\partial p}{\partial y} = -\frac{\partial r_2}{\partial t} - \left(V_2 + \sigma q^2\right)\frac{\partial r_2}{\partial x} - 2\sigma q\frac{\partial r_2}{\partial y} - \sigma\nu_2\frac{\partial q}{\partial y}. \tag{2.5}$$

Note also that the transformation (2.3) from $r_1,\ldots,r_3$ to $s_1,\ldots,s_3$ is not invertible. However, (2.4*a*) with $j=1$ is decoupled from the rest of the system, since s_1 does not appear in the remaining equations. Thus we can simply disregard it moving forward, since the four dependent variables r_1, s_2, s_3 and q, determined by the PDEs (2.2), plus (2.4*a*) with $j = 2,3$ and (2.4*b*), are a closed system. These dependent variables are sufficient to recover the solution of the KP equation.

Next, one can use (2.2) to eliminate r_1 from (2.4*b*), obtaining the following closed system of three PDEs for the three dependent variables s_2, s_3 and q:

$$\frac{\partial s_2}{\partial t} + 2\sigma q\frac{\partial s_2}{\partial y} + \sigma\left(\nu_3 - \nu_1\right)\frac{\partial q}{\partial y} = 0, \tag{2.6a}$$

$$\frac{\partial s_3}{\partial t} + 2\sigma q\frac{\partial s_3}{\partial y} + \sigma\left(\nu_2 - \nu_1\right)\frac{\partial q}{\partial y} = 0, \tag{2.6b}$$

$$\frac{\partial q}{\partial t} + \left(\nu_4 - 6\alpha + 2\right)\frac{\partial s_2}{\partial y} + 2\sigma q\frac{\partial q}{\partial y} = 0. \tag{2.6c}$$

All the coefficients appearing in (2.6) are completely determined by $s_2, s_3 \& q$, since

$$m = \frac{s_3}{s_2}, \qquad \frac{b}{m} = 2s_2, \tag{2.7}$$

Note that r_1 is also needed to recover the asymptotic solution of the KP equation, but its value, up to an integration constant determined by the initial conditions, can be obtained from s_2, s_3 by integrating (2.2). Introducing the vector $\mathbf{v} = (s_2, s_3, q)^T$, we can write the above system (2.6) in vector form as

$$\frac{\partial \mathbf{v}}{\partial t} + B_3\frac{\partial \mathbf{v}}{\partial y} = 0, \tag{2.8}$$

with

$$B_3 = \begin{pmatrix} 2\sigma q & 0 & \sigma\left(\nu_3 - \nu_1\right) \\ 0 & 2\sigma q & \sigma\left(\nu_2 - \nu_1\right) \\ \nu_4 - 6\alpha + 2 & 0 & 2\sigma q \end{pmatrix}. \tag{2.9}$$

The eigenvalues of B_3 are

$$\lambda_1 = 2\sigma q, \qquad \lambda_{2,3} = 2\sigma q \pm \sqrt{\Delta}, \tag{2.10a}$$

where

$$\Delta = \sigma\left(\nu_1 - \nu_3\right)\left(6\alpha - \nu_4 - 2\right) = 4\sigma s_2\frac{\left(\left(1-m\right)K_m - 2\left(2-m\right)E_mK_m + 3E_m^2\right)^2}{3E_mK_m\left(K_m - E_m\right)\left(E_m - \left(1-m\right)K_m\right)}. \tag{2.10b}$$

By properties of K_m and E_m, $\mathrm{sgn}\Delta = \mathrm{sgn}\sigma$ since $s_2 > 0$. Hence, if $\sigma = -1$, as for KPI, some of the eigenvalues are imaginary, implying that the initial value problem for the above system is ill-posed, confirming known results [4]. Incidentally, note that the PDEs for s_2 and q do not contain s_3 explicitly. However, the value of s_3 is required to determine m.

2.2. Integrability and Riemann invariants of the YT system

The three-component YT system (2.8) fails the Haantjes tensor test for integrability, as not all of the 27 components of the Haantjes tensor vanish in general. As already discussed above, this is because, to preserve integrability, it is necessary for (2.8) to be compatible with (1.11*d*) [i.e. $k_y = l_x$]. Conversely, we now show that, once this constraint is imposed, the corresponding reduction of the partial KPWS (1.14) becomes integrable, and that of the full KPWS (1.11) becomes compatible.

To see this, note that, if all variables are independent of x, the three conservation of waves equation (1.8) immediately yield $k_t = k_y = 0$, i.e. the local wavenumber k in the x direction is constant. At the same time, k is in fact a Riemann invariant of the three-component system (2.8) as we show next. Using the left eigenvector corresponding to the eigenvalue $\lambda = 2\sigma q$ in (2.10), we obtain the characteristic form

$$(\nu_2 - \nu_1)\,\mathrm{d}s_2 - (\nu_3 - \nu_1)\,\mathrm{d}s_3 = 0\,, \tag{2.11}$$

along $\mathrm{d}y/\mathrm{d}t = 2\sigma q$. The above differential form can be integrated by eliminating s_3 in favor of m using (2.7), to obtain

$$(\nu_2 - \nu_1 - m\,(\nu_3 - \nu_1))\,\mathrm{d}s_2 - s_2\,(\nu_3 - \nu_1)\,\mathrm{d}m = 0\,. \tag{2.12}$$

Multiplying by the integrating factor $1/[s_2(\nu_2 - \nu_1 - m(\nu_3 - \nu_1))]$ yields

$$\frac{1}{s_2}\mathrm{d}s_2 + \left(\frac{1}{m} - \frac{E_m}{m\,(1-m)\,K_m}\right)\mathrm{d}m = 0\,, \tag{2.13}$$

and recalling the derivative of K_m [cf (A.9*a*)], we express the above characteristic relation as

$$\mathrm{d}\left[\frac{1}{2}\log s_2 - \log K_m\right] = 0, \tag{2.14}$$

which yields the PDE

$$\frac{\partial k}{\partial t} + 2\sigma q\frac{\partial k}{\partial y} = 0\,, \tag{2.15}$$

demonstrating that k as defined by (1.5*a*) is in fact a Riemann invariant. The fact that k is a Riemann invariant is thus deeply connected with the integrability of the KPWS. On the one hand, enforcing the constancy of k is needed to ensure the compatibility of the full KPWS with the KP equation, as per the above discussion. On the other hand, when k is constant, the system (2.8) reduces to a two-component system. Any two-component system can always be reduced to Riemann invariant form and is therefore always locally integrable via the classical hodograph transform. We will see in sections 3 and 4 that a similar phenomenon also arises for the y-independent and t-independent reductions of the KPWS.

2.3. Further reduction of the YT system and its diagonalization

We now consider in detail the reduction of the YT system obtained when $k = k_0$ is constant. One can choose two different sets of dependent variables: a reduced system for the dependent variables $(s_2, q)^T$ or a reduced system for the dependent variables $(m, q)^T$. There is a one-to-one correspondence between the two sets of variables because their relationship in equation (1.5*a*)

$$\sqrt{s_2} = 2k_0 K_m\,, \tag{2.16}$$

is monotone ($\partial s_2/\partial m > 0$).

We begin by performing a change of variable from $\mathbf{v} = (s_2, s_3, q)^T$ to $\tilde{\mathbf{v}} = (s_2, q, k)^T$. We choose to keep s_2 as opposed to s_3 because s_2 never vanishes, whereas $s_3 \to 0$ in the harmonic limit. Then $\tilde{v}$ satisfies the system $\tilde{\mathbf{v}}_t + \tilde{B}_3 \tilde{\mathbf{v}}_y = 0$, with $\tilde{B}_3 = T^{-1} B_3 T$ and $T = \left(\partial v_i / \partial \tilde{v}_j\right)$. One can verify that the new coefficient matrix is block-diagonal, $\tilde{B}_3 = \mathrm{diag}(B_2, 2\sigma q)$, with the 2×2 matrix B_2 given by

$$B_2 = \begin{pmatrix} 2\sigma q & 2\sigma s_2 \dfrac{(1-m) K_m^2 - 2(2-m) K_m E_m + 3E_m^2}{3E_m (E_m - K_m)} \\ 2 + 2\left(\dfrac{m}{E_m - (1-m) K_m} - \dfrac{3}{K_m}\right) E_m & 2\sigma q \end{pmatrix}. \tag{2.17}$$

Since compatibility of the KPWS requires $k_t = k_y = 0$, we therefore must use the compatible solution $k \equiv k_0$, which simplifies the 3×3 system $\tilde{\mathbf{v}}_t + \tilde{B}_3 \tilde{\mathbf{v}}_y = 0$ to the following 2×2 system for the dependent variable $\mathbf{u} = (s_2, q)^T$:

$$\mathbf{u}_t + B_2 \mathbf{u}_x = 0. \tag{2.18}$$

The coefficient matrix B_2 in (2.17) contains the elliptic parameter m, which is defined implicitly in terms of s_2 and k_0 by the relation (2.16). We can solve for m by inverting K_m via $m = K_m^{-1}(\sqrt{s_2}/(2k_0), m) = 1 - \mathrm{dn}^2(\sqrt{s_2}/(2k_0), m)$, where dn is a Jacobi elliptic function.

In light of (2.16), we see that (2.18) is actually a one-parameter family of hydrodynamic type systems, parametrized by the constant value of k_0. Equivalently, one can use (2.16) to express s_2 as a function of m and k_0. Note however that q does not enter in the relation between s_2 and m. One can therefore replace (2.18) with the equivalent hydrodynamic system $\tilde{\mathbf{v}}_t + \tilde{B}_2 \tilde{\mathbf{v}}_x = 0$ for the modified dependent variable $\tilde{\mathbf{v}} = (m, q)^T$, i.e. in component form

$$\frac{\partial m}{\partial t} + 2\sigma q \frac{\partial m}{\partial y} + \sigma \Phi_1(m) \frac{\partial q}{\partial y} = 0, \tag{2.19a}$$

$$\frac{\partial q}{\partial t} + 2\sigma q \frac{\partial q}{\partial y} + k_0^2 \Phi_2(m) \frac{\partial m}{\partial y} = 0, \tag{2.19b}$$

where

$$\Phi_1(m) = \frac{2m(1-m) K_m \left(3E_m^2 - 2(2-m) E_m K_m + (1-m) K_m^2\right)}{3E_m (E_m - K_m)(E_m - (1-m) K_m)}, \tag{2.20a}$$

$$\Phi_2(m) = \frac{8\left(3E_m^2 - 2(2-m) E_m K_m + (1-m) K_m^2\right)}{m(1-m)}. \tag{2.20b}$$

Note that the constant parameter k_0 in the system (2.19) can be eliminated by the rescaling $q \mapsto k_0 q$ and $y \mapsto k_0 y$ when $k_0 \neq 0$.

Finally, we use the eigenvalues and eigenvectors of B_2 to complete the diagonalization of the YT system. The eigenvalues $\lambda_\pm$, which coincide with $\lambda_{2,3}$ in (2.10), are now expressed as

$$\lambda_\pm = 2\sigma q \pm k_0 \sqrt{\sigma \Phi_1(m) \Phi_2(m)}, \tag{2.21}$$

while the associated left eigenvectors are

$$\mathbf{w}_\pm = \left(\pm \sqrt{3\sigma E_m (K_m - E_m) / \left[s_2 K_m (E_m - (1-m) K_m)\right]}, 1\right), \tag{2.22}$$

leading to the characteristic relations

$$\mathrm{d}q \pm \sqrt{\frac{3\sigma E_m\left(K_m - E_m\right)}{s_2 K_m\left(E_m - (1-m)K_m\right)}}\mathrm{d}s_2 = 0 \tag{2.23a}$$

along the characteristic curves

$$\mathrm{d}y/\mathrm{d}t = 2\sigma q \mp \sqrt{\Delta}\,, \tag{2.23b}$$

with Δ as in (2.10). We now differentiate (2.16) and use the known differential equations for K_m, (A.9*a*), to express $\mathrm{d}s_2 = s_2[(E_m - (1-m)K_m)/(m(1-m)K_m)]\,\mathrm{d}m$, thereby simplifying (2.23*a*) to

$$\mathrm{d}q \pm \sqrt{\sigma}k_0 f(m)\,\mathrm{d}m = 0\,, \tag{2.24a}$$

where

$$f(m) = \frac{2\sqrt{3E_m\left(K_m - E_m\right)\left(E_m - (1-m)K_m\right)}}{m\left(1-m\right)\sqrt{K_m}}\,. \tag{2.24b}$$

Therefore, Riemann invariants for the two-component hydrodynamic system for $\tilde{u}$ are

$$R_\pm = q \pm \sqrt{\sigma}k_0 F(m)\,, \tag{2.25a}$$

with

$$F(m) = \int_0^m f(\mu)\,\mathrm{d}\mu\,. \tag{2.25b}$$

The above expression of the Riemann invariants is the same as those for the p-system modeling isentropic gas dynamics and nonlinear elasticity [35] where $f(m)$ is related to the sound speed of the medium. A plot of $f(m)$ is shown in figure 1. Note that $f(m) > 0$ for all $m \in [0,1)$, and $\lim_{m\to 0^+} f(m) = \pi/2$. However, $f(m) = 2[1 + 2/(\log(1-m) - 4\log 2)]^{1/2}/(1-m) + O(1)$ as $m \to 1^-$. As a result, we also have $F(m) \to +\infty$ and $R_\pm \to \pm\infty$ logarithmically in the limit $m \to 1^-$, implying that two-component system for m and q is singular in this limit. This is expected because $k = k_0 \neq 0$ is incompatible with the soliton limit, for which $k \to 0$ (cf (1.5*a*)). In figure 1 we also plot $F(m)$ as a function of m, which can be used to obtain the relationship between m and q satisfied by simple wave solutions of (2.18) with either R_+ or R_- identically constant.

It is well known that systems of the form (2.19) are integrable by the hodograph method, and the general solution $(R_+(y,t), R_-(y,t))$ is given locally in a neighborhood of points where $\partial R_\pm/\partial y \neq 0$ by

$$y + \lambda_\pm\left(R_+, R_-\right)t + W_\pm\left(R_+, R_-\right) = 0,$$

where $W_\pm(R_+, R_-)$ are solutions of the following system of linear PDEs:

$$\frac{1}{W_+ - W_-}\frac{\partial W_\pm}{\partial R_\mp} = \frac{1}{\lambda_+ - \lambda_-}\frac{\partial \lambda_\pm}{\partial R_\mp}\,.$$

In closing, we return our attention to the eigenvalues $\lambda_\pm$ of the system, given by $\lambda_\pm = 2\sigma q \pm \sqrt{\Delta(m)}$ (cf (2.10)), where in light of (2.16), we can now express Δ as

$$\Delta\left(m\right) = 16\sigma k_0^2 K_m \frac{\left(\left(1-m\right)K_m - 2\left(2-m\right)E_m K_m + 3E_m^2\right)^2}{3E_m\left(K_m - E_m\right)\left(E_m - (1-m)K_m\right)}\,. \tag{2.26}$$

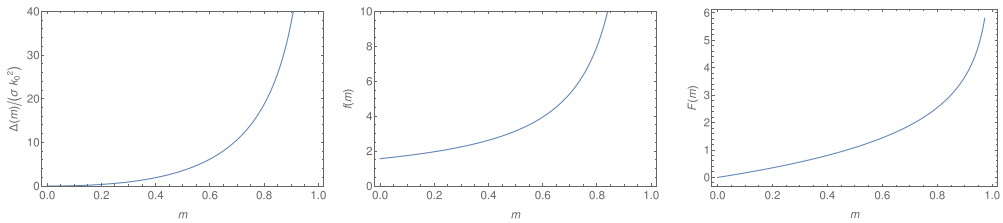

Figure 1. The quantities $\Delta(m)/(\sigma k_0^2)$ (left), $f(m)$ (center) and $F(m)$ (right) as functions of m.

Note that $\Delta(m)/(\sigma k_0^2) \to 0$ as $m \to 0$ and $\Delta(m)/(\sigma k_0^2) \to +\infty$ as $m \to 1$ (cf figure 1). When $m = 0$, $\lambda_\pm = 2\sigma q$ and $R_\pm = q$ so that (2.18) exhibits a one-component reduction in the harmonic limit to the inviscid Burgers equation. By monotonicity of $\Delta(m)$, the reduced 2×2 system is strictly hyperbolic according to the following definition of strict hyperbolicity: $\lambda_+ = \lambda_-$ if and only if $R_+ = R_-$ [18]. It can also be shown that $(\nabla_{m,q}\lambda_\pm) \cdot \mathbf{z}_\pm \neq 0$, where $\mathbf{z}_\pm$ are the right eigenvectors of the matrix for the hydrodynamic-type system (2.19) associated with eigenvalues $\lambda_\pm$, which implies that the system is genuinely nonlinear.

3. The XT system

Next we consider the reduction of the KPWS in which all fields are independent of y. Similarly to section 2, when solutions are independent of y the five-component KPWS (1.14), becomes what we call the 'XT system', which in vector form is

$$I_4 \frac{\partial \mathbf{r}}{\partial t} + A_5 \frac{\partial \mathbf{r}}{\partial x} = 0\,, \tag{3.1}$$

with $\mathbf{r} = (r_1, r_2, r_3, q, p)^T$ as before, and A_5 given in (A.6*a*). The corresponding reduction of the full KPWS (1.11) is given by (3.1) augmented with (1.11*d*). We will obtain analogous results to section 2 even though the analysis of the systems and the physics they describe are significantly different.

3.1. Reduction to a four-component system

In this case, reducing the size of the system is much easier than in section 2, because when all derivatives in y vanish, the last equation in (3.1) (equivalently, equation (1.11*c*)) determines p by direct integration of

$$\frac{\partial p}{\partial x} = -q\,(1-\alpha)\,\frac{\partial r_1}{\partial x} - q\alpha \frac{\partial r_3}{\partial x} - \nu_5 \frac{\partial q}{\partial x}. \tag{3.2}$$

Substituting the resulting expression into the PDEs for r_1, r_2, r_3 leads to the reduced system

$$\frac{\partial r_j}{\partial t} + \left(V_j - \sigma q^2\right) \frac{\partial r_j}{\partial x} + \sigma q^2\,(1-\alpha)\,\frac{\partial r_1}{\partial x} + \sigma q^2 \alpha \frac{\partial r_3}{\partial x} + \sigma q\,(\nu_5 - \nu_j)\,\frac{\partial q}{\partial x} = 0\,, \quad j = 1,2,3\,, \tag{3.3*a*}$$

$$\frac{\partial q}{\partial t} - q\,(4-\nu_4)\,\frac{\partial r_1}{\partial x} + q\,(2+\nu_4)\,\frac{\partial r_3}{\partial x} + \left(V_2 - \sigma q^2\right) \frac{\partial q}{\partial x} = 0\,, \tag{3.3*b*}$$

or equivalently, in vector form,

$$\frac{\partial \mathbf{r}_4}{\partial t} + A_4 \frac{\partial \mathbf{r}_4}{\partial x} = 0\,, \tag{3.4}$$

where now $\mathbf{r}_4 = (r_1, r_2, r_3, q)^T$ and

$$A_4 = \begin{pmatrix} V_1 - \sigma q^2 \alpha & 0 & \sigma q^2 \alpha & \sigma q\,(\nu_5 - \nu_1) \\ \sigma q^2\,(1-\alpha) & V_2 - \sigma q^2 & \sigma q^2 \alpha & \sigma q\,(\nu_5 - \nu_2) \\ \sigma q^2\,(1-\alpha) & 0 & V_3 + \sigma q^2\,(\alpha - 1) & \sigma q\,(\nu_5 - \nu_3) \\ -q\,(4 - \nu_4) & 0 & -q\,(2 + \nu_4) & V_2 - \sigma q^2 \end{pmatrix}. \tag{3.5}$$

3.2. Riemann invariant, integrability, further reduction and diagonalization of the XT system

The matrix A_4 has the eigenvalue

$$\lambda_o = V_2 - \sigma q^2\,, \tag{3.6}$$

with associated left eigenvector

$$\mathbf{w}_o = \left(q\,(1-m)\,(K_m - E_m)\,,\, q\,(E_m - (1-m)\,K_m)\,,\, qmE_m\,,\, -2m\,(1-m)\,(r_3 - r_1)\,K_m\right). \tag{3.7}$$

These expressions allow us to find a Riemann invariant and, in turn, to partially diagonalize the XT system (3.4). In this case, however, the calculations are more complicated than those of section 2. We begin by applying $\mathbf{w}_o^T$ to (3.4), obtaining the characteristic relation

$$\frac{1}{(r_3 - r_1)\,K_m}\left(\frac{K_m - E_m}{m}\mathrm{d}r_1 + \frac{E_m - (1-m)\,K_m}{(1-m)\,m}\mathrm{d}r_2 - \frac{E_m}{1-m}\mathrm{d}r_3\right) - 2\frac{\mathrm{d}q}{q} = 0 \tag{3.8}$$

along $\mathrm{d}x/\mathrm{d}t = V_2 - \sigma q^2$. To integrate this differential form and find the Riemann invariant, we first eliminate r_2 in favor of m using (1.4), implying $\mathrm{d}r_2 = (1-m)\,\mathrm{d}r_1 + m\,\mathrm{d}r_3 + (r_3 - r_1)\,\mathrm{d}m$, which yields

$$-\frac{\mathrm{d}\,(r_3 - r_1)}{r_3 - r_1} + \frac{E_m - (1-m)\,K_m}{m\,(1-m)\,K_m}\,\mathrm{d}m - 2\frac{\mathrm{d}q}{q} = 0\,. \tag{3.9}$$

Using equation (A.9*a*), this expression is the differential $-\mathrm{d}\log((r_3 - r_1)q^2/K_m^2) = 0$, which, upon substituting (1.9) and (1.5*a*), is equivalent to $\mathrm{d}l = 0$. The Riemann invariant R_o in this case is nothing other than the wavenumber in the y direction, $R_o = l$, which satisfies the PDE

$$\frac{\partial R_o}{\partial t} + \left(V_2 - \sigma q^2\right)\frac{\partial R_o}{\partial x} = 0\,. \tag{3.10}$$

The fact that l is a Riemann invariant for the XT system is not an accident, and is related to its compatibility with the full KPWS and with its integrability. This is because, when all fields are independent of y, the conservation of waves equations (1.8) yield $l_x = l_t = 0$, implying that, for one-phase solutions of the KP equation, l must be constant. We can use this relation to reduce the XT system (3.4) to a three-component system. To do this, we perform a change of dependent variable from $\mathbf{r}_4 = (r_1, r_2, r_3, q)^T$ to $\mathbf{v} = (r_1, r_2, r_3, l)^T$, which results in the partially decoupled system

$$\mathbf{v}_t + A_4'\,\mathbf{v}_x = 0\,, \tag{3.11}$$

with

$$A_4' = \begin{pmatrix} A_3 & \mathbf{a}_3 \\ 0_3^T & V_2 - \sigma q^2 \end{pmatrix}, \tag{3.12}$$

where the three-component vector $\mathbf{a}_3$ is immaterial for our purposes, $0_3 = (0,0,0)^T$ and

$$A_3 = A_{\mathrm{KdV}} + \frac{4\sigma K_m l_0^2}{3\,(r_2 - r_1)} A_3^{(2)}\,, \tag{3.13a}$$

$$A_{\mathrm{KdV}} = \mathrm{diag}\,(V_1, V_2, V_3)\,, \tag{3.13b}$$

$$A_3^{(2)} = \begin{pmatrix} 2a_3 & \frac{a_1 a_4}{(1-m)(K_m - E_m)} & \frac{m E_m a_2}{(1-m)(K_m - E_m)} \\ \frac{-(K_m - E_m) a_4}{a_1} & -\frac{2a_2}{1-m} & -\frac{E_m m a_3}{(1-m) a_1} \\ \frac{(K_m - E_m) a_2}{E_m} & \frac{a_1 a_3}{(1-m) E_m} & -\frac{2m a_4}{1-m} \end{pmatrix}, \tag{3.13c}$$

with $V_1, \ldots, V_3$ as in (1.12) and

$$a_1 = E_m - (1-m)\,K_m\,, \quad a_2 = (1-m)\,K_m + (2m-1)\,E_m\,, \tag{3.13d}$$

$$a_3 = (1-m)\,K_m - (1+m)\,E_m\,, \quad a_4 = 2\,(1-m)\,K_m - (2-m)\,E_m\,. \tag{3.13e}$$

Since l is constant for compatible solutions of the full KPWS, we can solve the fourth equation in (3.11) by taking $l \equiv l_0$, thereby arriving at the three-component system

$$\frac{\partial \mathbf{r}_3}{\partial t} + A_3 \frac{\partial \mathbf{r}_3}{\partial x} = 0\,, \tag{3.14}$$

where $\mathbf{r}_3 = (r_1, r_2, r_3)^T$ and the coefficient matrix A_3 is given by (3.13). The case $l_0 = 0$ yields $A_3 = A_{\mathrm{KdV}}$, so the system (3.14) reduces exactly to the KdV-Whitham system of Whitham modulation equations for the KdV equation [40]. We have therefore showed that, once compatibility is enforced, the XT system reduces to a one-parameter deformation of the KdV-Whitham system, parametrized by the value of the wavenumber l_0 along the transverse dimension. The transformation $r_j \mapsto l_0 r_j$, when $l_0 \neq 0$, eliminates l_0 from the system (3.14). Nevertheless, we will retain l_0 in what follows because it is helpful to highlight certain properties of the deformed system.

We now turn to the issue of the integrability of the XT system. The four-component XT system (3.3) fails the Haantjes test, since, on its own, the system is not compatible with the KP equation. On the other hand, the three-component reduced system (3.14) does pass the Haantjes test, in that all the terms of its Haantjes tensor associated with A_3 in (3.13) vanish identically. Thus, while the system (3.1) is not integrable, the conservative system (3.14) is an integrable, one-parameter family of deformations of the KdV-Whitham system.

The harmonic limit (i.e. $r_2 \to r_1$, implying $m \to 0$) of the deformed three-component system (3.14) yields the coefficient matrix

$$\lim_{r_2 \to r_1} A_3 = \begin{pmatrix} 12r_1 - 6r_3 - \sigma q^2 & 0 & \sigma q^2 \\ 0 & 12r_1 - 6r_3 - \sigma q^2 & \sigma q^2 \\ 0 & 0 & 6r_3 \end{pmatrix}, \tag{3.15}$$

with $q^2 = \pi^2 l_0^2/(r_3 - r_1)$. On the other hand, like with the reduced YT system, the soliton limit (i.e. $r_2 \to r_3$, implying $m \to 1$) of the reduced XT system (3.14) is singular for $l_0 \neq 0$, since some of the entries of $A_3^{(2)}$ diverge in that limit. Again, this is to be expected, because in the soliton limit one has $l_0 = 0$ [cf (1.5*a*)].

3.3. Deformed Riemann invariants

Since the one-parameter deformation (3.14) of the KdV-Whitham system is integrable, it can be written in diagonal form. We have not been able to determine the deformed Riemann invariants for all parameter values l_0 but we can obtain approximate Riemann invariants for small l_0 following standard methodology (e.g. see [21]). It is convenient to define the coefficient

$$c_2 = \frac{4K_m}{3m\left(r_2 - r_1\right)}\,. \tag{3.16}$$

so that (3.13) reads $A_3 = A_{\mathrm{KdV}} + A_{\mathrm{def}}$ and the 'deformation matrix' is simply $A_{\mathrm{def}} = \ell c_2 A_3^{(2)}$ with the (signed) deformation parameter

$$\ell \equiv \sigma l_0^2. \tag{3.17}$$

We seek an expansion of the deformed eigenvalues $\tilde{V}_j$ and corresponding right eigenvectors $\tilde{\mathbf{v}}_j$ in powers of ℓ near $\ell = 0$ as

$$\tilde{V}_j = V_j + \ell V_j^{(2)}\left(r_3\right) + O\left(\ell^2\right), \quad \tilde{\mathbf{v}}_j = \mathbf{e}_j + \ell v_j^{(2)}\left(r_3\right) + O\left(\ell^2\right), \quad j = 1,2,3, \tag{3.18}$$

where $\mathbf{r}_3 = (r_1, r_2, r_3)^T$ are the unperturbed KdV-Whitham Riemann invariants, the unperturbed speeds V_j are given in (1.12), and the unperturbed eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are simply the canonical basis in $\mathbb{R}^3$, i.e. $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{I}_3$, where $\mathbb{I}_n$ is the $n \times n$ identity matrix. We begin by computing the perturbation to the characteristic speeds. Since $A_{\mathrm{def}} = O(\ell)$, the first correction terms appear at $O(\ell)$. The deformed eigenvalue problem is

$$\left(A_{\mathrm{KdV}} + A_{\mathrm{def}}\right)\tilde{\mathbf{v}}_j = \tilde{V}_j \tilde{\mathbf{v}}_j\,, \qquad j = 1,2,3. \tag{3.19}$$

The unperturbed eigenvalue problem, obtained at $O(1)$, is simply $A_{\mathrm{KdV}}\mathbf{e}_i = V_i\mathbf{e}_i$, which is satisfied because the KdV-Whitham system is the $\ell = 0 = l_0$ reduction of the XT system (3.14). Collecting terms $O(\ell)$ in (3.19) yields

$$\left(A_{\mathrm{KdV}} - V_j \mathbb{I}_3\right)\mathbf{v}_j^{(2)} = \left(V_j^{(2)} - c_2 A_3^{(2)}\right)\mathbf{e}_j\,, \tag{3.20}$$

and multiplying from the left by $\mathbf{e}_j^T$ yields the first-order correction to the characteristic velocities as

$$V_j^{(2)} = \mathbf{e}_j^T c_2 A_3^{(2)} \mathbf{e}_j\,, \qquad j = 1,2,3\,, \tag{3.21}$$

since the $\mathbf{e}_j$ are orthonormal. Explicitly,

$$\tilde{V}_1 = V_1^{(0)} - 2c_2\ell\left((1+m)E_m - (1-m)K_m\right) + O\left(\ell^2\right), \tag{3.22a}$$

$$\tilde{V}_2 = V_2^{(0)} - 2c_2\ell\frac{(1-m)K_m - (1-2m)E_m}{1-m} + O\left(\ell^2\right), \tag{3.22b}$$

$$\tilde{V}_3 = V_3^{(0)} + 2c_2\ell m\frac{(2-m)E_m - 2(1-m)K_m}{1-m} + O\left(\ell^2\right). \tag{3.22c}$$

Note that if $\sigma = 1$, the corrections to the first and second characteristic velocities are negative, while the correction to the third characteristic velocity is positive (opposite if $\sigma = -1$). Note also that all three expansions remain well-ordered in the limit $m \to 0$, but diverge as $m \to 1$. Next, we use (3.20) and (3.21) to compute the correction to the eigenvectors. The

matrix $A_{\mathrm{KdV}} - V_j \mathbb{I}_3$ is singular for $j = 1,2,3$. The solvability condition (3.21) implies the existence of a solution to (3.20) in the form

$$\tilde{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \ell \begin{pmatrix} 0 \\ 1 \\ m \end{pmatrix}, \quad \tilde{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \ell \begin{pmatrix} m-1 \\ 0 \\ m \end{pmatrix}, \quad \tilde{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \ell \begin{pmatrix} m-1 \\ 1 \\ 0 \end{pmatrix}, \tag{3.23}$$

which is unique up to a linear combination of $\ell \mathbf{e}_j$ in $\tilde{v}_j$. The approximate eigenvectors in (3.23) are accurate to $O(\ell^2)$ corrections.

Next we compute the deformed Riemann invariants. It is non-trivial to find the correct integrating factor for the deformed Riemann invariants. To circumvent this issue, we take advantage of the fact that the total differential of each Riemann invariant is zero along the associated characteristic curve. We expand the deformed Riemann invariants $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3$ as

$$\tilde{R}_j = r_j + \ell R_j^{(2)}(\mathbf{r}_3) + O\left(\ell^2\right), \qquad j = 1,2,3. \tag{3.24}$$

We have

$$\mathrm{d}\tilde{R}_j = 0 \tag{3.25}$$

along the characteristic curve $\mathrm{d}x/\mathrm{d}t = \tilde{V}_j$, for $j = 1,2,3$. Expanding (3.25) yields

$$\mathrm{d}\tilde{R}_j = \boldsymbol{\nabla}_{\mathbf{r}} \tilde{R}_j \cdot \mathrm{d}\mathbf{r} = \boldsymbol{\nabla}_{\mathbf{r}} \tilde{R}_j \cdot \left(\frac{\partial \mathbf{r}_3}{\partial t} \mathrm{d}t + \frac{\partial \mathbf{r}_3}{\partial x} \mathrm{d}x \right) = 0, \qquad j = 1,2,3, \tag{3.26}$$

where $\boldsymbol{\nabla}_{\mathbf{r}} = (\partial_{r_1}, \partial_{r_2}, \partial_{r_3})^T$. Next we use (3.14) and (3.19) to rewrite (3.26) as

$$\left(\boldsymbol{\nabla}_{\mathbf{r}} \tilde{R}_j\right)^T \left(\tilde{V}_j \mathbb{I}_3 - A_3\right) \frac{\partial \mathbf{r}_3}{\partial x} \mathrm{d}t = 0, \qquad j = 1,2,3, \tag{3.27}$$

along the curve $\mathrm{d}x/\mathrm{d}t = \tilde{V}_j$. If the above differential must be zero for all $\partial \mathbf{r}_3/\partial x$, one can constrain each component to be zero, i.e.

$$\left(\boldsymbol{\nabla}_{\mathbf{r}} \tilde{R}_j\right)^T \left(\tilde{V}_j \mathbb{I}_3 - A_3\right) = 0^T, \qquad j = 1,2,3. \tag{3.28}$$

One can check that $\det(\tilde{V}_j \mathbb{I}_3 - A_3) = O(\ell^2)$ for $j = 1,2,3$, which allows for a nontrivial solution at $O(\ell)$. For each $j = 1,2,3$, (3.28) yields a system of three differential equations for $R_j^{(2)}$. Note that, even though it might not seem obvious *a priori*, these differential equations must necessarily be compatible since we know that the system is integrable and therefore admits Riemann invariants.

We present the calculations for $R_1^{(2)}$ in detail. Keeping terms up to $O(\ell)$, the first equation in (3.28) is trivially satisfied, while the remaining two equations are

$$\frac{\partial R_1^{(2)}}{\partial r_2} = \frac{\left(E_m - (1-m) K_m\right)^2}{3(1-m)(r_2 - r_1)^2}, \tag{3.29a}$$

$$\frac{\partial R_1^{(2)}}{\partial r_3} = -\frac{m^2 E_m^2}{3(1-m)(r_2 - r_1)^2}. \tag{3.29b}$$

and one can check that the equality of the mixed second derivatives, namely $\partial^2 R_1^{(2)}/\partial r_2 \partial r_3 = \partial^2 R_1^{(2)} \partial r_3 \partial r_2$, is indeed satisfied. Next, we need to integrate (3.29) to find $R_1^{(2)}$. We can integrate the equations manually, employing a process akin to that of finding a potential for a conservative vector field. We begin with (3.29*b*) since it is simpler. Because of the presence of

elliptic integrals, it is convenient to perform a change of variables from r_1, r_2, r_3 to r_1, r_2 and m. Solving (1.4) for r_3 as a function of m, we have

$$\frac{\partial R_1^{(2)}}{\partial m} = \frac{\partial R_1^{(2)}}{\partial r_3}\frac{\partial r_3}{\partial m} = \frac{E_m^2}{3(1-m)(r_2-r_1)}. \tag{3.30}$$

Integrating this equation (with r_1 and r_2 held constant) then yields $R_1^{(2)}$ as

$$R_1^{(2)} = \frac{g(m)}{3(r_2-r_1)}, \tag{3.31a}$$

with

$$g'(m) = \frac{E_m^2}{1-m}. \tag{3.31b}$$

Note that we have taken the arbitrary function of r_1 and r_2 in (3.31a) to be zero. By substituting (3.31a) into (3.29a) yields

$$g(m) = -(1-m)K_m^2 + 2E_mK_m - E_m^2, \tag{3.31c}$$

and one can confirm that (3.31c) is indeed compatible with (3.31b), which means we have successfully integrated (3.29), obtaining the first approximate deformed Riemann invariant of the XT system as

$$\tilde{R}_1 = r_1 + \frac{\ell}{3(r_2-r_1)}\left(2K_mE_m - E_m^2 - (1-m)K_m^2\right) + O\left(\ell^2\right). \tag{3.32a}$$

One can apply an identical process to find the remaining deformed Riemann invariants as

$$\tilde{R}_2 = r_2 + \frac{\ell}{4K_m}\left(2K_mE_m + \frac{E_m^2}{1-m} - (1-m)K_m^2\right) + O\left(\ell^2\right), \tag{3.32b}$$

$$\tilde{R}_3 = r_3 - \frac{\ell}{3(r_2-r_1)}\frac{m\left((1-m)K_m^2 - E_m^2\right)}{1-m} + O\left(\ell^2\right). \tag{3.32c}$$

The expressions of the deformed speeds and deformed Riemann invariants may prove to be useful when investigating the dynamics of weakly slanted wave fronts in the KP equation.

3.4. Hyperbolicity

The hyperbolicity of the three-component reduction (3.14) of the full KPWS can be determined by analyzing the eigenvalues of the coefficient matrix $A_3(\mathbf{r}_3)$, given in (3.13), which are the characteristic velocities $\tilde{V}_j, j = 1,2,3$. Because the characteristic polynomial

$$p(\lambda) = \det(A_3 - \lambda\mathbb{I}_3) = -\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0, \tag{3.33}$$

is a cubic with real coefficients, it has either three real roots or one real root and a complex conjugate pair. Equation (3.22) demonstrates that the $\tilde{V}_j$ are real for all $r_j \in \mathbb{R}$ with $r_1 \leqslant r_2 \leqslant r_3$ when the deformation parameter $\ell = \sigma l_0^2$ is sufficiently small in magnitude. Because the coefficients b_0, b_1, b_2 in (3.33) are smooth functions of $\mathbf{r}_3$, a bifurcation from all real roots to a complex conjugate pair can only occur if the discriminant of $p(\lambda)$,

$$D(\mathbf{r}_3) = 4b_1^3 + b_2^2b_1^2 - 18b_0b_2b_1 - 4b_0b_2^3 - 27b_0^2, \tag{3.34}$$

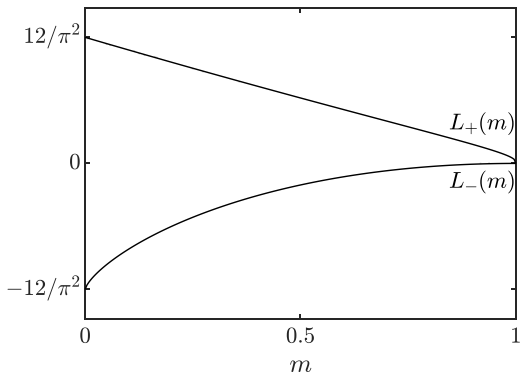

Figure 2. The functions $L_\pm(m)$ that determine the critical values $m = m_\pm$ at which two characteristic velocities of the XT system coalesce.

is zero. To evaluate $D(\mathbf{r}_3)$, we first simplify the calculation by restricting ourselves to the set

$$S = \left\{ \mathbf{r}_3 = (0, m, 1)^T \mid 0 \leqslant m \leqslant 1 \right\}, \tag{3.35}$$

and we simply write $D = D(m)$, which can be shown to be a quintic polynomial in ℓ with coefficients depending on m. Setting $D(m) = 0$ and solving for ℓ, one finds two complex conjugate solutions, which are not of interest, plus two real solutions $\ell = L_-(m)$ and $\ell = L_+(m)$, the latter of which is a double root. Explicitly,

$$L_+(m) = \frac{((2-m)E_m - 2(1-m)K_m)((1+m)E_m - (1-m)K_m)((1-m)K_m - (1-2m)E_m)}{3E_m(K_m - E_m)(E_m - (1-m)K_m)\left(2(2-m)K_mE_m - 3E_m^2 - (1-m)K_m^2\right)}, \tag{3.36}$$

while the expression for $L_-(m)$ is more complicated, so it is omitted for brevity. The expansions of $L_\pm(m)$ for small m are

$$L_+(m) = \frac{12}{\pi^2}(1-m) + O\left(m^2\right), \tag{3.37}$$

$$L_-(m) = -\frac{12}{\pi^2}\left(1 - \frac{4}{3}(2m)^{2/3} - m\right) + O\left(m^{4/3}\right). \tag{3.38}$$

Figure 2 shows that $L_+(m) > 0$ and $L_-(m) < 0$ for $m \in [0,1)$. Therefore, there are two critical values of m: $\ell = \pm l_0^2 = L_\pm(m_\pm)$. The fact that $\tilde{V}_j \in \mathbb{R}$ for $|\ell|$ sufficiently small implies that $D(m) > 0$ when $0 < m < m_\pm$, namely the XT system is (strictly) hyperbolic when $m \in (0, m_\pm)$. When $m = m_\pm$, two characteristic velocities coalesce.

In the case of the plus sign, the fact that $\ell = L_+(m_+)$ is a double root of $D(m) = 0$ implies that $\mathrm{d}D/\mathrm{d}m|_{m_+} = 0$. Then, in a neighborhood of $m = m_+$, the discriminant (3.34) exhibits parabolic behavior

$$D(m) = \frac{1}{2}\left.\frac{\mathrm{d}^2 D}{\mathrm{d}m^2}\right|_{m_+}(m - m_+)^2 + O(m - m_+)^3, \tag{3.39}$$

and it must be the case that $D(m) \geq 0$, i.e. $\mathrm{d}^2D/\mathrm{d}m^2|_{m_+} > 0$, because $D(m) > 0$ for $0 < m < m_+$. Since m_+ is the only point at which $D(m) = 0$, this implies $D(m) > 0$ for

$m \in [0, m_+) \cup (m_+, 1)$ and that the characteristic speeds are always real. Indeed, a direct calculation shows that, when $m = m_+$,

$$\tilde{V}_1 = \tilde{V}_2 = \frac{6\left((1+m)E(m)^2 - (1-m)^2 K(m)^2 - 2(1-m)mK(m)E(m)\right)}{(1-m)K(m)^2 - 2(2-m)K(m)E(m) + 3E(m)^2}, \tag{3.40a}$$

$$\tilde{V}_3 = \frac{2}{3}\left(3(m+1) + \frac{2mE(m)}{(m-1)K(m)+E(m)} + \frac{2mE(m)}{K(m)-E(m)} - \frac{2(1-m)K(m)}{E(m)}\right), \tag{3.40b}$$

and there are three corresponding linearly independent eigenvectors $\tilde{\mathbf{v}}_j$, $j = 1,2,3$. Consequently, we conclude that the XT system is hyperbolic for $m \in (0,1)$ and strictly hyperbolic when $m \neq m_+$.

In the case of the minus sign, the critical point m_- satisfying $\ell = -l_0^2 = L_-(m_-)$ is a simple root of $D(m)$. Since $D(m) > 0$ for $0 < m < m_-$ and D depends upon m smoothly, it necessarily is the case that $\mathrm{d}D/\mathrm{d}m|_{m_-} < 0$, so that the discriminant (3.34) becomes negative in a right neigborhood of $m = m_-$. This implies that, for $m_- < m < 1$, the XT system exhibits a complex conjugate pair of characteristic speeds and is not hyperbolic.

The above discussion of hyperbolicity of the XT system was limited to the set S defined in (3.35), where it was observed that a bifurcation occurs at the point $m = m_\sigma$. However, using the scaling symmetry $r_j(x,t) \to a^2 r_j(ax, a^3 t)$, $q(x,t) \to aq(ax, a^3 t)$ with $a = (r_3 - r_1)^{-1/2}$ and the Galilean symmetry $r_j(x,t) \to b + r_j(x - 6bt, t)$, $q(x,t) \to q(x - 6bt, t)$ with $b = -r_1$ [4], we can map any vector $\mathbf{r}_3 = (r_1, r_2, r_3)^T \in \mathbb{R}^3$ with $r_1 \leqslant r_2 \leqslant r_3$ to a vector $\tilde{\mathbf{r}}_3 = (0, m, 1)^T \in S$. For r_3, the bifurcation occurs on the surface

$$\Sigma_\pm = \left\{ \mathbf{r}_3 = (r_1, r_2, r_3)^T \;\middle|\; \pm l_0^2 = L_\pm(m_\pm)(r_3 - r_1)^2, \;\; m_\pm = \frac{r_2 - r_1}{r_3 - r_1} \right\}. \tag{3.41}$$

In the case of the plus sign, the XT system is hyperbolic, and strictly so for $\mathbf{r}_3 \notin \Sigma_+$. In the case of the minus sign, the XT system is hyperbolic so long as $(r_2 - r_1)/(r_3 - r_1) < m_-$ where, $L_-(m_-) = -l_0^2 (r_3 - r_1)^2$. When $(r_2 - r_1)/(r_3 - r_1) > m_-$, the XT system loses hyperbolicity.

4. The XY system

4.1. KPWS in a comoving frame and the XY system

The third and final class of reductions of the KPWS we consider is that of time-independent solutions, to be defined precisely below. While in sections 2 and 3 we considered reductions that are evolutionary, exhibiting well-posed initial value problems (at least when $\sigma = 1$), here we are considering a spatial problem, independent of t, for the modulations. As we will see, this does not preclude dynamics in the full solution to KP itself. In the previous sections we saw that, in order to ensure the compatibility of the XT and YT reductions with the KP equation, one must make sure that all three conservation of waves equations (1.8) are satisfied. We will see that this is also the case with stationary reductions of the KPWS.

Note that, even though one may think that a more general scenario is obtained by looking for traveling wave solutions, i.e. solutions that are stationary in a traveling frame of reference $(\tilde{x}, \tilde{y}, \tilde{t})$, with $\tilde{x} = x - ct$, $\tilde{y} = y - dt$ and $\tilde{t} = t$, this is not the case in practice. This is because the Galilean and pseudo-rotation invariance of the KP equation allow one to perform appropriate transformations of the dependent and independent variables to rewrite any traveling wave solution of the KP equation as stationary in a suitable reference frame. Since the KPWS preserves

these invariances, the same transformations will also work for the KPWS, see appendix (A.3) for details.

Based on the above discussion, consider situations in which the temporal derivatives in the partial KPWS (1.14) can be neglected, which then yields

$$A_5 \frac{\partial \mathbf{r}}{\partial x} + B_5 \frac{\partial \mathbf{r}}{\partial y} = 0\,. \tag{4.1}$$

The corresponding reduction of the full KPWS (1.11) is (4.1) augmented with (1.11*d*), which is unchanged in this reduction. Contrary to the reductions discussed in 2 and 3, here the independence from one of the coordinates does not automatically result in a reduction in the number of components. That is, all five dependent variables appear in (4.1). Assuming invertibility of A_5 and B_5, one could equivalently write (4.1) as an evolutionary system with respect to either x or y, e.g. as $\mathbf{r}_x + C\,\mathbf{r}_y = 0$. However, the resulting coefficient matrix $C = (A_5)^{-1}B_5$ is quite complicated, and therefore the resulting system is difficult to analyze.

4.2. Riemann invariant, reduction and integrability

In light of what we learned by studying the YT and XT systems, we expect that, when considering solutions that are independent of t, the frequency ω will be one of the Riemann invariants. Indeed, in this case the three compatibility conditions (1.8) yield immediately $\omega_x = \omega_y = 0$. We now show that this expectation is correct. In this case, however, the complexity of the system makes it impractical to use the direct approach based on the use of the characteristic relations and left eigenvectors that was used in the previous sections. We therefore use an alternative approach, based on calculating the total differential of $\omega = \omega(\mathbf{r})$ as

$$\mathrm{d}\omega = \boldsymbol{\nabla}_{\mathbf{r}}\omega \cdot \mathrm{d}r = \boldsymbol{\nabla}_{\mathbf{r}}\omega \cdot \left(\frac{\partial \mathbf{r}}{\partial x}\mathrm{d}x + \frac{\partial \mathbf{r}}{\partial y}\mathrm{d}y \right)\,, \tag{4.2a}$$

with $\boldsymbol{\nabla}_{\mathbf{r}} = (\partial_{r_1}, \partial_{r_2}, \partial_{r_3}, \partial_q, \partial_p)^T$ and directly verifying that ω is a Riemann invariant of (4.1). The evolution of ω as dictated by the system (4.1) along the characteristic coordinates $\mathrm{d}y/\mathrm{d}x = \lambda$ is then

$$\mathrm{d}\omega = \boldsymbol{\nabla}_{\mathbf{r}}\omega^T \left(\lambda \mathbb{I}_5 - A_5^{-1}B_5\right) \frac{\partial \mathbf{r}}{\partial y}\,\mathrm{d}x\,. \tag{4.2b}$$

Computing $\boldsymbol{\nabla}_{\mathbf{r}}\omega$, substituting in (4.2*b*) and setting $\mathrm{d}\omega = 0$ then yields a linear equation that determines the characteristic speed λ as

$$\lambda = \frac{2\sigma q}{V_2 - \sigma q^2}\,. \tag{4.3}$$

With V_2 given in (1.12), which confirms that ω is indeed a Riemann invariant for the system (4.1).

As per the above discussion, in order for the KPWS to be compatible, we must enforce $\omega_x = \omega_y = 0$. Following the procedures of sections 2 and 3, we partially diagonalize the *XY* system (4.1) by performing a dependent coordinate transformation so that ω is one of the new dependent variables. We then solve the resulting PDE by taking $\omega \equiv$ const and obtain a one-parameter family of reduced four-component systems, parametrized by the constant value of ω. Once again, however, the calculations are more involved than in the previous cases.

The complication is that the expression (1.6) for ω does not allow one to uniquely obtain any one of the dependent variables in terms of the others (recall (1.5*a*)). The best one can do is to solve for q, which entails a choice of sign:

$$q = \pm\sqrt{\sigma\left(\frac{\omega}{k} - V\right)}\,, \tag{4.4}$$

Following the same methods as in the previous sections, one can then obtain a four-component hydrodynamic system of equations for $\mathbf{v} = (r_1, r_2, r_3, p)^T$. A single coefficient matrix of the form

$$\frac{\partial \mathbf{v}}{\partial x} + C_4' \frac{\partial \mathbf{v}}{\partial y} = 0\,, \tag{4.5}$$

is quite complicated. However, the system (4.1) can be transformed, using the same methods, into the concise form

$$A' \frac{\partial \mathbf{u}}{\partial x} + B' \frac{\partial \mathbf{u}}{\partial y} = 0\,, \tag{4.6}$$

where $A' = AT$ and $B' = BT$, $\mathbf{u} = (r_1, r_2, r_3, \omega, p)^T$, and $T = \partial v/\partial u$. Once the PDE for ω is disregarded, since $\omega \equiv \omega_0$ solves it, we arrive at the system

$$A_4' \frac{\partial \mathbf{p}}{\partial x} + B_4' \frac{\partial \mathbf{p}}{\partial y} = 0\,, \tag{4.7}$$

where $\mathbf{p} = (r_1, r_2, r_3, p)^T$, the coefficient matrices are

$$A_4' = \begin{pmatrix} q\left((c_1+1)\nu_1 + q^2(-\sigma) + V_1\right) & -(c_1+c_2-1)\nu_1 q & (c_2+1)\nu_1 q & -q^2\sigma \\ (c_1+1)\nu_2 q & -q\left((c_1+c_2-1)\nu_2 + q^2\sigma - V_2\right) & (c_2+1)\nu_2 q & -q^2\sigma \\ (c_1+1)\nu_3 q & -(c_1+c_2-1)\nu_3 q & q\left((c_2+1)\nu_3 + q^2(-\sigma) + V_3\right) & -q^2\sigma \\ -\sigma(c_1+1)\nu_5 - \left((\alpha-1)q^2\right) & \sigma(c_1+c_2-1)\nu_5 & \alpha q^2 - \sigma(c_2+1)\nu_5 & q \end{pmatrix}, \tag{4.8}$$

and

$$B_4' = \begin{pmatrix} 2\sigma q^2 - (c_1+1)\nu_1 & (c_1+c_2-1)\nu_1 & -\left((c_2+1)\nu_1\right) & q\sigma \\ -\left((c_1+1)\nu_2\right) & 2\sigma q^2 + (c_1+c_2-1)\nu_2 & -\left((c_2+1)\nu_2\right) & q\sigma \\ -\left((c_1+1)\nu_3\right) & (c_1+c_2-1)\nu_3 & 2\sigma q^2 - (c_2+1)\nu_3 & q\sigma \\ (\alpha-1)q & 0 & -\alpha q & 0 \end{pmatrix}, \tag{4.9}$$

with

$$c_1 = \frac{\omega_0 (E_m - K_m)}{16 m k^3 K_m^3} \qquad \text{and} \qquad c_2 = \frac{\omega_0 E_m}{16 k^3 K_m^3 (1-m)}\,. \tag{4.10}$$

Note that q is present in the matrices for readability, however its definition is given in equations (1.9) and (4.4) in terms of the constant parameter ω_0 and the Riemann type variables r_j. Furthermore, one can use computer algebra software to perform the Haantjes tensor test on the resulting system. Doing so, we have verified that, as in the case of the XT and YT systems, the Haantjes tensor of the reduced XY system does indeed vanish identically, proving that the latter conservative system is integrable as well.

The above reduced XY system possesses a finite harmonic limit, similarly to those in the previous sections. Specifically, in the limit $r_2 \to r_1^+$, the PDEs for r_1 and r_2 coincide, and the

four-component system (4.7) reduces to a 3×3 system in the independent variables x and y for the three-component dependent variable $\mathbf{r}_3=(r_1,r_3,p)^T$, with coefficient matrices

$$A_3=\begin{pmatrix} (h_1+16)\,r_1-\frac{3}{8}\,(3h_1+8)\,r_3 & \frac{1}{4}\,(h_1+4)\,r_3 & -\sqrt{\sigma}h_2 \\ r_3-\frac{h_1 r_3}{8} & (h_1+4)\,r_1-\frac{3h_1 r_3}{4}+9r_3 & -\sqrt{\sigma}h_2 \\ \frac{(h_1-8)r_3}{8\sqrt{\sigma}h_2} & \frac{3(h_1-4)r_3-4(h_1+4)r_1}{4\sqrt{\sigma}h_2} & 1 \end{pmatrix}, \tag{4.11a}$$

$$B_3=\frac{1}{h_2}\begin{pmatrix} -\left(\sqrt{\sigma}\,(16\,(h_1+4)\,r_1+(40-17h_1)\,r_3)\right) & -2\,(h_1+4)\,r_3\sqrt{\sigma} & \sigma h_2 \\ (h_1-8)\,r_3\sqrt{\sigma} & -2\sqrt{\sigma}\,(8\,(h_1+4)\,r_1+(20-7h_1)\,r_3) & \sigma h_2 \\ 0 & -h_2 & 0 \end{pmatrix}, \tag{4.11b}$$

where

$$h_1=\frac{\pi\,\omega_0}{\sqrt{(r_3-r_1)^3}},\qquad h_2=8\sqrt{(h_1-2)\,r_3-(h_1+4)\,r_1}. \tag{4.12}$$

Like with the XT and YT reductions, however, the system (4.7) does not admit a finite soliton limit in general, since $\omega\to 0$ as $m\to 1$ (cf (1.5*a*)), which is incompatible with having $\omega=\omega_0\neq 0$ in (4.7).

4.3. Stationary solutions of the KP equation and Whitham modulation system for the Boussinesq equation

Importantly, even though the system (4.1) describes stationary solutions of the KPWS, the corresponding solutions of the KP equation are not stationary, unless $\omega_0=0$. On the other hand, if $\omega_0=0$, the modulated solutions of the KP equation described by (4.1) are also stationary. This point is relevant because stationary solutions of the KP equation (1.2) satisfy versions of the Boussinesq equation, namely [5, 6]

$$u_{\tau\tau}-c^2u_{xx}+\sigma\left(6uu_x+\varepsilon^2u_{xxx}\right)_x=0\,. \tag{4.13}$$

With $\tau=y$ and $c=0$. The case $\sigma=+1$ is the 'good' Boussinesq equation with real linear dispersion. Boussinesq derived the 'bad' version ($\sigma=-1$) as a long-wavelength model of water waves, whose linearized equation is ill-posed [16]. Therefore, the modulation system (4.7) with $\omega_0=0$ is also the genus-1 Whitham modulation equations for the above Boussinesq equations. This is noteworthy because the Boussinesq equations (4.13) are associated with a 3×3 Lax pair (e.g. see [5, 6]), which significantly complicates the analysis, and as a result, the development of Whitham modulation theory using finite gap integration has not been achieved yet.

We point out that, when $\omega_0=0$, the modulation system (4.7) greatly simplifies because $c_1=c_2=0$ and $q=\pm\sqrt{-\sigma V}$. Moreover, when $\omega_0=0$, the system (4.7) remains well-defined both in the harmonic and the soliton limits.

5. Concluding remarks

In this work we investigated the two-dimensional reductions of the KPWS obtained when all fields are independent of one of the spatial or temporal coordinates. We demonstrated that, on the one hand, the reductions of the partial, five-component KPWS (1.14) are not integrable. On the other hand, once all three conservation of waves are taken into account, i.e. when the

full KPWS (1.11) is considered, each of these reductions results in a constant of motion which makes each reduction compatible and integrable.

As far as the six-component KPWS (1.11) is concerned, in general, one does not expect an overdetermined quasi-linear system to be either compatible or integrable, so some mechanism of enforcing compatibility is required. In our previous work, we enforced the compatibility, and thereby obtained integrable systems, by considering the harmonic or soliton limit, either of which results in a reduction in the number of modulation equations. In this work, we added to the catalog of integrable reductions of the KPWS by characterizing two-dimensional reductions of the KPWS.

The results of this work and the above discussion lead to the natural issue of whether there are other integrable reductions of the KPWS, and whether it is possible to identify all such integrable reductions. In other words, the question is whether it is possible to identify suitable conditions that guarantee compatibility of the full KPWS. We plan to investigate this question in future work.

We reiterate that, even though both the reduced YT, XT and XY systems admit a finite harmonic limit, none of these systems admits a well-defined soliton limit in general. However, setting the constant values of the wavenumber (k or l) or frequency (ω) for the YT, XT, or XY system, respectively, to zero does result in well-defined soliton limits.

We should also mention that one could equivalently carry out all calculations by replacing the PDE for q with the following simplified PDE, as derived in [3, 4]:

$$\frac{\partial q}{\partial t} + \left(V + \sigma q^2\right) \frac{\partial q}{\partial x} + \frac{D}{Dy} \left(V + \sigma q^2\right) = 0 \,.$$

For brevity, however, we omit the details.

Finally, we reiterate that the XY reduction of the KPWS allowed us to explicitly obtain the Whitham modulation system for the Boussinesq equation, which had not been derived before. It is hoped that this novel system will prove to be as useful as the other reductions of the KPWS.

Another potential application of the results of this work are to situations in which initial or boundary data for the KPWS are chosen to be independent of one independent variable. In order to use the reduced YT, XT, or XY systems, the soliton limit will not be available except in specialized situations. Nevertheless, one interesting class of problems are generalized Riemann problems consisting of abrupt transitions between two periodic traveling waves. The reduced KPWS obtained here could be used to study certain generalized Riemann problems.

Data availability statement

No new data were created or analysed in this study.

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Appendix

A.1. Derivation of the KPWS via averaged conservation laws

Here the KPWS (1.11) is now derived by the method of averaged conservation laws [40]. Solutions (u, v) of the KP equation (1.2) satisfy the conservation laws

$$u_t + \left(3u^2 + \varepsilon^2 u_{xx}\right)_x + \sigma\left(v\right)_y = 0\,, \tag{A.1a}$$

$$\left(u^2\right)_t + \left(2\varepsilon^2 u u_{xx} - \varepsilon^2 u_x^2 + 4u^3 - \sigma v^2\right)_x + 2\sigma\left(uv\right)_y = 0\,, \tag{A.1b}$$

$$v_x - u_y = 0\,. \tag{A.1c}$$

Consider the expansion

$$u\left(x,y,t\right) = u_o\left(\theta,x,y,t\right) + \varepsilon u_1\left(\theta,x,y,t\right) + \dots\,, \tag{A.2}$$

and similarly for $v(x,y,t)$, where (u_o, v_o) is the cnoidal wave solution (1.3), the fast variable $\theta(x,y,t)$ satisfies (1.7), and each term in the expansion is periodic in θ with unit period. The derivatives are expanded as $\partial_x \mapsto k\partial_\theta/\varepsilon + \partial_x$, $\partial_y \mapsto l\partial_\theta/\varepsilon + \partial_y$, and $\partial_t \mapsto -\omega\partial_\theta/\varepsilon + \partial_t$. The period average of a function $F[u] \equiv F(u, u_x, u_y, u_t, u_{xy}, \dots)$ of u and its derivatives is

$$\overline{F}\left(x,y,t\right) = \overline{F\left[u_o\left(\cdot,x,y,t\right)\right]} \equiv \int_0^1 F\left[u_o\left(\theta,x,y,t\right)\right]\,\mathrm{d}\theta\,. \tag{A.3}$$

Moreover,

$$\overline{\frac{\partial F\left[u\right]}{\partial x}} = \overline{\frac{k}{\varepsilon}\frac{\partial F\left[u_o + \varepsilon u_1\right]}{\partial\theta} + \frac{\partial F\left[u_o + \varepsilon u_1\right]}{\partial x}} = \frac{\partial\overline{F\left[u_o\right]}}{\partial x} + \mathcal{O}\left(\varepsilon\right)\,, \tag{A.4a}$$

and, similarly,

$$\overline{\partial F\left[u\right]/\partial y} = \partial\overline{F\left[u_o\right]}/\partial y + \mathcal{O}\left(\varepsilon\right)\,, \qquad \overline{\partial F\left[u\right]/\partial t} = \partial\overline{F\left[u_o\right]}/\partial t + \mathcal{O}\left(\varepsilon\right)\,. \tag{A.4b}$$

Then, inserting the multiscale expansion (A.2) into the conservation laws (A.1) and averaging results in the averaged conservation laws at $\mathcal{O}(1)$

$$\left(\overline{u_o}\right)_t + \left(3\overline{u_o^2}\right)_x + \sigma\left(q\overline{u_o} + p\right)_y = 0\,, \tag{A.5a}$$

$$\left(\overline{u_o^2}\right)_t + \left(-3k^2\overline{u_{o,\theta}^2} + 4\overline{u_o^3} - \sigma q^2\overline{u_o^2} - 2\sigma q p\overline{u_o} - \sigma p^2\right)_x + 2\sigma\left(q\overline{u_o^2} + p\overline{u_o}\right)_y = 0\,, \tag{A.5b}$$

$$\left(q\overline{u_o} + p\right)_x - \left(\overline{u_o}\right)_y = 0\,. \tag{A.5c}$$

These three conservation laws, combined with the three conservation of waves equations (1.8), express the KPWS in conservative form. It is also straightforward to verify that (A.5*a*)–(A.5*c*) are respectively equivalent to the three solvability conditions (2.21a), (2.21b), and (2.21c), in [4].

A.2. Coefficients matrices, harmonic and soliton limits, relations between elliptic integrals

The coefficient matrices A_5 and B_5 of the KPWS (1.14) are:

$$A_5 = \begin{pmatrix} V_1 - \sigma q^2 & 0 & 0 & -\sigma\nu_1 q & -\sigma q \\ 0 & V_2 - \sigma q^2 & 0 & -\sigma\nu_2 q & -\sigma q \\ 0 & 0 & V_3 - \sigma q^2 & -\sigma\nu_3 q & -\sigma q \\ -(4-\nu_4)\,q & 0 & -(2+\nu_4)\,q & V_2 - \sigma q^2 & 0 \\ -(1-\alpha)\,q & 0 & \alpha q & 0 & 0 \end{pmatrix}, \tag{A.6a}$$

$$B_5 = \begin{pmatrix} 2\sigma q & 0 & 0 & \sigma\nu_1 & \sigma \\ 0 & 2\sigma q & 0 & \sigma\nu_2 & \sigma \\ 0 & 0 & 2\sigma q & \sigma\nu_3 & \sigma \\ 4-\nu_4 & 0 & 2+\nu_4 & 2\sigma q & 0 \\ 1-\alpha & 0 & \alpha & 0 & 0 \end{pmatrix}. \tag{A.6b}$$

The definitions of all the coefficients appearing in (A.6) are given in (1.12) through (1.13).

Next, for convenience, we list the limiting values of the coefficients appearing in the harmonic and soliton limits of the KPWS, since these coefficients appear in all reductions. Recall that, in the harmonic limit, the elliptic parameter m tends to 0 and $r_2 \mapsto r_1^+$. In this limit, the various coefficients then become

$$m = 0\,, \quad V = 4r_1 + 2r_3\,, \tag{A.7a}$$

$$V_1 = V_2 = 12r_1 - 6r_3\,, \quad V_3 = 6r_3\,, \tag{A.7b}$$

$$\nu_1 = \nu_2 = \nu_3 = \nu_5 = r_3, \quad \nu_4 = 4, \quad \alpha = 1\,. \tag{A.7c}$$

Conversely, in the soliton limit the elliptic parameter tends to 1 and $r_2 \mapsto r_3^-$. The limiting value of the various coefficients in this case is

$$m = 1\,, \quad V = r_1 + 2r_3\,, \tag{A.8a}$$

$$V_1 = 6r_1\,, \quad V_2 = V_3 = 2r_1 + 4r_3\,, \tag{A.8b}$$

$$\nu_1 = \nu_5 = r_1\,, \quad \nu_2 = \nu_3 = \frac{1}{3}\,(4r_3 - r_1)\,, \quad \nu_4 = 2\,, \quad \alpha = 0\,. \tag{A.8c}$$

In this work we use the elliptic parameter m as opposed to the elliptic modulus k. Recall that the two are related as $m = k^2$. The complementary modulus is then simply $k'^2 = 1 - k^2 = 1 - m$. While this choice is in line with modern works, it differs from the convention in [30] and its associated references. Thus the various ODEs from [30] must be transformed accordingly. Specifically, the derivatives of K and E with respect to the elliptic parameter m are

$$\frac{\mathrm{d}K_m}{\mathrm{d}m} = \frac{E_m - (1-m)\,K_m}{2m\,(1-m)}\,, \tag{A.9a}$$

$$\frac{\mathrm{d}E_m}{\mathrm{d}m} = \frac{E_m - K_m}{2m}\,. \tag{A.9b}$$

In addition, we have

$$\frac{\mathrm{d}^2E_m}{\mathrm{d}m^2} = -\frac{1}{2m}\frac{\mathrm{d}K_m}{\mathrm{d}m}\,. \tag{A.10}$$

A.3. Invariances, traveling wave and stationary solutions of the KP equation and KPWS

Here we show how, using the invariances of the KP equation and the KPWS, one can map all traveling wave solutions of the KP equation and the KPWS (i.e. solutions that are stationary in a comoving reference frame) into solutions that are stationary in a slanted but fixed reference frame.

To begin, it is useful to consider how the KPWS (1.14) with coefficient matrices I_4, A_5 and B_5 is affected by affine transformations of the independent variables. Recall first that the KP equation (1.2) is invariant under Galilean boosts,

$$u(x,y,t) \mapsto u'(x,y,t) = c + u(x',y,t)\,, \tag{A.11a}$$
$$v(x,y,t) \mapsto v'(x,y,t) = v(x',y,t)\,. \tag{A.11b}$$

With $x' = x - 6ct$, and 'pseudo-rotations',

$$u(x,y,t) \mapsto u'(x,y,t) = u(x',y',t)\,. \tag{A.12a}$$
$$v(x,y,t) \mapsto v'(x,y,t) = v(x',y',t) + au(x',y',t)\,. \tag{A.12b}$$

With $x' = x + ay - \sigma a^2 t$ and $y' = y - 2\sigma a t$, and with a and c arbitrary real parameters. Namely, if the $u(x,y,t)$ and $v(x,y,t)$ comprise a solution of the KP equation, so does the pair $u'(x,y,t)$ and $v'(x,y,t)$. Also recall that the above transformations are mapped respectively into

$$\mathbf{r}(x,y,t) \mapsto \mathbf{r}'(x,y,t) = 1_3 c + \mathbf{r}(x',y,t)\,, \tag{A.13a}$$
$$q(x,y,t) \mapsto q'(x,y,t) = q(x',y,t)\,, \tag{A.13b}$$
$$p(x,y,t) \mapsto p'(x,y,t) = p(x',y,t) - cq(x',y,t)\,, \tag{A.13c}$$

with $r_3 = (r_1, r_2, r_3)^T$, $1_3 = (1,1,1)^T$ and $x' = x - 6ct$, and

$$\mathbf{r}(x,y,t) \mapsto \mathbf{r}'(x,y,t) = \mathbf{r}(x',y',t)\,, \tag{A.13d}$$
$$q(x,y,t) \mapsto q'(x,y,t) = a + q(x',y',t)\,, \tag{A.13e}$$
$$p(x,y,t) \mapsto p'(x,y,t) = p(x',y',t)\,, \tag{A.13f}$$

with $x' = x + ay - \sigma a^2 t$ and $y' = y - 2\sigma a t$. Finally, recall that both of these transformations leave the partial KPWS (1.14) invariant [4]. Namely, if $\mathbf{r}(x,y,t)$, $q(x,y,t)$ and $p(x,y,t)$ are any solutions of the KPWS, so are $\mathbf{r}'(x,y,t)$, $q'(x,y,t)$ and $p'(x,y,t)$.

We now show that, using the above invariances, all one- and two-phase traveling wave solutions of the KP equation can be transformed to a stationary reference frame. These are the solutions of the KP equation that can be written in the form

$$u(x,y,t) = U(z_1, z_2)\,, \tag{A.14a}$$
$$z_n = k_n x + l_n y - \omega_n t, \quad n = 1,2\,. \tag{A.14b}$$

We show below that this formulation includes both classes of non-resonant elastic two-soliton solutions, the genus-2 solutions, as well as the Miles resonance solution, the one-soliton solutions and the genus-1 solutions as special cases. Starting with the two-phase solution (A.14), we apply a pseudo-rotation and Galilean boost, to obtain the new solution

$$u'(x,y,t) = c + U(z_1', z_2')\,, \tag{A.15a}$$

$$z_n' = k_n x + l_n' y - \omega_n' t\,, \quad l_n' = l_n - a k_n\,, \quad \omega_n' = \omega_n + \left(6c + \sigma a^2\right) k_n + 2\sigma a l_n\,, \tag{A.15b}$$

for $n = 1, 2$. The new solution $u'(x,y,t)$ is obviously stationary if $\omega_1' = \omega_2' = 0$. In turn, it is trivial to see that it is always possible to achieve $\omega_1' = \omega_2' = 0$ by choosing

$$a = -\frac{k_2\omega_1 - k_1\omega_2}{2\sigma\left(k_2 l_1 - k_1 l_2\right)}\,, \tag{A.16a}$$

$$c = \frac{4\sigma\left(k_1 l_2 - k_2 l_1\right)\left(l_2\omega_1 - l_1\omega_2\right) + \left(k_2\omega_1 - k_1\omega_2\right)^2}{24\sigma\left(k_2 l_1 - k_1 l_2\right)^2}\,. \tag{A.16b}$$

[Note that the denominators in (A.16) are always non-zero for genuine two-phase solutions. Conversely, if $k_2 l_1 = k_1 l_1$ the expression $u(x,y,t) = U(z_1, z_2)$ describes a one-phase solution, in which case it is sufficient to simply apply a Galilean boost.]

By definition, the two-phase representation (A.14) obviously includes all the genus-2 solutions of the KP equation (e.g. see [8]), of which the genus-1 solutions are a special case. It should then be clear that both of the non-resonant elastic two-soliton solutions as well as the Miles resonance solution and the one-soliton solutions are also included (since the former are obtained as a degeneration of the genus-2 solutions [1, 2], and the latter are in turn a degeneration of the former [9]). Nonetheless, we can give a simple proof of this fact. Recall that general soliton solutions of the KP equation can be obtained through the Wronskian formalism as [10]

$$u(x,y,t) = 6\frac{\partial^2}{\partial x^2}\left[\log\tau(x,y,t)\right]\,, \qquad \tau(x,y,t) = \mathrm{Wr}\left(f_1,\ldots,f_N\right)\,, \tag{A.17a}$$

$$f_n(x,y,t) = \sum_{m=1}^{M} C_{n,m}\mathrm{e}^{\theta_m}\,, \quad \theta_m(x,y,t) = K_m x + \sqrt{3}K_m^2 y - 4K_m^3 t\,, \quad m = 1,\ldots,M\,. \tag{A.17b}$$

In particular, the Miles resonance solution is obtained by taking $N = 1$ and $M = 3$, so that $\tau(x,y,t) = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3}$, and the two classes of non-resonant elastic two-soliton solutions are obtained by taking $N = 2$ and $M = 4$ and the following: (i) for the 'ordinary' two soliton solutions, $f_1 = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}$ and $f_2 = \mathrm{e}^{\theta_3} + \mathrm{e}^{\theta_4}$; (ii) for the 'asymmetric' two soliton solutions, $f_1 = \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_4}$ and $f_2 = \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3}$. The Miles resonance solution is then cast in the framework of (A.14) by simply writing $\tau(x,y,t) = \mathrm{e}^{\theta_1}\left(1 + \mathrm{e}^{z_1} + \mathrm{e}^{z_2}\right)$, with $z_1 = \theta_2 - \theta_1$ and $z_2 = \theta_3 - \theta_1$, since the common factor e^{θ_1} disappears from the solution (because the θ_j are linear in x) [10]. Similarly, for the ordinary two-soliton solution we have $\tau(x,y,t) = \mathrm{e}^{\theta_1+\theta_3} + \mathrm{e}^{\theta_1+\theta_4} + \mathrm{e}^{\theta_2+\theta_3} + \mathrm{e}^{\theta_2+\theta_4} = 2\mathrm{e}^{\frac{1}{2}(\theta_1+\theta_2+\theta_2+\theta_4)}(\cosh z_1 + \cosh z_2)$, where $z_1 = \frac{1}{2}(\theta_1 + \theta_3 - \theta_2 - \theta_4)$ and $z_2 = \frac{1}{2}(\theta_2 + \theta_3 - \theta_1 - \theta_4)$, and a similar representation works for the asymmetric two-soliton solution.

Finally, to complete our proof, we now show that no solutions containing more than two independent phases can be traveling wave solutions of the KP equation. (In fact, this statement applies to general nonlinear evolution equations in two spatial dimensions.) To see this, consider a generic N-phase solution $u(x,y,t) = U(z_1,\ldots,z_N)$, with z_n still given by (A.14b) for $n = 1,\ldots,N$. If $u(x,y,t)$ is a traveling wave solution, there exists a coordinate transformations $(x,y,t) \mapsto (X,Y,T)$ with $X = x - ct$, $Y = y - dt$ and $T = t$, such that $u(x,y,t) = u'(X,Y)$.

But the transformation yields $z_n = k_n(X+ct) + l_n(Y+dt) - \omega_n t$, so in order for $u'(X,Y)$ to be independent of T, we need c and d such that

$$k_n c + l_n d = \omega_n\,, \qquad n = 1,\ldots,N. \tag{A.18}$$

If $N=1$, there are an infinite number of solutions to (A.18). (In particular, one can set $d=0$ and take $c=\omega_1/k_1$.) If $N=2$, (A.18) admits a unique solution, given by $c=(\omega_1 l_2 - \omega_2 l_1)/(k_1 l_2 - k_2 l_1)$ and $d = -(\omega_1 k_2 - \omega_2 k_1)/(k_1 l_2 - k_2 l_1)$. If $N>2$, however, the system (A.18) is overdetermined, and no solution exists. (Here we assume that all phases are truly independent, which implies $k_n l_{n'} - k_{n'} l_n \neq 0$ for all $n,n' = 1,\ldots,N$ with $n \neq n'$. If this condition is violated, one can express the same solution with a smaller number of independent phases.)

A.4. Haantjes tensor test for integrability

An efficient criterion to test the diagonalizabiliy for a hydrodynamic system that does not require the computation of the eigenvalues and eigenvectors of the coefficient matrix was proposed in [20] and [19], involving the vanishing of the Haantjes tensor associated with the coefficient matrix. Specifically, for strictly hyperbolic systems, [19, 20] give the following theorem as a necessary condition for integrability: 'a hydrodynamic type system with mutually distinct characteristic speeds is diagonalizable if and only if the corresponding Haantjes tensor is identically zero.' For (1+1)- and (2+0)-dimensional conservative hydrodynamic type equations, the vanishing of the Haantjes tensor is also a sufficient condition for integrability.

The calculation of the Haantjes tensor requires calculation of the Nijenhuis tensor first. The Nijenhuis tensor of a matrix $M^i{}_j$ is defined as

$$N^i{}_{jk} = M^p_j \partial_{u^p} M^i{}_k - M^p_k \partial_{u^p} M^i{}_j - M^i{}_p \left(\partial_{u^j} M^p_k - \partial_{u^k} M^p_j\right) \tag{A.19}$$

where $\partial_{u^k} = \partial/\partial u^k$. In our case, the matrix $M^i{}_j$ is the corresponding coefficient matrix of the system for which diagonlizability is being tested. Once the Nijenhuis tensor is known, the Haantjes tensor can be obtained as

$$H^i{}_{jk} = N^i{}_{pr} M^p_j M^r_k - N^p_{jr} M^i{}_p M^r_k - N^p_{rk} M^i{}_p M^r_j + N^p_{jk} M^i{}_r M^r_p\,. \tag{A.20}$$

The calculation of the various tensors as applied to the various systems discussed in this work was performed using the Mathematica software package.

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