IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **55** (2022) 305701 (34pp) https://doi.org/10.1088/1751-8121/ac78db

Soliton resonance and web structure in the Davey–Stewartson system

Gino Biondini[1,*], **Dmitri Kireyev**[1] **and Ken-ichi Maruno**[2]

[1] Department of Mathematics, State University of New York, Buffalo, NY 14260, United States of America
[2] Department of Applied Mathematics, Waseda University, Shinjuku, Tokyo, Japan

E-mail: biondini@buffalo.edu

Received 13 April 2022
Accepted for publication 15 June 2022
Published 7 July 2022

Abstract

We write down and characterize a large class of nonsingular multi-soliton solutions of the defocusing Davey–Stewartson II equation. In particular we study their asymptotics at space infinities as well as their interaction patterns in the xy-plane, and we identify several subclasses of solutions. Many of these solutions describe phenomena of soliton resonance and web structure. We identify a subclass of solutions that is the analogue of the soliton solutions of the Kadomtsev–Petviashvili II equation. In addition to this subclass, however, we show that more general solutions exist, describing phenomena that have no counterpart in the Kadomtsev–Petviashvili equation, including V-shape solutions and soliton reconnection.

Keywords: soliton resonance, web structure, Davey–Stewartson system, Wronskian technique

(Some figures may appear in colour only in the online journal)

1. Introduction

Solutions of $(2+1)$-dimensional soliton equations have received renewed interest in recent years, and several works have showed that the kind of behavior described by them is much richer than for $(1+1)$-dimensional systems [7–9, 11, 14, 16, 28, 29, 33, 34, 38, 39, 43, 45]. In particular, the phenomenon of soliton resonance was first discovered for the Kadomtsev–Petviashvili (KP) equation [40] (see also references [41, 43, 45]). Recently, more general resonant solutions possessing a web-like structure have been observed [7–9, 11, 33, 39]. It was also conjectured in reference [11] that resonance and web structure are a generic feature of $(2+1)$-dimensional integrable systems whose solutions can be expressed in determinant form.

*Author to whom any correspondence should be addressed.

Indeed, resonant solutions with web structure have also recently been found in a coupled KP system [28, 29], in the so-called DKP equation [34] and in the two-dimensional Toda lattice and its fully discrete and ultra-discrete analogues [38].

In this paper we characterize a large class of soliton solutions of the Davey–Stewartson (DS) equation, we show that these solutions display phenoma of resonance and web structure, and we identify several subclasses. Line soliton solutions of the DSII equation were previously discussed in references [4, 20, 51] and the phenomenon of soliton resonance in the DS equation was also studied [21, 42]. It is also well-known that the DSII equation is in the KP hierarchy and is related to the KPII equation [27, 30, 44, 46]. Since the soliton solutions of the DSII equation are expressed by Wronskian or Grammian-type determinants [27, 44], once may expect that solutions similar to those of the KP equation may be found for the DSII equation. Indeed, in this work we show that an analogue to the solutions studied in references [7–9, 11, 33] exists for the defocusing DSII equation, and that a similar type of resonant solutions with web-like structure is produced as a result. To our knowledge, this is the first time that web structure is observed in the DSII equation. These results confirm that soliton resonance and web-like structures are general features of two-dimensional integrable systems whose solutions can be expressed via the determinant formalism. We also show, however, that, in addition to these solutions, there exist more general solutions of DSII, which describe phenomena that have no counterpart in the KP equation.

The outline of this work is the following. In section 2 we introduce the Wronskian formalism of solutions of the DS system and we give necessary conditions to obtain a large class of nonsingular solutions of the defocusing DSII system. In section 3 we discuss basic properties of multi-soliton solutions of the DSII system, which will be used in the remainder of the paper, and we characterize a 'restricted' class of solutions that are a direct counterpart of the soliton solutions of the KPII equation. In section 4 we begin to study unrestricted soliton solutions, starting from the simplest case: 'scalar' solutions, and we show that these solutions already give rise to novel behavior. In section 5 we discuss two further classes of solutions: non-resonant and fully resonant solutions, and in section 6 we then characterize the most general case. In section 7 we present further examples, and section 8 we end this work with a few concluding remarks.

2. The Davey–Stewartson system and its Wronskian solutions

The DS equation is the system [3, 17]

$$\mathrm{i}q_t + \frac{1}{2}\sigma q_{xx} - \frac{1}{2}q_{yy} + 2\sigma qQ + 4\sigma\nu|q|^2 q = 0, \tag{2.1a}$$

$$Q_{xx} + \sigma Q_{yy} = -4\nu(|q|^2)_{xx}, \tag{2.1b}$$

where subscripts x, y and t denote partial derivatives, with $Q = Q(x, y, t)$ and $q = q(x, y, t)$ being respectively a real-valued and a complex-valued function, and where $\sigma = \pm 1$ and $\nu = \pm 1$. The case $\sigma = -1$ identifies the DSI equation, the case $\sigma = 1$ the DSII equation; $\nu = -1$ identifies the focusing case and $\nu = 1$ the defocusing case. (The reason for this identification is that, for y-independent solutions, (2.1b) can be integrated to give $\mathrm{i}\sigma q_t + \frac{1}{2}q_{xx} - 4\nu|q|^2 q = 0$.) Equations of DS-type arise in a number of different mathematical and physical contexts, including water waves [17], plasma physics [42], optics [1, 2, 35] and ferro-magnetics [36]. The solutions of the DS system depend crucially on the signs of σ and ν. For the focusing DSI equation, exponentially localized solutions exist called dromions, corresponding to appropriate non-vanishing boundary conditions at infinity for the mean field

[13, 19, 22, 23]. The focusing DSII equation possesses line soliton solutions [51] which are unstable, algebraically decaying rational solutions called lumps [37, 51], as well as combinations of lumps and line solitons [18, 24]. For the defocusing DSII equation there are no lumps solutions [6], but dark soliton solutions exist [4, 42, 51]. Finally, the periodic solutions of all four variants were recently characterized in [10]. For these reasons, the DS system continues to be intensely studied in the literature (e.g., see [5, 31, 32, 47] and references therein).

In this work we are interested in studying and classifying a large class of soliton solutions of the defocusing DSII equation, that is, the system (2.1) with $\sigma = \nu = 1$. Consider the dependent variable transformation

$$q = \frac{G}{F}\,\mathrm{e}^{4\mathrm{i}q_o^2 t}, \quad Q = (\log F)_{xx}, \tag{2.2}$$

where F is a real function of x, y and t. The defocusing DS system is transformed via (2.2) into the bilinear forms [20, 51]

$$(2\mathrm{i}D_t + \sigma D_x^2 - D_y^2)\,G\cdot F + \frac{G}{F}((\sigma D_x^2 + D_y^2)\,F\cdot F + 8(\sigma\nu GG^* - q_0^2F^2)) = 0, \tag{2.3a}$$

$$((\sigma D_x^2 + D_y^2)\,F\cdot F + 8\sigma\nu GG^*) = (xC_1 + C_2)F^2. \tag{2.3b}$$

As we are interested in bounded solutions we take $C_1 = 0$. To simplify the system further we let $C_2 = 8q_0^2$, thus obtaining

$$(2\mathrm{i}D_t + \sigma D_x^2 - D_y^2)\,G\cdot F = 0, \tag{2.4a}$$

$$(\sigma D_x^2 + D_y^2)\,F\cdot F + 8\sigma\nu GG^* - 8q_o^2F^2 = 0, \tag{2.4b}$$

where the asterisks denotes complex conjugation, and D_x, D_y, etc are the Hirota derivatives [25]:

$$D_x^m f\cdot g = (\partial_x - \partial_{x'})^m f(x,y,t)g(x',y,t)|_{x'=x}$$

et simili. The real constant q_o is related to the background amplitude of the solution, as is most easily seen in the case of uniform fields, for which $q(x,y,t) = q_o\,\mathrm{e}^{4iq_o^2t}$. Owing to the scaling invariance of (2.1), without loss of generality hereafter we set $q_o = 1$. Now performing the change of independent variables

$$x_1 = \sigma' x + y, \qquad x_{-1} = \sigma(-\sigma' x + y), \tag{2.5a}$$

$$x_2 = -\mathrm{i}t, \qquad x_{-2} = \mathrm{i}t, \tag{2.5b}$$

where $\sigma' = \sqrt{-\sigma}$, one obtains [20, 44]

$$(D_{x_2} - D_{x_1}^2)\,G\cdot F = 0, \tag{2.6a}$$

$$(D_{x_{-2}} + D_{x_{-1}}^2)\,G\cdot F = 0, \tag{2.6b}$$

$$\sigma D_{x_1}D_{x_{-1}}F\cdot F + 2GH - 2F^2 = 0, \tag{2.6c}$$

with $H = G^*$. Note that the DS equation is related to the two-dimensional Toda lattice equation [26, 27, 44]; indeed, (2.6c) is the bilinear form of the two-dimensional Toda lattice

(see also [38]). Note also that equation (2.6) with $\sigma = 1$ (DSI defocusing) are a reduction of the two-component KP hierarchy [44]. Since in this work we are interested in studying solutions the DSII equation, hereafter we set $\sigma = 1$, which implies $\sigma' = \mathrm{i}$ and $x_{-1} = x_1^*$.

Lemma 2.1. *Solutions of the bilinear equation* (2.6) *can be obtained by the expressions*

$$F = \zeta\, \tau_N^{(s)}, \qquad G = \zeta\, \tau_N^{(s+1)}, \qquad H = \zeta\, \tau_N^{(s-1)}, \tag{2.7}$$

where $s \in \mathbb{Z}$ and $\zeta \in \mathbb{C}$ are arbitrary constants, with [20, 44]

$$\tau_N^{(n)} = \mathrm{Wr}_{x_1}(f_1^{(n)}, f_2^{(n)}, \ldots, f_N^{(n)}) = \det \begin{pmatrix} f_1^{(n)} & f_1^{(n+1)} & \cdots & f_1^{(n+N-1)} \\ f_2^{(n)} & f_2^{(n+1)} & \cdots & f_2^{(n+N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N^{(n)} & f_N^{(n+1)} & \cdots & f_N^{(n+N-1)} \end{pmatrix}, \tag{2.8}$$

and where $f_1(x_1, x_{-1}, x_2, x_{-2}), \ldots, f_N(x_1, x_{-1}, x_2, x_{-2})$ is any set of N linearly independent solutions of the linear equations

$$\frac{\partial f^{(j)}}{\partial x_1} = f^{(j+1)}, \qquad \frac{\partial f^{(j)}}{\partial x_{-1}} = f^{(j-1)}, \tag{2.9a}$$

$$\frac{\partial f^{(j)}}{\partial x_2} = f^{(j+2)}, \qquad \frac{\partial f^{(j)}}{\partial x_{-2}} = f^{(j-2)}, \tag{2.9b}$$

with $f^{(0)} = f$.

As with the KP equation, lemma 2.1 is a consequence of the Plücker relations for the determinants (2.8) [20]. It is crucial to realize, however, that these solution of equation (2.6) produce solutions of the DS equation (2.1) only when the corresponding functions F, G and H obtained from them via (2.7) satisfy the following two 'reality constraints': (i) F is real and (ii) $H = G^*$. These requirements impose a restriction on admissible sets of functions $f_1, \ldots, f_N$ as well as on the constants s and ζ. We next show how to satisfy the above-mentioned requirements and thus obtain general line-soliton solutions of the defocusing DSII equation:

Lemma 2.2. *Soliton solutions of the defocusing DSII equation can be obtained from* (2.7)–(2.9) *by taking $f_1, \ldots, f_N$ to be arbitrary real linear combinations of exponentials:*

$$f_n = \sum_{m=1}^{M} a_{n,m}\, \mathrm{e}^{\xi_m}, \tag{2.10}$$

with $a_{n,m} \in \mathbb{R}$, for $n = 1, \ldots, N$ and $m = 1, \ldots, M$ with $M > N$, and where the exponential phases $\xi_1, \ldots, \xi_M$ are linear functions of $(x_1, x_{-1}, x_2, x_{-2})$:

$$\xi_m = p_m x_1 + p_m^{-1} x_{-1} + p_m^2 x_2 + p_m^{-2} x_{-2} + \xi_{0,m}, \tag{2.11}$$

with $p_m = \mathrm{e}^{-\mathrm{i}\phi_m}$ and $\phi_m \in \mathbb{R}$ for $m = 1, \ldots, M$, and by taking

$$\zeta = 1/(2\mathrm{i})^{N(N-1)/2}, \qquad s = -(N-1)/2. \tag{2.12}$$

In particular, the first and second equations of (2.12) ensure that the first and second reality constraints are satisfied. The coefficients $a_{n,m}$ define the $N \times M$ *coefficient matrix* $A = (a_{n,m})$.

We will call the constants $\phi_1, \ldots, \phi_M$ the *phase parameters*. In terms of the original coordinates (x, y, t), the phases $\xi_1, \ldots, \xi_M$ become

$$\xi_m(x, y, t) = 2\ [x\ \sin\ \phi_m + y\ \cos\ \phi_m - t\ \sin(2\phi_m)] + \xi_{0,m}. \tag{2.13}$$

Without loss of generality we can take the phase parameters to be distinct and well-ordered, namely we assume $\phi_1 < \cdots < \phi_M$. Also, since $\xi_m(x, y, t)$ is a periodic function of ϕ_m, without loss of generality we can restrict the phase parameters $\phi_1, \ldots, \phi_M$ to be in the interval $[-\pi, \pi)$. Note that an equivalent way to obtain the same functions F, G and G^* from (2.7) and (2.10) is to set $s = 0$ in (2.7) while at the same time multiplying each exponential term in (2.10) by $\mathrm{e}^{-\mathrm{i}(N-1)\phi_m/2}$.

The proof of lemma 2.2 (that is, the fact that (2.12) and (2.11) indeed generate soliton solutions of the defocusing DSII equation via (2.2) and (2.7)) is an immediate consequence of the following:

Lemma 2.3. *Let $f_1, \ldots, f_N$ be given by (2.10), with $\xi_1, \ldots, \xi_M$ given by (2.13). Then, for $1 \leqslant N \leqslant M - 1$ the tau functions defined by (2.8) have the form*

$$\tau_{N,M}^{(n)} = (2\mathrm{i})^{N(N-1)/2} \sum_{1 \leqslant m_1 < \cdots < m_N \leqslant M} \Delta_{m_1,\ldots,m_N} A(m_1, \ldots, m_N)\, \mathrm{e}^{\xi_{m_1,\ldots,m_N} - \mathrm{i}[n+(N-1)/2]\phi_{m_1,\ldots,m_N}}, \tag{2.14}$$

where

$$\xi_{m_1,\ldots,m_N} = \xi_{m_1} + \cdots + \xi_{m_N}, \quad \phi_{m_1,\ldots,m_N} = \phi_{m_1} + \cdots + \phi_{m_N}, \tag{2.15a}$$

$$\Delta_{m_1,\ldots,m_N} = \prod_{1 \leqslant j < k \leqslant N} \sin\left[\frac{1}{2}(\phi_{m_j} - \phi_{m_k})\right], \tag{2.15b}$$

and $A(m_1, \ldots, m_N)$ denotes the $N \times N$ minor of A

$$A(m_1, \ldots, m_N) = \det(A[m_1, \ldots, m_N]). \tag{2.15c}$$

Above and throughout this work, $A[m_1, \ldots, m_k]$ denotes the $N \times k$ matrix obtained by selecting columns $m_1, \ldots, m_k$ of A, namely

$$A[m_1, \ldots, m_k] = \begin{pmatrix} a_{1,m_1} & \ldots & a_{1,m_k} \\ \vdots & & \vdots \\ a_{N,m_1} & \ldots & a_{N,m_k} \end{pmatrix}. \tag{2.16}$$

proof. Note first that, equations (2.9) and (2.11) allow us to write each of the columns appearing in the determinant on the right-hand side of (2.8) as

$$(f_1^{(n)}, \ldots, f_N^{(n)})^{\mathrm{t}} = A\,(\mathrm{e}^{\xi_1 - \mathrm{i}n\phi_1}, \ldots, \mathrm{e}^{\xi_M - \mathrm{i}n\phi_M})^{\mathrm{t}},$$

where the superscript t denotes matrix transpose. Combining all of these columns, we can therefore write the tau functions (2.8) as

$$\tau_{N,M}^{(n)} = \det\left(A\,\Theta\,K^{(n)}\right), \tag{2.17}$$

where $\Theta = \exp[\,\mathrm{diag}(\xi_1,\ldots,\xi_M)]$, and the $M\times N$ matrix $K^{(n)}$ is given by

$$K^{(n)} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}n\phi_1} & \mathrm{e}^{-\mathrm{i}(n+1)\phi_1} & \ldots & \mathrm{e}^{-\mathrm{i}(n+N-1)\phi_1} \\ \mathrm{e}^{-\mathrm{i}n\phi_2} & \mathrm{e}^{-\mathrm{i}(n+1)\phi_2} & \ldots & \mathrm{e}^{-\mathrm{i}(n+N-1)\phi_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{-\mathrm{i}n\phi_M} & \mathrm{e}^{-\mathrm{i}(n+1)\phi_M} & \ldots & \mathrm{e}^{-\mathrm{i}(n+N-1)\phi_M} \end{pmatrix}. \tag{2.18}$$

Equation (2.14) then follows from (2.17) via the Binet–Cauchy theorem, as in [8, 11, 33, 38]. □

At this point it is useful to make several remarks:

(a) An immediate consequence of lemma 2.3 is that F is real-valued, and that G and H are complex conjugates of each other. Thus, all of the solutions obtained from lemma 2.2 are indeed solutions of the DSII equations, as anticipated earlier.

(b) If $M < N$, the functions $f_1,\ldots,f_N$ are linearly dependent, and hence they generate the trivial solution of DS. Similarly, if $\mathrm{rank}\,A < N$ equation (2.14) produce trivial solutions of DS. Finally, if $M = N$ there is only one term in the summation in equation (2.14); as a consequence, q is constant, and Q is identically zero. Hence for all non-trivial and non-constant solutions it is $M > N$.

(c) Each term in the right-hand side of (2.14) contains combinations of N *distinct* phases $\xi_{m_1},\ldots,\xi_{m_N}$ chosen from $\xi_1,\ldots,\xi_M$. However, a given phase combination $\xi_{m_1,\ldots,m_N}$ is actually present in the tau functions if and only if the corresponding minor $A(m_1,\ldots,m_N)$ is non-zero.

(d) The only (x,y,t)-dependence of the tau functions comes from the exponential phases $\xi_1,\ldots,\xi_M$. Moreover, $\Delta_{m_1,\ldots,m_N}$ is strictly positive for all $1 \leqslant m_1 < m_2 < \cdots < m_N \leqslant M$, because $\phi_1,\ldots,\phi_M \in [-\pi,\pi)$. Hence, a sufficient condition to obtain non-singular solutions is that all minors of the coefficient matrix A be non-negative. (Indeed, this is enough to ensure that Q is positive definite, similarly as for the KP equation; cf reference [8].) It is not clear at present, however, whether this condition is also necessary.

(e) The fact that all these solutions are soliton solutions is a consequence of the exponential nature of $f_1,\ldots,f_N$, and can be shown in a similar way as for the KP equation [8], as we discuss in section 3.

(f) Transformations $A \to GA$ with $G \in \mathrm{GL}(N,\mathbb{R})$ (corresponding to performing elementary row operations on A, i.e., choosing N independent linear combinations of the functions $f_1,\ldots,f_N$ in equation (2.10)) amount to an overall rescaling $\tau_{N,M}^{(n)}(x,y,t) \to \det(G)\,\tau_{N,M}^{(n)}(x,y,t)$, which leaves the solution of DS invariant. This $\mathrm{GL}(N,\mathbb{R})$ gauge freedom can be exploited to choose the coefficient matrix A to be in reduced row-echelon form (RREF).

(g) In addition, the coefficient matrix A can be further reduced by normalizing each of its rows if desired, since a positive overall multiplicative constant in each column of the coefficient matrix A can be absorbed in the constants $\xi_{0,1},\ldots,\xi_{0,M}$.

In light of the above remarks, to avoid trivial and reducible cases from now on and throughout this work we will assume that: (a) $M > N$ and $\mathrm{rank}(A) = N$, (b) all non-zero $N\times N$ minors

of A are positive, (c) A is in RREF, (d) A is *irreducible*. We say that a matrix A of rank N is irreducible if, in RREF:

(a) Each column of A contains at least one non-zero element.
(b) Each row of A contains at least one non-zero element in addition to the pivot.

If condition (a) is violated, one of the exponential phases does not appear in the tau functions, which can then be rewritten in terms of linear combinations of $M-1$ phases. Also, if condition (b) is violated, one of the functions in (2.10) contains only one exponential term. That term can then be factorized in (2.14) and, similarly to the KP equation [8], one can therefore represent the solution in terms of $N-1$ functions containing linear combinations of $M-1$ phases.

3. Line solitons of the defocusing Davey–Stewartson II equation

Before studying general solutions displaying soliton resonance and web structure, it is useful to look at simpler solutions. We do so next, starting from one-soliton solutions, and then considering special subclasses of multi-soliton solutions.

3.1. One-soliton solutions

The simplest nontrivial solutions of the DS equation are obtained for $N=1$ and $M=2$. Choosing $\tau_1^{(n)} = \mathrm{e}^{\xi_1 - \mathrm{i}n\phi_1} + \mathrm{e}^{\xi_2 - \mathrm{i}n\phi_2}$, with ξ_1 and ξ_2 defined as in lemma 2.3, we have $s=0$ and $\zeta=1$, and (2.7) yield $F=\tau_1^{(0)}$ and $G=\tau_1^{(1)}$. In turn, (2.2) then leads to the one-soliton solution:

$$Q(x,y,t) = (\sin\,\phi_1 - \sin\,\phi_2)^2 \operatorname{sech}^2\left[\frac{1}{2}(\xi_1-\xi_2)\right]. \tag{3.1}$$

In the xy-plane this solution describes a plane wave $Q(x,y,t)=\Phi(\Xi_{i,j})$, with $\Xi_{i,j}(x,y,t)=\frac{1}{2}(\xi_i-\xi_j)=\mathbf{k}\cdot\mathbf{x}-\omega\,t$, where $\mathbf{x}=(x,y)$. The wavevector $\mathbf{k}=\mathbf{k}_{i,j}=(k_x,k_y)$ and frequency $\omega=\omega_{i,j}$ are given by

$$\mathbf{k}_{i,j} = (\sin\,\phi_i - \sin\,\phi_j, \cos\,\phi_i - \cos\,\phi_j), \tag{3.2a}$$

$$\omega_{i,j} = \sin(2\phi_i) - \sin(2\phi_j). \tag{3.2b}$$

These parameters satisfy the nonlinear dispersion relation

$$\omega^2 = (k_x^2-k_y^2)^2\left(\frac{4}{k_x^2+k_y^2}-1\right), \tag{3.3}$$

where both signs are admissible for ω. The above solution is localized around the (contour) line $\xi_1=\xi_2$ in the xy-plane, and is therefore referred to as a *line soliton*, like for the KP equation. When discussing the pattern of soliton solutions in the xy-plane, we will refer to $c=\mathrm{d}x/\mathrm{d}y$ as the *direction* of the line soliton. That is, $c=\tan\theta$ where θ is the angle between the line soliton and the positive y-axis, counted clockwise. Denoting by $c_{1,2}$ the direction of the solution (3.1), it is

$$c_{i,j} = -\frac{\cos\,\phi_i - \cos\,\phi_j}{\sin\,\phi_i - \sin\,\phi_j} = \tan(\theta_{i,j}), \tag{3.4}$$

$$\theta_{i,j} = \frac{1}{2}(\phi_i+\phi_j), \tag{3.5}$$

for all i, j such that $\theta_{i,j} \neq \pm\pi/2$. When $\theta_{i,j} = \pm\pi/2$, the line soliton is parallel to the x-axis, in which case we say that $c_{i,j} = \infty$.

The dependent variable $q(x, y, t)$ displays a similar type of behavior, namely:

$$\begin{aligned} q(x,y,t) &= \left(\mathrm{e}^{\xi_2 - \mathrm{i}\phi_2 + 4\mathrm{i}t} + \mathrm{e}^{\xi_1 - \mathrm{i}\phi_1 + 4\mathrm{i}t}\right) / \left(\mathrm{e}^{\xi_2} + \mathrm{e}^{\xi_1}\right) \\ &= \left(\cos\left[\frac{1}{2}(\phi_1 - \phi_2)\right] - \mathrm{i}\,\sin\left[\frac{1}{2}(\phi_1 - \phi_2)\right]\tanh\left[\frac{1}{2}(\xi_1 - \xi_2)\right]\right)\mathrm{e}^{-\frac{1}{2}\mathrm{i}(\phi_1 + \phi_2 - 8t)}. \end{aligned} \tag{3.6}$$

In particular, $|q(x, y, t)|^2$ is also a traveling wave, like $Q(x, y, t)$, localized around the line $\xi_1 = \xi_2$. There are important differences between the two components, however. The first difference is that q has dark-soliton behavior, while Q has bright-soliton behavior. For this reason we refer to q and Q respectively as the dark and bright components of the solution. The second difference is related to the dependence of the amplitude on the two solution components on the soliton direction. Equations (3.1) and (3.6) yield the amplitude of the bright and dark components of the solution respectively as

$$\max\, Q = (\sin\,\phi_i - \sin\,\phi_j)^2, \tag{3.7a}$$

$$1 - \min|q|^2 = \sin^2\left[\frac{1}{2}(\phi_i - \phi_j)\right]. \tag{3.7b}$$

(Since q has dark-soliton type behavior, its amplitude is defined as the maximum dip from the background value.) As in the KP equation we refer to the soliton amplitude and direction as the *soliton parameters*. The amplitude of $Q(x, y, t)$ vanishes whenever $\sin\phi_i = \sin\phi_j$, i.e., for horizontal solitons. More in general, it is easy to show that, for each value of θ, one can only realize solitons whose bright component has amplitude $0 \leqslant \max Q \leqslant 4\cos^2\theta$. On the other hand, the amplitude of the dark component never vanishes. Indeed, one can realize solutions where $1 - \min|q|^2$ takes any value between 0 and 1, independently of θ. In particular, the maximum amplitude is obtained for $\phi_{i,j} = \theta \pm \pi/2$.

The above discussion indicates that, even in the more complicated multi-solutions discussed later, all horizontal solitons will disappear from the bright component Q, but not from the dark component q. This difference between the bright and dark components stems from the fact that the bright component is obtained as a second logarithmic derivative of the tau function with respect to x, as in the KP equation (and unlike the dark component). Indeed, the case $c = \infty$ does not have a counterpart in solutions of the KP equation, where no horizontal solitons exist. As we will see in section 4.3, this feature gives rise to novel behavior compared to that of the soliton solutions of the KP equation.

Plots of the bright and dark components of a one-soliton solution are shown in figure 1. Note that each soliton direction can be realized with two possible values of θ in $[-\pi, \pi)$, and therefore to two possible choices of ϕ_i and ϕ_j (cf (3.4)). A similar statement holds for the soliton amplitude. This redundancy can be eliminated in the direct problem (i.e., when constructing a solution from a given set of soliton parameters, as in [7]), but not in the inverse problem (i.e., the problem of identifying the solitons corresponding to a given coefficient matrix and set of phase parameters). A related and important difference between the DSII equation and the KP equation is the existence of solitons with the same amplitude and direction, but opposite velocity. Equation (3.2) imply that the maps $\phi_n \mapsto \tilde{\phi}_{i,j} = \phi_n + \pi$ for $n = i, j$ results in $\mathbf{k}_{i,j} \mapsto \tilde{\mathbf{k}}_{i,j} = -\mathbf{k}_{i,j}$, but leaves $\omega_{i,j}$ invariant: $\tilde{\omega}_{i,j} = \omega_{i,j}$. Therefore, the transformation produces a soliton with same amplitude and direction as the original one, (i.e., $\tilde{a}_{i,j} = a_{i,j}$ and $\tilde{c}_{i,j} = c_{i,j}$)

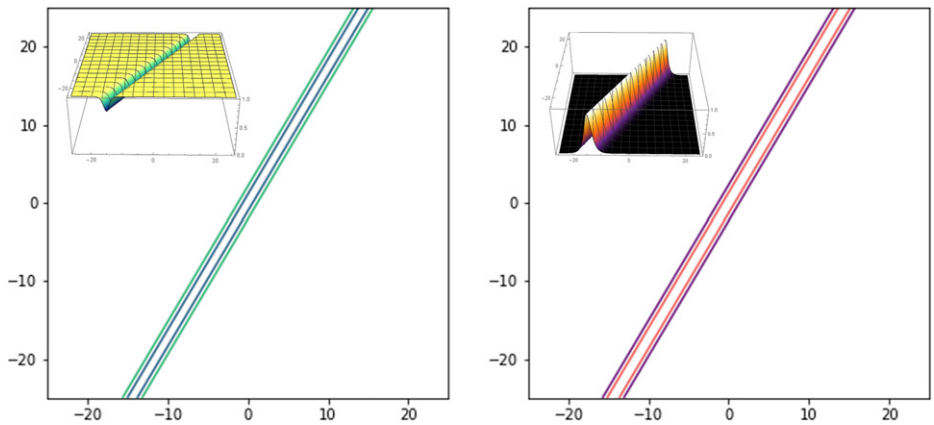

Figure 1. One-soliton solution of the defocusing DSII equation generated by $\tau_{1,2}^{(n)}$ at $t = 0$ with $(\phi_1, \phi_2) = (-2\pi/3, \pi)$. (Note one-soliton solutions are traveling wave solutions, so the time dependence in this case amounts to a simple translation.) Left column: $|q(x, y, 0)|$. Right column: $Q(x, y, 0)$. Here, and in all other subsequent figures, the horizontal and vertical axis are respectively the x-axis and y-axis, and the graph shows a contour plot of the solution in the xy-plane. Insets: the corresponding three-dimensional plots.

but opposite speed. As we will see in section 4.3, this feature also gives rise to novel behavior compared to that of the soliton solutions of the KP equation.

3.2. Multi-soliton solutions: dominant phase combination and asymptotic line solitons

We now begin to investigate the behavior of more general solutions of the DS equation obtained via lemma 2.2.

Definition 3.1. We say that an exponential term $\exp[\xi_{m_1,\ldots,m_N}]$ is *dominant* in some region $R \in \mathbb{R}^3$ if, for all $(x, y, t) \in R$, $\exp[\xi_{m_1,\ldots,m_N}] \geqslant \exp[\xi_{m'_1,\ldots,m'_N}]$ for all $\{m'_1, \ldots, m'_N\}$ appearing in the tau function—that is, for all $\{m'_1, \ldots, m'_N\}$ such that $A(m'_1, \ldots, m'_N) \neq 0$. The region R is called the dominant region of $\xi_{m_1,\ldots,m_N}$.

The above definition is the same as for the KP equation [8]. Similarly to the KP equation, we also have:

Lemma 3.2. *The solution $q(x, y, t)$ of the defocusing DSII equation generated by the tau function (2.17) is constant up to exponentially small terms, and $Q(x, y, t)$ is exponentially small, at every point in the interior of every dominant region.*

The proof of lemma 3.2 is essentially identical to that for the KP equation [8], and is therefore omitted. Lemma 3.2 implies that, like with the KP equation, the solution is localized at the boundaries of the dominant regions, where a *dominant balance* exists, i.e., a balance between exponential phase combinations that dominate over all others. These regions are characterized by the following:

Theorem 3.3. *Asymptotically as $x^2 + y^2 \to \infty$, and for generic values of the phase parameters $\phi_1, \ldots, \phi_M$:*

(*a*) *At finite times, the set of dominant phase combinations is invariant.*

(*b*) *In any two adjacent dominant regions, the dominant phase combinations contain* $N-1$ *common phases.*

The proof of theorem 3.3 is identical to that of theorem 2.5 in [8] and proposition 5 in [15]. An important consequence of theorem 3.3 is that solutions obtained from lemma 2.2 are all solitonic in character. That is, asymptotically, as $x^2+y^2\to\infty$, the solution of the DS system generated by lemma 2.1 is either locally constant or locally given by a suitable one-soliton solution corresponding to the transition between two indices i and j identifying two adjacent dominant phase combinations, i.e., the phases ξ_i and ξ_j appearing in (3.1) and (3.6). Accordingly, as in the KP equation [8, 33], we can introduce the following definition:

Definition 3.4. We call an asymptotic line soliton each local solution of the DS system identified by an *index pair* $[i,j]$ that labels the non-common phases between two adjacent dominant phase combinations as $x^2+y^2\to\infty$.

Remark 3.5. It is straightforward to see that, if all phase parameters are shifted by a common amount (mod 2π), i.e., if $\phi_j\mapsto\phi_j+\Delta\phi \bmod 2\pi$, the new phase parameters generate a solution in which all asymptotic line solitons are rotated by an angle $\Delta\phi$.

3.3. Restricted solutions and one-to-one correspondence with the KP equation

Before studying the most general soliton solutions generated by the tau function (2.17), it is instructive to first consider a subclass of solutions for which the phase parameters $\phi_1,\dots,\phi_M$ lie in the range

$$-\pi/2\leqslant\phi_1<\cdots<\phi_M<\pi/2. \tag{3.8}$$

We call these *restricted solutions* of the defocusing DSII system. Note that horizontal solitons are excluded from this class. Therefore, for these kinds of solutions we can unambiguously divide all line solitons into two categories: we will call *outgoing* solitons those extending out to infinity in any direction in the first and second quadrants of the xy-plane (i.e., as $y\to\infty$), and *incoming* solitons those extending out in any direction in the third and fourth quadrants (i.e., as $y\to-\infty$).

Moreover, when the phase parameters satisfy (3.8), if one lets $x=cy+x_o$ and considers the asymptotic behavior as $y\to\pm\infty$ with x_o fixed, the transition direction c becomes an increasing function of $\phi_i+\phi_j$, as can be seen from (3.4). With the above definitions, one can then apply to the class of restricted solutions the tools of analysis that were developed to study soliton solutions of the KP equation [7, 8, 11, 33], including in particular the methods of reference [8] for studying the asymptotics of the tau functions as $y\to\pm\infty$ as long as one takes into account that the roles of $y\to\infty$ and $y\to-\infty$ are reversed compared to what happens in the KP equation (e.g., compare the inequalities in lemma 6.1 with those in lemma 3.1 of [8].) Here we just state the main result. (Further details will be given when we present the analysis for more general solutions in section 6.) Let $P_{[i,j]}$ and $Q_{[i,j]}$ denote the sub-matrices obtained by selecting the following columns, with $i<j$ of A:

$$P_{[i,j]}:=A[1,2,\dots,i-1,j+1,\dots,M],\qquad Q_{[i,j]}:=A[i+1,\dots,j-1]. \tag{3.9}$$

Theorem 3.6. *Let* $\tau^{(n)}_{N,M}(x,y,t)$ *be the tau functions in* (2.14) *associated with a rank* N, *irreducible coefficient matrix* A *with non-negative minors. If the phase parameters* $\phi_1,\dots,\phi_M$ *satisfy* (3.8)*:*

(*a*) *For each pivot index e_n there exists a unique asymptotic line soliton as $y \to -\infty$, identified by an index pair $[e_n, j_n]$ with $n = 1, \ldots, N$ and $1 \leqslant i_n < j_n \leqslant M$.*

(*b*) *For each non-pivot index g_n there exists a unique asymptotic line soliton as $y \to \infty$, identified by an index pair $[i_n, j_n]$ with $n = 1, \ldots, M - N$ and $1 \leqslant i_n < g_n \leqslant M$.*

(*c*) *The index pairs that identify the asymptotic line solitons are uniquely determined by the following necessary and sufficient conditions. Let* $\operatorname{rank}(P_{[i,j]}) = k$ *and* $\operatorname{rank}(Q_{[i,j]}) = s$. *Then:*

1. *The pair $[i, j]$ identifies an asymptotic line-solitons as $y \to -\infty$ iff $k \leqslant N - 1$ and* $\operatorname{rank}(P_{[i+1,j]}) = \operatorname{rank}(P_{[i,j-1]}) = \operatorname{rank}(P_{[i+1,j-1]}) = k + 1$.
2. *The pair $[i, j]$ identifies an asymptotic line-soliton as $y \to \infty$ iff $s \leqslant N - 1$ and* $\operatorname{rank}(Q_{[i-1,j]}) = \operatorname{rank}(Q_{[i,j+1]}) = \operatorname{rank}(Q_{[i-1,j+1]}) = s + 1$.

A direct consequence of theorem 3.6 is that, if the phase parameters $\phi_1, \ldots, \phi_M$ satisfy (3.8), the solution of DS generated by the coefficient matrix A via equation (2.14) has exactly $N_- = M - N$ incoming line solitons and $N_+ = N$ outgoing line solitons. The proof of theorem 3.6 is exactly the same as for all the corresponding statements given in reference [8] for the KP equation, and is therefore omitted for brevity.

4. Unrestricted multi-soliton solutions: scalar solutions

We now begin to lift the restriction (3.8) on the phase parameters $\phi_1, \ldots, \phi_M$ and allow them to vary in the whole range $[-\pi, \pi)$, subject only to the ordering

$$-\pi \leqslant \phi_1 < \cdots < \phi_M < \pi. \tag{4.1}$$

We refer to solutions in this class as *unrestricted* solutions of the defocusing DSII system.

4.1. Transformation to radial coordinates

When the restriction (3.8) on the phase parameters $\phi_1, \ldots, \phi_M$ is removed, the asymptotic behavior of the τ functions as $y \to \pm\infty$ is not an appropriate tool to describe the solution, (unlike what happens with the KP equation and with restricted solutions of the DS system). Correspondingly, the distinction between incoming and outgoing solitons loses relevance, and must be abandoned. This complication is also related to the coming into play of new kinds of solutions, as we show below. At this point we need to recall that the definition of asymptotic soliton in section 3.2 allows for rays extending out to infinity in *any* direction in the xy-plane, including that of the x-axis. To this end, and to allow for the possibility of horizontal rays, hereafter it is convenient to convert from Cartesian coordinates to polar coordinates by letting

$$(x, y) = (r \sin\theta, r \cos\theta), \qquad r = \sqrt{x^2 + y^2}, \quad \theta = \arctan x/y. \tag{4.2}$$

We will then study the asymptotics of solutions as $r \to \infty$ for different values of θ, the angle from the positive y-axis, increasing clockwise. Using (4.2), we can rewrite the phases ξ_m as

$$\xi_m = 2r \cos(\theta - \phi_m) - 2 \sin(2\phi_m)t + \xi_{0,m}. \tag{4.3}$$

It will also be useful to write down explicitly the difference of two phases:

$$\xi_m - \xi_{m'} = 4\left[r\,\sin(\theta - \frac{1}{2}(\phi_m + \phi_{m'}))\sin\left(\frac{1}{2}(\phi_m - \phi_{m'})\right)\right.$$
$$\left. - t\,\cos(\phi_m + \phi_{m'})\sin(\phi_m - \phi_{m'})\right] + (\xi_{m,0} - \xi_{m',0}). \tag{4.4}$$

Anticipating the results of section 6, we will see that, in general, the tau functions $\tau_{N,M}^{(n)}$ generate solutions with a total of M asymptotic line solitons located along specific directions in the xy-plane, identified by sweeping a 2π range of values of θ.

4.2. Scalar soliton solutions

We begin by discussing the simplest possible unrestricted solutions, which is that of solutions obtained for $N = 1$. By analogy with the KP equation, we refer to this as the *scalar* case (since in this case there is only one function f in each of the determinants (2.8)). Later on, we will generalize the results to cases in which N is an arbitrary positive integer. However, we will see that, even in the case $N = 1$, there are significant differences between the solutions of the DS system and those of the KP equation.

Theorem 4.1. *Let $\tau_1^{(n)}(x, y, t)$ be the tau functions in (2.14) associated with a $1 \times M$ coefficient matrix A with positive entries and with phase parameters $\phi_1, \ldots, \phi_M$. As $r \to \infty$, the corresponding solution of the defocusing DSII system generated via lemmas 2.1 and 2.2 comprises M asymptotic line solitons, $M - 1$ of which localized along the directions $\theta = \theta_{i,j}$, with $[i, j]$ given by the index pairs $[1, 2], [2, 3], \ldots, [M - 1, M]$, with $\theta_{i,j}$ given by (3.5), plus one soliton localized along the direction $\theta = \theta_{i,j} + \pi$, with $[i, j] = [M, 1]$.*

proof. We begin by noting that, when $N = 1$ and $M > 1$, the entries of the coefficient matrix $A = (a_1, a_2, a_3, \ldots, a_M)$ can be absorbed in the constants $\xi_{0,m}$. Thus, without loss of generality, we can simply write the tau-functions as

$$\tau_{1,M}^{(n)}(x, y, t) = \sum_{m=1,\ldots,M} \mathrm{e}^{\xi_m - \mathrm{i}n\phi_m}, \quad n = -1, 0, 1. \tag{4.5}$$

Using polar coordinates as discussed above to write the phases ξ_m as in (4.3), the ordering (4.1) then allow us to find the dominant regions R_m corresponding to each of the dominant phases.

We first consider the case $m = 2, \ldots, M - 1$. By the definition of R_m, we seek all values $0 \leqslant \theta \leqslant 2\pi$ such that

$$\cos(\theta - \phi_s) \leqslant \cos(\theta - \phi_m) \quad \forall\ s = 1, \ldots, M. \tag{4.6}$$

Observe that $\cos(\theta - \phi_m)$ achieves its maximum at $\theta = \phi_m$. Moreover, the mth cosine curve intersects the sth cosine curve at $\theta = (\phi_m + \phi_s)/2$ in the interval $\phi_1 \leqslant \theta \leqslant \phi_M$ (e.g., see figure 2). Thus, the dominant region R_m is given by the following range of values of θ:

$$\frac{1}{2}(\phi_{m-1} + \phi_m) \leqslant \theta \leqslant \frac{1}{2}(\phi_m + \phi_{m+1}), \quad m = 2, \ldots, M - 1. \tag{4.7a}$$

The cases $m = 1$ and $m = M$ are studied by taking advantage of the periodicity of the solution with respect to θ. Following similar arguments as before, one can then show that the dominant regions R_1 and R_M are respectively given by the ranges

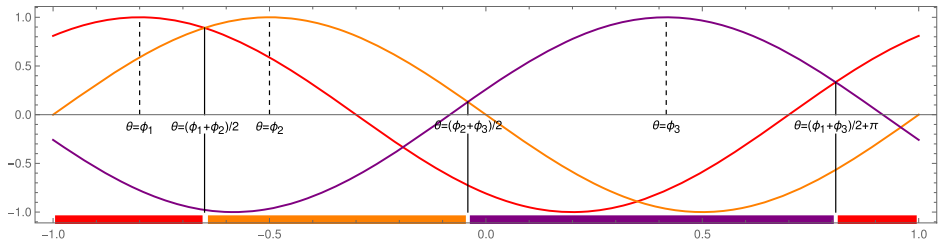

Figure 2. Cosine curves $\cos(\theta - \phi_m)$ for $\phi_1 = -4\pi/5$ (red), $\phi_2 = -\pi/2$ (orange) and $\phi_3 = 5\pi/12$ (purple) and their respective intersection points. The ranges of values of θ corresponding to the dominant region for each phase are also shown (with corresponding colors) at the bottom of the plot.

$$\frac{1}{2}(\phi_M + \phi_1) - \pi \leqslant \theta \leqslant \frac{1}{2}(\phi_1 + \phi_2), \tag{4.7b}$$

$$\frac{1}{2}(\phi_{M-1} + \phi_M) \leqslant \theta \leqslant \frac{1}{2}(\phi_M + \phi_1) + \pi. \tag{4.7c}$$

Note the $-\pi$ shift in the first inequality of (4.7b) and the π shift in the second inequality of (4.7c). These shifts arise because there are two values of θ such that $\cos(\phi_1 - \theta) = \cos(\phi_M - \theta)$. Unlike what happens when $m = 2, \ldots, M-1$, however, the value at which ϕ_1 and ϕ_M are the dominant phases occurs outside the range $\phi_1 \leqslant \theta \leqslant \phi_M$. This is the second time we encounter such a π shift (the first one being the shift in the directions of the solitons associated to the pivots in theorem 3.6), and a similar shift will reappear throughout this work (e.g., see part (b) of theorem 5.3, parts (c) and (4) of lemma 6.1 and part (b) of theorem 6.6).

Once the dominant regions are known, the transition lines are easily identified. Specifically, each dominant region R_m corresponds to the range $\theta_{m,-} < \theta < \theta_{m,+}$, with the angles $\theta_{m,\pm}$ (in increasing order) identifying the transition lines between these regions (and therefore their boundaries). In light of the above discussion, these angles are given by

$$\theta_{1,-} = \frac{1}{2}(\phi_M + \phi_1) - \pi, \tag{4.8a}$$

$$\theta_{m-1,+} = \theta_{m,-} = \frac{1}{2}(\phi_{m-1} + \phi_m), \quad m = 2, \ldots, M, \tag{4.8b}$$

$$\theta_{M,+} = \frac{1}{2}(\phi_M + \phi_1) + \pi. \tag{4.8c}$$

In conclusion, as θ increases within a 2π range, the following dominant phase transitions take place consecutively: $1 \to 2 \to \cdots \to M$ and $M \to 1$. For $m = 2, \ldots, M$, the angles $\theta_{m-1,+} = \theta_{m,-}$ are within the range $[-\pi, \pi)$. Moreover, $\theta_{M,+} = \theta_{1,-} + 2\pi$. Note however, that only one between $\theta_{1,-}$ and $\theta_{M,+}$ is within the range $[-\pi, \pi)$. For consistency, one needs to use the value within this range in the discussion.

□

Remark 4.2. The scalar case of soliton solutions of the defocusing DSII system is analog to the scalar case of the soliton solutions for the KP equation [39]. The crucial difference is the fact for the KP equation one can distinguish between outgoing line solitons (i.e., asymptotic

solitons as $y \to \infty$) and incoming line solitons (i.e., asymptotic solitons as $y \to -\infty$). In the DS system, in contrast, no such distinction is possible and one can have arbitrary numbers of asymptotic solitons above or below the x-axis depending on the value of the phase parameters (as will also be true in the general case $N > 1$), except for the following:

Lemma 4.3. *When $M > 2$, the solution generated by any collection of phase parameters $\phi_1, \ldots, \phi_M$ there is always at least one asymptotic soliton strictly above and at least one strictly below the* x*-axis.*

proof. We prove the result for the upper half-plane. By way of contradiction, suppose that we have a collection of phase parameters such that the resulting soliton configuration only has asymptotic line solitons in the lower half-plane (including the x-axis), thus implying that there is a single dominant region R_m, for some $m \in [1, M]$, containing all of the upper half-plane (i.e., the whole range $-\pi/2 \leqslant \theta \leqslant \pi/2$). Therefore, we must have that

$$-\pi \leqslant \theta_{m,-} \leqslant -\pi/2, \qquad \pi/2 \leqslant \theta_{m,+} < \pi. \tag{4.9}$$

If $2 \leqslant m \leqslant M-1$, recalling (4.8b), the conditions $\theta_{m,-} < \pi/2$ and $\theta_{m,+} \geqslant \pi/2$ imply $\phi_{m+1} - \phi_{m-1} > 2\pi$. But this is impossible, since $\phi_m \in [-\pi, \pi)$. Hence (4.9) can never be satisfied for $m = 2, \ldots, M-1$.

It remains to exclude the possibility that $m = 1$ or $m = M$. If $m = 1$, (4.9) and (4.8a) together imply $0 \leqslant \phi_1 + \phi_M \leqslant \pi$ and $\pi \leqslant \phi_1 + \phi_2 < 2\pi$. Since $\phi_2 < \phi_M$, however, it is impossible to have $\phi_1 + \phi_2 \geqslant \pi$ and $\phi_1 + \phi_M \leqslant \pi$. Hence (4.9) can never be satisfied for $m = 1$. Finally, if $m = M$, one can similarly show that (4.9) and (4.8c) imply $\phi_1 + \phi_M \geqslant -\pi$ and $\phi_{M-1} + \phi_M \leqslant -\pi$, which is impossible since $\phi_1 < \phi_{M-1}$.

□

Figure 3 shows temporal snapshots of a scalar soliton solution of the defocusing DSII equation, generated by $N = 1$ and $M = 4$. Note how, even though the direction and amplitude of each asymptotic line soliton remain unchanged, the overall spatial pattern of the solution is time dependent, demonstrating that, like in the KP equation, none of these are traveling wave solutions when $M > 2$. Note also how the horizontal soliton is absent from $Q(x, y, t)$ (i.e., the amplitude of its bright component is zero, consistently with the discussion in section 3.1), but is present in $q(x, y, t)$.

Figure 4 shows three further scalar soliton solutions generated by $N = 1$ and $M = 4$: a first one with three solitons in the upper-half plane and one in the lower-half plane (left), a second one with two solitons in the upper-half plane and two in the lower-half plane (center), and a third one with one soliton in the upper-half plane and three solitons in the lower-half plane (right). (A soliton solution with three asymptotic solitons above the x axis and one below can also be generated by simply adding or subtracting π to the phase parameters in the third panel.) These examples demonstrate that solutions with arbitrary nonzero numbers of solitons in the upper-half and lower-half plane, and therefore no statement stronger than lemma 4.3 is possible. Note also that the condition $M > 2$ in lemma 4.3 is needed, since $M = 2$ yields one-soliton solutions, which when $\phi_1 + \phi_2 = 0$ are horizontal, and therefore not strictly in either the upper-half or lower-half plane.

4.3. Y-shape, V-shape and L-shape solutions and soliton reconnection

As shown in the central column of figure 3 and in figure 4, several solitons can intersect simultaneously at certain discrete values of time. However, the general interaction pattern between the various solitons in all scalar solutions is a Y-shape vertex, in which exactly three solitons

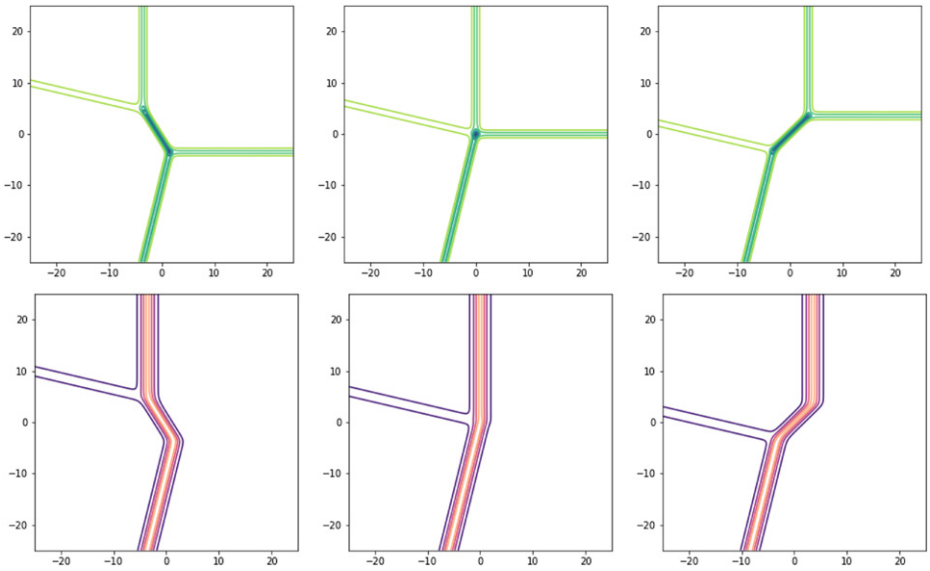

Figure 3. Temporal snapshots of a 'scalar' soliton solution of the defocusing DSII equation, generated by $N = 1$ and $M = 4$ with $(\phi_1, \phi_2, \phi_3, \phi_4) = (-3\pi/5, -\pi/4, \pi/4, 3\pi/4)$. Top row: $|q(x, y, t)|$. Bottom row: $Q(x, y, t)$. Left column: $t = -5$. Center column: $t = 0$. Right column: $t = 5$.

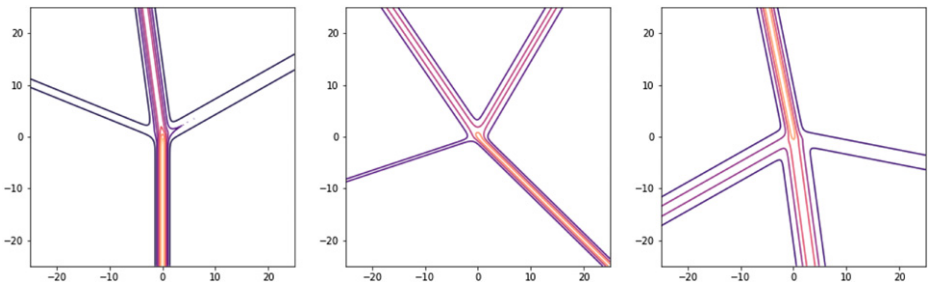

Figure 4. Three further soliton solutions of the defocusing DSII equation with $N = 1$ and $M = 4$ at $t = 0$. Here and in all subsequent figures, only the bright field $Q(x, y, t)$ is shown for brevity. Left: $(\phi_1, \ldots, \phi_4) = (-\pi/2, -\pi/4, \pi/6, \pi/2)$, resulting in three asymptotic solitons above the x-axis and one below. Center: $(\phi_1, \ldots, \phi_4) = (-5\pi/6, -3\pi/8, 0, \pi/3)$, resulting in two asymptotic solitons above the x-axis and two below. Right: $(\phi_1, \ldots, \phi_4) = (-5\pi/6, -\pi/2, 3\pi/8, 3\pi/4)$, resulting in one asymptotic line soliton above the x-axis and three below. Note that like in figure 3, none of these are traveling wave solutions of the DS system, and the relative positions of the individual solitons change in time, similarly to what happens in figures 3 and 6 and so on.

merge. This is similar to what happens for the KPII equation [11, 39]. A Y-shape solution, obtained with $N = 1$ and $M = 3$, is shown in the left panel of figure 5. This fundamental solution will be discussed more in detail in section 5.2, and the Y-shape interaction pattern makes up the building block of all fully resonant solutions, which are also discussed in section 5.2.

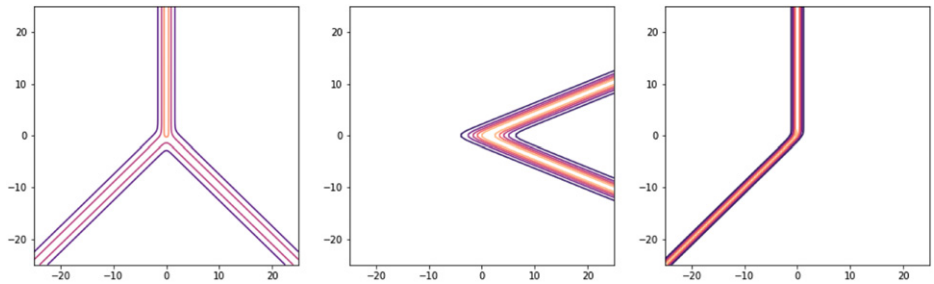

Figure 5. Soliton solutions with $N = 1$ and $M = 3$ displaying a Y-shape, a V-shape and an L-shape pattern, respectively: left: $(\varphi_1, \varphi_2, \varphi_3) = (-\pi/2, \pi/2, \pi)$. Center: $(\phi_1, \ldots, \phi_3) = (\pi/4, \pi/2, 3\pi/4)$. Right: $(\phi_1, \ldots, \phi_3) = (-\pi/4, \pi/4, 3\pi/4)$. As in figure 4 and in all subsequent figures, only contour plots of $Q(x, y, t)$ are shown for brevity. And like one-soliton solutions, all of these are traveling wave solutions of the DS system.

At the same time, recall that the amplitude of the bright component of any horizontal solitons is zero, as we saw in section 3.1. We now show that this feature gives rise to novel behavior in the solutions of the defocusing DSII system compared to the KP equation: V-shape and L-shape solutions. Consider solutions of the defocusing DSII system with $N = 1, M = 3$. When $\phi_i + \phi_j = \pi$ for any two indices i, j out of 1, 2 or 3, one of the three asymptotic line solitons [1, 2], [2, 3] and [1, 3] will be aligned horizontally, and will therefore be absent from the bright component. As a result, the bright component $Q(x, y, t)$ of the resulting solution will exhibit a V-shape pattern or an L-shape pattern (which the particular configuration depending on the particular choice of angles). Figure 5 shows one examples of each type. (Of course the dark field $q(x, y, t)$, not shown, will still exhibit a Y-shape dark-soliton pattern, since the amplitude of the dark field never vanishes.) When $M = 3$, it is not possible for more than one choice of indices i and j to satisfy the condition $\phi_i + \phi_j = \pi$. However, this is possible when $M \geqslant 4$. Later on we will see examples of solutions in which the condition is satisfied by more than one pair of indices.

Another novel phenomenon in the DSII system compared to the KP equation, partially related to the above, is that of 'soliton reconnection', an example of which was first presented by Nishinari *et al* [42]. This phenomenon occurs when there are two (or more) pairs of solitons with the same amplitude and direction but opposite velocity. For example, the simplest realization of this phenomenon occurs when there are solitons along directions $\theta = \theta_1$ and $\theta_1 + \pi$ with the same amplitude but opposite velocity, and similarly at $\theta = \theta_2$ and $\theta_2 + \pi$. Based on the above discussion, this requires $N = 1$ and $M = 4$, together with $\phi_3 = \phi_1 + \pi$ and $\phi_4 = \phi_2 + \pi$. In this case the asymptotic line solitons identified by the index pairs [1, 2], [3, 4], have the same amplitude and direction, but: (i) they are localized in opposite quadrants of the xy plane (since $\theta_{3,4} = \theta_{1,2} + \pi$), and (ii) their velocity is opposite (since $\mathbf{k}_{3,4} = -\mathbf{k}_{1,2}$ but $\omega_{3,4} = \omega_{1,2}$). The same is true for the asymptotic line solitons identified by index pairs [2, 3] and [1, 4] (since the soliton corresponding to the index pair [1, 4] is localized at a direction shifted by π, as per theorem 4.1, and again $\mathbf{k}_{1,4} = -\mathbf{k}_{2,3}$ while $\omega_{1,4} = \omega_{2,3}$). The resulting solution shows each pair of opposite solitons getting closer, reconnecting, and then separating from each other again, as shown in figure 6. Importantly, note that this is not a two-soliton solution (in the sense of the definition that will be given in section 5), since the asymptotic line soliton ray at a given value θ and the one localized at $\theta + \pi$ travel in opposite directions as functions of t (as is evident in figure 6).

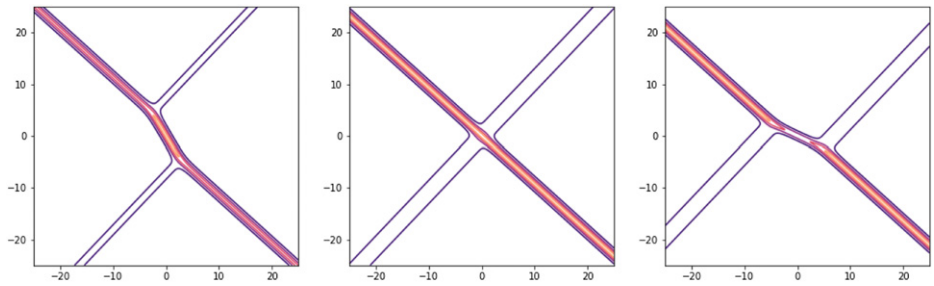

Figure 6. Soliton solutions with $N = 1$, $M = 4$ and $(\phi_1, \phi_2, \phi_3, \phi_4) = (\pi/7, \pi/3, 8\pi/7, 4\pi/3)$, illustrating the phenomenon of soliton reconnection: from left to right: $t = -35$, $t = 0$, $t = 35$.

5. Unrestricted solutions: fully non-resonant and fully resonant solutions

In section 6 we will show that the tau functions (2.17) generate solutions with M asymptotic line solitons as $r \to \infty$, independently of the value of N. Indeed, and similarly to the KP equation [8], there is exactly one asymptotic soliton associated to each of the pivot and non-pivot columns of the $N \times M$ coefficient matrix.

Remark 5.1. In what follows, it is convenient to deviate from the analysis for the KP equation and differentiate the index pairs that identify the asymptotic line solitons depending on whether they are associated to pivot or non-pivot indices. Specifically, we will use index pairs $[i, j]$ with $i < j$ to label asymptotic line solitons associated to a non-pivot index j and index pairs $[i, j]$ with $i > j$ label asymptotic line solitons associated to a pivot index i. In section 6 we will then show that: (a) for each non-pivot index g_n $(n = 1, \ldots, M - N)$ there is an asymptotic line soliton identified by the index pair $[i_n, g_n]$, with $i_n < g_n$, localized along the direction $\theta = \theta_{i_n, g_n}$. (b) For each pivot index e_n $(n = 1, \ldots, N)$ there is an asymptotic line soliton identified by an index pair $[j_n, e_n]$, with $j_n > e_n$, localized along the direction $\theta = \theta_{e_n, j_n} + \pi$.

By analogy with the KPII equation, we call *N-soliton solutions* those obtained when $M = 2N$. This definition is not as consequential for the defocusing DSII system as for the KP equation, because in the latter each N-soliton solution has exactly N solitons as $y \to \infty$ and N as $y \to -\infty$, whereas no such distinction exists for the former. However, there is a subclass of N-soliton solutions that has a precise analogy to the corresponding solutions of the KPII equation: these are the *elastic N-soliton solutions*, which, similarly to the KP equation, are defined as those N-soliton solutions such that, for each asymptotic line soliton along a given direction θ, there is also an asymptotic line soliton along the direction $\theta + \pi$, with the same amplitude and velocity [8]. (Like with the KP equation, we call *inelastic* solutions those N-soliton solutions for which the above conditions are not satisfied.)

It should be noted that in the KP equation, if two solitons have the same amplitude and direction, they also have the same velocity. We have already seen, however, that in the DS system this is not the case. Hence the need to also require that the velocities be the same, to exclude phenomena like the soliton reconnection discussed in section 4.3. Based on the above discussion, an equivalent way to characterize the elastic N-soliton solutions is as the solutions with the property that, for any index pair $[j, i]$ associated to a pivot index i, there is also a corresponding index pair $[i, j]$ associated to a non-pivot index j, and vice versa.

Before characterizing the most general multi-soliton solutions of the defocusing DSII system in section 6, it is useful to discuss two simpler subclasses, which also provide two particularly simple ways to generate elastic N-soliton solutions.

5.1. Non-resonant N-soliton solutions and reduction to the NLS equation

Apart from scalar solutions, the simplest soliton solutions of the DS system are generated when $M = 2N$ and the coefficient matrices have the smallest possible number of non-zero entries compatible with the irreducibility condition. As in the KP equation, these matrices generate fully *non-resonant* soliton solutions. The simplest such choice is

$$f_n = \mathrm{e}^{\xi_{2n-1}} + \mathrm{e}^{\xi_{2n}}, \quad n = 1, \ldots, N, \tag{5.1}$$

By analogy with the KP equation, we will call these *ordinary* N-soliton solutions. In the simplest case, $N = 1$, we recover the one-soliton solution. In the next simplest case, $N = 2$ and $M = 4$, the tau functions have four phase combinations:

$$\begin{aligned}\tau_{2,4}^{(n)} = {} & 2(\sin\,\phi_1 - \sin\,\phi_3)\,\mathrm{e}^{\xi_{1,3} - \mathrm{i}(n+1/2)\phi_{1,3}} \\ & + 2(\sin\,\phi_1 - \sin\,\phi_4)\,\mathrm{e}^{\xi_{1,4} - \mathrm{i}(n+1/2)\phi_{1,4}} \\ & + 2(\sin\,\phi_2 - \sin\,\phi_3)\,\mathrm{e}^{\xi_{2,3} - \mathrm{i}(n+1/2)\phi_{2,3}} \\ & + 2(\sin\,\phi_2 - \sin\,\phi_4)\,\mathrm{e}^{\xi_{2,4} - \mathrm{i}(n+1/2)\phi_{2,4}}.\end{aligned} \tag{5.2}$$

A corresponding solution will be shown in section 7.1. Similarly, figure 7 show an ordinary soliton solution obtained from $N = 3$, corresponding to the coefficient matrix

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{5.3}$$

The resulting asymptotic line solitons are identified by the index pairs [1, 2], [3, 4], [5, 6] and [2, 1], [4, 3], [6, 5], respectively. Even though the spatial pattern of the solution is not stationary, all the soliton interactions are non-resonant. That is, like with the corresponding solutions for the KP equation, none of the interacting solitons satisfy the resonance condition discussed in section 5.2, with the result that each interaction vertex is X-shaped instead of Y-shaped (like the solutions presented in sections 4.2 and 5.2).

However, (5.1) is not the only possible way to generate non-resonant soliton solutions. Another family of nonresonant solutions is obtained by choosing

$$f_n = \mathrm{e}^{\xi_n} + (-1)^{N-n}\,\mathrm{e}^{\xi_{2N-n+1}}, \quad n = 1, \ldots, N. \tag{5.4}$$

Figure 8 shows one such solution obtained with $N = 3$, corresponding to the coefficient matrix

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{5.5}$$

The corresponding asymptotic solitons are identified by the index pairs [1, 6], [2, 5], [3, 4] and [6, 1], [5, 2], [4, 3]. The choice of functions in (5.4) is important because it allows as a

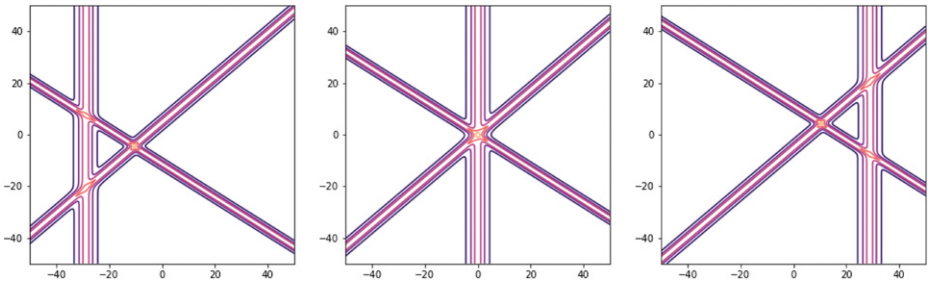

Figure 7. An ordinary three-soliton solution of the defocusing DSII equation generated with $N = 3$, $M = 6$, coefficient matrix A_1 from (5.3) and $(-16\pi/17, -5\pi/17, -\pi/17, 8\pi/17, 11\pi/17)$. From left to right: $t = -15$, $t = 0$, and $t = 15$.

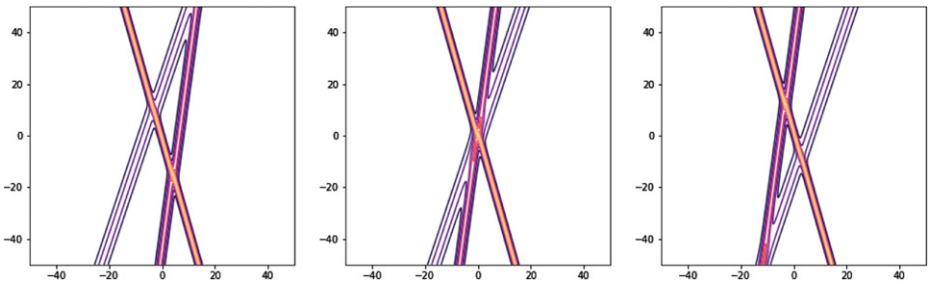

Figure 8. A non-resonant three-soliton solution of the defocusing DSII generated with $N = 3$, $M = 6$, coefficient matrix A_3 from (5.7) and parameters $(-3\pi/4, -4\pi/7, 0, \pi/5, 2\pi/5, 5\pi/6)$. From left to right: $t = -4$, $t = 0$, and $t = 3$.

special case the reduction to y-independent solutions, that is, to dark soliton solutions of the defocusing NLS equation. Indeed, by taking $\phi_n = -\phi_{N-n+1}$, we obtain

$$f_n = \mathrm{e}^{2y\,\cos\,\phi_n}\left[\mathrm{e}^{\tilde{\xi}_n+\xi_{0,n}} + (-1)^{N-n}\,\mathrm{e}^{-\tilde{\xi}_n+\xi_{0,2N-n+1}}\right], \quad n = 1, \ldots, N,$$

where $\tilde{\xi}_n = 2(x\,\sin\,\phi_n - t\,\sin\,2\phi_n)$. The only dependence of the resulting tau functions on y is then via the overall exponential factor $\exp[\cos\phi_1 + \cdots + \cos\phi_M]$, which however cancels out of (2.2). As a result, q and Q are both independent of y, and one recovers the well-known dark-soliton solutions of the NLS equation [52].

Still further ways to generate non-resonant N-soliton solutions exist. The general case is obtained by considering a subset of the partitions of the integers $1, \ldots, 2N$ into two disjoint sets of N elements each. It is straightforward to see that from any such partition one can define a set $\{i_n, j_n\}_{n=1}^N$ of N index pairs such that $1 \leqslant i_1 < i_2 < \cdots < i_N < 2N$ and $i_n < j_n$ for all $n = 1, \ldots, N$. One can generate a non-resonant N-soliton solution of the DS equation corresponding to any choice of index pairs which satisfies the condition $(j_n < i_{n'}) \vee (j_n > j_{n'})$ for all $1 \leqslant n < n' \leqslant N$. This is done by choosing

$$f_n = \mathrm{e}^{\xi_{i_n}} + (-1)^{\sigma_n}\mathrm{e}^{\xi_{j_n}}, \quad n = 1, \ldots, N, \tag{5.6}$$

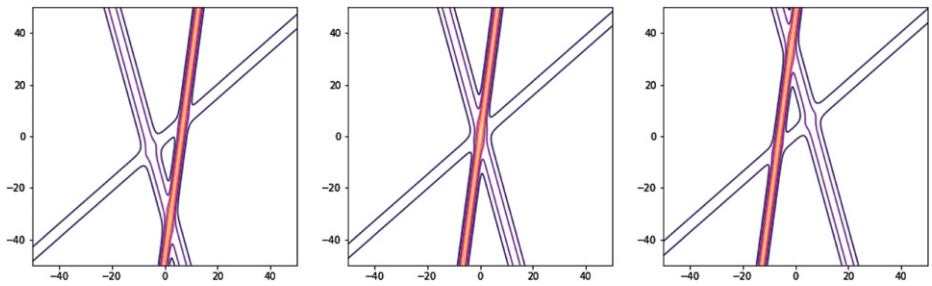

Figure 9. A non-resonant three-soliton solution of the defocusing DSII generated with $N = 3$, $M = 6$, coefficient matrix A_2 from (5.5) and parameters $(-3\pi/4, -\pi/6, 0, \pi/5, \pi/3, 5\pi/6)$. From left to right: $t = -4$, $t = 0$, and $t = 4$.

where σ_n is the number of values $n' = n+1, \ldots, N$ such that $j_{n'} < j_n$. All of these choices yield coefficient matrices A with only $2N$ nonzero entries, implying that the tau function has only 2^N nonzero corresponding terms. Also, all nonzero minors are equal to one. As a further example, figure 9 shows an additional nonresonant soliton solution obtained with $N = 3$ and coefficient matrix

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \tag{5.7}$$

The resulting asymptotic line solitons are identified by the index pairs [2, 3], [4, 5], [1, 6] as well as [3, 2], [5, 4], [6, 1]. Further examples of nonresonant solutions will be presented in section 7.1.

5.2. *Fully resonant soliton solutions*

Another distinguished class of solutions is that of *fully resonant* soliton solutions of the defocusing DSII equation. These are obtained when all minors of the coefficient matrix A are nonzero, and therefore the tau function has the maximum possible number of terms, which is $\binom{M}{N}$. One way to realize these solutions is to choose

$$f_n = f^{(-n+1)}, \quad n = 1, \ldots, N \tag{5.8}$$

(with $f^{(0)} = f$), which yields the analog of the solutions of the KP equation that also satisfy the finite Toda lattice hierarchy [11]. When $f_1, \ldots, f_N$ are chosen according to (5.8), the tau functions $\tau_{N,M}^{(n)}$ in (2.8) are given by Toeplitz determinants:

$$\tau_N^{(n)} = \det \begin{pmatrix} f^{(n)} & \cdots & f^{(n+N-1)} \\ \vdots & \ddots & \vdots \\ f^{(n-N+1)} & \cdots & f^{(n)} \end{pmatrix}. \tag{5.9}$$

In general, a $N \times 2N$ coefficient matrix A with all nonzero minors contains N^2 free parameters after reduction to RREF. Equation (5.8) yields one particular choice of these parameters. Indeed, choosing $f_1, \ldots, f_N$ according to equation (5.8) amounts to setting $A = \left(K^{(0)}\right)^{\mathrm{t}}$. We

then have $A(m_1, \ldots, m_N) = \Delta_{m_1,\ldots,m_N} \neq 0$ for all $1 \leqslant m_1 < m_2 < \cdots < m_N \leqslant M$, implying

$$\tau_{N,M}^{(n)} = (2\mathrm{i})^{N(N-1)/2} \sum_{1\leqslant m_1<\ldots<m_N\leqslant M} \Delta^2_{m_1,\ldots,m_N} \mathrm{e}^{\xi_{m_1,\ldots,m_N} + \mathrm{i}[n+(N-1)/2]\phi_{m_1,\ldots,m_N}}, \quad (5.10)$$

by virtue of lemma 2.3. All of these are nonsingular solutions of the DSII system which, when $N > 1$ and $M > 2$, describe fully resonant behavior between line solitons.

The simplest such solution is the Y-shape soliton solution, which is obtained when $N = 1$ (scalar case) and $M = 3$. The corresponding tau functions are given by

$$\tau_1^{(n)} = \mathrm{e}^{\xi_1 - \mathrm{i}n\phi_1} + \mathrm{e}^{\xi_2 - \mathrm{i}n\phi_2} + \mathrm{e}^{\xi_3 - \mathrm{i}n\phi_2}. \quad (5.11)$$

The corresponding Y-shape solution describes a resonant interaction of three line solitons, identified by the index pairs [1, 2], [1, 3] and [3, 1]. The resonance condition among three line solitons with wavevectors $\mathbf{k}_{s_j}$ and frequencies ω_{s_j}, for $j = 1, 2, 3$, is

$$\mathbf{k}_{s_1} + \mathbf{k}_{s_2} = \mathbf{k}_{s_3}, \qquad \omega_{s_1} + \omega_{s_2} = \omega_{s_3}, \quad (5.12)$$

and it is trivial to verify that this condition is satisfied for the three solitons generated by (5.11). An example of the traveling-wave Y-shaped solution generated by (5.11) is shown in the left plot of figure 5.

Similarly to what happens with the KP equation, the Y-shape solution is also a singular limit of the ordinary two-soliton solutions of the DS equation. As mentioned earlier, ordinary two-soliton solutions are given by the $N = 2$ tau functions obtained from (5.1). If $\phi_2 = \phi_3$, one can factorize the exponential term $\exp[\xi_1^{(n)} + \xi_2^{(n)} + \xi_4^{(n)}]$ from the tau functions (5.2). The term then gives zero contribution to the solution, and the remaining part of the tau functions are equivalent to the tau function $\tau_1^{(n)}$ in (5.11) up to a change of signs in two phases. Note also that the condition $\phi_2 = \phi_3$ is nothing else but the resonance condition among two solitons, and it describes the limiting case of an infinite phase shift in the ordinary two-soliton solution.

In more complicated solutions ($M > 3$), the relative spatial arrangement of the solitons in the finite xy-plane is not stationary (i.e., such solutions are not traveling wave solutions of the DS system), and a number of intermediate segments appear (e.g., as in figures 3–6 above as well as figures 10 and 11 below). Nonetheless, all of these solutions have in common the fact that each of these segments is described by the same soliton solution expression (3.1) and (3.6), in which the parameters $(\mathbf{k}, \omega)$ satisfy the nonlinear dispersion relation (3.3). Hence, these line segments are also true line solitons, even though that their spatial extension is finite. Moreover, each interaction vertex (i.e., each point in the xy-plane at which different solitons are joined) is locally Y-shaped, i.e., composed by three solitons whose parameters satisfy the resonance condition (5.12), similarly to what happens for the KPII equation [11]. This is the reason why solutions in which all minors of the coefficient matrix are called fully resonant.

In this work we are primarily concerned with characterizing the asymptotic line solitons, i.e., solitons that extend to infinity in some direction. For the fully resonant solutions, we begin with the following:

Lemma 5.2. *If all minors of the coefficient matrix A are nonzero, for each value of θ the dominant phase combination as $r \to \infty$ always contains consecutive phases* (*mod* M).

proof. The proof is constructive. Consider a dominant region R corresponding to some dominant phase combination $\xi_{m_1,\ldots,m_N}$. Fix a value of θ such that $(r, \theta) \in R$ for r large enough and consider the function $f(\phi) = \cos(\phi - \theta)$. Observe that f achieves its maximum at

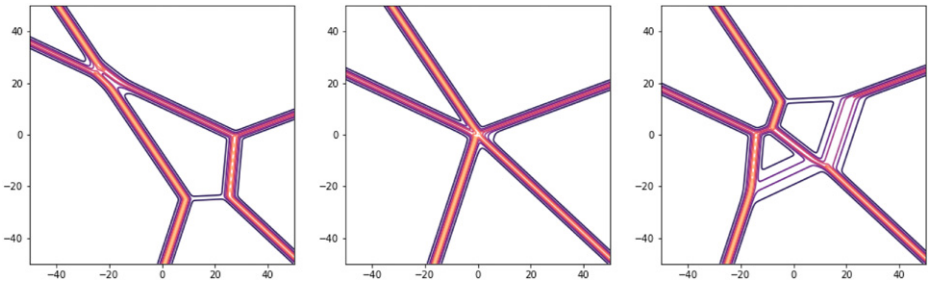

Figure 10. A fully resonant solution of the defocusing DSII equation generated with $N = 2$, $M = 5$ and $(-5\pi/7, -4\pi/7, 0, \pi/5, 10\pi/13)$. Left to right: $t = -20$, $t = 0$, and $t = 10$.

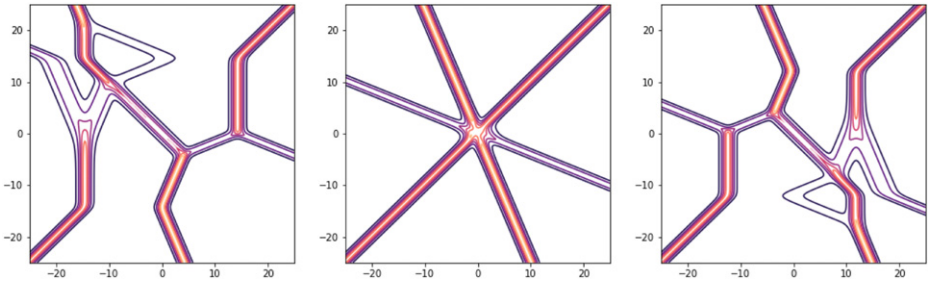

Figure 11. A fully resonant elastic three-solution of the defocusing DSII equation generated with $N = 3$, $M = 6$ and $(-3\pi/4, -\pi/2, -\pi/4, 0, \pi/4, 3\pi/4)$. Left to right: $t = -10$, $t = 0$, and $t = 8$.

$\theta = \phi$. Then if ϕ_j is the closest to θ (mod 2π) among all phases $\{\phi_1, \phi_2, \ldots, \phi_M\}$, we have that $f(\phi_j) \geqslant f(\phi_k)$ for all $k = 1, \ldots, j-1, j+1, \ldots, M$. Next, by the same argument, the phase ϕ_m with the next largest value of $f(\phi)$ will be either $j - 1 \bmod M$ or $j + 1 \bmod M$. Suppose it is ϕ_{j-1}. Once this phase is selected, the one with the next largest value of $f(\phi)$ will be labeled by either $j - 2 \bmod M$ or $j + 1 \bmod M$, and similarly if it is ϕ_{j+1}. Proceeding in this way, we keep choosing the next phase by always picking the largest possible value of f. We can continue this process recursively until we have collected N phases $\{\phi_i, \ldots, \phi_{i+N-1}\}$ (possibly with indices modulo M due to periodicity) for some i such that $f(\phi_j) < f(\phi_k)$ for all $\phi_j \in \{\phi_i, \ldots, \phi_{i+N-1}\}$ and for all $\phi_k \notin \{\phi_i, \ldots, \phi_{i+N-1}\}$. Thus we have shown that the dominant region R corresponds to the dominant phase combination $\xi_{i,\ldots,i+N-1}$.

□

Theorem 5.3. *If all minors of the coefficient matrix A are nonzero, the tau function* (2.17) *generates a solution with a total of M asymptotic line solitons as $r \to \infty$. Specifically:*

(*a*) *There are $M - N$ asymptotic line solitons identified by the index pairs $[m, m+N]$ for $m = 1, \ldots, M - N$ and localized along the direction $\theta = \theta_{m,m+N}$.*

(*b*) *There are N asymptotic line solitons identified by the index pairs $[m, m+N-M]$ for $m = M - N + 1, \ldots, M$ and localized along the direction $\theta = \theta_{m,m+M-N} + \pi$.*

Proof. To prove part (a), we begin by choosing θ so that the corresponding dominant region is $R_{1,\ldots,N}$. Note that such a value of θ can always be found, since all minors of A are nonzero.

As θ increases clockwise, we must eventually cross the boundary of $R_{1,\ldots,N}$. Since the adjacent dominant region must differ by only one phase and contain consecutive phase combinations, we then know that the adjacent dominant region must be $R_{2,\ldots,N+1}$. By the same reasoning as in the scalar case, we then also know that the boundary line must be given by $\theta = \theta_{1,N+1} = \frac{1}{2}(\phi_1 + \phi_{N+1})$. Proceeding in the same fashion, namely steadily increasing θ, we will encounter $M - N$ boundary crossings, each located at an angle $\theta = \theta_{m,m+N} = \frac{1}{2}(\phi_m + \phi_{m+N})$. Thus we have identified $M - N$ asymptotic line solitons labeled by the index pairs $[m, m + N]$ for $m = 1, \ldots, M - N$.

To prove part (b), we again begin by choosing a particular value of θ. This time however we take θ so that the corresponding dominant region is $R_{M-N+1,\ldots,M}$. As θ increases, we find that a phase transition between ϕ_{M-N+1} and ϕ_1 takes place at $\theta = \frac{1}{2}(\theta_{M-N+1} + \theta_1) + \pi$, where, similarly to what happens in the proof of theorem 4.1, the π shift occurs because phases ϕ_1 and ϕ_{M-N+1} are among the dominant phases when θ is outside the range $\phi_1 \leqslant \theta \leqslant \phi_{M-N+1}$. Therefore we have an asymptotic line soliton identified by the index pair $[M - N + 1, 1]$. Increasing θ progressively, consider the dominant region in which the N indices in the dominant phase combinations are $m, \ldots, M$ and $1, \ldots, m + N - M - 1$, for some $m = M - N, \ldots, M$. As θ increases, we have a phase transition when $\cos(\phi_m - \theta) = \cos(\phi_{m+N-M} - \theta)$. By similar arguments as before, we know that this transition, identified by the indices m and $N - M + m$, occurs when $\theta = \frac{1}{2}(\phi_m + \phi_{m+N-M}) + \pi$. There are N such phase transitions. Thus we have identified N asymptotic line solitons labeled by the index pairs $[m, m + N - M]$ for $m = 1, \ldots, N$.

□

Remark 5.4. Since all minors of A are nonzero, the pivot indices and non-pivot indices of A are respectively $1, \ldots, N$ and $N + 1, \ldots, M$. Thus, the indices $i + N$ for $i = 1, \ldots, M - N$ in part (a) of theorem 5.3 are precisely the non-pivot indices of A, while the indices $i + N - M$ for $i = M - N + 1, \ldots, M$ in part (b) are precisely the pivot indices. Note also that for the index pairs $[i, j]$ in part (a) we have $i < j$, while for the index pairs $[i, j]$ in part (b) we have $i > j$, in agreement with the labeling convention introduced earlier.

As an illustration of theorem 5.3, figures 10 and 11 show fully resonant solutions with $N = 2$ and $M = 5$ and with $N = 3$ and $M = 6$, respectively, generated by the coefficient matrices

$$A_4 = \begin{pmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 3 & 2 & 1 \end{pmatrix}, \qquad A_5 = \begin{pmatrix} 1 & 0 & 0 & 1 & 3 & 6 \\ 0 & 1 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{5.13}$$

The asymptotic solitons of the first resulting solution are identified by the index pairs [1, 3], [2, 4], [3, 5] (corresponding to the non-pivot indices) and [4, 1], [5, 2] (corresponding to the pivot indices). Those of the second solution by the index pairs [1, 4], [2, 5], [3, 6] and [4, 1], [5, 2], [6, 3] for the non-pivot and pivot indices, respectively. The second solution also provides an example of a fully resonant elastic three-soliton solution. An example of a fully resonant two-solution will be given in section 7.1, where all solutions obtained when $N = 2$ and $M = 4$ will be presented.

6. Unrestricted solutions: general case

We now turn to the most general case of unrestricted soliton solutions. Remarkably, after performing the change from Cartesian coordinates to polar coordinates, one can follow exactly the same analysis developed for the KP equation in [8]. For this reason, we limit ourselves

to stating the results, only presenting those parts of the proofs that differ appreciably from [8]. Note however that, even though the mathematical formalism carries over, the different parametric dependence of the solution yields very different physical behavior for the resulting solutions—as we already saw in the preceding sections.

Lemma 6.1. *As $r \to \infty$ along the direction $\theta = \theta_{i,j} = \frac{1}{2}(\phi_i + \phi_j)$, with $i < j$, the following inequalities hold for $\xi_o = \xi_i = \xi_j$:*

(*a*) *If $m \in \{i+1, \dots, j-1\}$, then $\xi_m > \xi_o$.*
(*b*) *If $m \in \{1, \dots, i-1, j+1, \dots, M\}$, then $\xi_m < \xi_o$.*

Similarly, as $r \to \infty$ along the direction $\theta = \theta_{i,j} + \pi$, with $i < j$, the following inequalities hold $\xi_o = \xi_i = \xi_j$:

(*c*) *If $m \in \{i+1, \dots, j-1\}$, then $\xi_m < \xi_o$.*
(*d*) *If $m \in \{1, \dots, i-1, j+1, \dots, M\}$, then $\xi_m > \xi_o$.*

proof. To prove the result, we express the difference of two phases $\xi_m - \xi_{m'}$ in polar coordinates using (4.4) along the direction $\theta = \theta_{i,j}$. We take r large enough so that the term proportional to r dominates, and we let $m' = i$, which yields $\theta_{i,j} - \frac{1}{2}(\phi_m + \phi_i) = \frac{1}{2}(\phi_j - \phi_m)$. For $m \in \{i+1, \dots, j-1\}$ we then have the inequalities $0 < \frac{1}{2}(\phi_m - \phi_i) < \pi$ and $0 < \frac{1}{2}(\phi_j - \phi_m) < \pi$, from which it follows that $\xi_m - \xi_{m'} > 0$ as $r \to \infty$ along $\theta = \theta_{i,j}$. By the same reasoning one may prove all remaining cases.

□

Similarly to the KP equation, it is useful to use the two submatrices $P_{[i,j]}$ and $Q_{[i,j]}$ with $i < j$ of the coefficient matrix A that were introduced in (3.9), which we rewrite here for convenience:

$$P_{[i,j]} = A[1, 2, \dots, i-1, j+1, \dots, M], \qquad Q_{[i,j]} = A[i+1, \dots, j-1]. \tag{6.1}$$

Indeed, as with the restricted solutions discussed in section 3.3 and with the KP equation, $P_{[i,j]}$ and $Q_{[i,j]}$ are instrumental in identifying the asymptotic line solitons of general line soliton solutions of the defocusing DSII system. Specifically:

Lemma 6.2 (Vanishing minor conditions). *Suppose that the index pair $[i, j]$ with $i < j$ identifies an asymptotic line soliton along the direction $\theta = \theta_{i,j}$ or $\theta = \theta_{i,j} + \pi$. Denote the two dominant phase combinations along the corresponding direction by $\xi_{i,p_1,\dots,p_k,q_1,\dots,q_s}$ and $\xi_{j,p_1,\dots,p_k,q_1,\dots,q_s}$, and let $A(i, p_1, \dots, p_k, q_1, \dots, q_s)$ and $A(j, p_1, \dots, p_k, q_1, \dots, q_s)$ be the corresponding nonzero minors, with $A[p_1], \dots, A[p_k] \in P_{[i,j]}$ and $A[q_1], \dots, A[q_s] \in Q_{[i,j]}$.*

(*a*) *If $[i, j]$ identifies an asymptotic line soliton as $r \to \infty$ along $\theta = \theta_{i,j} + \pi$, then:*

1. *All $N \times N$ minors obtained by replacing one of the columns $A[i], A[j], A[q_1], \dots, A[q_s]$ from either $A[i, p_1, \dots, p_k, q_1, \dots, q_s]$ or $A[j, p_1, \dots, p_k, q_1, \dots, q_s]$ with any column $A[p] \in P_{[i,j]}$ are zero;*
2. *All $N \times N$ minors obtained by replacing one of the columns $A[q_1], \dots, A[q_s]$ from either $A[i, p_1, \dots, p_k, q_1, \dots, q_s]$ or $A[j, p_1, \dots, p_k, q_1, \dots, q_s]$ with either $A[i]$ or $A[j]$, are zero.*

(*b*) *If $[i, j]$ identifies an asymptotic line soliton as $r \to \infty$ along $\theta = \theta_{i,j}$, then:*

1. *All $N \times N$ minors obtained by replacing one of the columns $A[i], A[j], A[p_1], \dots, A[p_k]$ from either $A[i, p_1, \dots, p_k, q_1, \dots, q_s]$ or $A[j, p_1, \dots, p_k, q_1, \dots, q_s]$ with any column $A[q] \in Q_{[i,j]}$, are zero;*

2. *All $N \times N$ minors obtained by replacing one of the columns $A[p_1], \ldots, A[p_k]$ from either $A[i, p_1, \ldots, p_k, q_1, \ldots, q_s]$ or $A[j, p_1, \ldots, p_k, q_1, \ldots, q_s]$ with either $A[i]$ or $A[j]$, are zero.*

proof. We prove part (a). Along the line $\theta_{i,j} + \pi$ the dominance of ξ_* implies that $\xi_* - \xi_m > 0$ for all $m \in \{1, \ldots, M\} - \{i, j\}$. However, by part (d) of lemma 6.1 we have $\xi_m - \xi_* > 0$. A contradiction unless the corresponding minors are zero. This proves part (1.) of (a). To prove part (2.) of (a), note that, if any of $A[q_1], \ldots, A[q_s]$ are replaced by $A[i]$ or $A[j]$, then the phase combinations corresponding to the resulting minors are greater than the dominant phase combinations along the direction $\theta_{i,j} + \pi$. This is a contradiction unless the corresponding minors are zero. Part (b) can be proved in a similar way using the first half of lemma 6.1.

□

Lemma 6.3 (Span). *Let $A[p_1], \ldots, A[p_k] \in P_{[i,j]}$ and $A[q_1], \ldots, A[q_s] \in Q_{[i,j]}$ be the columns in the minors associated with the dominant pair of phase combinations of lemma 6.2.*

(*a*) *If $[i, j]$ with $i < j$ identifies an asymptotic line soliton as $r \to \infty$ along $\theta = \theta_{i,j} + \pi$, the columns $A[p_1], \ldots, A[p_k]$ form a basis for the column space of the matrix $P_{[i,j]}$.*

(*b*) *If $[i, j]$ with $i < j$ identifies an asymptotic line soliton as $r \to \infty$ along $\theta = \theta_{i,j}$, the columns $A[q_1], \ldots, A[q_s]$ form a basis for the column space of the matrix $X^-[i, j]$.*

proof. We prove part (b). The proof of part (a) is similar and can be found in [8]. Since $[i, j]$ identifies a line soliton along $\theta_{i,j}$ then the minors $A[i, p_1, \ldots, p_k, q_1, \ldots, q_s]$ and $A[j, p_1, \ldots, p_k, q_1, \ldots, q_s]$ are non-zero. So $\{A[q_1], \ldots, A[q_s]\}$ is a linearly independent set. Thus any $A[q] \in Q_{[i,j]}$ can be written as

$$A[q] = aA[i] + \sum_{1 \leqslant m \leqslant k} b_m A[p_m] + \sum_{1 \leqslant m \leqslant s} c_m A[q_m]. \tag{6.2}$$

By part (1.) of (b) of lemma 6.2 replacing any of $A[i], A[p_m]$ with $A[q']$ yields a zero minor. Thus $a, b_m = 0$ and the set $\{A[q_1], \ldots, A[q_s]\}$ is a basis for $Q_{[i,j]}$

□

Lemma 6.4 (Rank conditions). *Let r be the number of columns from $P_{[i,j]}$ and let s be the number of columns from $Q_{[i,j]}$ in the minors associated with the dominant pair of phase combinations of lemma 6.1.*

(*a*) *If $[i, j]$ with $i < j$ identifies an asymptotic line soliton as $r \to \infty$ along $\theta = \theta_{i,j} + \pi$, with $i < j$, then $\operatorname{rank}(P_{[i,j]}) = k \leqslant N - 1$ and $\operatorname{rank}(P_{[i+1,j]}) = \operatorname{rank}(P_{[i,j-1]}) = \operatorname{rank}(P_{[i+1,j-1]}) = k + 1$.*

(*b*) *If $[i, j]$ with $i < j$ identifies an asymptotic line soliton as $r \to \infty$ along $\theta = \theta_{i,j}$, with $i < j$, then $\operatorname{rank}(Q_{[i,j]}) = s \leqslant N - 1$ and $\operatorname{rank}(Q_{[i-1,j]}) = \operatorname{rank}(Q_{[i,j+1]}) = \operatorname{rank}(Q_{[i-1,j+1]}) = s + 1$.*

Proof. We prove part (b), similarly to lemma 6.3. The rank of $Q_{[i,j]}$ is $s \leqslant N - 1$. Since $\{A[i], A[p_1], \ldots, A[p_k], A[q_1], \ldots, A[q_s]\}$ is a basis for $\mathbb{R}^N$ then $\operatorname{rank}(Q_{[i-1,j]}) = \operatorname{rank}(Q_{[i,j+1]}) = s + 1$. To see that $\operatorname{rank}(Q_{[i-1,j+1]}) = s + 1$ observe that

$$A[j] = aA[i] + \sum_{1 \leqslant m \leqslant k} b_m A[p_m] + \sum_{1 \leqslant m \leqslant s} c_m A[q_m]. \tag{6.3}$$

Replacing any of $A[i], A[p_m]$ with $A[q]$ results in a zero minor. Hence we can say $a = 0$ and $b_m = 0$ and $\text{rank}(Q_{[i-1,j+1]}) = s + 1$.

□

Lemma 6.5. *Consider an index pair $[i, j]$ with $1 \leqslant i < j \leqslant M$.*

(*a*) *If $[i, j]$ with $i < j$ identifies an asymptotic line soliton as $r \to \infty$ along the direction $\theta = \theta_{i,j} + \pi$, the index i labels a pivot column of the coefficient matrix A. That is $A[i] = A[e_n]$ with $1 \leqslant n \leqslant N$.*

(*b*) *If $[i, j]$ with $i < j$ identifies an asymptotic line soliton as $r \to \infty$ along the direction $\theta = \theta_{i,j}$, the index j labels a nonpivot column of the coefficient matrix A. That is, $A[j] = A[g_n]$ with $1 \leqslant n \leqslant M - N$.*

We are now ready to state the main result of this section, which provides the analogue for the defocusing DSII system of the characterization of the asymptotic line solitons of the KP equation in [8]. We do so by restoring the labeling convention that was introduced in remark 5.1 and that was already adopted in all the examples presented in sections 4 and 5, whereby index pairs $[i, j]$ with $i < j$ are used to label asymptotic line solitons associated to a non-pivot index and index pairs $[i, j]$ with $i > j$ are used to label asymptotic line solitons associated to a pivot index.

Theorem 6.6 (Asymptotic line solitons). *Let $\tau^{(n)}_{N,M}(x, y, t)$ be the tau functions in (2.14) associated with a rank N, irreducible coefficient matrix A with non-negative minors.*

(*a*) *For each pivot index e_n, with $n = 1, \ldots, N$, there exists a unique asymptotic soliton as $r \to \infty$ along the direction $\theta = \theta_{i,j} + \pi$, and identified by the index pair $[e_n, j_n]$ $1 \leqslant e_n < j_n \leqslant M$.*

(*b*) *For each nonpivot index g_n, with $n = 1, \ldots, M - N$, there exists a unique asymptotic soliton as $r \to \infty$ along the direction $\theta = \theta_{i,j}$, and identified by the index pair $[i_n, g_n]$ with $1 \leqslant i_n < g_n \leqslant M$.*

All the indices j_n for $n = 1, \ldots, N$ and i_n for $n = 1, \ldots, M - N$ are uniquely determined by the rank conditions in lemma 6.4.

We should point out that when all phase parameters $\phi_1, \ldots, \phi_M$ are restricted to the range $[-\pi/2, \pi/2)$, one also has $\theta_{i,j} \in [-\pi/2, \pi/2)$ for all $i, j = 1, \ldots, M$. In that case, the predictions of theorem 6.6 then reduce to those of theorem 3.6 for the restricted class of soliton solutions discussed in section 3.3. Unlike theorem 3.6, however, theorem 6.6 also holds when the restriction is lifted, and therefore applies to the full class of solutions presented in this work.

7. Further examples

The general results of section 6 were already illustrated through the various examples provided in sections 4 and 5. However, we now further elucidate those results by presenting several additional examples of soliton solutions of the defocusing DSII equation.

7.1. Elastic and inelastic soliton solutions with $N = 2$ and $M = 4$

The next simplest class of solutions after the scalar case (i.e., $N = 1$, which was discussed in section 4.2) is $N = 2$. Taking $M = N = 2$ generates solutions with q constant and Q identically zero (cf the remarks in section 2). It is also straightforward to see that solutions generated with $N = 2$ and $M = 3$ can be mapped into solutions generated with $N = 1$ and $M = 3$, since only

three phase combinations appear in the tau function in both cases. Therefore, the first novel case is $N = 2$ and $M = 4$, which is the subject of this section.

First of all, we note that the case $N = 2$ and $M = 4$ cannot be reduced to the case $N = 1$ and $M = 4$, since up to six distinct phase combinations can appear in the tau functions in the former, whereas only four arise in the latter. (At the same time, the case $N = 2$ and $M = 4$ is also inequivalent to the case $N = 1$ and $M = 6$ even though the same number of phase combinations appear in the tau function, because the former gives rise to only four asymptotic line solitons, compared to six asymptotic line solitons in the latter.) In light of this discussion, we classify all the solutions obtained with $N = 2$ and $M = 4$ on the basis of the number of nonzero minors (and therefore distinct phase combinations) appearing in the tau function as a result on the particular entries of the coefficient matrix. Since the smallest number of nonzero minors for an irreducible 2×4 matrix is 4, there are only three possibilities: 4, 5 or 6 nonzero minors. We discuss each of them in turn.

The cases of four and six nonzero minors give rise to three inequivalent classes of elastic two-soliton solutions, identified by the following three coefficient matrices:

$$
\begin{aligned}
A_{\text{ord}} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \\
A_{\text{asym}} &= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \\
A_{\text{res}} &= \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & a_{2,3} & a_{2,4} \end{pmatrix},
\end{aligned}
\tag{7.1}
$$

with $a_{2,3} > a_{2,4} > 0$. In particular, A_{ord} and A_{asym} give rise to tau functions with four phase combinations, with $A(1,2) = A(3,4) = 0$ for A_{ord} and $A_{(1,4)} = A(2,3) = 0$ for A_{asym}. The corresponding asymptotic line solitons for these two solutions are identified by the index pairs [1, 2], [3, 4] (as well as [2, 1] and [4, 3]) and [1, 4], [2, 3] (as well as [4, 1] and [3, 2]), respectively, where in each case the first two index pairs identify the solitons associated to the nonpivots and the last two index pairs those associated to the pivots. (Recall elastic N-soliton solutions are those for which the index pairs associated to the pivots coincide with those associated to the non-pivots.) In contrast, A_{res} gives rise to tau functions with six phase combinations, and the corresponding asymptotic line solitons are identified by the index pairs [1, 3], [2, 4] (as well as [3, 1] and [4, 2]). These three classes of solutions are the analogue of the 'ordinary', 'asymmetric' and 'resonant' elastic two-soliton solutions of the KP equation [7, 33]. Examples of the corresponding solutions are displayed in figures 12 and 13, respectively. Note that the solutions generated by A_{ord} and A_{asym} are non-resonant, and are traveling wave solutions of the DS equation, and therefore only a single snapshot for each solution is presented in figure 12. On the other hand, the resonant solution is not a traveling wave solution, and therefore figure 13 shows three temporal snapshots.

Solutions with five non-zero minors in the tau functions give rise to four inequivalent classes of inelastic two-soliton solutions, identified by the following coefficient matrices:

$$
A_{\text{I}} = \begin{pmatrix} 1 & 1 & 0 & -a \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad A_{\text{II}} = \begin{pmatrix} 1 & 0 & -a & -a \\ 0 & 1 & 1 & 1 \end{pmatrix}, \tag{7.2a}
$$

$$
A_{\text{III}} = \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 1 & 1 \end{pmatrix}, \qquad A_{\text{IV}} = \begin{pmatrix} 1 & 0 & -a & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \tag{7.2b}
$$

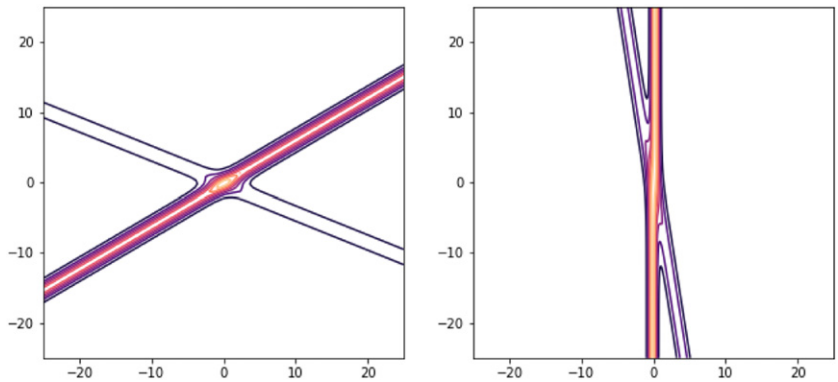

Figure 12. Soliton solutions of the defocusing DSII equation generated by $N = 2$ and $M = 4$ with $(-11\pi/13, -\pi/2, \pi/2, 3\pi/4)$. Left: 'ordinary' solution generated by the matrix A_{ord} in (7.1). Right: 'asymmetric' solution generated by the matrix A_{asym} in (7.1).

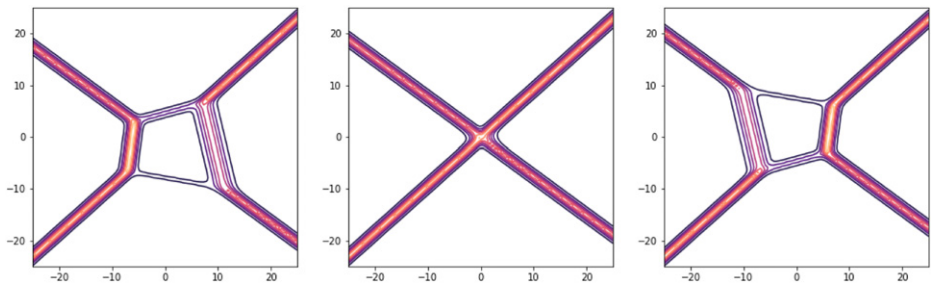

Figure 13. Resonant soliton solution of the defocusing DSII equation generated by $N = 2$ and $M = 4$ with $(-12\pi/13, -\pi/4, \pi/3, 7\pi/9)$ and coefficient matrix A_{res} in (7.1). Left: $t = -10$. Center: $t = 0$. Right: $t = 10$.

with $a > 0$. The corresponding zero minors are, respectively, $A(1,2)$, $A(3,4)$, $A(2,3)$ and $A(1,4)$. Examples of the corresponding solutions are displayed in figures 14–17. The resulting asymptotic line solitons are:

$$
\begin{aligned}
A_{\text{I}}: &\quad [1,2],\ [2,4], \quad [3,1],\ [4,3],\\
A_{\text{II}}: &\quad [1,3],\ [3,4], \quad [2,1],\ [4,2],\\
A_{\text{III}}: &\quad [2,3],\ [1,4], \quad [3,1],\ [4,2],\\
A_{\text{IV}}: &\quad [1,3],\ [2,4], \quad [4,1],\ [3,2],
\end{aligned}
$$

where in all four cases, the first two index pairs listed label the asymptotic line solitons associated with the pivots and the last two index pairs those associated with the nonpivots. As can be seen either from the above lists of index pairs or from figures 14–17, all four of these cases yield inelastic two-soliton solutions, i.e., two-soliton solutions for which the index pairs associated to the two pivot indices do not all coincide with those associated to the nonpivot indices, and in which, as a result, no symmetry is present between asymptotic line solitons along directions θ and $\theta + \pi$.

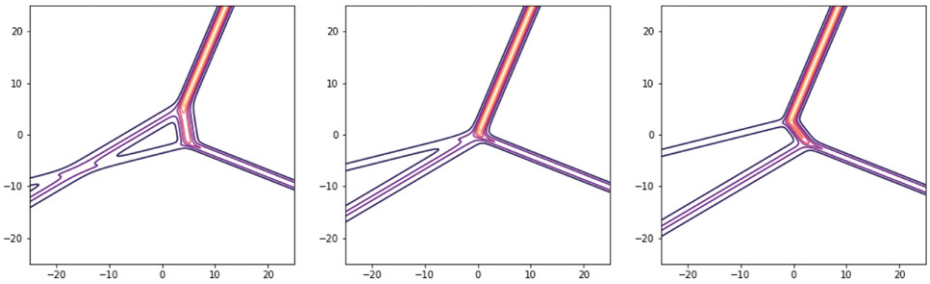

Figure 14. Inelastic two-soliton solution of the defocusing DSII equation generated by $N = 2$, $M = 4$ and coefficient matrix A_{I} in (7.2) with $a = 1$ and phase parameters $(-11\pi/13, \pi/2, \pi/11, 3\pi/4)$. Left to right: $t = -3$, $t = 0$, $t = 3$.

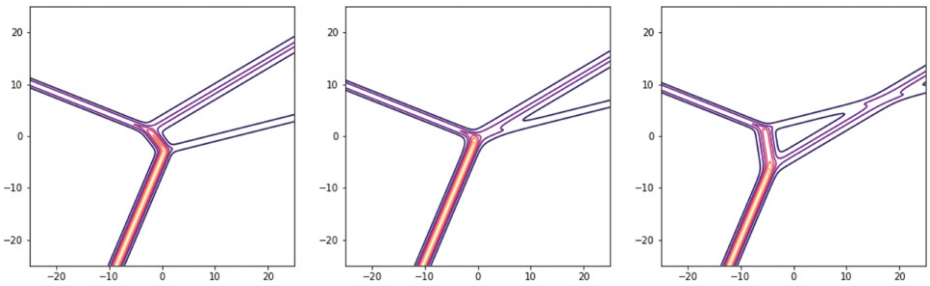

Figure 15. Same as figure 14, but for the coefficient matrix A_{II} in (7.2).

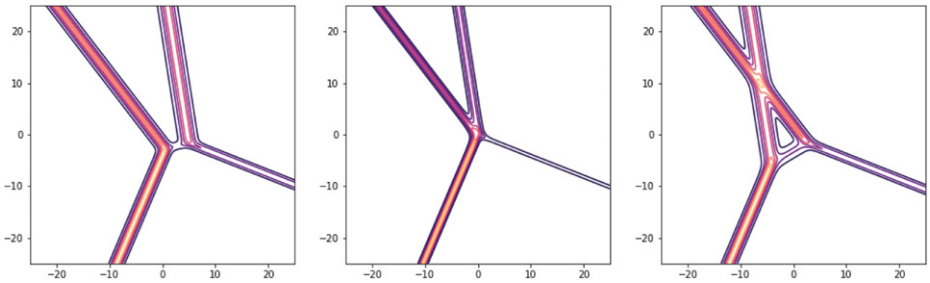

Figure 16. Same as figure 14, but for the coefficient matrix A_{III} in (7.2).

7.2. Partially resonant solutions

As our final set of examples, we present solutions that do not fall in any of the various categories discussed earlier: *partially resonant solutions*. These are defined as any solutions that are neither completely non-resonant (such as the solutions presented in section 5.1) nor fully resonant (such as the solutions presented in section 5.2). An equivalent characterization of partially resonant solutions is as the class of soliton solutions that are generated by coefficient matrices with more than 2^N nonzero minors but less than the maximum possible number $\binom{M}{N}$.

We have already seen examples of partially resonant solutions in the inelastic 2×4 solutions shown figures 14–17, for which one can indeed see that some of the interaction vertices

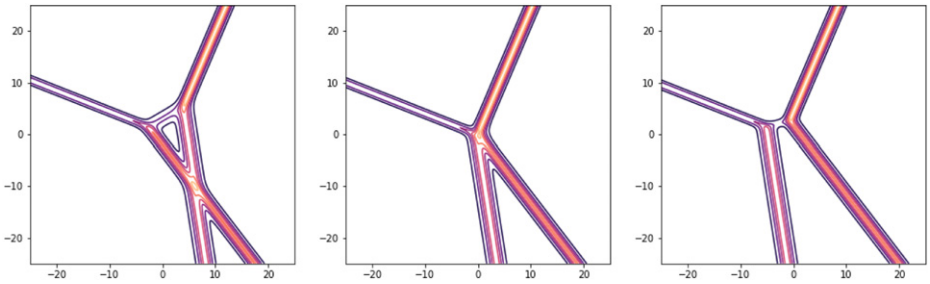

Figure 17. Same as figure 14, but for the coefficient matrix A_{IV} in (7.2).

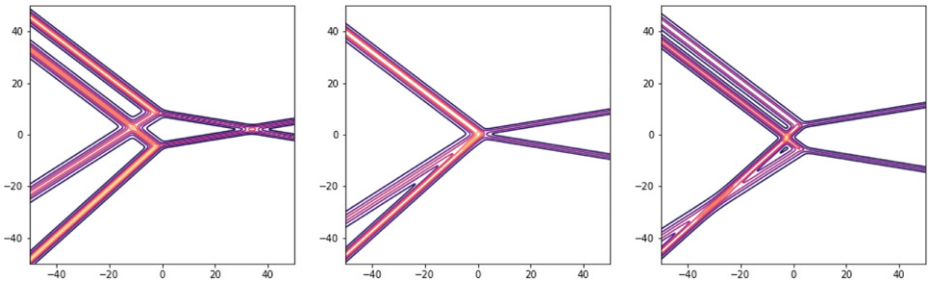

Figure 18. An inelastic solution of the defocusing DSII equation generated by $N = 3$ and $M = 6$ with $(-16\pi/19, -10\pi/19, -\pi/19, 0, 10\pi/19, 17\pi/19)$ and matrix $A_{3\times 6,\mathrm{I}}$ in (7.3a). Left to right: $t = -12$, $t = 0$, and $t = 8$.

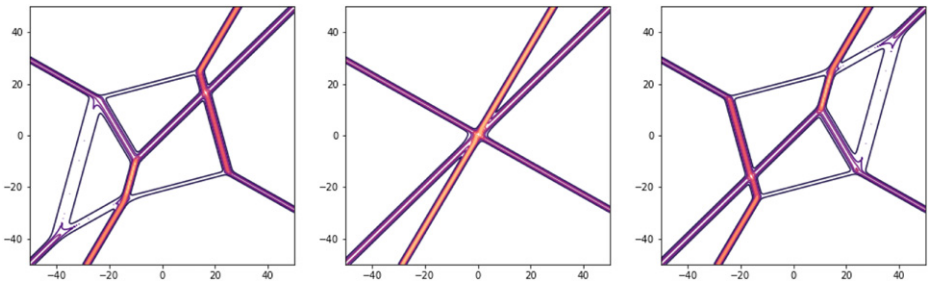

Figure 19. An elastic solution of the defocusing DSII equation generated by $N = 3$ and $M = 6$ with $(-5\pi/6, -\pi/3, 0, \pi/6, \pi/2, 4\pi/6)$ and matrix $A_{3\times 6,\mathrm{II}}$ in (7.3b). Left to right: $t = -16$, $t = 0$, and $t = 16$.

are X-shapes and some are Y-shapes. In a similar vein, figures 18–20 show three different partially resonant solutions with $N = 3$ and $M = 6$ generated by the following coefficient matrices:

$$A_{3\times 6,\mathrm{I}} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \tag{7.3a}$$

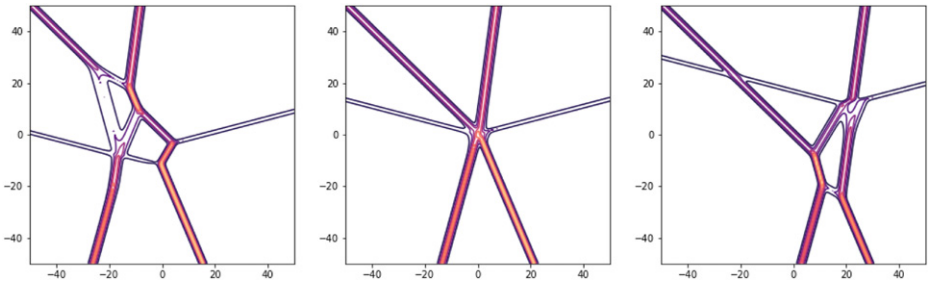

Figure 20. An inelastic solution of the defocusing DSII equation generated by $N = 3$ and $M = 6$ with $(-5\pi/6, -\pi/3, 0, \pi/6, \pi/2, 4\pi/6)$ and matrix $A_{3\times 6,\mathrm{III}}$ in (7.3c). Left to right: $t = -10$, $t = 0$, and $t = 13$.

$$A_{3\times 6,\mathrm{II}} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & -2 & -1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}, \tag{7.3b}$$

$$A_{3\times 6,\mathrm{III}} = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{7.3c}$$

The corresponding asymptotic line solitons are:

$$\begin{aligned} A_{3\times 6,\mathrm{I}}: &\quad [1,2],\ [2,3],\ [4,6],\quad [3,1],\ [5,4],\ [6,5], \\ A_{3\times 6,\mathrm{II}}: &\quad [1,4],\ [3,5],\ [2,6],\quad [4,1],\ [6,2],\ [5,3], \\ A_{3\times 6,\mathrm{III}}: &\quad [1,3],\ [2,4],\ [3,6],\quad [4,1],\ [5,2],\ [6,5]. \end{aligned}$$

The solution produced by $A_{3\times 6,\mathrm{I}}$, depicted in figure 18, is a non-interacting nonlinear superposition of 2 Y-shape solutions, resulting in an inelastic collection of six solitons. The solution produced by $A_{3\times 6,\mathrm{II}}$, depicted in figure 19, is a partially resonant elastic three-soliton solution. Finally, the solution produced by $A_{3\times 6,\mathrm{III}}$, depicted in figure 20, is another example of an inelastic interactions between six solitons. In all three of these cases, the local interaction patterns display a mix of Y-shape and X-shape vertices. For example, for the elastic solution produced by $A_{3\times 6,\mathrm{II}}$, the pairwise interaction among solitons [1, 4] and [2, 6] and that among solitons [1, 4] and [3, 5] are both resonant, but the pairwise interaction among solitons [2, 6] and [3, 5] is nonresonant.

8. Concluding remarks

In this work we studied the multi-soliton solutions of the defocusing DSII equation. In particular, we classified a large class of soliton solutions obtained from the Wronskian formalism. The classification is based on a relatively small number of key 'ingredients': (i) the Wronskian representation of the solutions, (ii) the transformation to polar coordinates, and (iii) the use of the methodology originally developed for the KP equation in [8]. We showed that a restricted class of solutions are in direct correspondence with the soliton solutions of the KPII equation. At the same time, we showed that there exist a large class of solutions of the defocusing DSII system that have no counterpart in the KP equation. In this sense, the solitonic sector of the

defocusing DSII equation can therefore be said to be richer than that of the KP equation. Still, once a determinant form for the solution has been obtained, and once appropriate coordinates have been identified (in our case polar coordinates), one finds a similar structure of solutions across many different systems, such as the KP equation [7, 8, 11, 14, 15], a coupled system of KP-type [28, 29], the DKP equation [34], and the two-dimensional Toda lattice [12, 38], demonstrating the existence of a universal structure for these solutions.

It might be worthwhile to point out that, even though the calculation of all the rank conditions that are necessary to identify the asymptotic line solitons produced by a given coefficient matrix might be a somewhat tedious task, it is nonetheless a task that is easily automatized and that can therefore be performed via a suitable symbolic computer software.

The classification framework presented in this work should make it possible to investigate in detail further properties of the multi-soliton solutions. For example, an open question is whether for $N > 1$ one can identify subclasses of solutions for which precise statements about the number of solitons in the upper-half plane and the lower-half plane. An obvious but important open question is whether these solutions are stable with respect to localized perturbations. A further open problem is the analogue of the so-called 'direct' problem for the KP equation [7], namely, the problem of identifying whether it is possible to construct multi-soliton solutions with a given, specified choice of soliton directions and amplitudes. In the case of the defocusing DSII equation, this problem is made more difficult by the more complicated dependence of the amplitude and direction on the phase parameters compared to the KP equation. Finally, a challenging problem will be that of characterizing the time evolution of 'essentially non-solitonic' initial conditions, i.e., initial conditions that are not simply a localized perturbation of an exact multi-soliton solution, such as a single 'bent' soliton, for example. Recent results for the KP equation indicate that Whitham modulation theory can be an effective tool to study this last problem [48–50]. We hope that the results of this work will motivate further study on these and other related questions.

Acknowledgments

GB was partially supported by the National Science Foundation under Grant Number DMS-2009487.

Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Gino Biondini https://orcid.org/0000-0003-3835-1343
Ken-ichi Maruno https://orcid.org/0000-0003-4197-8067

References

[1] Ablowitz M J, Biondini G and Blair S 1997 Multidimensional pulse propagation in non-resonant $\chi^{(2)}$ materials *Phys. Lett.* A **236** 520–4

[2] Ablowitz M J, Biondini G and Blair S 2001 Nonlinear Schrödinger equations with mean terms in non-resonant multi-dimensional quadratic materials *Phys. Rev.* E **63** 046605

[3] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)

[4] Anker D and Freeman N C 1978 On the soliton solution of the Davey–Stewartson equation by long waves *Proc. R. Soc.* A **360** 529–40

[5] Assainova O, Klein C, McLaughlin K D T-R and Miller P D 2019 A study of the direct spectral transform for the defocusing Davey–Stewartson II equation in the semiclassical limit *Commun. Pure Appl. Math.* **72** 1474–547

[6] Beals R and Coifman R R 1986 The D-bar approach to inverse scattering and nonlinear evolutions *Physica* D **18** 242–9

[7] Biondini G 2007 Line soliton interactions of the Kadomtsev–Petviashvili equation *Phys. Rev. Lett.* **99** 064103

[8] Biondini G and Chakravarty S 2006 Line soliton solutions of the Kadomtsev–Petviashvili equation *J. Math. Phys.* **47** 033514

[9] Biondini G and Chakravarty S 2007 Elastic and inelastic line-soliton solutions of the Kadomtsev–Petviashvili II equation *Math. Comput. Simul.* **74** 237–50

[10] Biondini G and Kireyev D 2022 Periodic solutions of the Davey–Stewartson equations *East Asian J. Appl. Math.* **12** 1

[11] Biondini G and Kodama Y 2003 On a family of solutions of the Kadomtsev–Petviashvili equation which also satisfy the Toda lattice hierarchy *J. Phys. A: Math. Gen.* **36** 10519–36

[12] Biondini G and Wang D 2010 On the soliton solutions of the two-dimensional Toda lattice *J. Phys. A: Math. Theor.* **43** 434007

[13] Boiti M, Leon J J P, Martina L and Pempinelli F 1989 Scattering of localized solitons in the plane *Phys. Lett.* A **132** 432–9

[14] Chakravarty S and Kodama Y 2008 Classification of the line-soliton solutions of KPII *J. Phys. A: Math. Theor.* **41** 275209

[15] Chakravarty S and Kodama Y 2007 Soliton solutions of the KP equation and application to shallow water waves *Stud. Appl. Math.* **123** 83–151

[16] Chakravarty S and Kodama Y 2008 A generating function for the *N*-soliton solutions of the Kadomtsev–Petviashvili II equation *Contemp. Math.* **471** 47

[17] Davey A and Stewartson K 1974 On three-dimensional packets of surface waves *Proc. R. Soc.* A **338** 101–10

[18] Fokas A S, Pelinovsky D E and Sulem C 2001 Interaction of lumps with a line soliton for the DSII equation *Physica* D **152–153** 189–98

[19] Fokas A S and Santini P M 1989 Coherent structures in multidimensions *Phys. Rev. Lett.* **63** 1329–33

[20] Freeman N C 1984 Soliton solutions of non-linear evolution equations *IMA J. Appl. Math.* **32** 125–45

[21] Gilson C R 1992 Resonant behaviour in the Davey–Stewartson equation *Phys. Lett.* A **161** 423–8

[22] Gilson C R and Nimmo J J C 1991 A direct method for dromion solutions of the Davey–Stewartson equations and their asymptotic properties *Proc. R. Soc.* A **435** 339–57

[23] Guil F and Mañas M 1995 Deformation of the dromion and Solitoff solutions of the Davey–Stewartson I equation *Phys. Lett.* A **209** 29–47

[24] Guo L, Chen L, Mihalache D and He J 2022 Dynamics of soliton interaction solutions of the Davey–Stewartson I equation *Phys. Rev.* E **105** 014218

[25] Hirota R 2004 *The Direct Method in Soliton* (*Theory, Cambridge Tracts in Mathematics*) (Cambridge: Cambridge University Press)

[26] Hirota R, Ito M and Kako F 1988 Two-dimensional Toda lattice equations *Prog. Theor. Phys. Suppl.* **94** 42–58

[27] Hirota R, Ohta Y and Satsuma J 1988 Wronskian structures of solitons for soliton equations *Prog. Theor. Phys. Suppl.* **94** 59–72

[28] Isojima S, Willox R and Satsuma J 2002 On various solution of the coupled KP equation *J. Phys. A: Math. Gen.* **35** 6893–909

[29] Isojima S, Willox R and Satsuma J 2003 Spider-web solutions of the coupled KP equation *J. Phys. A: Math. Gen.* **36** 9533–52

[30] Jimbo M and Miwa T 1983 Solitons and infinite-dimensional Lie algebras *Publ. Res. Inst. Math. Sci. Kyoto* **19** 943–1001

[31] Klein C and McLaughlin K D T-R 2017 Spectral approach to D-bar problems *Commun. Pure Appl. Math.* **70** 1052–83
[32] Klein C and Roidot K 2014 Numerical study of the semiclassical limit of the Davey–Stewartson II equations *Nonlinearity* **27** 2177–214
[33] Kodama Y 2004 Young diagrams and N-soliton solutions of the KP equation *J. Phys. A: Math. Gen.* **37** 11169–90
[34] Kodama Y and Maruno K-i 2006 N-soliton solutions to the DKP equation and Weyl group actions *J. Phys. A: Math. Gen.* **39** 4063–86
[35] Leblond H 1998 Bidimensional optical solitons in a quadratic medium *J. Phys. A: Math. Gen.* **31** 5129–43
[36] Leblond H 1999 Electromagnetic waves in ferromagnets: a Davey–Stewartson-type model *J. Phys. A: Math. Gen.* **32** 7907–32
[37] Mañas M and Santini P M 1997 Solutions of the Davey–Stewartson II equation with arbitrary rational localization and nontrivial interaction *Phys. Lett.* A **227** 325–34
[38] Maruno K-i and Biondini G 2004 Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete analogues *J. Phys. A: Math. Gen.* **37** 11819–39
Maruno K-i and Biondini G 2009 *J. Phys. A: Math. Theor.* **42** 029801 (erratum)
[39] Medina E 2002 An N soliton resonance solution for the KP equation: interaction with change of form and velocity *Lett. Math. Phys.* **62** 91–9
[40] Miles J W 1977 Resonantly interacting solitary waves *J. Fluid Mech.* **79** 171–9
[41] Newell A C and Redekopp L G 1977 Breakdown of Zakharov–Shabat theory and soliton creation *Phys. Rev. Lett.* **38** 377–80
[42] Nishinari K, Abe K and Satsuma J 1993 A new-type of soliton behavior in a two dimensional plasma system *J. Phys. Soc. Japan* **62** 2021–9
[43] Ohkuma K and Wadati M 1983 The Kadomtsev–Petviashvili equation: the trace method and the soliton resonances *J. Phys. Soc. Japan* **52** 749–60
[44] Ohta Y 1989 Wronskian solution of soliton equations *RIMS Kokyuroku* **684** 1–17 (in Japanese)
[45] Pashaev O K and Francisco M L Y 2005 Degenerate four-virtual-soliton resonance for the KP-II *Theor. Math. Phys.* **144** 1022–9
[46] Pelinovsky D 1995 On a structure of the explicit solutions to the Davey–Stewartson equations *Physica* D **87** 115–22
[47] Perry P A 2016 Global well-posedness and long-time asymptotics for the defocussing Davey–Stewartson II equation in $H^{1;1}(\mathbb{C})$ *J. Spectr. Theory* **6** 429–81
[48] Ryskamp S, Hoefer M A and Biondini G 2021 Oblique interactions between solitons and mean flow in the Kadomtsev–Petviashvili equation *Nonlinearity* **34** 3583
[49] Ryskamp S, Hoefer M A and Biondini G 2022 Modulation theory for soliton resonance and Mach reflection *Proc. R. Soc.* A **478** 20210823
[50] Ryskamp S, Maiden M, Biondini G and Hoefer M A 2021 Evolution of truncated and bent wave solitons: the Mach expansion problem *J. Fluid Mech.* **909** A24
[51] Satsuma J and Ablowitz J 1979 Two-dimensional lumps in nonlinear dispersive systems *J. Math. Phys.* **20** 1496–503
[52] Zakharov V E and Shabat A B 1973 Interaction between solitons in a stable medium *Sov. Phys. - JETP* **37** 823–8