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The Ablowitz–Ladik system with linearizable boundary conditions*

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Abstract

The boundary value problem (BVP) for the Ablowitz–Ladik (AL) system on the natural numbers with linearizable boundary conditions is studied. In particular: (i) a discrete analogue is derived of the Bäcklund transformation that was used to solved similar BVPs for the nonlinear Schrödinger equation; (ii) an explicit proof is given that the Bäcklund-transformed solution of AL remains within the class of solutions that can be studied by the inverse scattering transform; (iii) an explicit linearizing transformation for the Bäcklund transformation is provided; (iv) explicit relations are obtained among the norming constants associated with symmetric eigenvalues; (v) conditions for the existence of self-symmetric eigenvalues are obtained. The results are illustrated by several exact soliton solutions, which describe the soliton reflection at the boundary with or without the presence of self-symmetric eigenvalues.

Keywords: solitons, inverse scattering transform, boundary value problems, linearizable boundary conditions, Backlund transformations

1. Introduction

Discrete and continuous nonlinear Schrödinger (NLS) equations are universal models for the behavior of weakly nonlinear, quasi-monochromatic wave packets, and they arise in a variety of physical settings. In many situations, the natural formulation of the problem gives rise to boundary value problems (BVPs). In particular, BVPs for the NLS equation

* This article is dedicated to Mark Ablowitz on the occasion of his seventieth birthday.
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$$\mathrm{i}q_t + q_{xx} - 2\nu\,|q|^2 q = 0 \tag{1.1}$$

on the half line $0 < x < \infty$ (where $q = q(x, t)$ is a complex-valued function, subscripts x and t denote partial differentiation and the values $\nu = \pm 1$ denote respectively the defocusing and focusing cases, as usual) have been extensively studied using several approaches [4, 6–8, 10, 12, 14, 15, 17, 19, 24]. It is also well known that, in general, BVPs for integrable nonlinear evolution equations can only be linearized for special kinds of boundary conditions (BCs), which are then called *linearizable*. For the NLS equation, the linearizable BCs are homogeneous Robin BCs [23]

$$q_x(0, t) + \alpha\, q(0, t) = 0, \tag{1.2}$$

where $\alpha \in \mathbb{R}$ is an arbitrary constant. Limiting cases are Neumann and Dirichlet BCs, obtained for $\alpha = 0$ and as $\alpha \to \infty$ in (1.2), respectively. In [10] and [7], the BVP with linearizable BCs was considered using the extensions of the potential to the whole real line introduced in [14] and in [17, 24], respectively, and it was shown, that the discrete eigenvalues of the scattering problem appear in symmetric quartets—as opposed to pairs in the initial value problem (IVP). Moreover, the symmetries of the discrete spectrum, norming constants, reflection coefficients and scattering data were obtained, and the reflection experienced by the solitons at the boundary was explained as a special form of soliton interaction.

The purpose of this work is to obtain the discrete analogue of all the above results. The integrable discrete analogue of the NLS equation is the Ablowitz–Ladik (AL) system:

$$\mathrm{i}\dot{q}_n + \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \nu\left|q_n\right|^2\left(q_{n+1} + q_{n-1}\right) = 0, \tag{1.3}$$

where $q_n = q(n, t)$, the dot denotes derivative with respect to time, h is the system spacing, and again $\nu = \pm 1$. The BVP for (1.3) on the natural numbers $n \in \mathbb{N}$ with a generic BC at $n = 0$ was recently studied in [9]. The linearizable BCs for (1.3) are [9, 18]

$$q_0 - \chi\, q_{-1} = 0, \tag{1.4}$$

where $\chi \in \mathbb{R}$ is an arbitrary constant. In [11], the BVPs for AL system on the natural numbers with the discrete analogue of Dirichlet and Neumann BCs at the origin ($\chi = 0$ and $\chi = 1$, respectively) was solved using odd and even extensions of the potential to all integers. No explicit extension was found, however, for generic values of χ. Here we generalize those results by deriving the discrete analogue of the Bäcklund transformation (BT) used in the continuum limit. On one hand, this enables us to give cleaner proofs of some of the results obtained in [11]. Most importantly, we use the BT to extend the previous results to generic values of χ, thereby obtaining the discrete analogue of *all* the results available in the continuum case, concerning the relation betweeen symmetric eigenvalues, their norming constants, and the conditions for the existence of the self-symmetric eigenvalues.

We point out that the methodology of [9] (which follows the unified transform method of [16]) is more general, since it applies to BVPs with any BCs. On the other hand, in that work the analytic scattering coefficient $a_{22}(z)$ (defined in section 2) is assumed to be non-zero along the real and imaginary axes. Therefore the approach of [9] is limited to potentials for which no self-symmetric eigenvalues are present. Moreover, when linearizable BCs are given, the BVP posseses additional structure, which is more easily elucidated using the present approach [9].

Throughout this work, the notation $M = (M_1, M_2)$ is used to identify the columns of a 2×2 matrix M; subscripts i, j are used to denote the corresponding matrix entry, the superscript T denotes matrix transpose, the asterisk complex conjugation, a prime signifies

differentiation with respect to z, $[A, B] = AB - BA$ is the matrix commutator, $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Also, we will frequently use the shorthand notation f_n for $f(n, t)$ or $f(n, t, z)$. The intended meaning will be clear from the context. Finally, one can always make $h = 1$ in (1.3) without loss of generality by trivial rescalings [3]. We will assume that this has been done throughout. (Of course an inverse scaling can also be performed to recover the continuum limit, as discussed in detail in section 4, where such limit is considered.)

2. IST for the AL system on the integers

Here we briefly review the solution of the IVP for the AL system via IST, since it will be used in sections 4 and 6 to construct the BT and solve the BVP. We refer the reader to [2, 3] for all details. The AL system (1.3) with $h = 1$ is the compatibility of the matrix Lax pair

$$\Phi_{n+1} = \left(Z + Q_n\right)\Phi_n, \tag{2.1a}$$

$$\dot{\Phi}_n = \left(-\mathrm{i}\omega(z)\sigma_3 + H_n\right)\Phi_n, \tag{2.1b}$$

where $\Phi(n, t, z)$ is a 2×2 matrix eigenfunction, $\omega(z) = -(z - 1/z)^2/2$,

$$Z = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}, \qquad Q(n, t) = \begin{pmatrix} 0 & q_n \\ r_n & 0 \end{pmatrix}, \tag{2.2}$$

$$H(n, t, z) = \mathrm{i}\sigma_3\left(Q_n Z^{-1} - Q_{n-1}Z - Q_n Q_{n-1}\right), \tag{2.3}$$

and $r_n(t) = \nu q_n^*(t)$. The modified eigenfunctions are $\mu(n, t, z) = \Phi(n, t, z)\, Z^{-n}\mathrm{e}^{\mathrm{i}\omega(z)t\sigma_3}$, and they satisfy the modified Lax pair

$$\mu_{n+1} - Z\mu_n Z^{-1} = Q_n \mu_n Z^{-1}, \tag{2.4a}$$

$$\dot{\mu}_n + \mathrm{i}\omega(z)[\sigma_3, \mu_n] = H_n \mu_n, \tag{2.4b}$$

The Jost eigenfunctions are the solutions of (2.4) which tend to the identity as $n \to \mp\infty$. When $q_n \in \ell_1(\mathbb{Z})$, they are given by the solution of the following linear equations

$$\mu_-(n, t, z) = I + \sum_{m=-\infty}^{n-1} Z^{n-m-1} Q_m(t)\mu_-(m, z, t)\, Z^{m-n}, \tag{2.5a}$$

$$\mu_+(n, t, z) = I - \sum_{m=n}^{\infty} Z^{n-m-1} Q_m(t)\mu_+(m, z, t)\, Z^{m-n}. \tag{2.5b}$$

The eigenfunctions $\Phi_\pm(n, t, z)$ are defined accordingly. Analysis of (2.5) shows that the columns $\mu_{-,1}(n, t, z)$ and $\mu_{+,2}(n, t, z)$ can be analytically extended to the exterior ($|z| > 1$) of the unit circle, while $\mu_{-,2}(n, t, z)$ and $\mu_{+,1}(n, t, z)$ to its interior ($|z| < 1$). Let

$$C_n(t) = \prod_{m=n}^{\infty}\left(1 - q_m r_m\right), \qquad C_{-\infty} = \lim_{n\to-\infty} C_n(t). \tag{2.6}$$

with the usual assumptions that $C_{-\infty} < \infty$ and, in the defocusing case, $|q_n| < 1\ \forall\, n \in \mathbb{Z}$ [3]. One can show that $C_{-\infty}$ does not depend on time. Both $\Phi_-(n, t, z)$ and $\Phi_+(n, t, z)$ are fundamental matrix solution of the Lax pair (2.1), since $\det \Phi_\pm(n, t, z) = \det \mu_\pm(n, t, z)$ and

$$\det \mu_-(n, t, z) = C_{-\infty}/C_n(t), \qquad \det \mu_+(n, t, z) = 1/C_n(t).$$

One can then define the time-independent scattering matrix $A(z)$ for $|z| = 1$ as

$$\Phi_-(n, t, z) = \Phi_+(n, t, z)\, A(z). \tag{2.7}$$

The scattering coefficients can be expressed in terms of Wronskians:

$$a_{11}(z) = C_n \mathrm{Wr}\big(\Phi_{-,1}, \Phi_{+,2}\big), \qquad a_{22}(z) = -C_n \mathrm{Wr}\big(\Phi_{-,2}, \Phi_{+,1}\big),$$

etc, implying that $a_{11}(z)$ and $a_{22}(z)$ can be analytically extended to $|z| > 1$ and $|z| < 1$, respectively, while $a_{12}(z)$ and $a_{21}(z)$ are in general nowhere analytic. Moreover, the modified Jost solutions have the following asymptotic behavior as $z \to 0$ and $z \to \infty$:

$$\mu_-(n, t, z) = I + Q_{n-1}Z^{-1} + O\big(Z^{-2}\big), \qquad z \to (\infty, 0), \tag{2.8a}$$

$$\mu_+(n, t, z) = C_n^{-1} - C_n^{-1}Q_n Z + O\big(Z^2\big), \qquad z \to (0, \infty), \tag{2.8b}$$

where the notation $z \to (z_1, z_2)$ indicates $z \to z_1$ in the first column and $z \to z_2$ in the second column. In turn, (2.8) yield the asymptotic behavior of the scattering data. In particular

$$a_{22}(z) = 1 + z^2 \sum_{n=-\infty}^{\infty} r_n q_{n-1} + O\big(z^4\big) \qquad z \to 0. \tag{2.9}$$

The eigenfunctions satisfy the symmetries

$$\Phi_{\pm,1}(n, t, z) = \sigma_\nu \Phi^*_{\pm,2}\big(n, t, 1/z^*\big), \tag{2.10a}$$

$$\Phi_{\pm,2}(n, t, z) = \nu\sigma_\nu \Phi^*_{\pm,1}\big(n, t, 1/z^*\big), \tag{2.10b}$$

with

$$\sigma_\nu = \begin{pmatrix} 0 & 1 \\ \nu & 0 \end{pmatrix}, \tag{2.11}$$

implying

$$a^*_{11}\big(1/z^*\big) = a_{22}(z), \qquad |z| \leqslant 1, \tag{2.12a}$$

$$a^*_{21}\big(1/z^*\big) = \nu a_{12}(z) \qquad |z| = 1. \tag{2.12b}$$

Moreover, the scattering data satisfy the additional symmetry:

$$a_{22}(z) = a_{22}(-z), \qquad |z| \leqslant 1, \tag{2.13a}$$

$$a_{12}(z) = -a_{12}(-z), \qquad |z| = 1. \tag{2.13b}$$

The discrete eigenvalues of the scattering problem arise from the zeros of the analytic scattering coefficients $a_{11}(z)$ and $a_{22}(z)$. In the defocusing case there are no such zeros. In the focusing case we assume that there are a finite number of zeros and they are all simple. The above symmetries imply that the discrete eigenvalues appear in symmetric quartets. That is, let $\pm z_j$ for $j = 1,..., J$ be the zeros of $a_{22}(z)$ in $|z| < 1$ with $\arg z \in [0, \pi)$. Note that the total number of zeros inside the unit circle is $2J$. Moreover, equation (2.13a) implies that $\pm\bar{z}_j = \pm 1/z_j^*$ for $j = 1,..., J$ are the zeros of $a_{11}(z)$ in $|z| > 1$. In correspondence to all these zeros one has:

$$\mu_{-,2}(n, t, z_j) = b_j\, z_j^{2n} \mathrm{e}^{-2\mathrm{i}\omega(z_j)t} \mu_{+,1}(n, t, z_j),$$

$$\mu_{-,1}(n, t, \bar{z}_j) = \bar{b}_j\, \bar{z}_j^{-2n} \mathrm{e}^{2\mathrm{i}\omega(\bar{z}_j)t} \mu_{+,2}(n, t, \bar{z}_j),$$

and the corresponding residue relations are

$$\operatorname*{Res}_{z=z_j}\left[\frac{\mu_{-,2}(n,\,t,\,z)}{a_{22}(z)}\right]=K_j z_j^{2n}\mathrm{e}^{-2\mathrm{i}\omega(z_j)t}\mu_{+,2}(n,\,t,\,z_j),$$

$$\operatorname*{Res}_{z=\bar{z}_j}\left[\frac{\mu_{-,1}(n,\,t,\,z)}{a_{11}(z)}\right]=\bar{K}_j \bar{z}_j^{-2n}\mathrm{e}^{2\mathrm{i}\omega(\bar{z}_j)t}\mu_{+,2}(n,\,t,\,\bar{z}_j),$$

where $K_j = b_j/a_{22}'(z_j)$ and $\bar{K}_j = \bar{b}_j/a_{11}'(\bar{z}_j)$ are the norming constants. The norming constants for the eigenvalues at $\pm z_j$ are identical, and so are those for the eigenvalues at $\pm\bar{z}_j$. Moreover

$$\bar{b}_j = -b_j^*, \qquad \bar{K}_j = \left(z_j^*\right)^{-2}K_j^*.$$

The inverse problem is formulated in terms of the Riemann–Hilbert problem (RHP) defined by the jump condition

$$M^-(n,\,t,\,z) = M^+(n,\,t,\,z)(I - V(n,\,t,\,z)), \qquad |z| = 1, \tag{2.14}$$

where the sectionally meromorphic matrices are

$$M^+ = \operatorname{diag}\left(1,\,C_n\right)\left(\mu_{+,1},\,\mu_{-,2}/a_{22}\right), \qquad M^- = \operatorname{diag}\left(1,\,C_n\right)\left(\mu_{-,1}/a_{11},\,\mu_{+,2}\right), \tag{2.15}$$

the jump matrix is

$$V(n,\,t,\,z) = \begin{pmatrix} \nu\rho(z)\rho^*\left(1/z^*\right) & z^{2n}\mathrm{e}^{-2\mathrm{i}\omega(z)t}\rho(z) \\ -\nu z^{-2n}\mathrm{e}^{2\mathrm{i}\omega(z)t}\rho^*\left(1/z^*\right) & 0 \end{pmatrix}, \tag{2.16}$$

and $\rho(z) = a_{12}(z)/a_{22}(z)$ is the reflection coefficient. After regularization, the RHP can be formally solved via the Cauchy projectors over the unit circle. The asymptotic behavior of $M^\pm$ as $z \to 0$ or $z \to \infty$ then yields the reconstruction formula for the potential as

$$\begin{aligned} q_n(t) = &-2\sum_{j=1}^{J} K_j z_j^{2n}\mathrm{e}^{-2\mathrm{i}\omega(z_j)t}\mu_{+,11}\left(n+1,\,z_j,\,t\right) \\ &+\frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{2n}\mathrm{e}^{-2\mathrm{i}\omega(z)t}\rho(z)\mu_{+,11}(n+1,\,z,\,t)\,\mathrm{d}z. \end{aligned} \tag{2.17}$$

In the reflectionless case with $\nu = -1$, the RHP reduces to an algebraic system. Its solution yields the pure multi-soliton solution of the AL system as

$$q_n(t) = 2\det G^c/\det G, \tag{2.18}$$

where

$$G_{j,m} = \delta_{j,m} - 4K_m\, z_m^{2(n+1)}\mathrm{e}^{-2\mathrm{i}\omega(z_m)\,t}\sum_{p=1}^{J}\frac{K_p^*\,\left(z_p^*\right)^{2(n-1)}\mathrm{e}^{2\mathrm{i}\omega^*(z_p)\,t}}{\left(z_j^2-\bar{z}_p^2\right)\left(\bar{z}_p^2-z_m^2\right)},$$

for all $j,\,m = 1,\,\cdots,\,J$, and

$$G^c = \begin{pmatrix} 0 & \mathbf{y}^T \\ \mathbf{1} & G \end{pmatrix}, \qquad y_j = K_j\, z_j^{2n}\mathrm{e}^{-2\mathrm{i}\omega(z_j)t}.$$

For a single quartet (i.e., $J = 1$), one obtains the one-soliton solution of the AL system as

$$q(n,\,t) = \mathrm{e}^{\mathrm{i}(n\beta+wt+\varphi)}\sinh\alpha\ \operatorname{sech}\left[\alpha(n-vt-\delta)\right], \tag{2.19a}$$

with $z_1 = \mathrm{e}^{(\alpha+\mathrm{i}\beta)/2}$, $\varphi = \arg K_1 + \pi$, $\delta = [\log(\sinh\alpha) - \log|K_1|]/\alpha$, and

$$v = 2(\sin\beta\,\sinh\alpha)/\alpha, \qquad w = 2(\cos\beta\,\cosh\alpha - 1). \tag{2.19b}$$

Finally, the trace formula for the focusing case is: for $|z| < 1$,

$$\log a_{22}(z) = \sum_{m=1}^{J} \log\frac{z^2 - z_m^2}{z^2 - \bar{z}_m^2} + \log C_{-\infty} - \frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=1} \frac{\zeta\,\log\left(1 + |\rho(\zeta)|^2\right)}{\zeta^2 - z^2}\mathrm{d}\zeta. \tag{2.20}$$

The above theory was extended to potentials that tend to constant values as $n \to \pm\infty$ in [1, 20].

3. 'Half-line' scattering for the AL system

We now briefly review some relevant results about the IST for the AL system on the natural numbers [9], which will be used later. Define $\Phi_0(n, t, z)$ for all $n \in \mathbb{N}_0$ as the simultaneous solution of the Lax pair satisfying the BC $\Phi_0(0, 0, z) = I$. Also, let $\Phi_+(n, t, z)$ be the restriction to $n \in \mathbb{N}_0$ of the eigenfunction with the same name in the IVP. Thus, $\Phi_+(n, t, z)$ is still defined by (2.5), whereas $\Phi_0(n, t, z) = \mu_0(n, t, z)\, Z^n\, \mathrm{e}^{-\mathrm{i}\omega(z)t}$ is defined by

$$\begin{aligned}\mu_0(n, t, z) = I &+ \sum_{m=0}^{n-1} Z^{n-m-1} Q_m(t)\mu_0(m, t, z) Z^{m-n} \\ &+ \int_0^t Z^n \mathrm{e}^{-\mathrm{i}\omega(z)(t-\tau)\sigma_3} H(0, \tau, z)\mu_0(0, \tau, z)\, \mathrm{e}^{\mathrm{i}\omega(z)(t-\tau)\sigma_3} Z^{-n}\, \mathrm{d}\tau.\end{aligned} \tag{3.1}$$

As in the IVP, $\mu_{+,2}$ can be analytically extended to the exterior ($|z| > 1$) of the unit circle, while $\mu_{+,1}$ to its interior ($|z| < 1$), both with continuous extension to $|z| = 1$. On the other hand, both columns of $\mu_0(n, t, z)$ can be analytically extended to the punctured complex z-plane $\mathbb{C}\backslash\{0\}$. Moreover, $\mu_{0,1}(n, 0, z)$ is continuous and bounded on $|z| \geqslant 1$, while $\mu_{0,2}(n, 0, z)$ on $|z| \leqslant 1$. Since $\det\mu_0(n, t, z) = \prod_{m=0}^{n-1}(1 - q_m r_m)$, we can define the scattering matrix $s(z)$ of the half line problem as

$$\Phi_+(n, t, z) = \Phi_0(n, t, z)\, s(z), \tag{3.2}$$

which is also independent of time.

Evaluating (3.2) at $(n, t) = (0, 0)$, one has

$$s(z) = \mu_+(0, 0, z) = \Phi_+(0, 0, z), \tag{3.3}$$

implying

$$\det s(z) = 1/C_0. \tag{3.4}$$

From the analyticity of $\mu_+(n, t, z)$, we have that the first column $s_1(z)$ is analytic on $|z| < 1$, continuous and bounded on $|z| \leqslant 1$, while the second column $s_2(z)$ analytic on $|z| > 1$, continuous and bounded on $|z| \geqslant 1$. Moreover, the same symmetries as in the IVP allow us to obtain relations among the various entries of the scattering matrix and thereby write $s(z)$ as

$$s(z) = \begin{pmatrix} a(z) & \nu b^*\left(1/z^*\right) \\ b(z) & a^*\left(1/z^*\right)\end{pmatrix}, \tag{3.5}$$

with $a(z)$ and $b(z)$ analytic on $|z| < 1$. Comparing with the elements of the eigenfunctions, one has

$$a(z) = \mu_{+,11}(0, 0, z), \qquad b(z) = \mu_{+,21}(0, 0, z), \tag{3.6a}$$

$$b^*\left(1/z^*\right) = \nu\mu_{+,12}(0, 0, z), \qquad a^*\left(1/z^*\right) = \mu_{+,22}(0, 0, z). \tag{3.6b}$$

Equation (3.4) then implies

$$a(z)a^*\left(1/z^*\right) - \nu b(z)b^*\left(1/z^*\right) = 1/C_0, \qquad |z| \leqslant 1. \tag{3.7}$$

As a corollary

$$|a(z)|^2 - \nu\,|b(z)|^2 = 1/C_0, \qquad |z| = 1. \tag{3.8}$$

The problem with this approach is that in order to compute the evolution of $\mu_0(n, t, z)$ for $t > 0$ one needs to evaluate $H(0, t, z)$ (see (3.1)), which contains both $Q_0(t)$ and $Q_{-1}(t)$, whereas only the combination (1.4) is given as a BC. Moreover, the regions of boundedness of the columns $\mu_0(n, t, z)$ for $t > 0$ are smaller than those of $\mu_0(n, 0, z)$. Explicitly, $\mu_{0,1}(n, t, z)$ is bounded on $(|z| \geqslant 1) \cap (\mathrm{Im}(z^2) \leqslant 0)$, while $\mu_{0,2}(n, t, z)$ is bounded on $(|z| \leqslant 1) \cap (\mathrm{Im}(z^2) \geqslant 0)$. As a result, an additional eigenfunction is needed in order to properly formulate the inverse problem for $t > 0$, and considerable additional work is needed to eliminate the unknown boundary value $Q_{-1}(t)$ from the solution of the problem. A general approach to the BVP was presented in [9]. In the following sections, instead, we present an alternative approach that is more suitable for BVPs with linearizable BCs.

4. A 'linear' BT for the AL system

In this section we obtain a discrete analogue of the BT that was used in [6, 7, 12] to solve the BVP for the NLS equation with linearizable BCs. We begin with the following result, the proof of which is obtained by direct calculation:

Proposition 4.1. *Let $q_n(t)$ and $\tilde{q}_n(t)$ be two solutions of the AL system (1.3) for all $n \in \mathbb{Z}$, and suppose that there exists a matrix $B(n, t, z)$ such that the corresponding eigenfunctions $\Phi_n(n, t, z)$ and $\tilde{\Phi}_n(n, t, z)$ satisfy*

$$\tilde{\Phi}(n, t, z) = B(n, t, z)\Phi(n, t, z). \tag{4.1}$$

A necessary and sufficient condition for (4.1) to hold is

$$B_{n+1}\left(Z + Q_n\right) = \left(Z + \tilde{Q}_n\right) B_n. \tag{4.2}$$

where $\tilde{Q}_n(t)$ is given by (2.2) with $q_n(t)$ replaced by $\tilde{q}_n(t)$.

Similarly to the continuum case, the difference equation (4.2) yields an auto-BT (i.e., a Darboux transformation [21]) between the new potential and the original one. We will refer to $B(n, t, z)$ as the *Darboux matrix*. Note that, in principle, one also needs to consider the condition obtained from the second half of the Lax pairs:

$$\dot{B}_n = \tilde{T}_n B_n - B_n T_n, \tag{4.3}$$

where $T(n, t, z) = -\mathrm{i}\omega(z)\sigma_3 + H(n, t, z)$ and similarly for $\tilde{T}(n, t, z)$. This condition is not independent of (4.1) and (4.2), however. More precisely, if (4.1) and (4.2) hold at $t = 0$, one can use (4.1) as a *definition* of $B(n, t, z)$ for all $t > 0$. Since both $\Phi(n, t, z)$ and $\tilde{\Phi}(n, t, z)$ are simultaneous solutions of both parts of their respective Lax pairs, (4.1) implies that (4.3) is satisfied for all $t > 0$. Hence it is enough to consider only (4.2).

The case when the Darboux matrix $B(n, t, z)$ is independent of z is trivial. By analogy with the continuum case, here we therefore consider the next-simplest case. That is, recalling that $z = \mathrm{e}^{\mathrm{i}kh}$, we look for a solution $B(n, t, z)$ of (4.2) which is $O(z)$ as $z \to \infty$ and and $O(1/z)$ as $z \to 0$:

$$B(n, t, z) = z f_n + g_n/z + d_n, \tag{4.4}$$

where f_n, g_n and d_n are 2×2 matrices, possibly dependent on $q_n(t)$ and $\tilde{q}_n(t)$ but independent of z. Since the index n will play a key role in the following calculations, to simplify the notation, throughout this section we will denote the matrix entries of f_n, g_n and d_n by *superscripts* $i\,j$ (with $i, j = 1, 2$).

Lemma 4.2. *Let $B(n, t, z)$ be a solution of (4.2) in the form of (4.4). Then*

$$f_n = \begin{pmatrix} f^{11} & 0 \\ 0 & f_n^{22} \end{pmatrix}, \quad g_n = \begin{pmatrix} g_n^{11} & 0 \\ 0 & g^{22} \end{pmatrix}, \quad d_n = \begin{pmatrix} 0 & d_n^{12} \\ d_n^{21} & 0 \end{pmatrix}, \tag{4.5a}$$

where, $\forall\, n \in \mathbb{Z}$,

$$f_n^{22} = f_0^{22}\, p_n, \qquad g_n^{11} = g_0^{11}\, p_n, \tag{4.5b}$$

$$d_n^{12} = f^{11} q_n - f_n^{22}\, \tilde{q}_n = g^{22} \tilde{q}_{n-1} - g_n^{11} q_{n-1}, \tag{4.5c}$$

$$d_n^{21} = g^{22} r_n - g_n^{11} \tilde{r}_n = f^{11} \tilde{r}_{n-1} - f_n^{22}\, r_{n-1}, \tag{4.5d}$$

with

$$p_n = \prod_{m=0}^{n-1} \frac{1 - \tilde{q}_m \tilde{r}_m}{1 - q_m r_m}, \quad p_{-n} = \prod_{m=-n}^{-1} \frac{1 - q_m r_m}{1 - \tilde{q}_m \tilde{r}_m}, \tag{4.5e}$$

$\forall\, n \in \mathbb{N}$, and $p_0 = 1$. Moreover

$$g_0^{11} = \left(f_0^{22}\right)^* \mathrm{e}^{\mathrm{i}\gamma}, \qquad g^{22} = \left(f^{11}\right)^* \mathrm{e}^{\mathrm{i}\gamma}, \tag{4.6}$$

where $f^{11}, f_0^{22} \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ are arbitrary constants.

Proof. The result follows by substituting (4.4) into (4.2) and solving recursively at different powers of z. More precisely, from the terms at $O(z^2)$ and $O(1/z^2)$ we get, $\forall\, n \in \mathbb{Z}$, $f_{n+1}^{11} = f_n^{11} =: f^{11}$, $g_{n+1}^{22} = g_n^{22} =: g^{22}$, $f_n^{12} = f_n^{21} = 0$ and $g_n^{12} = g_n^{21} = 0$. From the terms at $O(z)$ and $O(1/z)$ not containing the potentials explicitly we get, $\forall\, n \in \mathbb{Z}$, $d_n^{11} =: d^{11}$ and $d_n^{22} =: d^{22}$. Collecting all other terms then yields

$$f_{n+1}^{22} - f_n^{22} = d_n^{12} \tilde{r}_n - d_{n+1}^{21} q_n, \qquad g_{n+1}^{11} - g_n^{11} = d_n^{21} \tilde{q}_n - d_{n+1}^{12} r_n, \tag{4.7}$$

together with (4.5*c*) and (4.5*d*) and $d^{11} = d^{22} = 0$. By substituting (4.5*c*) and (4.5*d*) respectively into the first and the second one of (4.7) one has

$$f_{n+1}^{22} = \frac{1 - \tilde{q}_n \tilde{r}_n}{1 - q_n r_n} f_n^{22}, \quad g_{n+1}^{11} = \frac{1 - \tilde{q}_n \tilde{r}_n}{1 - q_n r_n} g_n^{11}, \qquad \forall\, n \in \mathbb{Z},$$

the solutions of which yields (4.5*b*). Finally, relations (4.6) are obtained by taking the complex conjugate of (4.5*c*) and (4.5*d*) and comparing with the original equations. □

In other words, theorem 4.2 implies that the BT is specified completely in terms of five real constants. To choose the parameters of the BT in a way that is useful for the BVP, we

now establish a correspondence between the discrete and continuum case. Recall that the continuum limit of the AL system (1.3) with $h = 1$ is obtained by letting

$$q_{\mathrm{al}}(n, \tau) = h\, q_{\mathrm{nls}}\big(nh, h^2\tau\big), \qquad z = \mathrm{e}^{\mathrm{i}kh}, \tag{4.8}$$

and taking the limit $h \to 0$ with $x = nh$ and $t = h^2\tau$ fixed, under which (1.3) reduces to the NLS equation (1.1). (Alternatively, one can obtain the NLS equation in the limit $h \to 0$ by considering (1.3) with $h \neq 1$ and taking $q_{\mathrm{al}}(n, t) = q_{\mathrm{nls}}(nh, t)$.)

Lemma 4.3. (*BT for the AL system.*) *Suppose $q_n(t)$ and $\tilde{q}_n(t)$ solve the AL system and the corresponding eigenfunctions are related by the BT (4.1), where $B(n, t, z)$ is of the form (4.4) with $\mathrm{f_n}$, $\mathrm{g_n}$ and $\mathrm{d_n}$ as in lemma 4.2, and with*

$$f^{11} = -g^{22} = 1/h, \qquad f_o^{22} = -g_o^{11} = \chi/h. \tag{4.9}$$

Then $q_n(t)$ and $\tilde{q}_n(t)$ are related by the difference equation

$$\chi p_n\big(\tilde{q}_n + q_{n-1}\big) = q_n + \tilde{q}_{n-1}. \tag{4.10}$$

Proof. Equation (4.10) (as well as its complex conjugate) follow directly from the off-diagonal entries of (4.2) upon substituting (4.4) and (4.9). The diagonal entries of (4.2) are then shown to be consistent with (4.10) via the difference equation

$$p_{n+1} = p_n \frac{1 - \tilde{q}_n \tilde{r}_n}{1 - q_n r_n},$$

which in turn follows from (4.5*e*). □

Equation (4.10) is the desired BT between the original potential $q_n(t)$ and the new one $\tilde{q}_n(t)$. Summarizing, lemmas 4.2 and 4.3 imply that we can write the Darboux matrix $B_n = B(n, t, z)$ as

$$B_n = \sigma_3\big(Z - \chi p_n Z^{-1}\big)/h + \sigma_3\big(Q_n - \chi p_n \tilde{Q}_n\big)/h. \tag{4.11}$$

As in the continuum case, the BT for the AL system is now written entirely in terms of a single real parameter, in our case χ. And, similarly to the continuum case, (4.10) is a difference equation that determines the Backlund-transformed potential $\tilde{q}_n$ in terms of the original one q_n (or vice versa). Moreover, note that, as $h \to 0$,

$$p_n = 1 + h^2 \sum_{m=0}^{n-1} \big(q(x_m, t) r(x_m, t) - \tilde{q}(x_m, t)\tilde{r}(x_m, t)\big) + O\big(h^4\big) \tag{4.12}$$

(where $x_m = mh$ and $t = h^2\tau$ for brevity and the fields in the right-hand side solve the continuum limit as discussed above). We then have the following:

Lemma 4.4. *Let $B(n, t, z)$ be defined as in (4.11). If, in addition, (4.8) holds and*

$$\chi = 1 - \alpha h, \tag{4.13}$$

the Darboux matrix $B(n, t, z)$ for the AL system in (4.11) reduces to the one for the NLS equation in (C.15) in the limit $h \to 0$.

As in the continuum case, we now impose the mirror symmetry to obtain a BT for the BVP. Evaluating (4.10) at $n = 0$, one can show the following:

Lemma 4.5. (*Mirror symmetry.*) *Suppose that* $q_n(t)$ *and* $\tilde{q}_n(t)$ *solve the AL system* (*1.3*) $\forall\, n \in \mathbb{Z}$ *and that they are related by the BT in lemma 4.3. If, in addition,* $q_n(t)$ *and* $\tilde{q}_n(t)$ *satisfy the mirror symmetry*

$$\tilde{q}_n(t) = q_{-n-1}(t) \qquad \forall\, n \in \mathbb{Z}, \tag{4.14}$$

then $q_n(t)$ *satisfies the BC* (*1.4*).

The BCs (1.4) for the AL system reduce to the BCs (1.2) for the NLS equation in the limit $h \to 0$. In particular: (i) the Neumann BC in the continuum problem, obtained for $\alpha = 0$, corresponds to $\chi = 1$ for the AL system. (ii) The Dirichlet BC in the continuum problem is obtained in the limit $\alpha \to \infty$. The discrete analogue for the AL system is $\chi = 0$, corresponding to $\alpha = 1/h$, which, yields the correct limit as $h \to 0$.

In section 5 we show how the above BT can be linearized. Then in section 6 we show how it can be used to effectively solve the BVP for the AL system with the linearizable BC (1.4). Since we will not need to discuss the continuum limit further (except for a single reference after corollary 7.4), hereafter we take $h = 1$ in (4.11) for brevity. We also note for later reference that

$$\begin{aligned}
&\det B_n = z^2\chi p_n + \chi p_n/z^2 - 1 - \chi^2 p_n^2 + \big(q_n - \chi p_n \tilde{q}_n\big)\big(r_n - \chi p_n \tilde{r}_n\big),\\
&B_0(z) = B(0,\, t,\, z) = \sigma_3\big(Z - \chi Z^{-1}\big),\\
&\det B_0 = z^2\chi + \chi/z^2 - 1 - \chi^2.
\end{aligned}$$

5. Linearization of the BT

We now show how the BT introduced in section 4 can be explicitly linearized. Recall that the transformed potential is obtained from the original potential via (4.10), which we rewrite here:

$$\chi p_n\big(\tilde{q}_n + q_{n-1}\big) = q_n + \tilde{q}_{n-1}, \qquad n \in \mathbb{N}_0, \tag{5.1a}$$

together with the corresponding equation for $\tilde{r}_n$ in terms of r_n,

$$\chi p_n(\tilde{r}_n + r_{n-1}) = r_n + \tilde{r}_{n-1}, \qquad n \in \mathbb{N}_0, \tag{5.1b}$$

where

$$p_{n+1} = p_n\,\frac{1 - \tilde{q}_n\tilde{r}_n}{1 - q_n r_n}, \qquad n \in \mathbb{N}, \tag{5.1c}$$

plus the 'initial conditions' $\tilde{q}_{-1} = q_0$, $\tilde{r}_{-1} = r_0$ and $p_0 = 1$. Also recall that (5.1) obey the conservation law

$$\big(\chi p_n \tilde{q}_n - q_n\big)\big(\chi p_n \tilde{r}_n - r_n\big) - \chi^2 p_n^2 + p_n + \chi^2 = 1. \tag{5.2}$$

If $\chi = 0$, (5.1a) implies that $\tilde{q}_{n-1} = -q_n$, which is a linear relation, and the analogue of the odd extension in the continuum problem. For $\chi \neq 0$, we begin by introducing the change of variables

$$u_{n+1} = \tilde{q}_n + q_{n-1}, \qquad v_{n+1} = \tilde{r}_n + r_{n-1}, \qquad w_n = \chi p_n, \tag{5.3}$$

upon which (5.1) becomes

$$w_n u_{n+1} = u_n + \Delta q_n, \tag{5.4a}$$

$$w_n v_{n+1} = v_n + \Delta r_n, \tag{5.4b}$$

$$\left(1 - q_n r_n\right) w_{n+1} = \left[1 - u_{n+1}v_{n+1} + (r_{n-1}u_{n+1} + q_{n-1}v_{n+1}) - q_{n-1}r_{n-1}\right] w_n, \tag{5.4c}$$

where $\Delta f_n := f_n - f_{n-2}$, together with the initial conditions

$$u_1 = (2/\chi)\, q_0, \qquad v_1 = (2/\chi)\, r_0, \qquad w_1 = \frac{\chi - q_0 r_0/\chi}{1 - q_0 r_0}. \tag{5.5}$$

(The expression for w_1 follows from (5.1*c*) and the fact that (5.1*a*) and (5.1*b*) imply $\chi\tilde{q}_o = q_o$ and $\chi\tilde{r}_o = r_o$.) Moreover, using (5.3) in (5.2), solving for $u_{n+1}v_{n+1}$ and substituting in (5.4*c*) yields, after straightforward algebra

$$\begin{aligned}\left(1 - q_n r_n\right) w_n\, w_{n+1} = &\left(q_{n-1}r_n + q_n r_{n-1} + 1/\chi\right) w_n \\ &- r_n u_n - q_n v_n - \left(q_n r_n - q_{n-2}r_n - q_n r_{n-2} + 1 - \chi^2\right).\end{aligned} \tag{5.6}$$

The final step is the introduction of the further change of variable

$$u_n = U_n/Y_n, \qquad v_n = V_n/Y_n, \qquad w_n = W_n/Y_n, \tag{5.7}$$

which reduces the solution of (5.4*a*), (5.4*b*) and (5.6) to that of the system

$$U_{n+1} = U_n + \Delta q_n\, Y_n,\ V_{n+1} = V_n + \Delta r_n\, Y_n,\ Y_{n+1} = W_n, \tag{5.8a}$$

$$\begin{aligned}W_{n+1} = &\frac{q_{n-1}r_n + q_n r_{n-1} + 1/\chi}{1 - q_n r_n}\, W_n - \frac{r_n}{1 - q_n r_n}\, U_n \\ &- \frac{q_n}{1 - q_n r_n}\, V_n - \frac{q_n r_n - q_{n-2}r_n - q_n r_{n-2} + 1 - \chi^2}{1 - q_n r_n}\, Y_n.\end{aligned} \tag{5.8b}$$

for all $n \in \mathbb{N}$, together with the initial conditions

$$U_1 = u_1, \qquad V_1 = v_1, \qquad W_1 = w_1, \qquad Y_1 = 1. \tag{5.9}$$

Equations (5.8) are a system of homogeneous linear difference equations. The BT has therefore been completely linearized. We summarize the results in the following:

Theorem 5.1. *For all $n \geqslant 0$, the potential $\tilde{q}_n(t)$ defined by $q_n(t)$ via the BT (5.1) and the initial condition $\tilde{q}_{-1} = q_0$ can be obtained as $\tilde{q}_n = -q_{n-1} + U_{n+1}/Y_{n+1}$, where U_n and Y_n are given by the solution of the linear system (5.8) with initial conditions (5.9), and where u_1, v_1 and w_1 are given by (5.5).*

6. Solution of the BVP via BT

Similarly to the continuum case, the strategy for solving the BVP for the AL system (1.3) on $n \in \mathbb{N}$ with the linearizable BC (1.4) is to define the following extension:

Definition 6.1. (Extension.) Given $q_n(0)$ for $n \geqslant 0$, define $\tilde{q}_n(0)$ for all $n \geqslant 0$ via the difference equation (4.10) plus the initial condition $\tilde{q}_{-1}(0) = q_0(0)$. Then, given $\tilde{q}_n(0)$ for all $n \geqslant 0$, define $q_n(0)$ for all $n < 0$ via the mirror symmetry (4.14).

Since lemma 4.5 ensures that the BC (1.4) is satisfied for all $t > 0$, one can then use the IST of the IVP to solve the BVP. We call the potential obtained from the above steps the *extended* potential. As special cases of the above, we have the following:

Lemma 6.2. *Consider the extension in definition 6.1.*

(*i*) *If* $\chi = 0$, *the extension reduces to the odd extension of the original potential:* $q_{-n}(t) = -q_n(t)$.
(*ii*) *If* $\chi = 1$, *the extension reduces to the even extension of the potential with a shift:* $q_{-n}(t) = q_{n-1}(t)$. *Moreover,* $p_n(t) = 1 \ \forall\, n \in \mathbb{N}$.

Proof. It is easy to see that: (i) when $\chi = 0$, the above procedure yields $\tilde{q}_n(t) = -q_{n+1}(t)$, which turns via (4.14) into the odd extension of the potential. (ii) When $\chi = 1$, the above procedure yields $\tilde{q}_n(t) = q_n(t)$, which turns via (4.14) into the even extension (with a shift) of the potential. Finally, one can easily check from (4.5*e*) that $p_n(t) = 1$. □

The above two extensions above were used in [11] to solve the BVP for the AL system with those specific values of χ. Importantly, however, it was not possible to find an explicit extension for generic values of χ. This problem was resolved here through the introduction of the BT.

A technical but important issue in order for the above approach to be effective for generic values of χ is to show that the extension of $q_n(t)$ over negative integer values of n yields a potential that is still in the class of ICs amenable to the IST for the IVP. (Recall that for the NLS equation, a complete proof of the equivalent result was presented in [12].) in appendix B we show that this indeed is the case, by proving the following:

Theorem 6.3. *Let* $\tilde{q}_n(t)$ *be defined for all* $n \in \mathbb{N}$ *by the BT* (4.10). *If* $q_n(t) \in \ell_1(\mathbb{N})$, *then* $\tilde{q}_n(t) \in \ell_1(\mathbb{N})$.

As a byproduct, the proof of theorem 6.3 also yields the following important result:

Corollary 6.4. *Given* $q_n(t)$ *for* $n \in \mathbb{N}$, *let* $\tilde{q}_n(t)$ *and* $q_n(t)$ *for all* $n \in \mathbb{Z}$ *be defined by the BT* (4.10) *and the mirror symmetry* (4.14). *Also, let* $a(z)$ *and* $b(z)$ *the half-line scattering coefficients, defined by* (3.5). *The constant* p_∞ *in the BT can be expressed as a function of* χ *as follows:*

(*i*) *If* $\chi = 0$ *or* $\chi = \pm 1$, *then* $p_\infty = 1$.
(*ii*) *For* $\chi > 1$, *if* $a(1/\sqrt{\chi}) = 0$ *then* $p_\infty = 1/\chi^2$, *while if* $a(1/\sqrt{\chi}) \neq 0$ *then* $p_\infty = 1$.
(*iii*) *For* $0 < \chi < 1$, *if* $b(\sqrt{\chi}) = 0$ *then* $p_\infty = 1$, *while if* $b(\sqrt{\chi}) \neq 0$ *then* $p_\infty = 1/\chi^2$.
(*iv*) *For* $-1 < \chi < 0$, *if* $b(\mathrm{i}\sqrt{|\chi|}) = 0$ *then* $p_\infty = 1$, *while if* $b(\mathrm{i}\sqrt{|\chi|}) \neq 0$ *then* $p_\infty = 1/\chi^2$.
(*v*) *For* $\chi < -1$, *if* $a(\mathrm{i}/\sqrt{|\chi|}) = 0$ *then* $p_\infty = 1/\chi^2$, *while if* $a(\mathrm{i}/\sqrt{|\chi|}) \neq 0$ *then* $p_\infty = 1$.

As we will see below, corollary 6.4 will be a key piece in the characterizaztion of the symmetries of the BVP.

7. Symmetries of the BVP

We now use the approach outlined in section 6 to characterize the solutions of the BVP for generic values of χ. Note first that the mirror symmetry (4.14) implies

$$\tilde{C}_n(t) C_{-n}(t) = C_{-\infty} = \tilde{C}_{-\infty}, \tag{7.1a}$$

$$\tilde{p}_n p_{-n} = 1, \tag{7.1b}$$

where $\tilde{C}_n(t)$ and $\tilde{p}_n$ are defined as in (2.6) and (4.5*e*), but with q_n and r_n replaced by $\tilde{q}_n$ and $\tilde{r}_n$, respectively. Note also that (4.14) yields $p_n = p_{-n}\ \forall\ n \in \mathbb{N}$, implying

$$\lim_{n\to\pm\infty} B_n = \sigma_3\left(Z - \chi p_\infty Z^{-1}\right) =: B_\infty(z), \tag{7.2}$$

with $p_\infty = \lim_{n\to\pm\infty} p_n$, and where the off-diagonal terms vanish thanks to theorem 6.3.

Lemma 7.1. (*Symmetries via BT.*) *Let $\Phi_\pm(n, t, z)$ and $\tilde{\Phi}_\pm(n, t, z)$ be respectively the Jost solutions of the original and transformed scattering problem, related by the BT (4.1). For all $|z| = 1$ one has*

$$\tilde{\Phi}_\pm(n, t, z) B_\infty(z) = B(n, t, z)\Phi_\pm(n, t, z), \tag{7.3}$$

and the corresponding scattering matrices satisfy

$$\tilde{A}(z) B_\infty(z) = B_\infty(z) A(z). \tag{7.4}$$

Proof. Since $\tilde{\Phi}_\pm(n, t, z)$ and $B(n, t, z)\Phi_\pm(n, t, z)$ are both fundamental solutions of the same scattering problem, they can be expressed as linear combinations of each other, namely $\tilde{\Phi}_\pm(n, t, z) N_\pm(z) = B(n, t, z)\Phi_\pm(n, t, z)$, for some invertible matrices $N_\pm(z)$ independent of n and t. Comparing the asymptotic behavior of the left-hand side and right-hand side of this expression as $n \to -\infty$ one then obtains (7.3). Finally, (7.4) follows trivially by combining the two relations in (7.3). □

Lemma 7.2. (*Symmetries via mirror transformation.*) *Let $\Phi_\pm(n, t, z)$ and $\tilde{\Phi}_\pm(n, t, z)$ be the Jost solutions of the scattering problems corresponding to $q_n(t)$ and $\tilde{q}_n(t)$, respectively. If the mirror symmetry (4.14) holds, for all $|z| = 1$ one has*

$$\tilde{\Phi}_-(n, t, z) = C_{-n}(t)\, \sigma_3\, \Phi_+(-n, 1/z, t)\, \sigma_3, \tag{7.5a}$$

$$\tilde{\Phi}_+(n, t, z) = \left(1/\tilde{C}_n(t)\right) \sigma_3\, \Phi_-(-n, 1/z, t)\, \sigma_3, \tag{7.5b}$$

and the corresponding scattering matrices satisfy

$$A(1/z) = C_{-\infty}\, \sigma_3\, \tilde{A}^{-1}(z)\, \sigma_3. \tag{7.6}$$

Proof. Note first that (4.14) implies $(Z^{-1} + Q_{-n})^{-1} = (Z - \tilde{Q}_{n-1})/(1 - \tilde{q}_{n-1}\tilde{r}_{n-1})$. Using this relation, one can show that if $\Phi(n, t, z)$ is an eigenfunction of the original scattering problem, the matrix

$$\Psi(n, t, z) = \left(1/\tilde{C}_n\right) \sigma_3 \Phi(-n, 1/z, t) \tag{7.7}$$

is an eigenfunction of the transformed one. Now let $\Phi = \Phi_-$ in the above. Letting $n \to \infty$ and comparing the asymptotic behaviors of $\Psi(n, t, z)$ and $\tilde{\Phi}(n, t, z)$ one obtains the second of (7.5). Similarly, by taking $\Phi = \Phi_+$ and letting $n \to -\infty$ one gets the first of (7.5). Finally, (7.6) follows by combining the two equations in (7.5) □

Combining lemmas 7.1 and 7.2 we have:

Theorem 7.3. (*Symmetries of the BVP.*) *Let $\Phi_\pm(n, t, z)$ be the Jost solutions of the scattering problem corresponding to the potential $q_n(t)$ satisfying the AL system (1.3) and the BC (1.4), extended to all $n \in \mathbb{Z}$ via the BT (4.1) with the mirror symmetry (4.14). For all $|z| = 1$ one has*

$$\Phi_-(-n,\,1/z,\,t)\sigma_3 B_\infty(z) = \frac{C_{-\infty}}{C_{-n}(t)}\,\sigma_3\,B(n,\,t,\,z)\,\Phi_+(n,\,t,\,z), \tag{7.8a}$$

$$\Phi_+(-n,\,1/z,\,t)\sigma_3 B_\infty(z) = \frac{1}{C_{-n}(t)}\,\sigma_3\,B(n,\,t,\,z)\,\Phi_-(n,\,t,\,z), \tag{7.8b}$$

as well as

$$A(1/z)\sigma_3 B_\infty(z) = C_{-\infty}\,\sigma_3\,B_\infty(z)\,A^{-1}(z). \tag{7.9}$$

with $B_\infty(z)$ as in (7.2).

It will be useful to decompose (7.9) componentwise. Explicitly, taking into account the symmetries (2.12*a*), we have

$$a_{22}(z) = a_{11}(1/z) = \left(a_{22}\left(z^*\right)\right)^*, \qquad |z| \leqslant 1, \tag{7.10}$$

$$a_{12}(z) = f(z)\,a_{12}(1/z) = \nu f(z)\left(a_{21}\left(z^*\right)\right)^*, \qquad |z| = 1, \tag{7.11}$$

with

$$f(z) = \frac{z\chi p_\infty - 1/z}{z - \chi p_\infty/z}. \tag{7.12}$$

Recalling corollary 6.4, we can then write an explicit expression for $f(z)$:

Corollary 7.4. *Let $q_n(t)$ be a solution of the AL system (1.3) for $n \in \mathbb{N}$, and let $q_n(t)$ for $n < 0$ be defined by the BT (4.10) plus the mirror symmetry (4.14). Let*

$$f_1(z) = \frac{z - z^{-1}\chi}{z\chi - z^{-1}}, \qquad f_2(z) = \frac{z\chi - z^{-1}}{z - z^{-1}\chi}. \tag{7.13}$$

The function $f(z)$ in (7.12) simplifies as follows:

(*i*) *If $\chi = 0$, then $f(z) = -1/z^2$.*
(*ii*) *If $\chi = \pm 1$, then $f(z) = \pm 1$.*
(*iii*) *If $\chi > 1 \wedge a(1/\sqrt{\chi}) = 0$ or $\chi < -1 \wedge a(\mathrm{i}/\sqrt{|\chi|}) = 0$, then $f(z) = f_1(z)$.*
(*iv*) *If $\chi > 1 \wedge a(1/\sqrt{\chi}) \neq 0$ or $\chi < -1 \wedge a(\mathrm{i}/\sqrt{|\chi|}) \neq 0$, then $f(z) = f_2(z)$.*
(*v*) *If $0 < \chi < 1 \wedge b(\sqrt{\chi}) \neq 0$ or $-1 < \chi < 0 \wedge b(\mathrm{i}\sqrt{|\chi|}) \neq 0$, then $f(z) = f_1(z)$.*
(*vi*) *If $0 < \chi < 1 \wedge b(\sqrt{\chi}) = 0$ or $-1 < \chi < 0 \wedge b(\mathrm{i}\sqrt{|\chi|}) = 0$, then $f(z) = f_2(z)$.*

In comparison with [11], note that the $f(z)$ above is the same as that in [11] when $\chi = 0$, but different when $\chi = 1$. This is due to the fact that in [11] the BC for $\chi = 1$ was $q_0(t) - q_1(t) = 0$ while in this work $\chi = 1$ implies $q_0(t) - q_{-1}(t) = 0$. Conversely, in comparison with the continuum problem (i.e., the NLS equation), $f(z)$ is the same as $f(k)$ for the 'Neumann-like' BCs $\chi = 1$, but different for the 'Dirichlet-like' BCs $\chi = 0$.

Also, with respect to the continuum limit, it is easy to show that $f(z)$ in (7.12) equals $f(k)$ in (C.22) plus $O(h)$ terms as $h \to 0$. Indeed, both of (7.11) reduce to (C.21) as $h \to 0$. In particular, for $\chi = 0$ one has $p_n = 1$, and for $\chi = 1$ one has $p_n = 1 - q_n r_n$, implying $p_\infty = 1$ in both cases.

Recall that the half-line scattering matrix $s(z)$ defined in (3.2) is the matrix that relates the Jost eigenfunctions $\Phi_+(n,\,t,\,z)$ and $\Phi_0(n,\,t,\,z)$, where $\Phi_+(n,\,t,\,z)$ is defined as in the IVP and $\Phi_0(n,\,t,\,z)$ is the simultaneous solution of the Lax pair such that $\Phi_0(0,\,0,\,z) = I$ (see section 3). The above results allow us to obtain $A(z)$ from $s(z)$:

Corollary 7.5. *For all $|z| = 1$, the scattering matrix $A(z)$ of the extended problem is obtained from the half-line scattering matrix $s(z)$ as*

(*i*) *If $\chi \neq \pm 1$,*

$$A(z) = C_0\, s^{-1}(z)\, B_0^{-1}(z)\sigma_3\, s(1/z)\, \sigma_3\, B_\infty(z), \tag{7.14a}$$

(*ii*) *If $\chi = 1$,*

$$A(z) = C_0\, s^{-1}(z)\, \sigma_3\, s(1/z)\, \sigma_3, \tag{7.14b}$$

(*iii*) *If $\chi = -1$,*

$$A(z) = C_0\, s^{-1}(z)\, s(1/z). \tag{7.14c}$$

with $B_0(z) = B(0, t, z)$ and $B_\infty(z)$ is as in (7.2).

Proof. We start by evaluating (7.8*b*) at $n = 0$. Recall that $B_0(z)$ and $B_\infty(z)$ are invertible for all $z \neq \pm\sqrt{\chi}, \pm 1/\sqrt{\chi}$ (see (B.2)). Thus, if $\chi \neq \pm 1$, $B_0(z)$ and $B_\infty(z)$ are invertible $\forall\ |z| = 1$. We then substitute the scattering relation (2.7) and its half-line analogue (3.3), to obtain (7.14*a*). When $\chi = \pm 1$, the singular values $z = \pm\sqrt{\chi}, \pm 1/\sqrt{\chi}$ lie on the unit circle, and the matrices $B_0(z)$ and $B_\infty(z)$ are not invertible at $z = \pm 1$. On the other hand, for $\chi = 1$ we have $B_0(z) = B_\infty(z) = (z - 1/z)\, I$, and therefore the two singular factors cancel each other out. Similarly, for $\chi = -1$ we have $B_0(z) = B_\infty(z) = (z + 1/z)\, \sigma_3$. □

8. Discrete spectrum and self-symmetric eigenvalues

Importantly, (7.10) implies that an additional symmetry exists for the discrete spectrum in the BVP compared to the IVP, similarly to in the continuum case. Recalling that the discrete eigenvalues are the zeros of $a_{11}(z)$ and $a_{22}(z)$, we have:

Theorem 8.1. *Let $q_n(t)$ be a potential satisfying the AL system (1.3) and the BC (1.4), extended to all $n \in \mathbb{Z}$ via the BT (4.1) with the mirror symmetry (4.14). The discrete eigenvalues of the corresponding scattering problem appear in symmetric octets:*

$$\pm z_j,\ \pm z_j^*,\ \pm 1/z_j^*,\ \pm 1/z_j, \qquad j = 1, \ldots, J. \tag{8.1}$$

Similarly to the continuum case [10], we denote the eigenvalue symmetric to z_j as $z_{j'} = z_j^*$, (where however the eigenvalue symmetric to k_j was $k_{j'} = -k_j^*$).

There are two situations in which an eigenvalue octet degenerates into a quartet: (i) $\operatorname{Im} z_j = 0$, i.e., $z_{j'} = z_j$; (ii) $\operatorname{Re} z_j = 0$, i.e., $z_{j'} = -z_j$. In the first case (i.e., real eigenvalues), these eigenvalues correspond to the self-symmetric eigenvalues of the continuum problem [7], and we will refer to them as *self-symmetric eigenvalues of the AL system*. In the second case (i.e., purely imaginary eigenvalues), these eigenvalues arise as an artifact of the discretization, and have no correspondence in the continuum case.

Definition 8.2. We denote by S_1 the number of non-self-symmetric eigenvalues, i.e., the number of discrete eigenvalues in the interior of the first quadrant inside the unit circle,

(which is also the number of non-degenerate octets), by S_i be the number of imaginary eigenvalues (or more precisely the number of discrete eigenvalues on the positive imaginary axis inside the unit circle), and by S_0 the number of self-symmetric eigenvalues, i.e., the number of discrete eigenvalues on the positive real axis inside the unit circle.

The total number of discrete eigenvalues is therefore

$$2J = 4S_1 + 2S_i + 2S_0.$$

Without loss of generality, we label the distinct eigenvalues inside the unit circle as

(i) Non-degenerate eigenvalue octets: $z_1, \ldots, z_{S_1}$ and $z_{J-S_1+1}, \ldots, z_J$ with $z_{J-S_1+j} = -z_j^*$ for $j = 1, \ldots, S_1$, plus their complex conjugates.
(ii) Imaginary eigenvalues: $z_{S_1+1}, \ldots, z_{S_1+S_i}$, plus their complex conjugates.
(iii) Real self-symmetric eigenvalues: $z_{S_1+S_i+1}, \ldots, z_{J-S_1}$, plus their opposites.

Recall that, in the continuum case, there are certain constraints on the existence of self-symmetric eigenvalues (see appendix C.2), as well as on the associated norming constants. Our next task is to obtain the discrete analogue of these constraints. This will be considerably more complicated than in the continuum case.

Lemma 8.3. *Let $C_n(t)$ and $\tilde{C}_n(t)$ be defined as in (2.6) for the original potential $q_n(t)$ and the transformed one $\tilde{q}_n(t)$, respectively. We have*

$$C_{-\infty} = \left(\frac{1 - \chi p_\infty}{1 - \chi}\right)^2 C_0^2, \qquad \chi \neq 1, \tag{8.2a}$$

$$C_{-\infty} = C_0^2, \qquad \chi = 1. \tag{8.2b}$$

Proof. Recall that, under our assumptions, C_n for all $n \in \mathbb{Z}$ and p_∞ are all positive quantities. From (4.11) we have

$$B(0, t, 1) = (1 - \chi)\,\sigma_3, \qquad B_\infty(1) = \left(1 - \chi p_\infty\right)\sigma_3. \tag{8.3}$$

Also, for $\chi \neq 1$ corollary 6.4 implies $p_\infty \neq 1/\chi$, which in turn ensures that both $B(0, t, 1)$ and $B_\infty(1)$ are non-singular, Evaluating both of the symmetries 7.8 at $(n, z) = (0, 1)$ and eliminating $\Phi_\pm(0, t, 1)$ using (7.8*b*) then yields the first of (8.2*a*). The corresponding expression for $\chi = 1$ is easily obtained using the even extension. □

In particular, lemma 8.3 implies that C_0 is actually independent of t in the BVP with linearizable BCs (since $C_{-\infty}$ is a conserved quantity for the AL system). Using the symmetries (7.5) for $\tilde{\Phi}_\pm(n, t, z)$, we can also obtain

$$\tilde{C}_0 = C_0 p_\infty, \qquad \chi \in \mathbb{R}. \tag{8.4}$$

Next we show that lemma 8.3 provides an alternative way to determine the value of p_∞. Indeed, since both C_∞ and p_∞ are positive, using (8.2*a*) and (8.4), from (7.1*a*) one obtains

$$\left(\chi^2 p_\infty - 1\right)\left(p_\infty - 1\right) = 0, \qquad \chi \in \mathbb{R}. \tag{8.5}$$

(Note however that (8.5) admits two roots, however, only one of which is correct.)

We now obtain a representation for $C_{-\infty}$ in terms of the scattering data of the extended problem. Recall first the trace formula (2.20). Since $\lim_{z\to 0} a_{22}(z) = 1$, the limit of (2.20) as $z \to 0$ yields:

$$C_{-\infty} = \frac{1}{\left|z_1 \cdots z_J\right|^4} \exp\left\{ \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \frac{\log\left(1 + |\rho(\zeta)|^2\right)}{\zeta} \mathrm{d}\zeta \right\}. \tag{8.6}$$

Note that the integral in the exponential function is purely imaginary, hence (8.6) is consistent with the fact that $C_{-\infty} > 0$. Equation (8.6) will be a key tool to identify necessary conditions for the existence of self-symmetric eigenvalues.

Lemma 8.4. *Let $a_{22}(z)$ be the coefficient defined by the scattering relation (2.7) in the IVP obtained from the BVP through the BT (4.10) and the mirror symmetry (4.14). The following two representations hold:*

$$a_{22}(1) = (-1)^{S_0} \sqrt{C_{-\infty}}\, \mathrm{e}^{-\frac{1}{\pi}\int_0^{\pi/2} \log\left(1+|\rho(\zeta)|^2\right)\mathrm{d}\zeta}, \qquad \chi \in \mathbb{R}, \tag{8.7a}$$

$$a_{22}(1) = \operatorname{sign}\left(\frac{1 - \chi p_\infty}{1 - \chi}\right)\sqrt{C_{-\infty}}, \qquad \chi \in \mathbb{R} \ \wedge \ \chi \neq 1. \tag{8.7b}$$

Proof. We start by deriving (8.7*b*). For $\chi \neq 1$, evaluating (7.8*b*) at $(n, z) = (0, 1)$ and recalling (8.3) and (2.7), one has

$$A(1) = C_0 \frac{1 - \chi p_\infty}{1 - \chi} I. \tag{8.8}$$

Then, by using (8.2*a*) one obtains (8.7*b*). Next we derive (8.7*a*). The trace formula (2.20), evaluated at $z = 1$, is

$$a_{22}(1) = C_{-\infty} \prod_{m=1}^{J} \frac{1 - z_m^2}{1 - \bar{z}_m^2}\, \mathrm{e}^{-\frac{1}{2\pi \mathrm{i}}\int_{|\zeta|=1} \frac{\zeta \log\left(1+|\rho(\zeta)|^2\right)}{\zeta^2-1}\mathrm{d}\zeta}, \tag{8.9}$$

where the integral is defined as the limiting value of that in (2.20) as $z \to 1$ from inside the unit circle. For the IBVP, we can break down the product in (8.9) into the product of three parts, each corresponding to one of the types of eigenvalues:

$$\prod_{m=1}^{S_1} \frac{\left(1 - z_m^2\right)\left(1 - \left(z_m^*\right)^2\right)}{\left(1 - \left(z_m^*\right)^{-2}\right)\left(1 - z_m^{-2}\right)} = \prod_{m=1}^{S_1} \left|z_m\right|^4,$$

$$\prod_{m=S_1+1}^{S_1+S_i} \frac{1 - z_m^2}{1 - \left(z_m^*\right)^{-2}} = \prod_{m=S_1+1}^{S_1+S_i} \left|z_m\right|^2,$$

$$\prod_{m=S_1+S_i+1}^{J} \frac{1 - z_m^2}{1 - z_m^{-2}} = (-1)^{S_0} \prod_{m=S_1+S_i+1}^{J} \left|z_m\right|^2.$$

Moreover, the symmetries (2.13*a*) and (7.11) imply

$$\frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \frac{\zeta \log\left(1 + |\rho(\zeta)|^2\right)}{\zeta^2 - 1}\mathrm{d}\zeta = \frac{1}{\pi}\int_0^{\pi/2} \log\left(1 + |\rho(\zeta)|^2\right)\mathrm{d}\zeta$$

By substituting these expressions into (8.9), one obtains (8.7*a*). □

Since the exponential term in (8.7) is real, we then have

Corollary 8.5. *For all $\chi \neq 1$,*

$$\operatorname{sign}\left(\frac{1-\chi p_\infty}{1-\chi}\right) = (-1)^{S_0}. \tag{8.10}$$

Combining (8.10) and corollary 6.4, we have:

Theorem 8.6. *Let $\tilde{q}_n(t)$ be defined by the BT (4.10) and satisfy the mirror symmetry (4.14), where $a(z)$ and $b(z)$ are defined as in (3.5). The number of self-symmetric eigenvalues S_0 is as follows:*

(i) If $\chi \leqslant 0$, then S_0 is even.
(ii) For $\chi > 1$, if $a(1/\sqrt{\chi}) = 0$, then S_0 is odd, whereas if $a(1/\sqrt{\chi}) \neq 0$, then S_0 is even.
(iii) For $0 < \chi < 1$, if $b(\sqrt{\chi}) = 0$, then S_0 is even, whereas if $b(\sqrt{\chi}) \neq 0$, then S_0 is odd.

Note that theorem 8.6 imposes no constraints on the number of self-symmetric eigenvalues when $\chi = 1$. In section 9 we will show that there can be any number of self-symmetric eigenvalues for $\chi = 1$. We will also show that no self-symmetric eigenvalues can be present when $\chi < 0$.

9. Norming constants and further conditions on self-symmetric eigenvalues

Lemma 9.1. *The norming constants of symmetric eigenvalues satisfy the relations*

$$b_{s'}b_s^* = C_{-\infty} f(z_{s'}), \qquad K_{s'}K_s^* = C_{-\infty}\frac{f(z_{s'})}{\left[\left(a_{22}'(z_s)\right)^*\right]^2}, \tag{9.1}$$

where $f(z)$ is as in (7.12).

Proof. The result follows from (7.8) and noting that (7.11) imples $a_{22}'(z_{s'}) = (a_{22}'(z_s))^*$. □

Explicitly, writing the norming constant associated to z_j as $K_j = \mathrm{e}^{\xi_j + \mathrm{i}\eta_j}$ for $j = 1,\ldots,2J$, we have

$$\xi_{s'} + \xi_s = \log C_{-\infty} + \log\left|f(z_s)\right| - 2\log\left|a_{22}'(z_s)\right|, \tag{9.2a}$$

$$\eta_{s'} - \eta_s = 2\arg\left[a_{22}'(z_s)\right] - \arg\left[f(z_s)\right]. \tag{9.2b}$$

Lemma 9.2. *In the IBVP, $S_i = 0$, i.e., there are no purely imaginary eigenvalues.*

Proof. Let z_s be a purely imaginary eigenvalue. Since $z_{s'} = -z_s$ since b_s is the same for $\pm z_s$, we have $b_{s'} = b_s$. Equation (9.1) becomes

$$\left|b_s\right|^2 = C_{-\infty} f(z_s), \quad \left|K_s\right|^2 = C_{-\infty} f(z_s)\Big/\left[\left(a_{22}'(z_s)\right)^*\right]^2. \tag{9.3}$$

In the reflectionless case

$$a_{22}'(z_s) = C_{-\infty}\frac{2z_s}{z_s^2-\left(z_s^*\right)^{-2}}\prod_{m=S_1+1}^{S_1+S_i}{}' \frac{\left|z_s\right|^2-\left|z_m\right|^2}{\left|z_s\right|^2-\left|z_m\right|^{-2}}$$
$$\prod_{m=S_1+S_i+1}^{J-S_1}\frac{\left|z_s\right|^2+z_m^2}{\left|z_s\right|^2+z_m^{-2}}\prod_{m=1}^{S_1}\frac{\left|z_s\right|^2+z_m^2}{\left|z_s\right|^2+\left(z_m^*\right)^{-2}}\frac{\left|z_s\right|^2+\left(z_m^*\right)^2}{\left|z_s\right|^2+z_m^{-2}}, \tag{9.4}$$

Since C_∞ and the last three products are real, and the remaining fraction is a purely imaginary number, we have $(a_{22}'(z_s))^2 < 0$. For the first of (9.3) to hold, one needs $f(z_s) > 0$, while for the second of (9.3) one needs $f(z_s) < 0$. Thus there can be no purely imaginary eigenvalues.

In the non-reflectionless case, the right-hand side of (9.4) contains the additional factor

$$\exp\left\{-\frac{1}{2\pi \mathrm{i}}\int_{|\zeta|=1}\frac{\zeta\log\left(1+|\rho(\zeta)|^2\right)}{\zeta^2-z_s^2}\,\mathrm{d}\zeta\right\}.$$

On the other hand, it is straightforward to show that the above quantity is real. Thus $(a_{22}'(z_s))^2$ is also negative, implying that the above result holds for non-reflectionless potentials. □

Lemma 9.3. *Real self-symmetric eigenvalue* z_s *for the AL system exists if* $\chi > 0$ *and if*

$$0 < z_s < \sqrt{\chi}, \qquad \chi \leqslant 1, \tag{9.5a}$$

$$0 < z_s < 1/\sqrt{\chi}, \qquad \chi > 1. \tag{9.5b}$$

The parameters of the norming constants of the self-symmetric eigenvalues then satisfy

$$\xi_s = \frac{1}{2}\log C_{-\infty} + \frac{1}{2}\log f(z_s) - \log\left|a_{22}'(z_s)\right|. \tag{9.6}$$

Proof. Recall that from (9.1), for self-symmetric eigenvalues we also obtain (9.3). Equation (10.1) for the self-symemtric eigenvalues in the reflectionless case is

$$a_{22}'(z_s) = C_{-\infty}\frac{2z_s}{z_s^2-z_s^{-2}}\prod_{m=S_1+1}^{J-S_1}{}'\frac{z_s^2-z_m^2}{z_s^2-z_m^{-2}}\prod_{m=1}^{S_1}\frac{z_s^2-z_m^2}{z_s^2-\left(z_m^*\right)^{-2}}\frac{z_s^2-\left(z_m^*\right)^2}{z_s^2-z_m^{-2}}, \tag{9.7}$$

Since $z_s \in \mathbb{R}$, for z_m in the first quarter, $z_s^2 - z_m^2 = r\mathrm{e}^{\mathrm{i}\gamma}$ and $z_s^2 - (z_m^*)^2 = r\mathrm{e}^{-\mathrm{i}\gamma}$ with r and γ real. Thus $(z_s^2 - z_m^2)(z_s^2 - (z_m^*)^2) = r^2 \in \mathbb{R}$ and the last product in (9.7) is then real. It is easy to see that the first part of (9.7) is also real, hence $[(a_{22}'(z_s))^*]^2 > 0$. On the other hand, $C_{-\infty} > 0$ by definition. Thus for self-symmetric eigenvalue to exist, from (9.3) we need $f(z_s) > 0$. We then have the following scenarios:

(i) $\chi = 0$: in this case, $f(z_s) = -1/z_s^2 < 0$, thus no self-symmetric eigenvalues can exist.
(ii) $\chi = 1$: in this case, $f(z_s) = 1 > 0$, so self-symmetric eigenvalues can exist. Note that there are no further constraints on the self-symmetric eigenvalue, which can therefore be located arbirarily in (0, 1).
(iii) $\chi > 0$ and $\chi \neq 1$: in this case $f(z_s)$ assumes special values at the points $0, \pm\sqrt{\chi}, \pm\sqrt{1/\chi}$. For $f(z_s)$ to be positive and z_s in the first quadrant, one needs $0 < z_s < \sqrt{\chi}$ if $\chi < 1$ and $0 < z_s < 1/\sqrt{\chi}$ if $\chi > 1$.
(iv) If $\chi < 0$, $f(z_s) < 0\ \forall\, z_s$, no self-symmetric eigenvalues exist since (9.5) also holds for the Dirichlet and Neumann BCs, it can be used for all cases.

Similarly to lemma 9.2, in the case of non-reflectionless potentials, the right-hand-side of (9.7) contains the additional term (9.5), which however can be shown to be positive. Thus the above results also hold for general potentials. □

The proof of lemma 9.3 finally completes the picture about the number of self-symmetric eigenvalues obtained in theorem 8.6:

Corollary 9.4. *Let $\tilde{q}_n$ be defined by the BT (4.10) and satisfy the mirror symmetry (4.14). If $\chi \leqslant 0$, the number of self-symmetric eigenvalues S_0 is zero.*

Conversely, note that when $\chi = 1$, no restriction is placed on the value of S_0. Note also that, for $\chi \geqslant 0$, (9.6) only imposes a constraint on the absolute value of K_s. There are no conditions on the phase of K_s, which in fact can be arbitrary.

10. Examples

We now produce some explicit solutions to illustrate the formalism presented in the previous sections. We point out that, even though all the examples presented below are, for simplicity, reflectionless solutions of the focusing case, the results of the previous sections apply also in the defocusing case, and also when the reflection coefficient is not identically zero.

We begin by writing some expressions that will prove useful and which hold for arbitrary potentials. From (2.20), the derivative of $a_{22}(z)$ evaluated at z_m is, for $|z| \leqslant 1$

$$a'_{22}(z_j) = C_{-\infty} \frac{2z_j}{z_j^2 - \bar{z}_j^2} \prod_{m=1}^{J}{}' \frac{z_j^2 - z_m^2}{z_j^2 - \bar{z}_m^2}\, \mathrm{e}^{-\frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \frac{\zeta \log\left(1+|\rho(\zeta)|^2\right)}{\zeta^2 - z_j^2} \mathrm{d}\zeta}. \tag{10.1}$$

Recall that the eigenvalues are ordered as mentioned in section 7. Note first that, for the self-symmetric eigenvalues ($s = S_1 + 1, \cdots, J$), (10.1) yields

$$a'_{22}(z_s) = C_{-\infty} \frac{2z_s}{z_s^2 - z_s^{-2}} \prod_{m=S_1+1}^{J-S_1}{}' \frac{z_s^2 - z_m^2}{z_s^2 - z_m^{-2}}$$
$$\prod_{m=1}^{S_1} \frac{z_s^2 - z_m^2}{z_s^2 - \left(z_m^*\right)^{-2}} \frac{z_s^2 - \left(z_m^*\right)^2}{z_s^2 - z_m^{-2}} \mathrm{e}^{-\frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \frac{\zeta \log\left(1+|\rho_1(\zeta)|^2\right)}{\zeta^2 - z_s^2} \mathrm{d}\zeta},$$

while for the non-self-symmetric eigenvalues ($s = 1, \cdots, S_1$), one has

$$a'_{22}(z_s) = C_{-\infty} \frac{2z_s\left(z_s^2 - \left(z_s^*\right)^2\right)}{\left(z_s^2 - \left(z_s^*\right)^{-2}\right)\left(z_s^2 - z_s^{-2}\right)} \prod_{m=S_1+1}^{J-S_1} \frac{z_s^2 - z_m^2}{z_s^2 - z_m^{-2}}$$
$$\prod_{m=1}^{S_1}{}' \frac{z_s^2 - z_m^2}{z_s^2 - \left(z_m^*\right)^{-2}} \frac{z_s^2 - \left(z_m^*\right)^2}{z_s^2 - z_m^{-2}} \mathrm{e}^{-\frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \frac{\zeta \log\left(1+|\rho_1(\zeta)|^2\right)}{\zeta^2 - z_s^2} \mathrm{d}\zeta}.$$

10.1. One non-self-symmetric eigenvalue

Here $S_0 = 0$, $S = S_1 = 1$, implying $J = 2$. Let $z_1 = \mathrm{e}^{(\alpha + \mathrm{i}\beta)/2}$ with $\alpha < 0$ and $0 < \beta < \pi$. Let us denote the norming constants as $K_1 = \mathrm{e}^{\xi_1 + \mathrm{i}\eta_1}$, $K_2 = K_{1'} = \mathrm{e}^{\xi_2 + \mathrm{i}\eta_2}$, similarly to [11].

Equation (9.2) is then

$$\xi_2 + \xi_1 = \log C_{-\infty} + \log \left|f(z_1)\right| - 2\log \left|a'_{22}(z_1)\right|, \tag{10.2a}$$

$$\eta_2 - \eta_1 = 2\arg\left[a'_{22}(z_1)\right] - \arg\left[f(z_1)\right]. \tag{10.2b}$$

By (8.6) in the reflectionless case, $C_{-\infty} = 1/|z_1 z_2|^4 = \mathrm{e}^{-4\alpha}$, hence

$$a'_{22}(z_1) = C_{-\infty}\frac{2z_1\left(z_1^2 - \left(z_1^*\right)^2\right)}{\left(z_1^2 - \left(z_1^*\right)^{-2}\right)\left(z_1^2 - z_1^{-2}\right)},$$

thus

$$2\log\left|a'_{22}(z_1)\right| = 2\log C_{-\infty} + 3\alpha_1 \log\left|\sinh\alpha_1 \sinh\left(\alpha_1 + \mathrm{i}\beta_1\right)/\sin\beta_1\right|,$$
$$2\arg\left(a'_{22}(z_1)\right) = -2\arg\left[\sinh\left(\alpha_1 + \mathrm{i}\beta_1\right)\right] - \beta_1 + \pi.$$

We now consider specific cases. For *Dirichlet-like BC*, since $f(z_1) = -1/z_1^2$, one has $\log|f(z_1)| = -\alpha_1$, $\arg[f(z_1)] = -\beta_1 + \pi$, and the symmetry (10.2) becomes

$$\xi_2 + \xi_1 = 2\log\left|\sinh\alpha_1 \sinh\left(\alpha_1 + \mathrm{i}\beta_1/\sin\beta_1\right.\right|,$$
$$\eta_2 - \eta_1 = -2\arg\left[\sinh\left(\alpha_1 + \mathrm{i}\beta_1\right)\right].$$

For *Neumann-like BC*, since $f(z_1) = 1$, one has $\log|f(z_1)| = \arg[f(z_1)] = 0$, the symmetry (10.2) becomes

$$\xi_2 + \xi_1 = \alpha_1 + 2\log\left|\sinh\alpha_1 \sinh\left(\alpha_1 + \mathrm{i}\beta_1\right)/\sin\beta_1\right|,$$
$$\eta_2 - \eta_1 = -2\arg\left[\sinh\left(\alpha_1 + \mathrm{i}\beta_1\right)\right] - \beta_1 + \pi.$$

Finally, for *Robin-like BC*, the symmetry (10.2) is

$$\xi_2 + \xi_1 = \alpha_1 + 2\log\left|\sinh\alpha_1 \sinh\left(\alpha_1 + \mathrm{i}\beta_1\right)/\sin\beta_1\right| + \log\left|f(z_1)\right|,$$
$$\eta_2 - \eta_1 = -2\arg\left[\sinh\left(\alpha_1 + \mathrm{i}\beta_1\right)\right] - \beta_1 + \pi - \arg\left[f(z_1)\right],$$

where $f(z_1) = (z_1\chi - z_1^{-1})/(z_1 - z_1^{-1}\chi)$. The behavior of such kinds of solutions was discussed in [11].

10.2. One and two self-symmetric eigenvalues

In the case of one self-symmetric eigenvalue, i.e., $S_0 = 1$, $S_1 = 0$, one gets

$$q_n(t) = \mathrm{e}^{2\mathrm{i}(\cosh\alpha_1 - 1)t + \mathrm{i}\arg K_1}\sinh\alpha_1 \operatorname{sech}\left[\frac{1}{2}\log f(z_1) + \frac{1}{2}\alpha_1(2n+1)\right],$$

where $\arg K_1$ is arbitrary, and it simply produces an overall rotation of the solution. In the case of two self-symmetric eigenvalues, instead, i.e., $S_0 = 2$, $S_1 = 0$, one gets an ordinary two-soliton solution of the AL system where, for $j = 1, 2$, the norming constants satisfy the following constraints:

$$\xi_j = \frac{1}{2}\alpha_j + \frac{1}{2}\log f(z_j) + \log\left|\frac{\sinh\alpha_j \sinh\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\sinh\left(\frac{\alpha_1-\alpha_2}{2}\right)}\right|.$$

Similarly to the continuum problem [7], such a solution describes the nonlinear superposition of two stationary solitons.

11. Concluding remarks

The main results of this work are represented by: (i) lemmas 4.3–4.5, which summarize the construction of the appropriate BT for the BVP for the AL system; (ii) theorem 5.1, which summarizes the linearization of such transformation; (iii) theorem 6.3, which guarantees that the extension via BT belongs to the class of potentials for which the IVP is amenable to solution by the classical IST; (iv) theorem 7.3, which provides the symmetries of the eigenfunctions and scattering matrix of the extended problem; (v) theorem 8.1, which shows that the discrete eigenvalues of the extended problem appear in symmetric octets; (vi) theorem 8.6, lemmas 9.2 and 9.3 and corollary 9.4, which provide constraints on the number and location of imaginary or self-symmetric eigenvalues; and (vii) lemma 9.1, which relates the norming constants of symmetric eigenvalues.

We emphasize that the solution of the original problem (namely, the BVP on the natural numbers) concides with the restriction to the natural numbers of the solution of the extended problem (where 'extended problem' identifies the IVP obtained by extending the original potential to all integers via the BT). Thus, once the IVP for the extended problem has been constructed, the solution of the BVP is recovered from the formalism of the IVP, namely the solution of the usual RHP and the evaluation of its asymptotics as $z \to 0$ or $z \to \infty$ (see section 2 and [3]). Note that the constant χ that defines the BCs does not appear explicitly in the formulation of the RHP for the IVP, but it determines the scattering data through the initial condition.

A natural question that could be asked is whether the symmetries of the extended problem provide any insight on the solution of the original problem. The reason for such a question should be obvious. If the answer were negative, one could rightfully wonder whether there is any real usefulness in the extension approach, since the linearization of the BVP can also generically be achieved using the more general discrete analogue of the unified transform method of Fokas [16], presented in [9]. Importantly, however, the answer to the above question is affirmative, as we discuss next.

Recall that the IST allows one to compute the long-time asymptotic behavior of the solution. In particular, as $t \to \infty$ the solution of the IVP decays along all directions $n = vt$ except for those values of v which coincide with the velocity of one of the solitons (see (2.19)). For the restriction to $n \geqslant 0$, one can readily see that the only solitons that survive as $t \to \infty$ are those for which $v \geqslant 0$, corresponding to discrete eigenvalues in the first quadrant. On the other hand, the IVP and BVP are both time-reversible, implying that one can also run time backwards. By computing the long-time asymptotics as $t \to -\infty$, one sees that the only solitons that survive in the restriction to $n \geqslant 0$ are those with $v \leqslant 0$, corresponding to discrete eigenvalues in the second quadrant. But the symmetries of the discrete eigenvalues in the extended problem (see theorem 8.1) imply that to each soliton in the problem there corresponds a symmetric soliton with equal amplitude and opposite velocity. Exactly one of these solitons shows up in the long-time asymptotics as $t \to \infty$, while the other shows up in the asymptotics as $t \to -\infty$. A similar, but more subtle, symmetry exists for the radiative components of the solution.

Thus, the symmetries of the extended problem reflect a corresponding symmetry in the solution of the BVP: the symmetry between the long-time asymptotic behaviors of the solution as $t \to -\infty$ and as $t \to \infty$. In other words, the solution of BVPs with linearizable

BCs possesses additional structure compared to that of BVPs with generic BCs. The discrete version of the Fokas method presented in [9] has a broader scope of applicability than the method of extension via BT presented here (since the latter only applies to problems with linearizable BCs, while the former applies to a wider class of BCs). At the present time, however, the extra structure present in BVPs with linearizable BCs can only be elucidated using the extension method presented here, which is specifically tailored for such BVPs.

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Appendix A. Asymptotics of the scattering eigenfunctions of the 'half-line' problem

Throughout this appendix, we denote respectively by M_D and M_O the diagonal and off-diagonal parts of a 2×2 matrix M. Note $Z^n M_D = M_D Z^n$ and $Z^n M_O = M_O Z^{-n}$,

A.1. Asymptotics of $\Phi_+(n,t,z)$ as $n \to \infty$

Equation (2.5*b*), implies that the diagonal and off-diagonal parts of μ_+ satisfy

$$\mu_{+,D}(n,\,t,\,z) = I - \sum_{m=n}^{\infty} Q_m \mu_O(m,\,z,\,t) Z^{-1},$$
$$\mu_{+,O}(n,\,t,\,z) = -\sum_{m=n}^{\infty} Q_m \mu_D(m,\,z,\,t) Z^{-2(n-m)+1}.$$

Introducing the expansion

$$\mu = \sum_{k=0}^{\infty} \mu^{(k)}, \qquad \mu^{(0)} = I, \qquad \mu^{(k+1)} = -\sum_{m=n}^{\infty} Z^{n-m-1} Q_m(t) \mu^{(k)} Z^{m-n}.$$

we have

$$\mu_D^{(k+1)}(n) = -\sum_{m=n}^{\infty} Q_m \mu_O^{(k)} Z^{-1}, \qquad \mu_O^{(k+1)}(n) = -\sum_{m=n}^{\infty} Q_m \mu_D^{(k)} Z^{-2(n-m)+1}, \tag{A.1}$$

Since $Q \in L^1$, one has $Q_m = O(1/m)$. It is then easy show that, for all $k \in \mathbb{Z}$,

$$\mu_D^{(2k)}(n,\,t,\,z) = O\left(1/n^{2k}\right), \quad \mu_O^{(2k+1)}(n,\,t,\,z) = O\left(1/n^{2k+1}\right), \quad \mu_D^{(2k+1)} = \mu_O^{(2k)} = O. \tag{A.2}$$

Indeed, one has $\mu_D^{(0)} = I$ and $\mu_O^{(0)} = O$, and the induction step follows directly from (A.1). In particular

$$\mu_+(n,\,t,\,z) = I - Q_n Z + O\left(1/n^2\right) \qquad n \to \infty, \tag{A.3}$$

or, columnwise

$$\mu_{+,1}(n,\,t,\,z) = \begin{pmatrix} 1 \\ -z\, r_n \end{pmatrix} + O\left(1/n^2\right), \tag{A.4a}$$

$$\mu_{+,2}(n,\,z) = \begin{pmatrix} -\,q_n/z \\ 1 \end{pmatrix} + O\big(1/n^2\big), \tag{A.4b}$$

as $n \to \infty$.

A.2. Asymptotics of $\Phi_0(n, t, z)$

From (3.2), one has $\Phi_0(n,\,t,\,z) = \Phi_+(n,\,t,\,z)s^{-1}(z)$. Then by using (3.4) and (3.5), we have

$$\Phi_{0,1}(n,\,t,\,z) = C_0\,a^*\big(1/z^*\big)\,\Phi_{+,1}(n,\,t,\,z) - C_0\,b(z)\,\Phi_{+,2}(n,\,t,\,z), \tag{A.5a}$$

$$\Phi_{0,2}(n,\,t,\,z) = C_0\,b^*\big(1/z^*\big)\,\Phi_{+,1}(n,\,t,\,z) + C_0\,a(z)\,\Phi_{+,2}(n,\,t,\,z). \tag{A.5b}$$

Equation (A.5) is valid on the circle $|z| = 1$. For the purposes of this section, however, we will assume that $b(z)$ admits analytic extension off the circle $|z| = 1$. As $n \to \infty$, on $|z| = 1$, and for $t = 0$, we then have

$$\begin{aligned}\Phi_{0,1}(n,\,z,\,0) = {}& C_0\,a^*\big(1/z^*\big)\left[\begin{pmatrix} z^n \\ -z^{n+1}\,r_n \end{pmatrix} + O\big(1/n^2\big)\right] \\ & - C_0\,b(z)\left[\begin{pmatrix} -q_n/z^{n+1} \\ 1/z^n \end{pmatrix} + O\big(1/n^2\big)\right],\end{aligned} \tag{A.6a}$$

$$\begin{aligned}\Phi_{0,2}(n,\,z,\,0) = {}& C_0\,a(z)\left[\begin{pmatrix} -q_n/z^{n+1} \\ 1/z^n \end{pmatrix} + O\big(1/n^2\big)\right] \\ & + C_0\,b^*\big(1/z^*\big)\left[\begin{pmatrix} z^n \\ -z^{n+1}\,r_n \end{pmatrix} + O\big(1/n^2\big)\right].\end{aligned} \tag{A.6b}$$

Now we extend (A.6*a*) to $|z| \geqslant 1$ and (A.6*b*) to $|z| \leqslant 1$, noting that the second term in each of (A.6*a*) and (A.6*a*) is either bounded (if $|z| = 1$) or vanish as $n \to \infty$ (if $|z| \neq 1$). The above relations of course yield the corresponding asymptotic behavior of the modified eigenfunction $\mu_0(n,\,t,\,z)$.

Appendix B. Convergence of the Bäcklund-transformed potential

Let us define

$$\Psi(n,\,t,\,z) = \begin{cases} \Psi_-(n,\,t,\,z) = \big(\Phi_{0,1}(n,\,t,\,z),\,\Phi_{+,2}(n,\,t,\,z)\big), & |z| \geqslant 1, \\ \Psi_+(n,\,t,\,z) = \big(\Phi_{+,1}(n,\,t,\,z),\,\Phi_{0,2}(n,\,t,\,z)\big), & |z| \leqslant 1. \end{cases} \tag{B.1}$$

From (2.1) we have

$$\det\Phi_n = \prod_{m=0}^{n-1}\big(1 - q_m r_m\big)\det\Phi_0, \qquad \det\tilde{\Phi}_n = \prod_{m=0}^{n-1}\big(1 - \tilde{q}_m\tilde{r}_m\big)\det\tilde{\Phi}_0.$$

Then (4.1) implies

$$\det B(n,\,t,\,z) = p_n\,\frac{\det\tilde{\Phi}_0}{\det\Phi_0} = p_n\,\det B(0,\,t,\,z), \tag{B.2}$$

which in turns implies that either $\det B(n,\,t,\,z) = 0\ \forall\ n \in \mathbb{Z}$ or $\det B(n,\,t,\,z) \neq 0\ \forall\ n \in \mathbb{Z}$. Thus it is enough to consider just the zeros of $\det B(0,\,t,\,z)$. Here we will work out the proof

in detail the two major cases: $0 < \chi < 1$ and $\chi > 1$. The remaining cases, including $\chi < 0$, $\chi = 0, \pm 1$ will be discussed more briefly.

For $0 < \chi < 1$ or $\chi > 1$, $\det B(0, t, z) = 0$ at $z = \pm\sqrt{\chi}, \pm 1/\sqrt{\chi}$. At each of these values, there is a constant vector $\mathbf{c}$ such that $B(0, t, z)\,\Psi(0, t, z)\,\mathbf{c} = 0$. Since $B(0, t, z)\,\Psi(0, t, z)\,\mathbf{c}$ is itself an eigenfunction, we also have

$$B(n, t, z)\,\Psi(n, t, z)\,\mathbf{c} = 0, \qquad \forall\, n > 0, \tag{B.3}$$

Finally, note

$$B\left(0, t, \pm\sqrt{\chi}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \pm\left(\chi\sqrt{\chi} - 1/\sqrt{\chi}\right) \end{pmatrix}, \tag{B.4a}$$

$$B\left(0, t, \pm 1/\sqrt{\chi}\right) = \begin{pmatrix} \pm\left(1/\sqrt{\chi} - \chi\sqrt{\chi}\right) & 0 \\ 0 & 0 \end{pmatrix}, \tag{B.4b}$$

so

$$B\left(0, t, \pm\sqrt{\chi}\right)\begin{pmatrix} * \\ 0 \end{pmatrix} = 0, \qquad B\left(0, t, \pm\, 1/\sqrt{\chi}\right)\begin{pmatrix} 0 \\ * \end{pmatrix} = 0, \tag{B.5}$$

where the asterisks denote arbitrary non-zero entries.

B.1. The case $\chi > 1$

Let $z = \sqrt{\chi} > 1$, so that $\Psi(n, t, z) = \Psi_{-}(n, t, z)$ in (B.1). From (B.4a), there are two sub-cases: either $\Phi_{+,22}(0, \sqrt{\chi}, t) = 0$ or $\Phi_{+,22}(0, \sqrt{\chi}, t) \neq 0$. From (3.6), these two sub-cases correspond to $a(1/\sqrt{\chi}) = 0$ or $a(1/\sqrt{\chi}) \neq 0$. Note also that, as $n \to \infty$, (B.3) yields $B(n, \sqrt{\chi}, 0)\Psi(n, \sqrt{\chi}, 0)\,\mathbf{c} = 0$, with

$$\Psi\left(n, \sqrt{\chi}, 0\right) = \left[C_0\, a^*\left(1/\sqrt{\chi}\right)\left(\begin{pmatrix} \sqrt{\chi}^n \\ 0 \end{pmatrix} + o(1)\right) + O(1), \left(\begin{pmatrix} 0 \\ 1/\sqrt{\chi}^n \end{pmatrix} + o(1)\right)\right],$$

from which we also obtain the two same sub-cases $a(1/\sqrt{\chi}) = 0$ or $a(1/\sqrt{\chi}) \neq 0$.

B.1.1. The sub-case $a\left(1/\sqrt{\chi}\right) = 0$. In this case we have $B(0, \sqrt{\chi}, t)\,\Phi_{+,2}(0, \sqrt{\chi}, t) = 0$, which then implies $B(n, \sqrt{\chi}, t)\,\Phi_{+,2}(n, \sqrt{\chi}, t) = 0$ for all $n > 0$. Multiplying both sides by $\sqrt{\chi}^n \mathrm{e}^{-\mathrm{i}\omega t}$, one has $B(n, \sqrt{\chi}, t)\,\mu_{+,2}(n, \sqrt{\chi}, t) = 0\ \forall\, n > 0$, the asymptotic expansion of which yields

$$\begin{pmatrix} \sqrt{\chi}\left(1 - p_\infty\right) & q - \chi\tilde{q} \\ -\left(r - \chi\tilde{r}\right) & \chi\sqrt{\chi}p_\infty - 1/\sqrt{\chi} \end{pmatrix}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)\right] = 0, \tag{B.6}$$

implying

$$(q - \chi\tilde{q})(1 + o(1)) = 0, \qquad \chi\sqrt{\chi}p_\infty - 1/\sqrt{\chi} + o(1) = 0. \tag{B.7}$$

as $n \to \infty$. The first and second of (B.7) yield $\tilde{q} = O(q)$ as $n \to \infty$ and $p_\infty = 1/\chi^2$, respectively. From $\tilde{q} = O(q)$, one has $\tilde{q} \in L^1$.

B.1.2. The sub-case $a\left(1/\sqrt{\chi}\right) \neq 0$. Letting $\mathbf{c} = (1,0)^T$, we have

$$B\left(0, \sqrt{\chi}, t\right)\Psi_{-}\left(0, \sqrt{\chi}, t\right)\mathbf{c} = 0,$$

which then implies $B(n, \sqrt{\chi}, t)\,\Psi_{-}(n, \sqrt{\chi}, t)\,\mathbf{c} = 0 \quad \forall\, n > 0$. Multiplying both sides by $\mathrm{e}^{\mathrm{i}\omega t}/\sqrt{\chi}^n$, we then have, for all $n > 0$,

$$B\big(n,\ \sqrt{\chi},\ t\big)\Big[\mu_{0,1}\big(n,\ \sqrt{\chi},\ t\big),\ \mu_{+,2}\big(n,\ \sqrt{\chi},\ t\big)\,\mathrm{e}^{2\mathrm{i}\omega t}/\chi^n\Big]\,\mathbf{c}=0. \tag{B.8}$$

As $n\to\infty$, (B.8) yields

$$\sqrt{\chi}\big(1-p_\infty\big)C_0\,a^*\big(1/\sqrt{\chi}\big)(1+o(1))+(q-\chi\tilde{q})o(1)=0, \tag{B.9a}$$

$$-(r-\chi\tilde{r})\,C_0\,a^*\big(1/\sqrt{\chi}\big)(1+o(1))+\big(\chi\sqrt{\chi}p_\infty-1\sqrt{\chi}\big)o(1)=0. \tag{B.9b}$$

Equation (B.9b) implies $\tilde{r}=O(r)\to 0$ as $n\to\infty$, using which in equation (B.9a) one gets $p_\infty=1$. Also $\tilde{r}=O(r)$ implies $\tilde{q}\in L^1$.

B.2. The case $0<\chi<1$

Let $z=\sqrt{\chi}<1$ as before, implying $\Psi(n,t,z)=\Psi_+(n,t,z)$ in (B.1). From (B.4a), there are two cases: $\Phi_{+,21}(0,t,\sqrt{\chi})=0$ or $\Phi_{+,21}(0,t,\sqrt{\chi})\neq 0$. From (3.6), these two cases correspond to $b(\sqrt{\chi})=0$ or $b(\sqrt{\chi})\neq 0$. On the other hand, from the limit of (B.3) as $n\to\infty$ we have two more cases $a(\sqrt{\chi})=0$ or $a(\sqrt{\chi})\neq 0$. Since $a(z)$ and $b(z)$ can not be simultaneously zero, we have three cases: (i) $b(\sqrt{\chi})=0,\ a(\sqrt{\chi})\neq 0$, (ii) $b(\sqrt{\chi})\neq 0$, $a(\sqrt{\chi})\neq 0$, (iii) $b(\sqrt{\chi})\neq 0$, $a(\sqrt{\chi})=0$.

B.2.1. The sub-case $b(\sqrt{\chi})=0$ and $a(\sqrt{\chi})\neq 0$. Since $b(\sqrt{\chi})=0$, we have

$$B\big(0,\ \sqrt{\chi},\ t\big)\,\Phi_{+,1}\big(0,\ \sqrt{\chi},\ t\big)=0,$$

which then implies

$$B\big(n,\ \sqrt{\chi},\ t\big)\,\Phi_{+,1}\big(n,\ \sqrt{\chi},\ t\big)=0\qquad\forall\ n>0$$

Multiplying both sides by $\mathrm{e}^{\mathrm{i}\omega t}/\sqrt{\chi}^n$, one has $B(n,\sqrt{\chi},t)\,\mu_{+,1}(n,\sqrt{\chi},t)=0\quad\forall\ n>0$, the asymptotic expansion of which implies $p_\infty=1$ and $\tilde{q}\in L^1$ as $n\to\infty$.

B.2.2. The sub-case $b(\sqrt{\chi})\,a(\sqrt{\chi})\neq 0$. Choosing $\mathbf{c}=(-1,\varphi_{+,21}(0,t,\sqrt{\chi}))^T$, we have $B(0,\sqrt{\chi},t)\,\Psi_+(0,\sqrt{\chi},t)\,\mathbf{c}=0$, implying

$$B\big(n,\ \sqrt{\chi},\ t\big)\,\Psi_+\big(n,\ \sqrt{\chi},\ t\big)\,\mathbf{c}=0,\qquad\forall\ n>0.$$

Multiplying both sides by $\sqrt{\chi}^n\mathrm{e}^{-\mathrm{i}\omega t}$, we have

$$B\big(n,\ \sqrt{\chi},\ t\big)\Big[\mu_{+,1}\big(n,\ \sqrt{\chi},\ t\big)\chi^n\mathrm{e}^{-2\mathrm{i}\omega t},\ \mu_{0,2}\big(n,\ t,\ \sqrt{\chi}\big)\Big]\,\mathbf{c}=0,\quad\forall\ n>0. \tag{B.10}$$

The limit of (B.10) yields

$$(q-\chi\tilde{q})\Big(C_0\,a\big(\sqrt{\chi}\big)\Phi_{+,21}\big(0,\ \sqrt{\chi},\ t\big)+o(1)\Big)=0, \tag{B.11a}$$

$$\begin{aligned}&C_0\,a\big(\sqrt{\chi}\big)\big(\chi\sqrt{\chi}p_\infty-1/\sqrt{\chi}\big)\Phi_{+,21}\big(0,\ \sqrt{\chi},\ t\big)\\&\quad-(r-\chi\tilde{r})o(1)+\ +o(1)=0.\end{aligned} \tag{B.11b}$$

Equation (B.11a) implies $\tilde{q}\in L^1$, using which in (B.11b) we get $p_\infty=1/\chi^2$.

The sub-case $b(\sqrt{\chi})\neq 0$ and $a(\sqrt{\chi})=0$. From (A.4) and (A.6b), one has

$$\mu_+^{(1)}\big(n,\ \sqrt{\chi},\ 0\big)\,\chi^n=\begin{pmatrix}\chi^n\\-\chi^n\sqrt{\chi}\,r_n\end{pmatrix}+O\big(1/n^2\big),$$

$$\mu_0^{(2)}\left(n,\ \sqrt{\chi},\ 0\right) = \frac{1}{b\left(\sqrt{\chi}\right)}\left(\begin{pmatrix} \chi^n \\ -\chi^n\sqrt{\chi}\ r_n \end{pmatrix} + O\left(1/n^2\right)\right) + O\left(1/n^2\right).$$

Since $b(\sqrt{\chi}) \neq 0$, following the same steps as in the previous subcase, we obtain equation (B.10). As $n \to \infty$, one has

$$\sqrt{\chi}\left(1 - p_\infty\right) + \left(q_n - \chi\tilde{q}_n\right)\left(-\sqrt{\chi}r_n + o(1)\right) + o(1) = 0,$$

$$-(r_n - \chi\tilde{r}_n)(1 + o(1/n)) + \left(\chi\sqrt{\chi}p_\infty - \frac{1}{\sqrt{\chi}}\right)\left(-\sqrt{\chi}r_n + o(1)\right) = 0,$$

Recalling that $q_n = o(1/n)$, the first of these equations implies $p_\infty = 1$, while the second one implies $\tilde{q} \in L^1$ as before.

B.3. The cases $-1 < \chi < 0$ *and* $\chi < -1$

For $-1 < \chi < 0$ and $\chi < -1$, the zeros of $\det B(0, t, z)$ are $z = \pm\mathrm{i}\sqrt{|\chi|}, \pm\mathrm{i}/\sqrt{|\chi|}$, with

$$B\left(0,\ \pm\mathrm{i}\sqrt{|\chi|},\ t\right) = \begin{pmatrix} 0 & 0 \\ 0 & \pm\,\mathrm{i}\left(\chi\sqrt{|\chi|} + 1/\sqrt{|\chi|}\right) \end{pmatrix}, \tag{B.12a}$$

$$B\left(0,\ \pm\mathrm{i}/\sqrt{|\chi|},\ t\right) = \begin{pmatrix} \pm\,\mathrm{i}\left(1/\sqrt{|\chi|} + \chi\sqrt{|\chi|}\right) & 0 \\ 0 & 0 \end{pmatrix}. \tag{B.12b}$$

When $\chi < -1$, there are two cases, $a(\mathrm{i}/\sqrt{|\chi|}) = 0$ and $a(\mathrm{i}/\sqrt{|\chi|}) \neq 0$. Following the same steps as in section B.1 we can show that $p_\infty = 1/\chi^2$ for $a(\mathrm{i}/\sqrt{|\chi|}) = 0$, $p_\infty = 1$ for $a(\mathrm{i}/\sqrt{|\chi|}) \neq 0$, and $\tilde{q} \in L^1$ in both cases.

When $-1 < \chi < 0$ there are three cases: (i) $b(\mathrm{i}\sqrt{|\chi|}) = 0$, $a(\mathrm{i}\sqrt{|\chi|}) \neq 0$, (ii) $b(\mathrm{i}\sqrt{|\chi|}) \neq 0$, $a(\mathrm{i}\sqrt{|\chi|}) \neq 0$, and (iii) $b(\mathrm{i}\sqrt{|\chi|}) \neq 0$, $a(\mathrm{i}\sqrt{|\chi|}) = 0$. Similar arguments as in section B.2 will give similar results as well.

B.4. The special points $\chi = 0, \pm 1$

At $\chi = 0$, one has $\det B(n, 0, t) = -1 + q_n r_n$ and $\det B(0, 0, t) = 1$. Then by (B.2), $p_n = 1 - q_n r_n$. Since $q_n \to 0$ as $n \to \infty$, $p_\infty = \lim_{n\to\infty} p_n = 1$. On other hand, recall that (5.1a) implies that $q_n = \tilde{q}_{n-1}$. Since $q_n \in L^1$, it is clear that $\tilde{q}_{n-1} \in L^1$.

At $\chi = 1$, $\det B_0 = 0$ at $z = \pm 1$. Since $B(0, \pm 1, t) \equiv 0$, one has $B(0, \pm 1, t)\tilde{\Phi}(0, \pm 1, t) \equiv 0$, implying $B(n, \pm 1, t)\tilde{\Phi}(n, \pm 1, t) \equiv 0\ \forall\ n$. As $\tilde{\Phi}(n, \pm 1, t)$ is invertible, we get $B(n, \pm 1, t) \equiv 0\ \forall\ n$, hence $p_n = 1\ \forall\ n$ and $q_n = \tilde{q}_n$, implying $p_\infty = 1$ and $\tilde{q}_n \in L^1$.

Similar arguments apply for the case $\chi = -1$ with the special values $z = \pm i$, implying $B(n, \pm 1, t) \equiv 0\ \forall\ n$, which in turns imply that $p_\infty = 1$ and $\tilde{q}_n = q_n \in L^1$.

Appendix C. The NLS equation with linearizable BCs

Here we briefly recall the relevant formulae for the BVP for the NLS equation, in order to compare with the continuum limit of the results for the AL system. We refer the reader to [3, 5, 13, 22, 25] for details on the IVP and to [4, 6–8, 10, 12, 14, 15, 17, 19, 24] for details on the BVP with linearizable BCs.

C.1. IST for the NLS equation

The Lax pair for the NLS equation (1.1) is

$$\Phi_x - \mathrm{i}k\sigma_3\Phi = Q\,\Phi, \qquad \Phi_t + 2\mathrm{i}k^2\sigma_3\Phi = H\,\Phi, \tag{C.1}$$

where $\Phi = \Phi(x, t, k)$ is a 2×2 matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x, t) = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \tag{C.2}$$

$$H(x, t, k) = \mathrm{i}\sigma_3\left(Q_x - Q^2\right) - 2kQ, \tag{C.3}$$

and $r(x, t) = \nu q^*(x, t)$. The modified eigenfunctions with constant limits as $x \to \pm\infty$ are $\mu(x, t, k) = \Phi(x, t, k)\,\mathrm{e}^{-\mathrm{i}\theta(x,t,k)\sigma_3}$, where $\theta(x, t, k) = kx - 2k^2t$. If $q(x, t)$ vanishes sufficiently fast as $x \to \pm\infty$, the Jost solutions are defined $\forall\, k \in \mathbb{R}$ by Volterra integral equations as

$$\mu_-(x, t, k) = I + \int_{-\infty}^{x} \mathrm{e}^{\mathrm{i}k(x-y)\sigma_3} Q(y, t)\mu_-(y, t, k)\,\mathrm{e}^{-\mathrm{i}k(x-y)\sigma_3}\mathrm{d}y,$$
$$\mu_+(x, t, k) = I - \int_{x}^{\infty} \mathrm{e}^{\mathrm{i}k(x-y)\sigma_3} Q(y, t)\mu_+(y, t, k)\,\mathrm{e}^{-\mathrm{i}k(x-y)\sigma_3}\mathrm{d}y.$$

The columns $\mu_{-,1}$ and $\mu_{+,2}$ can be analytically extended into the lower-half of the complex the k-plane, $\mu_{-,2}$ and $\mu_{+,1}$ in the upper-half plane. The time-independent scattering matrix $A(k)$ is defined as

$$\Phi_-(x, t, k) = \Phi_+(x, t, k)\,A(k), \qquad k \in \mathbb{R} \tag{C.4}$$

Correspondingly, $a_{11}(k)$ and $a_{22}(k)$ can be analytically continued respectively on $\operatorname{Im} k < 0$ and $\operatorname{Im} k > 0$ (but $a_{12}(k)$ and $a_{21}(k)$ are in general nowhere analytic). The eigenfunctions and scattering data satisfy the symmetry relations

$$\Phi_{\pm,1}(x, t, k) = \sigma_\nu\left(\Phi_{\pm,2}\left(x, t, k^*\right)\right)^*, \tag{C.5a}$$

$$\Phi_{\pm,2}(x, t, k) = \nu\sigma_\nu\left(\Phi_{\pm,1}\left(x, t, k^*\right)\right)^*, \tag{C.5b}$$

with σ_ν as before, implying

$$a_{22}(k) = a_{11}^*\left(k^*\right), \qquad a_{21}(k) = \nu a_{12}^*(k). \tag{C.6}$$

The asymptotic behavior of the eigenfunctions and scattering data as $k \to \infty$ is

$$\mu_- = I - \frac{1}{2\mathrm{i}k}\sigma_3 Q + \frac{1}{2\mathrm{i}k}\sigma_3 \int_{-\infty}^{x} q(y, t)r(y, t)\,\mathrm{d}y + O\left(1/k^2\right),$$
$$\mu_+ = I - \frac{1}{2\mathrm{i}k}\sigma_3 Q - \frac{1}{2\mathrm{i}k}\sigma_3 \int_{x}^{\infty} q(y, t)r(y, t)\,\mathrm{d}y + O\left(1/k^2\right),$$

and $A(k) = I + O(1/k)$. Discrete eigenvalues are values of $k \in \mathbb{C}$ such that $a_{22}(k) = 0$ or $a_{11}(k) = 0$. In the defocusing case, there are no such values. In the focusing case, we assume that there exist a finite number of such zeros, and that they are simple. Note that $a_{22}(k_j) = 0$ if and only if $a_{11}(k_j^*) = 0$. Let k_j and $\bar{k}_j = k_j^*$, for $j = 1, \ldots, J$, be the zeros of a_{22} and a_{11} respectively, with $\operatorname{Im} k_j > 0$. Then

$$\mu_{-,2}\left(x, t, k_j\right) = b_j\,\mathrm{e}^{2\mathrm{i}\theta\left(x,t,k_j\right)}\mu_{+,1}\left(x, t, k_j\right), \tag{C.7}$$

$$\mu_{-,1}\left(x,\, t,\, \bar{k}_j\right) = \bar{b}_j\, \mathrm{e}^{-2\mathrm{i}\theta\left(x,t,\bar{k}_j\right)}\mu_{+,2}\left(x,\, t,\, \bar{k}_j\right). \tag{C.8}$$

In turn, these provide the residue relations that will be used in the inverse problem:

$$\operatorname*{Res}_{k=k_j}\left[\frac{\mu_{-,2}}{a_{22}}\right] = C_j\, \mathrm{e}^{2\mathrm{i}\theta\left(x,t,k_j\right)}\mu_{+,1}\left(x,\, t,\, k_j\right), \tag{C.9}$$

$$\operatorname*{Res}_{k=\bar{k}_j}\left[\frac{\mu_{-,1}}{a_{11}}\right] = \bar{C}_j\, \mathrm{e}^{-2\mathrm{i}\theta\left(x,t,\bar{k}_j\right)}\mu_{+,2}\left(x,\, t,\, \bar{k}_j\right), \tag{C.10}$$

where the norming constants are $C_j = b_j/a_{22}'(k_j)$ and $\bar{C}_j = \bar{b}_j/a_{11}'(\bar{k}_j)$, and satisfy the symmetry relations $\bar{b}_j = -b_j^*$ and $\bar{C}_j = -C_j^*$. The inverse problem is defined in terms of the matrix RHP: $\forall\, k \in \mathbb{R}$,

$$M^-(x,\, t,\, k) = M^+(x,\, t,\, k)(I - J(x,\, t,\, k)), \tag{C.11}$$

where the sectionally meromorphic functions are $M^+(x,\, t,\, k) = (\mu_{+,1},\, \mu_{-,2}/a_{22})$ and $M^-(x,\, t,\, k) = (\mu_{-,1}/a_{11},\, \mu_{+,2})$, the jump matrix is

$$J(x,\, t,\, k) = \begin{pmatrix} \nu\, |\rho(k)|^2 & \mathrm{e}^{2\mathrm{i}\theta(x,t,k)}\rho(k) \\ -\,\nu \mathrm{e}^{-2\mathrm{i}\theta(x,t,k)}\rho^*(k) & 0 \end{pmatrix},$$

and the reflection coefficient is $\rho(k) = a_{12}(k)/a_{22}(k)$. The matrices $M^{\pm}(x,\, t,\, k) - I$ are $O(1/k)$ as $k \to \infty$. The reconstruction formula is

$$q(x,\, t) = -2\mathrm{i}\sum_{j=1}^{J} C_j\ \mathrm{e}^{2\mathrm{i}\theta\left(x,t,k_j\right)}\mu_{+,11}\left(x,\, t,\, k_j\right) + \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{e}^{2\mathrm{i}\theta(x,t,k)}\rho(k)\mu_{+,11}(x,\, t,\, k)\,\mathrm{d}k. \tag{C.12}$$

In the reflectionless case $[\rho(k) = 0 \quad \forall\, k \in \mathbb{R}]$ with $\nu = -1$, the RHP reduces to an algebraic system, whose solution yields the pure multi-soliton solution of the NLS equation as $q(x,\, t) = 2\mathrm{i}\det G^c/\det G$, where $G = (G_{j,j'})$,

$$G^c = \begin{pmatrix} 0 & \mathbf{y}^T \\ \mathbf{1} & G \end{pmatrix}, \quad G_{j,j'} = \delta_{j,j'} + \mathrm{e}^{2\mathrm{i}\theta\left(x,t,k_j'\right)}C_{j'}\sum_{p=1}^{J}\frac{C_p^*\mathrm{e}^{-2\mathrm{i}\theta\left(x,t,k_p^*\right)}}{\left(k_j - k_p^*\right)\left(k_p^* - k_{j'}\right)},$$

$\mathbf{y} = (y_1,\ldots,y_J)^T$, $\mathbf{1} = (1,\, \cdots,\, 1)^T$ and $y_j = C_j\, \mathrm{e}^{2\mathrm{i}\theta\left(x,t,k_j\right)}$ for $j,\, j' = 1,\ldots,J$. In particular, the one-soliton solution, i.e., $J = 1$ with $k_1 = (V + \mathrm{i}A)/2$ and $C_1 = A\, \mathrm{e}^{A\xi + \mathrm{i}(\varphi + \pi/2)}$, is $q(x,\, t) = A\, \mathrm{e}^{\mathrm{i}[Vx + (A^2 - V^2)\, t + \varphi]}\operatorname{sech}[A(x - 2Vt - \xi)]$.

C.2. BT and BVP for the NLS equation

Suppose $q(x,\, t)$ and $\tilde{q}(x,\, t)$ both solve the NLS equation (1.1) and the corresponding eigenfunctions $\Phi(x,\, t,\, k)$ and $\tilde{\Phi}(x,\, t,\, k)$ are related by

$$\tilde{\Phi}(x,\, t,\, k) = B(x,\, t,\, k)\, \Phi(x,\, t,\, k). \tag{C.13}$$

If $\Phi(x,\, t,\, k)$ is invertible, a necessary and sufficient condition for (C.13) is

$$B_x = \mathrm{i}k\left[\sigma_3,\, B\right] + \tilde{Q}B - BQ. \tag{C.14}$$

If $B(x, t, k)$ is linear in k, (C.14) yields

$$B(x, t, k) = 2\mathrm{i}k\ I + b\,\sigma_3 + \left(\tilde{Q} - Q\right)\sigma_3, \tag{C.15a}$$

$$b(x, t) = \alpha + \int_0^x \left(\tilde{q}(y, t)\tilde{r}(y, t) - q(y, t)r(y, t)\right) \mathrm{d}y, \tag{C.15b}$$

where α is an arbitrary constant. In particular, (C.14) yields the BT between the original potential and the new one:

$$\tilde{q}_x(x, t) - q_x(x, t) = \left(\tilde{q}(x, t) + q(x, t)\right) b(x, t). \tag{C.16}$$

If we now impose the *mirror* symmetry

$$\tilde{q}(x, t) = q(-x, t), \tag{C.17}$$

(C.16), evaluated at $x = 0$, yields the Robin BC (1.2). The mirror symmetry induces the following additional symmetries: for the eigenfunctions and the scattering matrix:

$$\Phi_\pm(x, t, k) = B^{-1}(x, t, k)\sigma_3\, \Phi_\mp(-x, t, -k)\, \sigma_3 B_\infty(k). \tag{C.18}$$

In turn, the scattering matrix $A(k)$ satisfies:

$$A(-k) = \sigma_3 B_\infty(k) A^{-1}(k) B_\infty^{-1}(k)\sigma_3, \qquad k \in \mathbb{R}, \tag{C.19}$$

or, in component form:

$$a_{22}(k) = a_{22}^*\left(-k^*\right), \quad \operatorname{Im} k \geqslant 0, \tag{C.20}$$

$$a_{12}(k) = f(k) a_{12}(-k), \quad k \in \mathbb{R}, \tag{C.21}$$

where

$$f(k) = \frac{2\mathrm{i}k - b_\infty}{2\mathrm{i}k + b_\infty}, \tag{C.22}$$

where $b_\infty = \lim_{x\to\pm\infty} b(x, t)$. As a result, an additional symmetry exists for the discrete spectrum: for each discrete eigenvalue k_n there exist a symmetric eigenvalue

$$k_{n'} = -k_n^*. \tag{C.23}$$

We say that k_n is *self-symmetric* if $k_{n'} = k_n$, i.e., if $k_n = \mathrm{i}A_n/2$. Denoting by S the number of self-symmetric eigenvalues in the UHP, one has

$$b_\infty = a_{22}(0)\,\alpha = (-1)^S \alpha. \tag{C.24}$$

Writing the discrete eigenvalues as $k_j = (V_j + \mathrm{i}A_j)/2$, one has

$$b_n b_{n'}^* = f\left(k_n\right), \tag{C.25}$$

In particular, a self-symmetric eigenvalue $k_n = \mathrm{i}A_n/2$ can exist if and only if $\alpha < \infty$ and $A_n > |\alpha|$.

References

[1] Ablowitz M J, Biondini G and Prinari B 2007 Inverse scattering transform for the integrable discrete nonlinear Schrödinger equation with nonvanishing boundary conditions *J. Phys. A: Math. Theor.* **23** 1711–58

[2] Ablowitz M J and Ladik F 1975 Nonlinear differential-difference equations *J. Math. Phys.* **16** 598–603

[3] Ablowitz M J, Prinari B and Trubatch A D 2003 *Discrete and Continuous Nonlinear Schrödinger Systems* (*London Mathematical Society Lecture Note Series* vol 302) (Cambridge: Cambridge University Press)
[4] Ablowitz M J and Segur H 1975 The inverse scattering transform: semi-infinite interval *J. Math. Phys.* **16** 1054–6
[5] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, Pa: SIAM)
[6] Bikbaev R F and Tarasov V O 1991 Initial boundary value problem for the nonlinear Schrödinger equation *J. Phys. A: Math. Gen.* **24** 2507–16
[7] Biondini G and Bui A 2012 On the nonlinear Schrödinger equation on the half line with homogeneous Robin boundary conditions *Stud. Appl. Math.* **129** 249–71
[8] Biondini G, Fokas A S and Shepelsky D 2014 Comparison of two approaches to the initial-boundary value problem for the nonlinear Schrödinger equation on the half line with Robin boundary conditions *Unified Transform Method for Boundary Value Problems: Applications and Advances* ed A S Fokas and B Pelloni (Philadelphia, Pa: SIAM)
[9] Biondini G and Hwang G 2008 Initial-boundary value problems for discrete evolution equations: discrete linear Schr¨odinger and integrable discrete nonlinear Schrödinger equations *Inverse Problems* **24** 065011
[10] Biondini G and Hwang G 2009 Solitons, boundary value problems and a nonlinear method of images *J. Phys. A: Math. Theor.* **42** 205207
[11] Biondini G and Hwang G 2010 The Ablowitz–Ladik system on the natural number with certain linearizable boundary conditions *Appl. Anal.* **89** 627–44
[12] Deift P and Park 2011 Long-time asymptotics for solutions of the NLS equation with a delta potential and even initial data *Int. Math. Res. Not.* **2011** 5505–624
[13] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)
[14] Fokas A S 1989 An initial-boundary value problem for the nonlinear Schrödinger equation *Physica* D **35** 167–85
[15] Fokas A S, Its A R and Sung L-Y 2005 The nonlinear Schrödinger equation on the half-line *Nonlinearity* **18** 1771–822
[16] Fokas A S 2008 *A Unified Approach to Boundary Value Problems* (Philadelphia, Pa: SIAM)
[17] Habibullin I T 1991 Integrable initial boundary value problem *Theor. Math. Phys.* **86** 28–36
[18] Habibullin I T 1995 Boundary conditions for integrable chains *Phys. Lett.* A **207** 263–8
[19] Its A R and Shepelsky D 2013 Initial boundary value problem for the focusing NLS equation with Robin boundary condition: half-line approach *Proc. R. Soc.* A **469** 20120199
[20] Vekslerchik V E and Konotop V V 1992 Discrete nonlinear Schrödinger equation under non-vanishing boundary conditions *Inverse Problems* **8** 889–909
[21] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[22] Novikov S P, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons: The Inverse Scattering Method* (New York: Plenum)
[23] Sklyanin E K 1987 Boundary conditions for integrable equations *Funct. Anal. Appl.* **21** 86–87
[24] Tarasov V O 1991 The integratble initial-boundary value problem on a semiline: nonlinear Schrödinger equation and sine-Gordon equations *Inverse Problems* **7** 435–49
[25] Zakharov V E and Shabat A B 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media *Sov. Phys.—JETP* **34** 62–69