Soliton solutions of the Kadomtsev-Petviashvili II equation

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We study a general class of line-soliton solutions of the Kadomtsev-Petviashvili II (KPII) equation by investigating the Wronskian form of its tau-function. We show that, in addition to the previously known line soliton solutions of KPII, this class also contains a large variety of multisoliton solutions, many of which exhibit nontrivial spatial interaction patterns. We also show that, in general, such solutions consist of unequal numbers of incoming and outgoing line solitons. From the asymptotic analysis of the tau function, we explicitly characterize the incoming and outgoing line solitons of this class of solutions. We illustrate these results by discussing several examples. © *2006 American Institute of Physics.*
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I. INTRODUCTION

The Kadomtsev-Petviashvili (KP) equation

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+6u\frac{\partial u}{\partial x}\right)+3\sigma^2\frac{\partial^2 u}{\partial y^2}=0, \tag{1.1}$$

where $u=u(x,y,t)$ and $\sigma^2=\pm 1$, is one of the prototypical $(2+1)$-dimensional integrable nonlinear partial differential equations. The case $\sigma^2=-1$ is known as the KPI equation, and $\sigma^2=1$ as the KPII equation. Originally derived[11] as a model for small-amplitude, long-wavelength, weakly two-dimensional (y-variation much slower than the x-variation) solitary waves in a weakly dispersive medium, the KP equation arises in disparate physical settings including water waves and plasmas, astrophysics, cosmology, optics, magnetics, anisotropic two-dimensional lattices, and Bose-Einstein condensation. The remarkably rich mathematical structure underlying the KP equation, its integrability and large classes of exact solutions have been studied extensively for the past 30 years, and are documented in several monographs.[1,3,8,15,18,21]

In this paper we study a large class of solitary wave solutions of the KPII equation. It is well known (e.g., see Refs. 5 and 15) that solutions of the KPII equation can be expressed as

$$u(x,y,t)=2\frac{\partial^2}{\partial x^2}\ln\,\tau(x,y,t), \tag{1.2}$$

where the tau function $\tau(x,y,t)$ is given in terms of the Wronskian determinant[7,15]

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$$\tau(x,y,t) = \mathrm{Wr}(f_1, \ldots ,f_N) = \det\begin{pmatrix} f_1 & f_2 & \cdots & f_N \\ f_1' & f_2' & \cdots & f_N' \\ \vdots & \vdots & & \vdots \\ f_1^{(N-1)} & f_2^{(N-1)} & \cdots & f_N^{(N-1)} \end{pmatrix}. \tag{1.3}$$

with $f^{(i)}=\partial^i f/\partial x^i$, and where the functions $f_1,\ldots,f_N$ are a set of linearly independent solutions of the linear system

$$\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial f}{\partial t} = \frac{\partial^3 f}{\partial x^3}. \tag{1.4}$$

Equations (1.2) and (1.3) can also be obtained as the N-fold Darboux transformation for KPII (Ref. 15) starting from a seed solution $u=0$. In fact, the functions $f_1,\ldots,f_N$ in Eqs. (1.3) are precisely N independent solutions of the KPII Lax pair: $\partial_y f - \partial_x^2 f + uf = 0$ and $\partial_t f - \partial_x^3 f + 6u(\partial_x u) f + 3(\partial_x^{-1}\partial_y u) f = 0$, with $u=0$. A one-soliton solution of the KPII equation is obtained by choosing $N=1$ and $f(x,y,t)=e^{\theta_1}+e^{\theta_2}$, where

$$\theta_m(x,y,t) = k_m x + k_m^2 y + k_m^3 t + \theta_{m,0} \tag{1.5}$$

with $k_m, \theta_{m,0} \in \mathbb{R}$, $m=1,2$ and with $k_1 \neq k_2$ for nontrivial solutions. Without loss of generality, one can order the parameters as $k_1 < k_2$. The above choice yields the following traveling-wave solution:

$$u(x,y,t) = \tfrac{1}{2}(k_2 - k_1)^2 \operatorname{sech}^2 \tfrac{1}{2}(\theta_2 - \theta_1) = \Phi(\mathbf{k}\cdot\mathbf{x} + \omega t), \tag{1.6}$$

where $\mathbf{x}=(x,y)$. The wave-vector $\mathbf{k}=(l_x,l_y)$ and the frequency ω are given by

$$\mathbf{k} = (k_1 - k_2, k_1^2 - k_2^2), \quad \omega = k_1^3 - k_2^3, \tag{1.7}$$

and they satisfy the nonlinear dispersion relation

$$-4\omega l_x + l_x^4 + 3l_y^2 = 0. \tag{1.8}$$

The solution in Eq. (1.6) is localized along points satisfying $\theta_1=\theta_2$, which defines a line in the xy plane, for fixed t. Such solitary wave solutions of the KPII equation are thus called *line solitons*. They are stable with respect to transverse perturbations unlike the KPI [Eq. (1.1) with $\sigma^2=-1$] line-soliton solutions which are not stable with respect to small transverse perturbations. Equation (1.6) also implies that, apart from the constant $\theta_{1,0}-\theta_{2,0}$ corresponding to an overall translation of the solution, a line soliton of KPII is characterized by either the phase parameters k_1,k_2, or the physical parameters, namely, the *soliton amplitude a* and the *soliton direction c*, defined, respectively, as

$$a = k_2 - k_1, \quad c = k_1 + k_2. \tag{1.9}$$

Note that $c=\tan\alpha$, where α is the angle, measured counterclockwise, between the line soliton and the positive y axis. Hence, the soliton direction c can also be viewed as the "velocity" of the soliton in the xy plane, $c=-\mathrm{d}x/\mathrm{d}y=l_y/l_x$. For any given choice of amplitude and direction of the soliton, one obtains the phase parameters $k_{1,2}$ uniquely as $k_1=\frac{1}{2}(c-a)$ and $k_2=\frac{1}{2}(c+a)$.

When $c=0$ (equivalently, $k_1=-k_2$), the solution in Eq. (1.6) becomes y-independent and reduces to the one-soliton solution of the Korteweg-de Vries (KdV) equation. Similar to KdV, it is also possible to obtain multisoliton solutions of the KPII equation. Each of the multisoliton solutions decay exponentially in the xy plane, except along a number of rays or line solitons as $y\to\pm\infty$. These line solitons are sorted according to their directions, with increasing values of c from left to right as $y\to-\infty$ and increasing values of c from right to left as $y\to\infty$. However, the multisoliton solution space of the KPII equation turns out to be much richer than that of the (1 +1)-dimensional KdV equation due to the dependence of the KPII solutions on the additional spatial variable y. It is possible to construct a general family of multisoliton solutions via the Wronskian of Eq. (1.3) by choosing M phases $\theta_1, \ldots, \theta_M$ defined as in Eq. (1.5) with distinct real *phase parameters* $k_1 < k_2 < \cdots < k_M$ and then defining the functions $f_1, \ldots, f_N$ in Eq. (1.3) by

$$f_n(x,y,t) = \sum_{m=1}^{M} a_{n,m} e^{\theta_m}, \quad n = 1,2, \ldots, N. \tag{1.10}$$

The constant coefficients $a_{n,m}$ define the $N \times M$ *coefficient matrix* $A := (a_{n,m})$, which is required to be of full rank [i.e., rank$(A)=N$] and all of whose nonzero $N \times N$ minors must be sign definite. The full rank condition is necessary and sufficient for the functions f_n in Eq. (1.10) to be linearly independent. The sign definiteness of the nonzero minors is sufficient to ensure that the tau function $\tau(x,y,t)$ has no zeros in the xy plane for all t, so that the KPII solution $u(x,y,t)$ resulting from Eq. (1.2) is nonsingular.

One of the main results of this work (cf. Theorem 3.6) is to show that, when the coefficient matrix A satisfies certain conditions (cf. Definition 2.2), Eq. (1.10) leads to a multisoliton configuration with N_- asymptotic line solitons as $y \to -\infty$ and N_+ asymptotic line solitons as $y \to \infty$, where $N_- = M - N$ and $N_+ = N$. Furthermore, each of the asymptotic line solitons has the form of a plane wave similar to the one-soliton solution in Eq. (1.6). We refer to these multisoliton configurations the (N_-, N_+)-*soliton solutions* of KPII, and call the asymptotic line solitons as $y \to -\infty$ and as $y \to \infty$ the *incoming* and *outgoing* line solitons, respectively. The amplitudes, directions and even the number of incoming solitons are in general different from those of the outgoing ones, depending on the values of M, N, the phase parameters $k_1, \ldots, k_M$ and the coefficient matrix A. We note that a special family of KPII (N_-, N_+)-soliton solutions which also satisfy the finite Toda lattice hierarchy, was found earlier in Ref. 2. In this paper, we generalize the results of Ref. 2 to the entire class of (N_-, N_+)-soliton solutions of KPII generated by arbitrary coefficient matrices A. These solutions exhibit a variety of spatial interaction patterns which include the formation of intermediate line solitons and web structures in the xy plane.[2,12,16,23] In contrast, the line solitons for the previously known[5,15,24] *ordinary* soliton solutions of KPII (cf. Sec. IV) and the KdV solitons experience only a phase shift after collision. The existence of these nontrivial spatial features was found to be related to the presence of *resonant* soliton interactions in some earlier studies.[4,17,19,22] Several examples of these (N_-, N_+)-soliton solutions of KPII are discussed throughout this work (e.g., see Figs. 1–4). If $M = 2N$, it follows from Theorem 3.6 that $N_- = N_+ = N$, i.e., the numbers incoming and outgoing asymptotic line solitons are the same. We call the resulting solutions the *N-soliton solutions* of KPII. Among these, there is an important subclass called the *elastic* N-soliton solutions, for which the amplitudes and directions of the out-going line solitons coincide with those of the incoming line solitons. Elastic N-soliton solutions possess a number of interesting features, some of which have been studied in Ref. 12. A detailed study of the specific properties of the elastic N-solutions will be reported in a future presentation.

We note that multisoliton solutions exhibiting nontrivial spatial structures and interaction patterns were also recently found in other (2+1)-dimensional integrable equations. For example, solutions with soliton resonance and web structure were presented in Refs. 9 and 10 for a coupled KP system, and similar solutions were also found in Ref. 14 in discrete soliton systems such as the two-dimensional Toda lattice, together with its fully discrete and ultradiscrete analogues. From these works, the existence of these solutions appears to be a rather common feature of (2+1)-dimensional integrable systems. Thus, we expect that the scope of the results described in this paper will not be limited to the KP equation alone, but will also be applicable to a variety of other (2+1)-dimensional integrable systems.

II. THE TAU FUNCTION AND THE ASYMPTOTIC LINE SOLITONS

In this section we investigate the properties of the tau function given by Eq. (1.3) when the N functions $f_1, \ldots, f_N$ are chosen according to Eq. (1.10) as linear combinations of M exponentials $e^{\theta_1}, \ldots, e^{\theta_M}$. We should emphasize that Eq. (1.10) represents the most general form for the functions involving linear combinations of exponential phases. Since the elements of the $N \times M$ coefficient matrix $A=(a_{n,m})$ are the linear combination coefficients of the functions $f_1, \ldots, f_N$, one can naturally identify each f_n with one of the rows of A and each phase θ_m with one of the columns of A, and vice versa. Next, we examine the asymptotic behavior of the tau function in the xy plane as $y \to \pm\infty$. It is clear that, with the above choice of functions, the tau function is a linear combination of exponentials. Consequently, the leading order behavior of the tau function as $y \to \pm\infty$ in a given asymptotic sector of the xy plane is governed by those exponential terms which are dominant in that sector. A systematic analysis of the dominant exponential phases allows us to characterize the incoming and outgoing line solitons of (N_-, N_+)-soliton solutions of KPII.

A. Basic properties of the tau function

We present here some general properties of the tau function. Without loss of generality, throughout this work we choose the phase parameters k_m to be distinct and well ordered as $k_1 < k_2 < \cdots < k_M$.

Lemma 2.1: Suppose $\tau_{N,M} = \mathrm{Wr}(f_1, \ldots, f_N)$ *as in Eq.* (1.3) *with the functions* $f_1, \ldots, f_N$ *given by Eq.* (1.10). *Then*

$$\tau_{N,M}(x,y,t) = \det(A \Theta K^T), \tag{2.1}$$

where $A=(a_{n,m})$ *is the* $N \times M$ *coefficient matrix,* $\Theta = \mathrm{diag}(e^{\theta_1}, \ldots, e^{\theta_M})$, *and the* $N \times M$ *matrix* K *is given by*

$$K = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ k_1 & k_2 & \cdots & k_M \\ \vdots & \vdots & & \vdots \\ k_1^{N-1} & k_2^{N-1} & \cdots & k_M^{N-1} \end{pmatrix},$$

where the superscript T *denotes matrix transpose. Moreover,* $\tau_{N,M}$ *can be expressed as*

$$\tau_{N,M}(x,y,t) = \sum_{1 \le m_1 < m_2 < \cdots < m_N \le M} V(m_1, \ldots, m_N) A(m_1, \ldots, m_N) \exp[\theta_{m_1,\ldots,m_N}], \tag{2.2}$$

where $\theta_{m_1,\ldots,m_N}$ *denotes the phase combination*

$$\theta_{m_1,\ldots,m_N}(x,y,t) = \theta_{m_1}(x,y,t) + \cdots + \theta_{m_N}(x,y,t), \tag{2.3}$$

$A(m_1, \ldots, m_N)$ *denotes the* $N \times N$ *minor of* A *obtained by selecting columns* $m_1, \ldots, m_N$, *and* $V(m_1, \ldots, m_N)$ *denotes the Van der Monde determinant*

$$V(m_1, \ldots, m_N) = \prod_{1 \le s_1 \le s_2 \le N} (k_{m_{s_2}} - k_{m_{s_1}}). \tag{2.4}$$

Proof: Equation (2.1) follows by direct computation of the Wronskian determinant (1.3). Next, to prove Eq. (2.2) apply the Binet-Cauchy theorem to expand the determinant in Eq. (2.1) and note that the $N \times N$ minor of K obtained by selecting columns $1 \le m_1 < \cdots < m_N \le M$ is given by the Van der Monde determinant $V(m_1, \ldots, m_N)$. □

From Lemma 2.1 we have the following basic properties of the tau function:

(i) The spatiotemporal dependence of the tau function in Eq. (2.2) is confined to a sum of exponential phase combinations $\theta_{m_1,\ldots,m_N}$ which according to Eq. (2.3) are linear in x, y, t. Moreover, all the Van der Monde determinants $V(m_1, \ldots, m_N)$ are positive, as the phase parameters $k_1, \ldots, k_M$ are well ordered. A sufficient condition for the tau function in Eq. (2.2) to generate a nonsingular solution of KPII is that it is sign-definite for all $(x,y,t) \in \mathbb{R}^3$. In turn, a sufficient condition for the sign-definiteness of the tau-function is that the minors of the coefficient matrix A are either all non-negative or all nonpositive. However, it is not clear at present whether these conditions are also necessary. If the tau function in Eq. (2.2) is taken as a sum of exponential phase combinations with non-negative coefficients, the solution $u(x,y,t)$ in Eq. (1.2) can be expressed as a ratio of two sums, each containing the *same* set of exponential terms, and with non-negative coefficients. Consequently, the resulting solution of KPII is bounded and positive definite for all $(x,y,t) \in \mathbb{R}^3$.

(ii) Each exponential term in the tau function of Eq. (2.2) contains combinations of N *distinct* phases $\theta_{m_1},\dots,\theta_{m_N}$ identified by integers $m_1,\dots,m_N$ chosen from $\{1,\dots,M\}$. Thus, the maximum number of terms in the tau function is given by the binomial coefficient $\binom{M}{N}$. However, a given phase combination $\theta_{m_1,\dots,m_N}$ is actually *present* in the tau function if and only if the corresponding minor $A(m_1,\dots,m_N)$ is nonzero.

(iii) If $M<N$ the functions $f_1,\dots,f_N$ are linearly dependent; in this case there are no terms in the summation in Eq. (2.2), and therefore the tau function $\tau_{N,M}(x,y,t)$ is identically zero. Also, if $M=N$, there is only one term in the summation corresponding to the determinant of A; then $\tau_{N,M}(x,y,t)$ depends linearly on x and therefore it generates the trivial solution $u(x,y,t)=0$. Finally, if $\mathrm{rank}(A)<N$, all $N\times N$ minors of A vanish identically, leading once again to $\tau_{N,M}(x,y,t)=0$. Therefore, for nontrivial solutions one needs $M>N$ and $\mathrm{rank}(A)=N$.

(iv) The transformation $A\to A'=GA$ with $G\in \mathrm{GL}(N,\mathbb{R})$ (corresponding to elementary row operations on A) amounts to an overall rescaling $\tau(x,y,t)\to\tau'(x,y,t)=\det(G)\tau(x,y,t)$ of the tau function (2.1). Such rescaling leaves the solution $u(x,y,t)$ in Eq. (1.2) invariant. This reflects the fact that N independent linear combinations of the functions $f_1,\dots,f_N$ in Eq. (1.10) generate equivalent tau functions. This $\mathrm{GL}(N,\mathbb{R})$ gauge freedom can be exploited to choose the coefficient matrix A in Eq. (2.1) to be in reduced row-echelon form (RREF). The $\mathrm{GL}(N,\mathbb{R})$ invariance means that the tau function (2.1) represents a point in the real Grassmannian $\mathrm{Gr}(N,M)$.[12]

(v) Suppose that one of the functions in Eq. (1.10) contains only one exponential term, and is given by $f_p=a_{p,q}e^{\theta_q}$ with $a_{p,m}=0\ \forall m\neq q$. Then, the minors $A(m_1,\dots,m_N)=0$ whenever $q\notin\{m_1,\dots,m_N\}$. As a result, the tau function in Eq. (2.2) can be expressed as $\tau_{N,M}(x,y,t)=e^{\theta_q}\tau'(x,y,t)$, and $\tau'(x,y,t)$ is a sum of exponential phase combinations, where each combination consists of $N-1$ distinct phases chosen from all M phases except θ_q. From Eq. (1.2) it is evident that $\tau_{N,M}(x,y,t)$ and $\tau'(x,y,t)$ generate the same solution of KPII. Moreover, the function $\tau'(x,y,t)$ is effectively equivalent to a tau function $\tau_{N-1,M-1}(x,y,t)$ with a coefficient matrix obtained by deleting the pth row and qth column of A. Hence in this case the tau function $\tau_{N,M}(x,y,t)$ is reducible to another tau function $\tau_{N-1,M-1}(x,y,t)$ obtained from a Wronskian of $N-1$ functions with $M-1$ distinct phases.

In accordance with the above remarks, throughout this work we consider the coefficient matrix A to be in RREF. Also, to avoid trivial and singular cases, from now on we assume that $M>N$ and $\mathrm{rank}(A)=N$, and that all nonzero $N\times N$ minors of A are positive. Finally, we assume that A satisfies the following irreducibility conditions.

Definition 2.2 (Irreducibility): A matrix A of rank N is said to be irreducible if, in RREF:

(i) *Each column of A contains at least one nonzero element.*

(ii) *Each row of A contains at least one nonzero element in addition to the pivot.*

Condition (i) in Definition 2.2 requires that each exponential phase appear in at least one of the functions $f_1,\dots,f_N$. If a particular phase is absent, then the corresponding tau function $\tau_{N,M}$ can be reexpressed in terms of a reduced tau function $\tau_{N,M-1}$. Condition (ii) requires that each function contains at least two exponential phases in order to avoid reducible situations like those in part (v) of the above remarks. Note also that if an $N\times M$ matrix A is irreducible, then $M>N$.

B. Dominant phase combinations and index pairs

We now study the asymptotic behavior of the tau function in the xy-plane for large values of $|y|$ and finite values of t. Let Θ denote the set of all phase combinations $\theta_{m_1,\ldots,m_N}$ such that $A(m_1,\ldots,m_N)\neq 0$, that is, the set of phase combinations that are actually present in the tau function $\tau(x,y,t)$.

Definition 2.3 (Dominant phase): A given phase combination $\theta_{m_1,\ldots,m_N}\in\Theta$ *is said to be dominant for the tau function* $\tau(x,y,t)$ *of Eq.* (2.2) *in a region* $R\in\mathbb{R}^3$ *if* $\theta_{m'_1,\ldots,m'_N}(x,y,t)\leq\theta_{m_1,\ldots,m_N}(x,y,t)$ *for all* $\theta_{m'_1,\ldots,m'_N}\in\Theta$ *and for all* $(x,y,t)\in R$. *The region R is called the dominant region of* $\theta_{m_1,\ldots,m_N}$.

The phase combinations $\theta_{m_1,\ldots,m_N}(x,y,t)$ are linear functions of x, y, and t. So, each of the inequalities in Definition 2.3 defines a convex subset of $\mathbb{R}^3$. The dominant region R associated with each phase combination is also convex because it is defined by the intersection of finitely many convex subsets. Furthermore, since the phase combinations are defined globally on $\mathbb{R}^3$, each point $(x,y,t)\in\mathbb{R}^3$ belongs to some dominant region R. As a result, we obtain a partition of the entire $\mathbb{R}^3$ into a finite number of convex dominant regions, intersecting only at points on the boundaries of each region. It is important to note that such boundaries always exist whenever there is more than one phase combination in the tau function, because then there are more than one dominant region in $\mathbb{R}^3$. The significant of the dominant regions lies in the following:

Lemma 2.4: The solution $u(x,y,t)$ *of the KPII equation generated by the tau function* (2.2) *is exponentially small at all points in the interior of any dominant region. Thus, the solution is localized only at the boundaries of the dominant regions, where a balance exists between two or more dominant phase combinations in the tau function of Eq.* (2.2).

Proof: Let R be the dominant region of $\theta_{m_1,\ldots,m_N}$, which is therefore the *only* dominant phase in the interior of R. Then from Eq. (2.2), $\tau_{N,M}(x,y,t)\sim O(e^{\theta_{m_1,\ldots,m_N}})$ in the interior of R. As a result, $\ln\tau_{N,M}(x,y,t)$ locally becomes a linear function of x apart from exponentially small terms. Hence, it follows from Eq. (1.2) that the solution $u(x,y,t)$ of KPII is exponentially small at all such interior points of R. □

The boundary between any two adjacent dominant regions is the set of points across which *a transition* from one dominant phase combination $\theta_{m_1,\ldots,m_N}$ to another dominant phase combination $\theta_{m'_1,\ldots,m'_N}$ takes place. Such boundary is therefore identified by the equation $\theta_{m_1,\ldots,m_N}=\theta_{m'_1,\ldots,m'_N}$, which defines a line in the xy plane for fixed values of t. The simplest instance of a transition between dominant phase combinations arises for the one-soliton solution (1.6), which is localized along the line $\theta_1=\theta_2$ defining the boundary of the two regions of the xy plane where θ_1 and θ_2 dominate. In the one-soliton case, these two regions are simply half-planes. But in the general case the dominant regions are more complicated, although the solution $u(x,y,t)$ is still localized along the boundaries of these regions, corresponding to similar phase transitions. For example, Fig. 1(a) illustrates a (2,1)-soliton known as a *Miles resonance*[17] (also called a Y junction), generated by the tau function $\tau_{1,2}=e^{\theta_1}+e^{\theta_2}+e^{\theta_3}$. In this case, the xy plane is partitioned into three dominant regions corresponding to each of the dominant phases θ_1, θ_2, and θ_3. Once again, the solution $u(x,y,t)$ is exponentially small in the interior of each dominant regions, and is localized along the phase transition boundaries: here, $\theta_1=\theta_2$, $\theta_1=\theta_3$, and $\theta_2=\theta_3$. It should also be noted that some of these regions have infinite extension in the xy plane, while others are bounded, as in the case of resonant soliton solutions, described in Sec. IV and Ref. 2. Each phase transition which occurs asymptotically as $y\to\pm\infty$ defines an *asymptotic line soliton*, which is infinitely extended in the xy plane.

When studying the asymptotics of the tau function for large $|y|$ it is useful to consider the limit $y\to\pm\infty$ along the straight lines

$$L_c: x+cy=\xi, \tag{2.5}$$

parametrized by the direction c. Note that c increases counterclockwise, namely from the positive x axis to the negative x axis for $y>0$ and from the negative x axis to the positive x axis for y

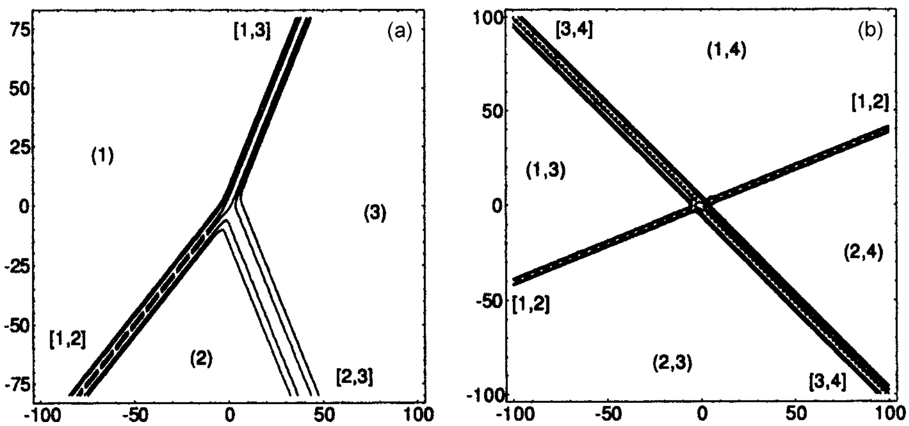

FIG. 1. Dominant phase combinations in the different regions of the xy plane (labeled by the indices in parentheses) and the asymptotic line solitons (labeled by the indices in square braces) for two different line soliton solutions: (a) a fundamental Miles resonance (Y junction) produced by the tau function with $N=1$, $M=3$ and $(k_1,k_2,k_3)=(-1,0,\frac{1}{2})$ at $t=0$; (b) an ordinary two-soliton solution, produced by the coefficient matrix in Example 2.7 with $(k_1,\ldots,k_4)=(-\frac{3}{2},-\frac{1}{2},0,1)$ at $t=0$ (see text for details). Here and in all of the following figures, the horizontal and vertical axes are, respectively, x and y, and the graphs show contour lines of $\ln u(x,y,t)$ at a fixed value of t.

<0. From Eqs. (1.5) and (2.5), each exponential phases along L_c is $\theta_m=k_m(k_m-c)y+k_m\xi+k_m^3t+\theta_{m,0}$. The difference between two such phases along L_c then becomes

$$\theta_m-\theta_{m'}=(k_m-k_{m'})(k_m+k_{m'}-c)y+(k_m-k_{m'})\xi+(k_m^3-k_{m'}^3)t+\theta_{m,0}-\theta_{m',0}, \tag{2.6a}$$

and the difference between any two phase combinations along L_c is given by

$$\theta_{m_1,\ldots,m_N}-\theta_{m'_1,\ldots,m'_N}=\left(\sum_{j=1}^{N}(k_{mj}-k_{m'_j})(k_{m_j}+k_{m'_j}-c)\right)y+\delta(\xi,t), \tag{2.6b}$$

where $\delta(\xi,t)=\Sigma_{j=1}^N[(k_{m_j}-k_{m'_j})\xi+(k_{m_j}^3-k_{m'_j}^3)t+\theta_{m_j,0}-\theta_{m'_j,0}]$. In particular, the *single-phase-transition* line $L_{m,m'}$: $\theta_m=\theta_{m'}$ given by Eq. (2.5) with $c_{m,m'}=k_m+k_{m'}$, will play an important role below.

It is also convenient at this point to introduce the following notations which will be employed throughout this paper. We denote by $A[m]\in\mathbb{R}^N$ the mth column of a matrix A, and we denote by $A[m_1,\ldots,m_r]$ the $N\times r$ submatrix obtained by selecting the r columns $A[m_1],\ldots,A[m_r]$. We also label the N pivot columns of an irreducible, $N\times M$ matrix A by $A[e_1],\ldots,A[e_N]$, with $1=e_1<e_2<\cdots<e_N<M$, and we label the $M-N$ nonpivot columns by $A[g_1],\ldots,A[g_{M-N}]$, where $1<g_1<g_2<\cdots<g_{M-N}=M$. Note that A has N pivot columns because it is rank N; also, $e_1=1$ since A is in RREF, and $e_N<M$ since it is irreducible. We now establish a result that will be useful in order to characterize the asymptotics of the tau function.

Theorem 2.5: *(Single-phase transition) Asymptotically as* $y\to\pm\infty$, *and for generic values of the phase parameters* $k_1,\ldots,k_M$, *the dominant phase combinations in the tau function* (2.2) *exhibit the following behaviors in the xy plane.*

(i) *For finite values of* t, *the set of dominant phase combinations remains invariant in time.*
(ii) *The dominant phase combinations in any two adjacent dominant regions contain* $N-1$ *common phases.*

We discuss below several consequences of Theorem 2.5 which is proved in the Appendix.

Consider the single-phase transition as $y\to\pm\infty$ in which a phase θ_i from the dominant phase combination in one region is replaced by another phase θ_j to produce the dominant phase combination in the adjacent region. We refer to this transition as an $i\to j$ transition, which takes place along the line L_{ij}: $\theta_i=\theta_j$ whose direction in the xy plane is given by $c_{ij}=k_i+k_j$. As $y\to\infty$, it is clear from Eq. (2.6a) that, if $k_i < k_j$, the transition $i \to j$ takes place from the left of the line $L_{i,j}$ to its right, while if $k_i > k_j$ the transition $i \to j$ takes place from the right of the line $L_{i,j}$ to its left. Thus, as $y \to \infty$, each dominant phase region R is bounded on the left by the transition line $L_{i,j}$ given by the *minimum* value of $c_{i,j}$ that corresponds to an allowed transition and, on the right by the transition line $L_{i,j}$ given by the *maximum* value of $c_{i,j}$ that corresponds to an allowed transition. Here, an *allowed* transition from one dominant phase combination to another means that the minors associated with those phase combinations in the tau function of Eq. (2.2), are both nonzero. In turn, these nonvanishing minors determine the values of c_{ij} corresponding to the allowed single-phase transitions. A similar statement can be made for transitions occurring as $y \to -\infty$. So, each dominant phase region R as $y \to \pm\infty$ has boundaries defined by a counterclockwise and a clockwise single-phase transitions which can be determined in the following way.

Corollary 2.6: Suppose that $\theta_{m_1,\ldots,m_N}$ *is the dominant phase combination on a region* R *asymptotically as* $y \to \pm\infty$. *Let* J *be the complement of the set of indices* $\{m_1, m_2, \ldots, m_N\}$ *in* $\{1, 2, \ldots, M\}$. *For each* $j \in J$, *define* $I_j \subseteq \{m_1, m_2, \ldots, m_N\}$ *as the set of all indices* $m_r \in \{m_1, m_2, \ldots, m_N\}$ *such that the minor* $A(m_1, \ldots, m_{r-1}, j, m_{r+1}, \ldots, m_N) \neq 0$. *Then, the following hold.*

(i) *As* $y \to \infty$, *the directions of the counterclockwise and clockwise transition boundaries of* R *are, respectively, given by*

$$c_+ = \min_{i \in I_j, j \in J} [c_{i,j}] \quad \text{with } k_i > k_j, \quad c_- = \max_{i \in I_j, j \in J} [c_{i,j}] \quad \text{with } k_i < k_j. \tag{2.7a}$$

(ii) *As* $y \to -\infty$, *the directions of the counterclockwise and clockwise transition boundaries of* R *are, respectively, given by*

$$c_+ = \min_{i \in I_j, j \in J} [c_{i,j}] \quad \text{with } k_i > k_j, \quad c_- = \max_{i \in I_j, j \in J} [c_{i,j}] \quad \text{with } k_i > k_j. \tag{2.7b}$$

The results of Theorem 2.5 and Corollary 2.6 can be used to determine the asymptotic behavior of the tau function, thereby obtaining an important characterization of the asymptotic line solitons corresponding to (N_-, N_+)-soliton solutions of the KPII equation. Namely, for the tau function $\tau_{N,M}(x,y,t)$ of Eq. (2.2) with generic values of the phase parameters $k_1, \ldots, k_M$ we have the following:

(i) As $y \to \pm\infty$, the dominant phase combinations of the tau function in adjacent regions of the xy plane contain $N-1$ common phases and differ by only a single phase. The transition between any two such dominant phase combinations $\theta_{i,m_2,\ldots,m_N}$ and $\theta_{j,m_2,\ldots,m_N}$ occurs along the line $L_{i,j}: \theta_i = \theta_j$, where a single phase θ_i in the dominant phase combination is replaced by a phase θ_j. Moreover, if the dominant phase combination $\theta_{i,m_2,\ldots,m_N}$ in a given region is known, the transition line $L_{i,j}$ and the dominant phase combination $\theta_{j,m_2,\ldots,m_N}$ are determined via Corollary 2.6. In particular, Eqs. (2.7) for $c_\pm$ determine explicitly the pair of phase parameters k_i and k_j corresponding to the single-phase transition $i \to j$ across each boundary $L_{i,j}$ of a given dominant phase region.
(ii) As $y \to \pm\infty$ along the line $L_{i,j}$, the asymptotic behavior of the tau function is determined by the balance between the two dominant phase combinations $\theta_{i,m_2,\ldots,m_N}$ and $\theta_{j,m_2,\ldots,m_N}$, and is given by

$$\tau_{N,M}(x,y,t) \sim V_i A(i, m_2, \ldots, m_N) e^{\theta_{i,m_2,\ldots,m_N}} + V_j A(j, m_2, \ldots, m_N) e^{\theta_{j,m_2,\ldots,m_N}},$$

where $V_i := V(i, m_2, \ldots, m_N)$ and $V_j := V(j, m_2, \ldots, m_N)$ are Van der Monde determinants defined in Eq. (2.4), and where the minors $A(i, m_2, \ldots, m_N)$ and $A(j, m_2, \ldots, m_N)$ of the coefficient matrix A are both nonzero. The solution $u(x,y,t)$ of the KPII equation in a neighborhood of such a single-phase transition is then obtained from Eq. (1.2) as

$$u(x,y,t) \sim \tfrac{1}{2}(k_i - k_j)^2 \operatorname{sech}^2\left[\tfrac{1}{2}(\theta_i - \theta_j)\right]. \tag{2.8}$$

Moreover, Lemma 2.4 and Theorem 2.5 together imply that the solution of the KPII equation is exponentially small everywhere in the xy-plane except at the locations of such single-phase transitions. Equation (2.8), which is a traveling wave solution satisfying the dispersion relation in Eq. (1.8), coincides with the one-soliton solution in Eq. (1.6). Thus, it defines an asymptotic line soliton associated with the single-phase transition $i \to j$. The phase parameters k_i and k_j associated with the single-phase transition $i \to j$ are determined by Eqs. (2.7). Then, the soliton amplitude is given by $a_{i,j}=|k_i-k_j|$, and the soliton direction is given by the direction of $L_{i,j}$, which is $c_{i,j}=k_i+k_j$.

(iii) All of the asymptotic line solitons resulting from the single-phase transitions described above are invariant in time, in the sense that their number, amplitudes, and directions are constants.

Motivated by these results, *we label each asymptotic line soliton by the index pair* $[i,j]$ *which uniquely identifies the phase parameters* k_i *and* k_j *in the ordered set* $\{k_1, \ldots, k_M\}$. The results summarized in the above remarks can be applied to explicitly delineate the dominant phase combinations and the asymptotic line solitons associated with the tau function of a given (N_-,N_+)-soliton solution of the KPII equation, as illustrated by the following example.

Example 2.7: When $N=2$ and $M=4$, Lemma 2.1 implies that the tau function $\tau(x,y,t)$ is given by

$$\tau(x,y,t) = \mathrm{Wr}(f_1,f_2) = \sum_{1\le m<m'\le 4} (k_{m'} - k_m)A(m,m')e^{\theta_m+\theta_{m'}}, \tag{2.9}$$

where the phases are given by $\theta_m=k_m x+k_m^2 y+k_m^3 t+\theta_{m,0}$ for $m=1,\ldots,4$, as in Eq. (1.5), and where the phase parameters are ordered as $k_1<\cdots<k_4$. We consider the line-soliton solution constructed from the functions $f_1=e^{\theta_1}+e^{\theta_2}$ and $f_2=e^{\theta_3}+e^{\theta_4}$, so that the associated 2×4 coefficient matrix is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{2.10}$$

Then $A(1,2)=A(3,4)=0$, and the remaining four minors are all equal to 1. We apply Corollary 2.6 to determine the asymptotic line solitons associated with the tau function in Eq. (2.9). First note from the expression $\theta_{m,m'}=(k_m+k_{m'})x+(k_m^2+k_{m'}^2)y+(k_m+k_{m'}^3)t+(\theta_{m,0}+\theta_{m',0})$ that for every finite value of y the dominant phase combination as $x\to-\infty$ is given by $\theta_{1,3}$, which corresponds to the *minimum* value of $k_m+k_{m'}$ such that $A(m,m')\neq 0$ (cf. Definition 2.3). We denote by $R_{1,3}$ the region of the xy plane where $\theta_{1,3}$ is the dominant phase. The transition boundaries of $R_{1,3}$ are determined by applying Corollary 2.6 as follows: The complement of the index set $\{1,3\}$ is $J=\{2,4\}$. When $j=2\in J$, we have $A(1,2)=0$ but $A(2,3)\neq 0$; hence $I_2=\{1\}$. Similarly, when $j=4$ we have $I_4=\{3\}$ because $A(1,4)\neq 0$ but $A(4,3)=0$. Thus the possible transitions $i\to j$ from $R_{1,3}$ are $1\to 2$ and $3\to 4$. As $y\to\infty$, the second of Eqs. (2.7a) implies that the clockwise transition boundary of $R_{1,3}$ is given by the transition line $L_{3,4}$, whose direction $c_{3,4}=k_3+k_4$ is greater than the direction $c_{1,2}=k_1+k_2$ of the line $L_{1,2}$. Across the transition line $L_{3,4}$, the dominant phase combination switches from $\theta_{1,3}$ to $\theta_{1,4}$, onto the corresponding dominant region $R_{1,4}$. Similarly, as $y\to-\infty$, the first of Eqs. (2.7b) implies that the counterclockwise transition boundary of $R_{1,3}$ is given by the transition line $L_{1,2}$, whose direction $c_{1,2}$ is less than the direction $c_{3,4}$ of the line $L_{3,4}$. This implies that the dominant phase combination and dominant region change to $\theta_{2,3}$ and $R_{2,3}$, respectively. Applying Corollary 2.6 again to the region $R_{2,3}$ as $y\to-\infty$, one finds $J=\{1,4\}$ with $I_1=\{2\}$ and $I_4=\{3\}$, so the possible transitions from $R_{2,3}$ are $2\to 1$ and $3\to 4$. The $2\to 1$ transition corresponds to a clockwise transition from $R_{2,3}$ back to $R_{1,3}$, whereas the $3\to 4$ transition corresponds to a counterclockwise transition from $R_{2,3}$ to the region $R_{2,4}$, where $\theta_{2,4}$ is the dominant phase combination. Continuing counterclockwise from $R_{1,3}$ we finally obtain the following dominant phase regions asymptotically as $y\to\pm\infty$, together with the associated single-phase transitions:

$$R_{1,3} \xrightarrow{1\to 2} R_{2,3} \xrightarrow{3\to 4} R_{3,4} \xrightarrow{2\to 1} R_{1,4} \xrightarrow{4\to 3} R_{1,3}. \tag{2.11}$$

It is then clear that there are two asymptotic line solitons as $y \to -\infty$ as well as $y \to \infty$, and in both cases they correspond to the lines $\theta_1 = \theta_2$ and $\theta_3 = \theta_4$. The dominant phase regions, denoted by indices (m, m'), and the asymptotic line solitons, identified by the index pairs $[i,j]$, are illustrated in Fig. 1(b). The corresponding solution is called an ordinary 2-soliton solution. The ordinary N-soliton solutions are described in Sec. IV.

In the following section we obtain several results that will allow us to identify more precisely the index pairs corresponding to each asymptotic line soliton. In addition, we will prove a general result concerning the numbers of asymptotic line solitons present in an (N_-, N_+)-soliton solution corresponding to the tau function of Eq. (2.2).

III. ASYMPTOTIC LINE SOLITONS AND THE COEFFICIENT MATRIX

In this section we continue our investigation of the tau function in the general setting introduced in Sec. II. We have seen in the preceding section that an asymptotic line soliton corresponds to a dominant balance between two phase combinations in the tau function. But we still need to identify which phase combinations in a given tau function are indeed dominant as $y \to \pm\infty$. This requires a detailed study of the structure of the $N \times M$ coefficient matrix A associated with the tau function. In this section we carry out this analysis, which enables us to explicitly identify all the asymptotic line solitons of a given tau function in an algorithmic fashion. One of our main results of this section will be to establish that, for arbitrary values of N and M, and for irreducible coefficient matrices (cf. Definition 2.2) with non-negative $N \times N$ minors, the tau function (2.2) produces an (N_-, N_+)-soliton solution with $N_- = M - N$ and $N_+ = N$, i.e., a solution in which there are $N_- = M - N$ asymptotic line solitons as $y \to -\infty$ and $N_+ = N$ asymptotic line solitons as $y \to \infty$.

A. Dominant phases and the structure of the coefficient matrix

We begin by presenting a simple yet useful result (see also Ref. 2, Lemma 2.4) that will be frequently used to determine the dominant phase combinations in the tau function as $y \to \pm\infty$.

Lemma 3.1 (Dominant phase conditions): As $y \to \pm\infty$ along the line $L_{i,j}$: $\theta_i = \theta_j$ with $i < j$, the exponential phases $\theta_1, \ldots, \theta_M$ satisfy the following relations.

(i) *As $y \to \infty$, $\theta_m < \theta_*$, $\forall m \in \{i+1, \ldots, j-1\}$, and $\theta_m > \theta_*$, $\forall m \in \{1, \ldots, i-1, j+1, \ldots, M\}$, where $\theta_* := \theta_i = \theta_j$.*

(ii) *As $y \to -\infty$, $\theta_m > \theta_*$, $\forall m \in \{i+1, \ldots, j-1\}$, while $\theta_m < \theta_*$, $\forall m \in \{1, \ldots, i-1, j+1, \ldots, M\}$.*

Proof: It follows from Eq. (2.6a) that, along the line $L_{i,j}$ whose direction is $c_{i,j} = k_i + k_j$, the difference between any two exponential phases θ_m and $\theta_{m'}$ is given by

$$\theta_m - \theta_{m'} = (k_m - k_{m'})[(k_m + k_{m'}) - (k_i + k_j)]y + \delta'(\xi, t), \tag{3.1}$$

where $\delta'(\xi, t)$ is a linear function of ξ and t and which also depends on the constants $\theta_{m,0}$, $\theta_{m',0}$, $\theta_{i,0}$, and $\theta_{j,0}$. It is clear that the sign of $\theta_m - \theta_{m'}$ as $y \to \pm\infty$ and for finite values of ξ and t is determined by the coefficient of y on the right-hand side of Eq. (3.1). Then, setting $m' = i$ (or $m' = j$) in Eq. (3.1) one obtains the desired inequalities. □

Lemma 3.1, which is illustrated in Fig. 2, will be used to obtain a set of conditions that are necessary for a given pair of phase combinations in the tau-function to be dominant. These conditions are given in terms of the vanishing of certain $N \times N$ minors of the coefficient matrix A, and they determine which phase combinations are present (or absent) in the tau function of Eq. (1.3). In order to derive these conditions, it is convenient to introduce two submatrices $P_{i,j}$ and $Q_{i,j}$ associated with any index pair $[i,j]$ with $1 \leq i < j \leq M$, and given by

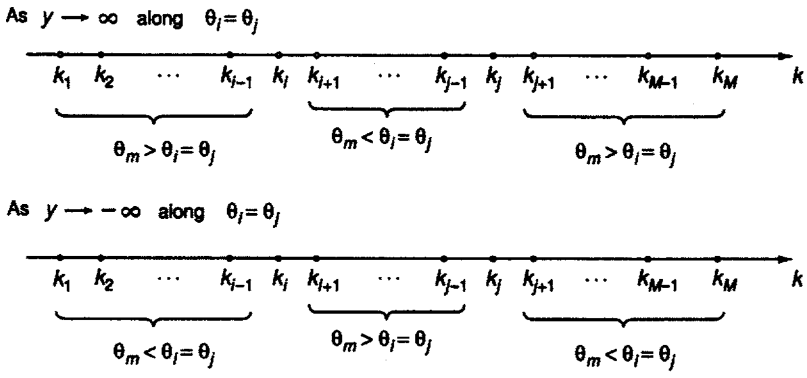

FIG. 2. Relations among the exponential phases as $y \to \pm\infty$ along the direction $L_{i,j}: \theta_i = \theta_j$.

$$P_{i,j} = A[1,2,\ldots,i-1,j+1,\ldots,M], \quad Q_{i,j} = A[i+1,\ldots,j-1]. \tag{3.2}$$

The matrix $P_{i,j}$ is formed by the consecutive columns of A to the left of column $A[i]$ and those to the right of column $A[j]$, while $Q_{i,j}$ is formed by the consecutive columns of A between columns $A[i]$ and $A[j]$. Using the matrices $P_{i,j}$ and $Q_{i,j}$ and the dominant phase conditions in Lemma 3.1 we then have the following.

Lemma 3.2 (Vanishing minor conditions): Suppose that the index pair $[i,j]$ identifies an asymptotic line soliton. Let the two dominant phase combinations along the line $L_{i,j}: \theta_i = \theta_j$ be given by $\theta_{i,p_1,\ldots,p_r,q_1,\ldots,q_s}$ and $\theta_{j,p_1,\ldots,p_r,q_1,\ldots,q_s}$, and let $A(i,p_1,\ldots,p_r,q_1,\ldots,q_s)$, $A(j,p_1,\ldots,p_r,q_1,\ldots,q_s)$ be the corresponding nonzero minors where $A[p_1],\ldots,A[p_r] \in P_{i,j}$ and $A[q_1],\ldots,A[q_s] \in Q_{i,j}$.

(i) *If $[i,j]$ identifies an asymptotic line soliton as $y \to \infty$, then*

(a) *all $N \times N$ minors obtained by replacing one of the columns $A[i],A[j],A[q_1],\ldots,A[q_s]$ from either $A(i,p_1,\ldots,p_r,q_1,\ldots,q_s)$ or $A(j,p_1,\ldots,p_r,q_1,\ldots,q_s)$ with any column $A[p] \in P_{i,j}$, are zero*;

(b) *all $N \times N$ minors obtained by replacing one of the columns $A[q_1],\ldots,A[q_s]$ from either $A(i,p_1,\ldots,p_r,q_1,\ldots,q_s)$ or $A(j,p_1,\ldots,p_r,q_1,\ldots,q_s)$ with either $A[i]$ or $A[j]$, are zero.*

(ii) *If $[i,j]$ identifies an asymptotic line soliton as $y \to -\infty$, then*

(a) *all $N \times N$ minors obtained by replacing one of the columns $A[i],A[j],A[p_1],\ldots,A[p_r]$ from either $A(i,p_1,\ldots,p_r,q_1,\ldots,q_s)$ or $A(j,p_1,\ldots,p_r,q_1,\ldots,q_s)$ with any column $A[q] \in Q_{i,j}$, are zero;*

(b) *all $N \times N$ minors obtained by replacing one of the columns $A[p_1],\ldots,A[p_r]$ from either $A(i,p_1,\ldots,p_r,q_1,\ldots,q_s)$ or $A(j,p_1,\ldots,p_r,q_1,\ldots,q_s)$ with either $A[i]$ or $A[j]$, are zero.*

Proof: All of the above conditions follow from the repeated use of the dominant phase conditions in Lemma 3.1. For example, as $y \to \infty$ along the line $L_{i,j}$, Lemma 3.1 implies $\theta_p > \theta_m$ for all $p \in \{1,\ldots,i-1,j+1,\ldots,M\}$ and for all $m \in \{i,j,q_1,\ldots,q_s\}$. Consequently, if condition (b) in part (i) of the Lemma does not hold, each of the phase combinations obtained by replacing θ_m with θ_p in either $\theta_{i,p_1,\ldots,p_r,q_1,\ldots,q_s}$ or $\theta_{j,p_1,\ldots,p_r,q_1,\ldots,q_s}$ will be greater than both $\theta_{i,p_1,\ldots,p_r,q_1,\ldots,q_s}$ and $\theta_{j,p_1,\ldots,p_r,q_1,\ldots,q_s}$. But this contradicts the hypothesis that $\theta_{i,p_1,\ldots,p_r,q_1,\ldots,q_s}$ and $\theta_{j,p_1,\ldots,p_r,q_1,\ldots,q_s}$ are the dominant phase combinations as $y \to \infty$ along $L_{i,j}$. The other conditions follow in a similar fashion. □

We should emphasize that in general, the asymptotic solitons and the index pairs labeling them as $y \to \infty$ are different from those as $y \to -\infty$. Lemma 3.2 allows us to determine the ranks of the submatrices P_{ij} and Q_{ij} associated with each asymptotic line soliton $[i,j]$. This information will be exploited later in Theorem 3.6 to identify explicitly the asymptotic line solitons produced by any given tau function. The next two results are direct consequences of the conditions specified in Lemma 3.2.

Lemma 3.3 (Span): Let $A[p_1],\ldots,A[p_r]\in P_{i,j}$ *and* $A[q_1],\ldots,A[q_s]\in Q_{i,j}$ *be the columns in the minors associated with the dominant pair of phase combinations of Lemma 3.2.*

(i) *If* $[i,j]$ *identifies an asymptotic line soliton as* $y\to\infty$, *the columns* $A[p_1],\ldots,A[p_r]$ *form a basis for the column space of the matrix* $P_{i,j}$.

(ii) *If* $[i,j]$ *identifies an asymptotic line soliton as* $y\to-\infty$, *the columns* $A[q_1],\ldots,A[q_s]$ *form a basis for the column space of the matrix* $Q_{i,j}$.

Proof: We prove part (i). Since $A(i,p_1,\ldots,p_r,q_1,\ldots,q_s)\neq 0$ by Lemma 3.2, the set of columns $\mathcal{A}=\{A[i],A[p_1],\ldots,A[p_r],A[q_1],\ldots,A[q_s]\}$ is a basis for $\mathbb{R}^N$. Hence the set $\{A[p_1],\ldots,A[p_r]\}\subset\mathcal{A}$ is linearly independent. Moreover, any $A[p]\in P_{i,j}$ can be expanded with respect to the basis $\mathcal{A}$ as

$$A[p]=aA[i]+\sum_{m=1}^{r}b_mA[p_m]+\sum_{m=1}^{s}c_mA[q_m]. \tag{3.3}$$

Replacing one of the columns $A[i],A[q_1],\ldots,A[q_s]$ in $A(i,p_1,\ldots,p_r,q_1,\ldots,q_s)$ with $A[p]\in P_{ij}$, we have from Lemma 3.2(i)(a) that

$$A(p,p_1,\ldots,p_r,q_1,\ldots,q_s)=0,\quad A(i,p_1,\ldots,p_r,q_1,\ldots q_{m-1},p,q_{m+1},\ldots q_s)=0.$$

Hence in Eq. (3.3) we have $a=0$ and $c_m=0\,\forall\, m=1,\ldots,s$. Therefore $A[p]\in\mathrm{span}(\{A[p_1\ ,\ldots,A[p_r]\})$ for all $A[p]\in P_{i,j}$. Similarly, part (ii) follows from the conditions in Lemma 3.2(ii)(a). □

Lemma 3.4 (Rank conditions): Let r be the number of columns from $P_{i,j}$ *and let s be the number of columns from* $Q_{i,j}$ *in the minors associated with the dominant pair of phase combinations of Lemma 3.2.*

(i) *If* $[i,j]$ *identifies an asymptotic line soliton as* $y\to\infty$, *then* $\mathrm{rank}(P_{i,j})=r\leqslant N-1$ *and* $\mathrm{rank}(P_{i,j}|A[i])=\mathrm{rank}(P_{i,j}|A[j])=\mathrm{rank}(P_{i,j}|A[i,j])=r+1$.

(ii) *If* $[i,j]$ *identifies an asymptotic line soliton as* $y\to-\infty$, *then* $\mathrm{rank}(Q_{i,j})=s\leqslant N-1$ *and* $\mathrm{rank}(Q_{i,j}|A[i])=\mathrm{rank}(Q_{i,j}|A[j])=\mathrm{rank}(Q_{i,j}|A[i,j])=s+1$.

Above and hereafter, $(A|B)$ *denotes the matrix A augmented by the matrix B.*

Proof: Let us prove part (i). Since the columns $A[p_1],\ldots,A[p_r]$ form a basis for the column space of $P_{i,j}$, from Lemma 3.3(i) we immediately have $\mathrm{rank}(P_{i,j})=r$. Moreover, since $\mathcal{A}=\{A[i],A[p_1],\ldots,A[p_r],A[q_1],\ldots,A[q_s]\}$ is a basis for $\mathbb{R}^N$, the vectors $A[i],A[p_1],\ldots,A[p_r]$ are linearly independent, and therefore $\mathrm{rank}(P_{i,j}|A[i])=r+1$. Similarly, replacing $A[i]$ with $A[j]$ in the previous statement we have $\mathrm{rank}(P_{i,j}|A[j])=r+1$. It remains to prove that $\mathrm{rank}(P_{i,j}|A[i,j])=r+1$. Expanding the jth column of A in terms of $\mathcal{A}$ as in Lemma 3.3 we have

$$A[j]=aA[i]+\sum_{m=1}^{r}b_mA[p_m]+\sum_{m=1}^{s}c_mA[q_m]. \tag{3.4}$$

By replacing one of the columns $A[q_1],\ldots,A[q_s]$ in $A(i,p_1,\ldots,p_r,q_1,\ldots,q_s)$ with $A[j]$, we have from Lemma 3.2(i)(b) that $A(i,p_1,\ldots,p_r,q_1,\ldots,q_{m-1},j,q_{m+1},\ldots,q_s)=0$. Therefore $c_m=0$ for all $m=1,\ldots,s$. Consequently we have $A[j]\in\mathrm{span}(\{A[i],A[p_1],\ldots,A[p_r]\})$, which implies that $\mathrm{rank}(P_{i,j}|A[i,j])=r+1$. Similarly, using Lemma 3.2(ii)(b) one can establish the corresponding results in part (ii) for the asymptotic line solitons as $y\to-\infty$. □

It is important to note that, even though Lemmas 3.3–3.4 were proved by using the vanishing minor conditions in Lemma 3.2, they provide additional information on the structure of the coefficient matrix A. For example, when $r<N-1$ for an asymptotic line soliton as $y\to\infty$, Lemma

3.4 yields $\mathrm{rank}(P_{i,j}|A[i,j])<N$, and when $s<N-1$ for an asymptotic line soliton as $y\to-\infty$, Lemma 3.4 yields $\mathrm{rank}(Q_{i,j}|A[i,j])<N$. As a consequence, we immediately have the following additional vanishing minor conditions:

(i) If $[i,j]$ identifies an asymptotic line soliton as $y\to\infty$, then

$$A(i,j,p_1,\ldots,p_r,m_1,\ldots,m_{N-r-2})=0 \quad \forall\{m_1,\ldots,m_{N-r-2}\}\subset\{1,\ldots,M\}. \tag{3.5a}$$

(ii) If $[i,j]$ identifies an asymptotic line soliton as $y\to-\infty$, then

$$A(i,j,q_1,\ldots,q_s,m_1,\ldots,m_{N-s-2})=0 \quad \forall\{m_1,\ldots,m_{N-s-2}\}\subset\{1,\ldots,M\}. \tag{3.5b}$$

We remark that conditions (3.5) were also introduced (without proof) in Ref. 12 (cf. Definition 4.2) in order to characterize the tau functions of the elastic N-soliton solutions which correspond to the special case $M=2N$. It should also be noted that, when $[i,j]$ identifies an asymptotic line soliton as $y\to\infty$, Lemma 3.4(i) only provides information on $P_{i,j}$, and the only condition on $Q_{i,j}$ is that $\mathrm{rank}(Q_{i,j})\geq s$. Similarly, when $[i,j]$ identifies an asymptotic line soliton as $y\to-\infty$, all is known about $P_{i,j}$ is that $\mathrm{rank}(P_{i,j})\geq r$.

B. Characterization of the asymptotic line solitons from the coefficient matrix

In the preceding section we derived several conditions that an index pair $[i,j]$ must satisfy in order to identify an asymptotic line soliton. Those results are now applied to obtain a complete characterization of the incoming and outgoing asymptotic line solitons of a generic line-soliton solution of the KPII equation.

Lemma 3.5 (Pivots and nonpivots): Consider an index pair $[i,j]$ *with* $1\leq i<j\leq M$.

(i) *If* $[i,j]$ *identifies an asymptotic line soliton as* $y\to\infty$, *the index* i *labels a pivot column of the coefficient matrix A. That is,* $A[i]=A[e_n]$ *with* $1\leq n\leq N$.
(ii) *If* $[i,j]$ *identifies an asymptotic line soliton as* $y\to-\infty$, *the index* j *labels a nonpivot column of the coefficient matrix A. That is,* $A[j]=A[g_n]$ *with* $1\leq n\leq M-N$.

Proof: We first prove part (i). Suppose that $\theta_{i,m_2,\ldots,m_N}$ is one of the dominant phase combinations corresponding to the asymptotic line soliton $[i,j]$ as $y\to\infty$. The corresponding minor $A(i,m_2,\ldots,m_N)$ is nonzero. Since A is in RREF, we have $A[i]=\Sigma_{r=1}^{n}c_rA[e_r]$ for some $n\leq N$, where $e_1<\cdots<e_n\leq i$. Therefore $A(i,m_2,\ldots,m_N)=\Sigma_{r=1}^{n}c_rA(e_r,m_2,\ldots,m_N)$. If $e_n<i$, we have $A[e_1],\ldots,A[e_n]\in P_{i,j}$, where $P_{i,j}$ is the submatrix of A defined in Eq. (3.2). Then from condition (a) in Lemma 3.2(i) we have $A(e_r,m_2,\ldots,m_N)=0\,\forall r=1,\ldots,n$, implying that $A(i,m_2,\ldots,m_N)=0$. But this is impossible, since $\theta_{i,m_2,\ldots,m_N}$ is a dominant phase combination. Therefore we must have $i=e_n$, meaning that $A[i]$ is a pivot column.

Part (ii) follows from the rank conditions in Lemma 3.4(ii). In particular, $\mathrm{rank}(Q_{i,j}|A[i])=\mathrm{rank}(Q_{i,j}|A[i,j])=s+1$ implies that $A[j]\in\mathrm{span}(\{A[i],\ldots,A[j-1]\})$. Since A is in RREF, none of its pivot column can be spanned by the preceding columns. Hence $A[j]$ is not a pivot column. □

Lemma 3.5 identifies outgoing and incoming asymptotic line solitons, respectively, with the pivot and the nonpivot columns of A. It is then natural to ask if in fact each of the N pivot columns and each of the $M-N$ nonpivot columns identifies an outgoing or incoming line soliton, and whether such identification is unique. Both of these questions can be answered affirmatively by the following theorem which constitutes one of the main results of this work, and is proved in the Appendix.

Theorem 3.6: *(Asymptotic line solitons) Let* $\tau_{N,M}(x,y,t)$ *be the tau function in Eq. (2.1) associated with a rank N, irreducible coefficient matrix A with non-negative minors.*

(i) *For each pivot index* e_n *there exists a unique asymptotic line soliton as* $y\to\infty$, *identified by an index pair* $[e_n,j_n]$ *with* $n=1,\ldots,N$ *and* $1\leq e_n<j_n\leq M$.
(ii) *For each nonpivot index* g_n *there exists a unique asymptotic line soliton as* $y\to-\infty$, *identified by an index pair* $[i_n,g_n]$ *with* $n=1,\ldots,M-N$ *and* $1\leq i_n<g_n\leq M$.

Thus, the solution of KPII generated by the coefficient matrix A via Eq. (2.1) *has exactly* $N_+=N$ *asymptotic line solitons as* $y\to\infty$ *and* $N_-=M-N$ *asymptotic line solitons as* $y\to-\infty$.

Part (i) of Theorem 3.6 uniquely identifies the asymptotic line solitons as $y\to\infty$ by the index pairs $[e_n,j_n]$ where $e_n<j_n$. The indices $e_1,\dots,e_N$ label the N pivot columns of A, however, the j_n's may correspond to either pivot or nonpivot columns, and indeed both cases appear in examples. Moreover, when the pivot indices are sorted in increasing order $1=e_1<e_2<\cdots<e_N<M$, the indices $j_1,\dots,j_N$ in general *are not sorted* in any specific order. For example, the line solitons as $y\to\infty$ generated by the matrix A in Eq. (4.5) of Sec. IV have $j_1<j_3<j_2$. In fact, the indices $j_1,\dots,j_N$ need not necessarily even be distinct. Similarly, part (ii) of Theorem 3.6 uniquely identifies the asymptotic line solitons as $y\to-\infty$ by index pairs $[i_n,g_n]$, where $i_n<g_n$. In this case, the indices $g_1,\dots,g_{M-N}$ label the $M-N$ nonpivot columns of A, but the i_n's may correspond to either pivot or nonpivot columns. Moreover, when the nonpivot indices are sorted in increasing order $1<g_1<\cdots<g_{M-N}=M$, the indices $i_1,\dots,i_{M-N}$ are not in general sorted, and need not be distinct. Theorem 3.6 yields an important characterization of the solution via the associated coefficient matrix A, and it provides a concrete method to identify the asymptotic line solitons as $y\to\pm\infty$, as illustrated with the examples below. Further examples are discussed in Sec. IV.

Example 3.7: Consider the tau function $\tau_{N,M}$ with $N=2$ and $M=5$ generated by the coefficient matrix

$$A=\begin{pmatrix}1 & 1 & 0 & -1 & -2\\ 0 & 0 & 1 & 1 & 1\end{pmatrix}. \tag{3.6}$$

The pivot columns of A are labeled by the indices $\{e_1,e_2\}=\{1,3\}$, and the nonpivot columns by the indices $\{g_1,g_2,g_3\}=\{2,4,5\}$. It follows from Theorem 3.6 that there will be $N_+=N=2$ asymptotic line solitons as $y\to\infty$, identified by the index pairs $[1,j_1]$ and $[3,j_2]$ for some $j_1>1$ and $j_2>3$, and that there will be $N_-=M-N=3$ asymptotic line solitons as $y\to-\infty$, identified by the index pairs $[i_1,2]$, $[i_2,4]$, and $[i_3,5]$, for some $i_1<2$, $i_2<4$, and $i_3<5$. We first determine the asymptotic line solitons as $y\to\infty$ using part (i) of Theorem 3.6 together with the rank conditions in Lemma 3.4(i). The we find the asymptotic line solitons as $y\to-\infty$ using part (ii) of Theorem 3.6 and the rank conditions in Lemma 3.4(ii).

For the first pivot column, $e_1=1$, we start with $j=2$ and consider the submatrix $P_{1,2}=\left(\begin{smallmatrix}0 & -1 & -2\\ 1 & 1 & 1\end{smallmatrix}\right)$. Since $\mathrm{rank}(P_{1,2})=2>1=N-1$, from Lemma 3.4(i) we conclude that the pair [1,2] cannot identify an asymptotic line soliton as $y\to\infty$. Incrementing j to $j=3,4,5$ and checking the rank of each submatrix $P_{1,j}$ we find that the rank conditions in Lemma 3.4(i) are satisfied when $j=4$, and $P_{1,4}=\left(\begin{smallmatrix}-2\\ 1\end{smallmatrix}\right)=A[5]$. So, $\mathrm{rank}(P_{1,4})=1$ and $\mathrm{rank}(P_{1,4}|A[1])=\mathrm{rank}(P_{1,4}|A[4])=2$. The condition $\mathrm{rank}(P_{1,4}|A[1,4])=2$ is trivial here, since any three columns are linearly dependent. Thus, the first asymptotic line soliton as $y\to\infty$ is identified by the index pair [1,4]. For the second pivot, $e_2=3$, proceeding in a similar manner we find that $j=4$ does not satisfy the rank conditions because $P_{3,4}$ has rank 2. But $j=5$ satisfies Lemma 3.4(i), since $P_{3,5}=\left(\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right)$, which yields $\mathrm{rank}(P_{3,5})=1$ and $\mathrm{rank}(P_{3,5}|A[3])=\mathrm{rank}(P_{3,5}|A[5])=2$. Again, $\mathrm{rank}(P_{3,5}|A[3,5])=2$ is trivially satisfied here. So the asymptotic line solitons as $y\to\infty$ are given by the index pairs [1,4] and [3,5], and the associated phase transition diagram (cf. Corollary 2.6) is given by

$$R_{1,3}\xrightarrow{3\to5}R_{1,5}\xrightarrow{1\to4}R_{4,5}.$$

We now consider the asymptotics for $y\to-\infty$. Starting with the nonpivot column $g_1=2$, the only column to its left is $i=1$. We have $Q_{1,2}=\varnothing$, and $\mathrm{rank}(Q_{1,2}|A[1])=\mathrm{rank}(Q_{1,2}|A[2])=\mathrm{rank}(Q_{1,2}|A[1,2])=1$. Consequently, the pair [1,2] identifies an asymptotic line soliton as $y\to-\infty$. For $g_2=4$ we consider $i=1,2,3$ and find that the rank conditions in Lemma 3.4(ii) are satisfied only for $i=2$. In this case, $Q_{2,4}=\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)=A[3]$, so $\mathrm{rank}(Q_{2,4})=1=N-1$ and $\mathrm{rank}(Q_{2,4}|A[2])=\mathrm{rank}(Q_{2,4}|A[4])=2$, while $\mathrm{rank}(Q_{2,4}|A[2,4])=2$ is trivially satisfied. Hence [2,4] is the unique

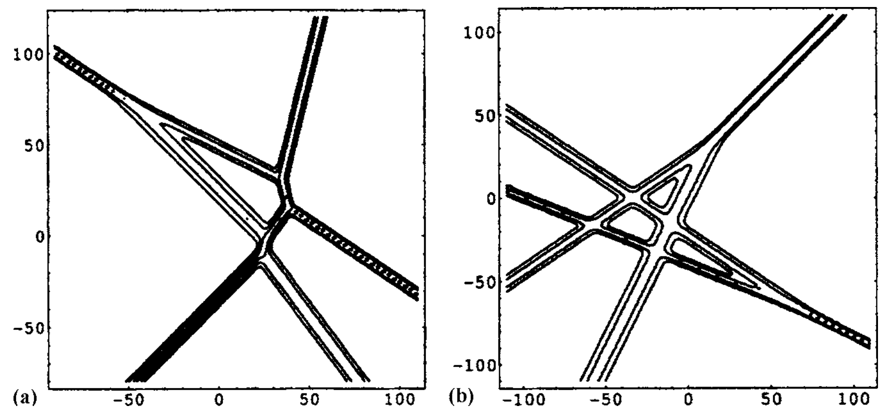

FIG. 3. Line soliton solutions of KPII: (a) the (3,2)-soliton solution generated by the coefficient matrix A in Example 3.7 with $(k_1,\ldots,k_5)=(-1,0,\frac{1}{4},\frac{3}{4},\frac{5}{4})$ at $t=-32$; (b) the inelastic 3-soliton solution generated by the coefficient matrix A in Example 3.8 with $(k_1,\ldots,k_6)=(-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2})$ at $t=20$ (see text for details).

asymptotic line soliton as $y\to-\infty$ associated to the nonpivot column $g_2=4$. In a similar way we can uniquely identify the last asymptotic line soliton as $y\to-\infty$ as given by the indices [3,5]. The phase transition diagram for $y\to-\infty$ is thus given by

$$R_{1,3}\xrightarrow{1\to2}R_{2,3}\xrightarrow{2\to4}R_{3,4}\xrightarrow{3\to5}R_{4,5}.$$

To summarize, there are $N_+=2$ outgoing line solitons, each associated with one of the pivot columns $e_1=1$ and $e_2=3$, given by the index pairs [1,4] and [3,5], and there are $N_-=3$ incoming line solitons, each associated with one of the nonpivot columns $g_1=2$, $g_2=4$, and $g_3=5$, given by the index pairs [1,2], [2,4], and [3,5]. A snapshot of the solution at $t=-32$ is shown in Fig. 3(a).

Example 3.8: Consider the tau function with $N=3$ and $M=6$ generated by the coefficient matrix in RREF,

$$A=\begin{pmatrix}1&1&1&0&0&0\\0&0&0&1&0&-1\\0&0&0&0&1&2\end{pmatrix}. \tag{3.7}$$

Again, we first determine the asymptotic line solitons as $y\to\infty$, and then the asymptotic line solitons as $y\to-\infty$.

The pivot columns of A are labeled by the indices $e_1=1$, $e_2=4$, and $e_3=5$. Thus, we know that the asymptotic line solitons as $y\to\infty$ will be given by the index pairs $[1,j_1]$, $[4,j_2]$, and $[5,j_3]$ for some $j_1>1$, $j_2>4$, and $j_3>5$. Starting with the first pivot, $e_1=1$, we take $j=2,3,\ldots$ and check the rank of the submatrix $P_{i,j}$ in each case. When $j=2$ we have $P_{1,2}=\begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&2\end{pmatrix}$, and $\mathrm{rank}(P_{1,2})=3>N-1$. So, by Lemma 3.4(i), the index pair [1,2] does not correspond to an asymptotic line soliton as $y\to\infty$. In fact, using Lemma 3.1 it can be verified that $\theta_{3,5,6}$ is the only dominant phase combination along the line $\theta_1=\theta_2$ as $y\to\infty$. Next, we consider $j=3$. In this case we have $P_{1,3}=\begin{pmatrix}0&0&0\\1&0&-1\\0&1&2\end{pmatrix}$, with $\mathrm{rank}(P_{1,3})=2=:r$ and $\mathrm{rank}(P_{1,3}|A[1])=\mathrm{rank}(P_{1,3}|A[3])=\mathrm{rank}(P_{1,3}|A[1,3])=3=r+1$. So the rank conditions in Lemma 3.4(i) are satisfied. Therefore the index pair [1,3] corresponds to an asymptotic line soliton as $y\to\infty$. Moreover, by considering $j=4,5,6$ one can easily check that the rank conditions are no longer satisfied. Thus [1,3] is the *unique* asymptotic line soliton associated with the pivot index $e_1=1$ as $y\to\infty$, in agreement with Theorem 3.6. Let us now consider the second pivot column, $e_2=4$. In this case we find that the rank conditions are only satisfied when $j=5$, since $P_{4,5}=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, with $\text{rank}(P_{4,5})=2=:r$ and $\text{rank}(P_{4,5}|A[4]) =\text{rank}(P_{4,5}|A[5])=\text{rank}(P_{4,5}|A[4,5])=3=r+1$. Therefore, the index pair $[4,5]$ corresponds to an asymptotic line soliton as $y\to\infty$. Finally, for $e_3=5$, since we know from Theorem 3.6 that $j>e_3$, we immediately find that the third asymptotic line soliton as $y\to\infty$ is given by the index pair [5,6]. From Corollary 2.6, the phase transition diagram as $y\to\infty$ is given by

$$R_{1,4,5} \xrightarrow{5\to6} R_{1,4,6} \xrightarrow{4\to5} R_{1,5,6} \xrightarrow{1\to3} R_{3,5,6}.$$

The nonpivot columns of the coefficient matrix A are labeled by the indices $g_1=2$, $g_2=3$, and $g_3=6$. For $g_1=2$, the only possible value of $i<j$ is $i=1$. In this case $Q_{1,2}=\emptyset$, so $\text{rank}(Q_{1,2})=0$ and $\text{rank}(Q_{1,2}|A[1])=\text{rank}(Q_{1,2}|A[2])=\text{rank}(Q_{1,2}|A[1,2])=1$. Thus the pair [1,2] identifies an asymptotic line soliton as $y\to-\infty$. For $g_2=3$ we consider $i=2,1$. When $i=2$, the rank conditions in Lemma 3.4(ii) are satisfied, leading to the asymptotic line soliton [2,3] as $y\to-\infty$. We can check that the soliton associated with the nonpivot column $g_2=3$ is unique by considering $i=1$ and verifying that the rank conditions are not satisfied. Similarly, it is easy to verify that for $g_3=6$ the index pair [4,6] uniquely identifies the asymptotic line soliton as $y\to-\infty$. The phase transition diagram as $y\to-\infty$ reads as follows:

$$R_{1,4,5} \xrightarrow{1\to2} R_{2,4,5} \xrightarrow{2\to3} R_{3,4,5} \xrightarrow{4\to6} R_{3,5,6}.$$

Summarizing, there are $N_+=3$ asymptotic line solitons as $y\to\infty$ identified by the index pairs [1,3], [4,5], and [5,6], and there are $N_-=3$ asymptotic line solitons as $y\to-\infty$ identified by the index pairs [1,2], [2,3], and [4,6]. A snapshot of the solution at $t=-20$ is shown in Fig. 3(b).

Examples 3.7 and 3.8 illustrate the fact that, starting from any given coefficient matrix A in RREF, the asymptotic line solitons as $y\to\pm\infty$ can be identified in an algorithmic way by applying Theorem 3.6 together with the rank conditions in Lemma 3.4.

IV. FURTHER EXAMPLES

In this section we present a variety of line-soliton solutions of KPII generated by the tau function (2.2) with different choices of coefficient matrices.

Ordinary N-soliton solutions: These are constructed by taking $M=2N$ and choosing the functions $\{f_n\}_{n=1}^N$ in Eq. (1.10) as (e.g., see Refs. 5 and 15)

$$f_n(x,y,t) = e^{\theta_{2n-1}} + e^{\theta_{2n}}, \quad n = 1, \ldots ,N. \tag{4.1}$$

The corresponding coefficient matrix is thus given by

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix},$$

with N pairs of identical columns at positions $\{2n-1,2n\}$, $n=1,\ldots,N$. There are only 2^N nonzero minors of A, which are given by $A(m_1,m_2,\ldots,m_N)=1$ where, for each $n=1,\ldots,N$, either $m_n=2n-1$ or $m_n=2n$. The asymptotic analysis of the preceding section implies that the nth asymptotic line soliton as $y\to\pm\infty$ is identified by the index pair $[2n-1,2n]$ for $n=1,\ldots,N$, where $i_n=2n-1$ and $j_n=2n$ label, respectively, the pivot and nonpivot columns of A. Therefore the amplitude and direction are given by $a_n=k_{2n}-k_{2n-1}$ and $c_n=k_{2n-1}+k_{2n}$. Moreover, the dominant pair of phase combinations for the nth soliton as $y\to\infty$ is given by $\theta_{1,3,\ldots,2n-1,2n+2,2n+4,\ldots,2N}$ and $\theta_{1,3,\ldots,2n-3,2n,2n+2,\ldots,2N}$, while the dominant phase combinations for the same soliton as $y\to-\infty$ by

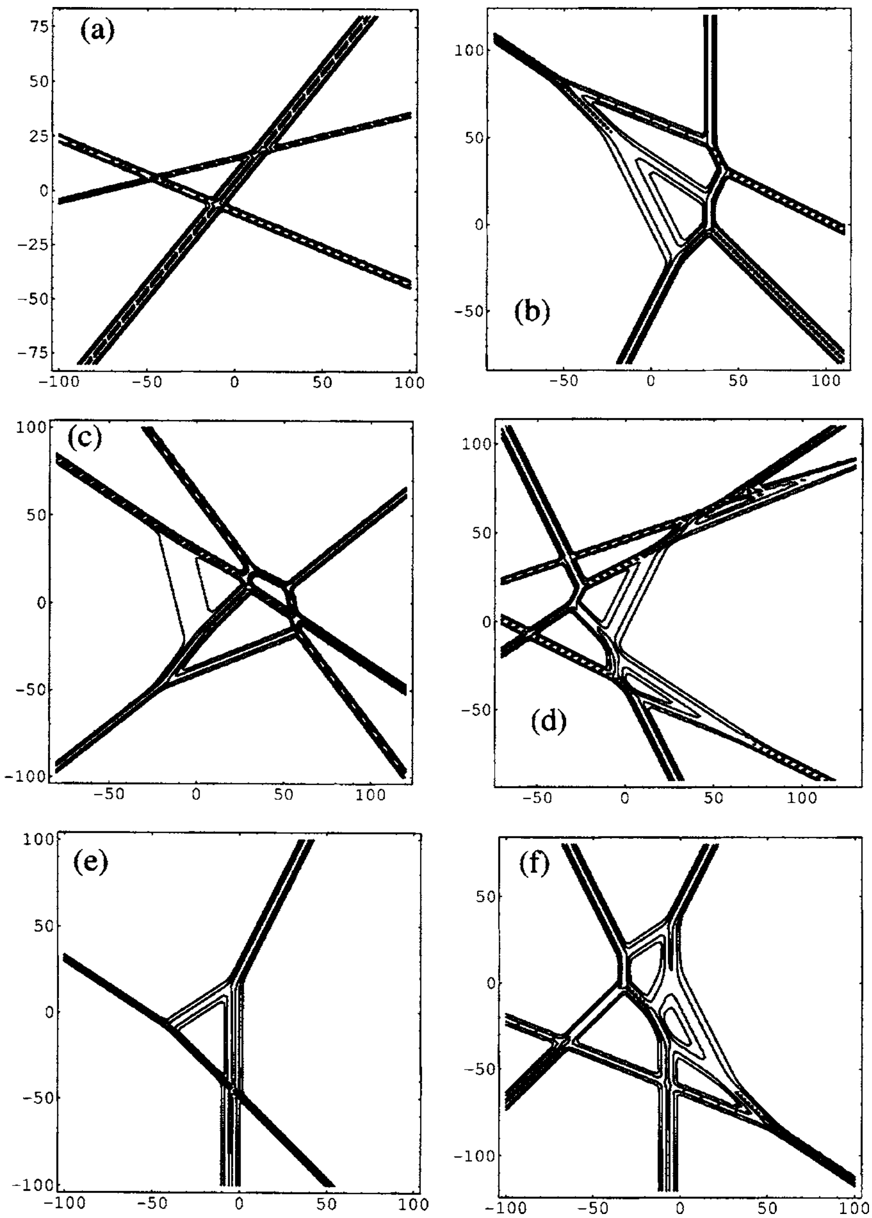

FIG. 4. Line-soliton solutions of KPII: (a) an ordinary 3-soliton solution with $(k_1,\ldots,k_6)=(-3,-2,0,1,\frac{3}{2},2)$ at $t=4$; (b) a fully resonant (3,2)-soliton solution with $(k_1,\ldots,k_5)=(-1,0,\frac{1}{2},1,\frac{3}{2})$ at $t=-32$; (c) an elastic, partially resonant 3-soliton solution with A given by Eq. (4.5) and $(k_1,\ldots,k_6)=(-\frac{3}{2},-1,0,\frac{1}{4},\frac{3}{2},\frac{7}{4})$ at $t=-20$; (d) an elastic, partially resonant 4-soliton solution with A given by Eq. (4.6) and $(k_1,\ldots,k_8)=(-2,-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2})$ at $t=20$; (e) an inelastic 2-soliton solution with A given by Eq. (4.7) and $(k_1,\ldots,k_4)=(-1,-\frac{1}{2},\frac{1}{2},2)$ at $t=16$; (f) an inelastic 3-soliton solution with A given by Eq. (4.8) and $(k_1,\ldots,k_6)=(-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2})$ at $t=32$.

$\theta_{2,4,\ldots,2n,2n+1,2n+3,\ldots,2N-1}$ and $\theta_{2,4,\ldots,2n-2,2n-1,2n+1,\ldots,2N-1}$. Apart from the phase shift of each line soliton, the interaction gives rise to a pattern of N intersecting lines in the xy plane, as shown in Fig. 4(a).

Solutions of KPII which also satisfy the finite Toda lattice hierarchy: Another class of (N_-,N_+)-soliton solutions of KPII is given by the following choice of functions $\{f_n\}_{n=1}^{N}$ in Eq. (1.10):

$$f_n = f^{(n-1)}, \quad n = 1, \ldots, N. \tag{4.2}$$

In addition to generating solutions of KPII, the set of tau functions $\tau_{N,M}$ for $N=1,\ldots,M$ also satisfy the Plücker relations for the finite Toda lattice hierarchy.[2] Choosing $f(x,y,t)=\Sigma_{m=1}^{M} e^{\theta_m}$ yields the following coefficient matrix:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ k_1 & k_2 & \cdots & k_M \\ \vdots & \vdots & \ddots & \vdots \\ k_1^{N-1} & k_2^{N-1} & \cdots & k_M^{N-1} \end{pmatrix}. \tag{4.3}$$

Note that A in Eq. (4.3) is not in RREF, and coincides with the matrix K in Lemma 2.1. The pivot columns of A are labeled by indices $1,\ldots,N$. Furthermore, all the $N\times N$ minors of A are nonzero, and coincide with the Van der Monde determinants in Eq. (2.4). The corresponding class of KPII solutions was studied in Ref. 2, where it was shown that the N asymptotic line solitons as $y\to\infty$ are identified by the index pairs $[n,n+M-N]$ for $n=1,\ldots,N$, while the $M-N$ asymptotic line solitons as $y\to-\infty$ are identified by the index pairs $[n,n+N]$ for $n=1,\ldots,M-N$. These pairings can also be easily verified using Theorem 3.6. The dominant pair of phase combinations for the nth soliton as $y\to\infty$ is given by $\theta_{1,\ldots,n,M-N+n+1,\ldots,M}$ and $\theta_{1,\ldots,n-1,M-N+n,\ldots,M}$, while the dominant pair of phase combinations for the nth soliton as $y\to-\infty$ by $\theta_{n,\ldots,N+n-1}$ and $\theta_{n+1,\ldots,N+n}$. The solution displays phenomena of soliton resonance and web structure [e.g., see Fig. 4(b)]. More precisely, the interaction of the asymptotic line solitons results in a pattern with $(2N_- -1)N_+$ interaction vertices, $(3N_- -2)N_+$ intermediate interaction segments and $(N_- -1)(N_+ -1)$ "holes" in the xy plane. Each of the intermediate interaction segments can be effectively regarded as a line soliton since it satisfies the dispersion relation (1.8). Furthermore, all of the asymptotic and intermediate line solitons interact via a collection of *fundamental resonances*. A fundamental resonance, also called a Y junction, describes an interaction of three line solitons whose wave numbers $\mathbf{k}_a$ and frequencies ω_a $(a=1,2,3)$ satisfy the three-wave resonance conditions[17,19]

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3, \quad \omega_1 + \omega_2 = \omega_3. \tag{4.4}$$

Such a solution is shown in Fig. 1(a).

Elastic N-soliton solutions: As mentioned in Sec. I and in the Appendix, the elastic N-soliton solutions are those for which the sets of incoming and outgoing asymptotic line solitons are the same. In this case we necessarily have $M=2N$. Ordinary N-soliton solutions and solutions of KPII which also satisfy the finite Toda lattice hierarchy with $M=2N$ are two special classes of elastic N-soliton solutions. However, a large variety of other elastic N-soliton solutions do also exist, and were recently investigated in Ref. 12. For example, Fig. 4(c) shows an elastic 3-soliton solution generated by the coefficient matrix,

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & -2 & -1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}. \tag{4.5}$$

In this case the pivot columns are labeled by indices 1, 2, and 3. So, from Lemma 3.5 we know that the asymptotic line solitons as $y\to\infty$ will be identified by index pairs $[1,j_1]$, $[2,j_2]$, and $[3,j_3]$, while those as $y\to-\infty$ by index pairs $[i_1,4]$, $[i_2,5]$, and $[i_3,6]$, for certain values of $i_1,\ldots,i_3$ and $j_1,\ldots,j_3$. Indeed, from the results developed in Sec. III one can verify that both the incoming and the outgoing asymptotic line solitons are given by the *same* index pairs [1,4], [2,6], and [3,5]. The soliton interactions in this case are *partially resonant*, in the sense that the pairwise interaction among solitons [1,4] and [2,6] and that among solitons [1,4] and [3,5] are both resonant, but the pairwise interaction among solitons [2,6] and [3,5] is nonresonant. Similarly, Fig. 4(d) shows an elastic, partially resonant 4-soliton solution generated by the coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}. \quad (4.6)$$

In this case the pivot columns are labeled by the indices 1, 2, 4, and 6 and the nonpivot columns by the indices 3, 5, 7, and 8. The asymptotic line solitons as $y \to \pm\infty$ are identified by the index pairs [1,3], [2,5], [4,7], and [6,8]. As can be seen from Fig. 4(f), the pairwise interaction of solitons [1,3] and [2,5], solitons [2,5] and [4,7], and [4,7] and [6,8] are resonant, but the remaining pairwise interactions between solitons [1,3] and [4,7], [1,3] and [6,8], [2,5] and [6,8], are nonresonant. It should be clear from these examples that a large variety of elastic N-soliton solutions with resonant, partially resonant and nonresonant interactions is possible.

Inelastic N-soliton solutions: There also exist a large class of N-soliton solutions that are not elastic. We have already seen such solutions in Examples 3.7 and 3.8 [cf. Figs. 3(a) and 3(b)] of Sec. III. As a further example, Fig. 4(e) shows an inelastic 2-soliton solution generated by the coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \quad (4.7)$$

In this case the pivot columns are labeled by indices 1 and 2. The asymptotic line solitons as $y \to -\infty$ are identified by the index pairs [1,4] and [2,3], while those as $y \to \infty$ by the index pairs [1,3] and [2,4]. Notice that the outgoing solitons interact resonantly via two Y junctions, while the incoming soliton pair interact nonresonantly. This is in contrast with an elastic 2-soliton solution, where both incoming and outgoing pairs of solitons exhibit the same kind of interaction. Similarly, Fig. 4(f) shows inelastic 3-soliton solution generated by the coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \quad (4.8)$$

Here the pivot columns are labeled by indices 1, 2, and 5. The asymptotic line solitons as $y \to \infty$ are identified by the index pairs [1,3], [2,5], and [5,6], while those as $y \to -\infty$ by the index pairs [1,3], [2,4], and [3,6].

Finally, we remark that in the generic case $M \neq 2N$, the numbers of asymptotic line solitons as $y \to \pm\infty$ are different, as in the solutions shown in Figs. 3(a) and 4(b). Also, note that the one-soliton solutions, the ordinary two-soliton solutions and the Y junction solutions have the property that their time evolution is just an overall translation of a fixed spatial pattern. However, for all other solutions discussed above, the interaction patterns formed by the asymptotic line solitons, and the relative positions of the interaction vertices in the xy plane are in general time dependent.

V. CONCLUSIONS

In this paper we have studied a class of line-soliton solutions of the Kadomtsev-Petviashvili II equation by expressing the tau function as the Wronskian of N linearly independent combinations of M exponentials. From the asymptotics of the tau function as $y \to \pm\infty$ we showed that each of these solutions of KPII is composed of asymptotic line solitons which are defined by the transition between two dominant phase combinations with $N-1$ common phases. Moreover, the number, amplitudes and directions of the asymptotic line solitons are invariant in time. We also derived an algorithmic method to identify these asymptotic line solitons in a given solution by examining the $N \times M$ coefficient matrix A associated with the corresponding tau function. In particular, we proved that every $N \times M$, irreducible coefficient matrix A produces an (N_-, N_+)-soliton solution of KPII in which there are $N_+ = N$ asymptotic line solitons as $y \to \infty$, labeled by the pivot columns of A, and $N_- = M - N$ asymptotic line solitons as $y \to -\infty$, labeled by the nonpivot columns of A. Such solutions exhibit a rich variety of time-dependent spatial patterns which include resonant soliton interactions and web structure. Finally, we discussed a number of examples of such (N_-,N_+)-soliton solutions in order to illustrate the above results.

It is remarkable that the KPII equation possesses such a rich structure of line-soliton solutions generated by a simple form of the tau function. In this work we have primarily focused on the asymptotic behavior of the solutions as $y\to\pm\infty$, but not on their interactions in the xy plane. A full characterization of the interaction patterns of the general (N_-,N_+)-soliton solutions is an important open problem, which is left for further study. Nonetheless, we believe that our results will provide a key step toward that endeavor. We point out that resonant interaction described by the line solitons of KPII is a physical phenomenon that has been observed experimentally in ion-acoustic waves (see e.g., Refs. 20 and 13). Hence, we expect that the resonant solutions considered in this work are likely to be stable with respect to small perturbations and physically relevant. However, a formal stability analysis of these (2+1)-dimensional solutions is a highly nontrivial task, and has not yet been carried out to the best of our knowledge. Finally, we note that soliton solutions exhibiting phenomena of soliton resonance and web structure have been found for several other (2+1)-dimensional integrable systems, and those solutions can also be described by direct algebraic methods similar to the ones used here. Therefore we expect that the results presented in this work will also be useful to study solitonic solutions in a variety of other (2+1)-dimensional integrable systems.

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APPENDIX A: PROOF OF THEOREM 2.5

To prove part (i) of Theorem 2.5, it is sufficient to show that, along each line L_c, the sign of the inequalities among the phase combinations in Definition 2.3 remain unchanged in time as $y\to\pm\infty$. For this purpose, note that the sign of $\theta_{m_1,\ldots,m_N}-\theta_{m_1',\ldots,m_N'}$ in Eq. (2.6b) is determined by the coefficient of y on the right-hand side as $y\to\pm\infty$ and for finite ξ and t, if this coefficient is nonzero. For generic values of the phase parameters $k_1,\ldots,k_M$ this coefficient is indeed nonvanishing, and its sign depends only on the direction c of the line L_c. Consequently, the dominant phase combinations asymptotically as $y\to\pm\infty$ are determined only by the constant c for finite time.

Part (ii) of the theorem is proved by showing that the only possible phase transitions are those in which a single phase, say θ_m changes to $\theta_{m'}$ between the two dominant phase combinations across adjacent regions, and that no other type of transitions can occur. We first prove that single-phase transitions are allowed; then we show that no other type of transitions are allowed. In the following, we will assume t to be finite so that the dominant phase combinations remain invariant, according to part (i). Suppose that $\theta_{m_1,\ldots,m_N}$ is the dominant phase combination in a region R asymptotically for large values of $|y|$. Since R is a proper subset of $\mathbb{R}^3$, it *must* have a boundary, across which a transition will take place from $\theta_{m_1,\ldots,m_N}$ to some other dominant phase combination. Since $\theta_{m_1,\ldots,m_N}$ is dominant, $A(m_1,\ldots,m_N)\neq 0$ according to Definition 2.3. Therefore, the columns $A[m_1],\ldots,A[m_N]$ of the coefficient matrix form a basis of $\mathbb{R}^N$, and for all $j\notin\{m_1,\ldots,m_N\}$ we have that $A[j]$ is in the span of $A[m_1],\ldots,A[m_N]$. Thus there exists at least one column $A[m_s]$ such that the coefficient of $A[m_s]$ in the expansion of $A[j]$ is nonzero. We then have $A(m_1,\ldots,m_{s-1},j,m_{s+1},\ldots,m_N)\neq 0$, implying that the phase combination $\theta_{m_1,\ldots,m_{s-1},j,m_{s+1},\ldots,m_N}$ is actually present in the tau function. Then, for *any* $j\notin\{m_1,\ldots,m_N\}$ it is possible to have a single-phase transition from R to the adjacent region R' across the line $\theta_{m_s}=\theta_j$, since the sign of $\theta_{m_s}-\theta_j$ changes across this line, implying that $\theta_{m_1,\ldots,m_{s-1},j,m_{s+1},\ldots,m_N}$ is larger than $\theta_{m_1,\ldots,m_N}$ in R'.

We next show that no *other* type of transitions can occur apart from single-phase transitions; we do so by *reductio ad absurdum*. Suppose that at least two phases $\theta_{m_1}, \theta_{m_2}$ from the dominant phase combination $\theta_{m_1,\ldots,m_N}$ in a region R are replaced with phases $\theta_{m_1'}, \theta_{m_2'}$ during the transition from R to an adjacent region R'. This transition occurs along the common boundary of R and R', which is given by line $L:(\theta_{m_1}+\theta_{m_2})-(\theta_{m_1'}+\theta_{m_2'})=0$. Thus, along L, the differences $\theta_{m_1}-\theta_{m_1'}$ and $\theta_{m_2}-\theta_{m_2'}$ (or, equivalently, the differences $\theta_{m_1}-\theta_{m_2'}$ and $\theta_{m_2}-\theta_{m_1'}$) must have opposite signs or be both zero.

If both differences are zero along L, the lines $\theta_{m_1}=\theta_{m_1'}$ and $\theta_{m_2}=\theta_{m_2'}$ (or, equivalently, the lines $\theta_{m_1}=\theta_{m_2'}$ and $\theta_{m_1'}=\theta_{m_2}$) must both coincide with the line L in the xy plane. This is possible only at a given instant of time *and* if the directions of the two lines are the same, i.e., if $k_{m_1}+k_{m_1'}=k_{m_2}+k_{m_2'}$ (or, equivalently, $k_{m_1}+k_{m_2'}=k_{m_1'}+k_{m_2}$). So for generic values of the phase parameters, or for generic values of time, this exceptional case can be excluded. Hence, we assume that $\theta_{m_1}-\theta_{m_1'}$ and $\theta_{m_2}-\theta_{m_2'}$ are of opposite signs.

Note that $\theta_{m_1}-\theta_{m_1'}=\theta_{m_1,\ldots,m_N}-\theta_{m_1',m_2,\ldots,m_N}$ and $\theta_{m_2}-\theta_{m_2'}=\theta_{m_1,\ldots,m_N}-\theta_{m_1,m_2',m_3\ldots,m_N}$. Moreover, since $\theta_{m_1,\ldots,m_N}$ is the dominant phase in R, both of these phase differences must be *positive* in the interior of R if the minors $A(m_1',m_2,\ldots,m_N)$ and $A(m_1,m_2',m_3\ldots,m_N)$ are nonzero. Hence, we must conclude that $\theta_{m_1}-\theta_{m_1'}$ and $\theta_{m_2}-\theta_{m_2'}$ cannot have opposite signs unless one or both of the phase combinations $\theta_{m_1',m_2,\ldots,m_N}$ and $\theta_{m_1,m_2',m_3\ldots,m_N}$ is absent from the tau function. This requires that either $A(m_1',m_2,\ldots,m_N)$ or $A(m_1,m_2',m_3\ldots,m_N)$ must be zero. A similar argument applied to the phase differences $\theta_{m_1}-\theta_{m_2'}$ and $\theta_{m_2}-\theta_{m_1'}$ leads to the conclusion that one or both of the minors $A(m_2',m_2,\ldots,m_N)$ and $A(m_1,m_1',m_3\ldots,m_N)$ must vanish. However, from the Plücker relations among the $N\times N$ minors of A we have

$$A(m_1,m_2\ldots,m_N)A(m_1',m_2',\ldots m_N)=A(m_1,m_2',m_3,\ldots m_N)A(m_1',m_2,\ldots,m_N) - A(m_1,m_1',m_3\ldots,m_N)A(m_2',m_2,\ldots,m_N).$$

It follows from above that either $A(m_1,\ldots,m_N)=0$ or $A(m_1',m_2',m_3,\ldots,m_N)=0$. But this is impossible since by assumption both minors on the left-hand-side are associated with dominant phase combinations. Thus, they are both nonzero. Hence we have reached a contradiction which implies that as $y\to\pm\infty$, phase transitions where more than one phase changes simultaneously across adjacent dominant phase regions, are impossible.

APPENDIX B: PROOF OF THEOREM 3.6

First we need to establish the following Lemma that will be useful in proving the theorem.

Lemma B.1: If P_{ij} is the submatrix defined in Eq. (3.2) *and e_n labels the nth pivot column of an irreducible coefficient matrix A, then $N-1\leqslant \mathrm{rank}(P_{e_ne_n+1})\leqslant N$, $\forall n=1,\ldots,N$.*

Proof: Recall that the pivot indices are ordered as $1=e_1<e_2<\cdots<e_N<M$ for an irreducible matrix A. Then it follows from Definition 2.2(ii) that, corresponding to each pivot column $A[e_n]$ of an irreducible matrix A, there exists at least one nonpivot column $A[j_*]$, with $j_*>e_n$, that has a nonzero entry in its nth row. Hence we have $A(e_1,\ldots,e_{n-1},j_*,e_{n+1},\ldots,e_N)\neq 0$. This implies that the matrix $A[1,\ldots,e_n-1,e_n+1,\ldots,M]=(P_{e_ne_n+1}|A[e_n+1])$ which contains the columns $A[e_1],\ldots,A[e_{n-1}],A[j_*],A[e_{n+1}],\ldots,A[e_N]$, has rank N. Thus, the rank of $P_{e_ne_n+1}$ is *at least* $N-1$, and this yields the desired result. □

We are now ready to prove Theorem 3.6. We prove part (i) here; the proof of part (ii) follows similar steps. The proof of part (i) is divided in two parts. First we show that for each pivot index e_n, $n=1,\ldots,N$, there exists an index $j_n>e_n$ with the necessary and sufficient properties for $[e_n,j_n]$ to identify an asymptotic line soliton as $y\to\infty$; then we prove that such a j_n is unique.

Existence: The proof is constructive. For each pivot index e_n, and for any $j>e_n$, we consider the rank of the matrix $P_{e_n,j}=A[1,2,\ldots,e_n-1,j+1,\ldots,M]$ starting from $j=e_n+1$. When $j=e_n+1$ we have $P_{e_n,j}=P_{e_n,e_n+1}$, and therefore $N-1\leqslant\mathrm{rank}(P_{e_n,e_n+1})\leqslant N$ from Lemma B.1. If $\mathrm{rank}(P_{e_n,e_n+1})=N$, then Lemma 3.4(i) implies that the pair $[e_n,e_n+1]$ *does not* identify an asymptotic line soliton as $y\to\infty$. In this case, we increment the value of j successively from e_n+1, until $\mathrm{rank}(P_{e_n,j})$ decreases from N to $N-1$. Note that a value of j such that $\mathrm{rank}(P_{e_n,j})=N-1$ always exists because if $j=M$, then $\mathrm{rank}(P_{e_n,M})=\mathrm{rank}(A[1,\ldots,e_n-1])=n-1\leqslant N-1$, since A is in RREF. Suppose $j=j_*$ is the smallest index such that $\mathrm{rank}(P_{e_n,j_*})=N-1$ and $\mathrm{rank}(P_{e_n,j_*}|A[j_*])=N$. We next check the rank of $\mathrm{rank}(P_{e_n,j_*}|A[e_n])$. Since $\mathrm{rank}(P_{e_n,j_*})=N-1$, two cases are possible: either (a) $\mathrm{rank}(P_{e_n,j_*}|A[e_n])=N$ or (b) $\mathrm{rank}(P_{e_n,j_*}|A[e_n])=N-1$. We discuss these two cases separately.

(a) Suppose that $\mathrm{rank}(P_{e_n,j_*}|A[e_n])=N$. By construction we have $\mathrm{rank}(P_{e_n,j_*}|A[j_*])=N$, and since $N=\mathrm{rank}(A)$ one also has $\mathrm{rank}(P_{i_n,j_*}|A[e_n,j_*])=N$. In this case we set $j_*=j_n$. It follows from Lemma 3.4 that the pair $[e_n,j_n]$ satisfies the necessary rank conditions to identify an asymptotic line soliton as $y\to\infty$. Next we show that these rank conditions are also *sufficient* in order to determine a pair of dominant phase combinations in the tau function corresponding to the single-phase transition $e_n\to j_n$. Since $\mathrm{rank}(P_{e_n,j_n})=N-1$, it is possible to choose $N-1$ linearly independent columns $A[p_1],\ldots,A[p_{N-1}]$ from the matrix P_{e_n,j_n} so that for all choices of linearly independent columns $A[l_1],\ldots,A[l_{N-1}]\in P_{e_n,j_n}$ one has $\theta_{p_1,\ldots,p_{N-1}}\geqslant\theta_{l_1,\ldots,l_{N-1}}$ as $y\to\infty$ along the transition line L_{e_n,j_n}. The existence of such a set is guaranteed because part (i) of the dominant phase condition 3.1 implies that, as $y\to\infty$ in the $[e_n,j_n]$ direction, the phases corresponding to the index set P_{e_n,j_n} are ordered as $\theta_1>\theta_2>\cdots>\theta_{e_n-1}$ and $\theta_{j_n+1}<\theta_{j_n+2}<\cdots<\theta_M$. Therefore, it is possible to select the top $N-1$ phases from the above two lists so that the corresponding columns are linearly independent. Furthermore, the conditions $\mathrm{rank}(P_{e_n,j_n}|A[e_n])=\mathrm{rank}(P_{e_n,j_n}|A[j_n])=N$ imply that the minors $A(e_n,p_1,\ldots,p_{N-1})$ and $A(j_n,p_1,\ldots,p_{N-1})$ are both nonzero, and thus $\theta_{e_n,p_1,\ldots,p_{N-1}}$ and $\theta_{j_n,p_1,\ldots,p_{N-1}}$ form a dominant pair of phase combinations as $y\to\infty$ along the direction of L_{e_n,j_n}.

(b) Suppose that $\mathrm{rank}(P_{e_n,j_*}|A[e_n])=N-1$. Note that this is possible only for $n<N$, because when $n=N$ the submatrix $P_{e_N,j}$ for any $j>e_N$ contains the pivot columns $A[e_1],\ldots,A[e_{N-1}]$. Hence, $\mathrm{rank}(P_{e_N,j})=N-1$ and $\mathrm{rank}(P_{e_N,j}|A[e_N])=N$. Consequently, $n=N$ always belongs to case (a) above and not to case (b). So we consider only the case $n<N$ below.

Since $\mathrm{rank}(P_{e_n,j_*})=\mathrm{rank}(P_{e_n,j_*}|A[e_n])=N-1$, this means that $A[e_n]\in\mathrm{span}(P_{e_n,j_*})$. However, since $A[e_n]$ is a pivot column, it cannot be spanned only by its preceding columns $A[1],\ldots,A[e_n-1]$. Hence the spanning set of $A[e_n]$ from P_{e_n,j_*} must contain at least one column from $A[j_*+1],\ldots,A[M]$. In this case we continue incrementing the value of j starting from j_* until the pivot column $A[e_n]$ is no longer in the span of the columns of the resulting submatrix $P_{e_n,j}$. Let j_n be the smallest index such that $A[e_n]$ is spanned by the columns of the submatrix $P_{e_n,j_n}|A[j_n]$ but *not* by those of P_{e_n,j_n}. Then, by construction we have $\mathrm{rank}(P_{e_n,j_n})=:r<N-1$, and $\mathrm{rank}(P_{e_n,j_n}|A[e_n])=\mathrm{rank}(P_{e_n,j_n}|A[j_n])=\mathrm{rank}(P_{e_n,j_n}|A[e_n,j_n])=r+1$. The rank conditions in Lemma 3.4(i) are once again satisfied for the index pair $[e_n,j_n]$ thus found. The sufficiency of these conditions can then be established by following similar steps as in case (a). Namely, it is possible to choose a set of linearly independent vectors $A[l_1],\ldots,A[l_r]\in P_{e_n,j_n}$ and extend this set to a basis of $\mathbb{R}^N$ as follows: $\{A[e_n],A[l_1],\ldots,A[l_r],A[m_1],\ldots,A[m_s]\}$, where $A[m_1],\ldots,A[m_s]\in Q_{e_n,j_n}$ and $r+s=N-1$. We then have $A(e_n,l_1,\ldots,l_r,m_1,\ldots,m_s)\neq0$, which also implies $A(j_n,l_1,\ldots,l_r,m_1,\ldots,m_s)\neq0$ since $A[e_n]\in\mathrm{span}(P_{e_n,j_n}|A[j_n])$. As in case (a), we can now maximize the phase combinations over all such sets $\{l_1,\ldots,l_r,m_1,\ldots,m_s\}$, and find a set of indices $\{p_1,\ldots,p_r,q_1,\ldots,q_s\}$ such that $\theta_{e_n,p_1,\ldots,p_r,q_1,\ldots,q_s}$ and $\theta_{j_n,p_1,\ldots,p_r,q_1,\ldots,q_s}$ form a dominant pair of phase combinations as $y\to\infty$ along the direction of L_{e_n,j_n}. Summarizing, we have shown that for each pivot index $e_n, n=1,2,\ldots,N$, there exists at least one asymptotic line soliton $[e_n,j_n]$ with $j_n>e_n$ as $y\to\infty$. Next we prove uniqueness.

Uniqueness: Suppose that $[e_n,j_n]$ and $[e_n,j_n']$ are two asymptotic line solitons identified by the same pivot index e_n as $y\to\infty$. Without loss of generality, assume that $j_n'>j_n$, and consider the line soliton $[e_n,j_n']$. Lemma 3.4(i) implies that $\mathrm{rank}(P_{e_n,j_n'}|A[j_n'])=\mathrm{rank}(P_{e_n,j_n}|A[e_n,j_n'])$. Hence the pivot column $A[e_n]$ is spanned by the columns of the submatrix $(P_{e_n,j_n'}|A[j_n'])$. But by assumption we have $(P_{e_n,j_n'}|A[j_n'])\subseteq P_{e_n,j_n}$, since $j_n'>j_n$. Hence $A[e_n]$ is also spanned by the columns of P_{e_n,j_n}. This however implies that $\mathrm{rank}(P_{e_n,j_n}|A[e_n])=\mathrm{rank}(P_{e_n,j_n})$, which contradicts the necessary rank conditions in Lemma 3.4(i) for $[e_n, j_n]$ to identify an asymptotic line soliton as $y \to \infty$. Therefore we must have $j_n = j_{n'}$. Thus, it is not possible to have two distinct asymptotic line solitons as $y \to \infty$ associated with the same pivot index e_n. Part (i) of Theorem 3.6 is now proved.

APPENDIX C: EQUIVALENCE CLASSES AND DUALITY OF SOLUTIONS

In this appendix, we investigate the relationship between two classes of KPII multisoliton solutions with complementary sets of asymptotic line solitons. Note that the KPII equation (1.1) is invariant under the inversion symmetry $(x,y,t) \to (-x,-y,-t)$. As a result, if $u(x,y,t)$ is an $(M-N,N)$-soliton solution of KPII with $M-N$ incoming and N outgoing line solitons, then $u(-x,-y,-t)$ is a $(N,M-N)$-soliton solution of KPII where the numbers of incoming and outgoing line solitons are reversed. It follows from Theorem 3.6 that the solution $u(-x,-y,-t)$ should correspond to *some* tau function $\tau_{M-N,M}(x,y,t)$ associated with an $M-N \times M$ coefficient matrix whose pivot and nonpivot columns uniquely identify the asymptotic line solitons of $u(-x,-y,-t)$.

Before proceeding further, we introduce the notion of an equivalence class which plays an important role in subsequent discussions. Let Θ as in Definition 2.3 denote the set of all phase combinations $\theta_{m_1,\ldots,m_N}$ which appear with nonvanishing coefficients in the tau function $\tau(x,y,t)$ of Eq. (2.2).

Definition C.2 (Equivalence class): Two tau functions are defined to be in the same equivalence class if (up to an overall exponential phase factor) the set Θ is the same for both. The set of (N_-, N_+)-soliton solutions of KPII generated by an equivalence class of tau functions defines an equivalence class of solutions.

It is clear from the above definition that tau functions in a given equivalence class can be viewed as positive-definite sums of the *same* exponential phase combinations but with different sets of coefficients. They are parametrized by the same set of phase parameters $k_1, \ldots, k_M$, but the constants θ_{m0} in the phase θ_m are different. Moreover, the irreducible coefficient matrices associated with the tau functions have exactly the same sets of vanishing and nonvanishing minors, but the magnitudes of the nonvanishing minors are different for different matrices. Thus, it is evident from the remarks following Corollary 2.6 in Sec. II that the asymptotic line solitons of each solution in an equivalence class arise from the *same* $i \to j$ single phase transition, and are therefore labeled by the same index pair $[i,j]$. Theorem 3.6 then implies that the coefficient matrices associated with the tau functions in the same equivalence class have identical sets of pivot and nonpivot indices labeling the asymptotic line solitons as $y \to \infty$ and as $y \to -\infty$, respectively. Thus, solutions in the same equivalence class can differ only in the position of each asymptotic line solitons and in the location of each interaction vertex. As a result, any (N_-, N_+)-soliton solution of KPII can be transformed into any other solution in the same equivalence class by spatio-temporal translations of the individual asymptotic line solitons. We refer to the two tau functions $\tau_{N,M}(x,y,t)$ and $\tau_{M-N,M}(x,y,t)$ as *dual* to each other if the solution $u(-x,-y,-t)$ obtained from the function $\tau_{N,M}(-x,-y,-t)$ and the solution generated by $\tau_{M-N,M}(x,y,t)$ are in the same equivalence class. Note that $\tau_{N,M}(-x,-y,-t)$ is not exactly a tau function according to Eq. (2.2), but it is possible to construct from it a dual tau function $\tau_{M-N,M}(x,y,t)$ whose coefficient matrix B can be derived from the coefficient matrix A associated with the tau function $\tau_{N,M}(x,y,t)$. We describe this construction below.

Since A is of rank N and in RRFF, it can be expressed as $A = [I_N, G]P$, where I_N is the $N \times N$ identify matrix of pivot columns, G is the $N \times (M-N)$ matrix of nonpivot columns, and P denotes the $M \times M$ permutation matrix of M columns of A. We augment A with $M-N$ additional rows to form the invertible $M \times M$ matrix,

$$S = \begin{pmatrix} I_N & G \\ O & I_{M-N} \end{pmatrix} P, \tag{C1}$$

where O is the $(M-N) \times N$ zero matrix and I_{M-N} is the $(M-N) \times (M-N)$ identify matrix. Let A' be the $(M-N) \times M$ matrix obtained by selecting the last $M-N$ rows of $(S^{-1})^T$. The rank of A' is $M-N$, and the following complementarity relation holds between A and A'.

Lemma C.3: The pivot columns of A' are labeled by exactly the same set of indices which label the nonpivot columns of A, and vice versa. Moreover, if A is irreducible then A' is also irreducible.

Proof: From Eq. (C1) and the fact that $P^{-1}=P^T$ for a permutation matrix, we obtain

$$(S^{-1})^T = \begin{pmatrix} I_N & O^T \\ -G^T & I_{M-N} \end{pmatrix} P, \tag{C2}$$

which implies that $A'=[-G^T, I_{M-N}]P$. Then (by performing row reduction in *reverse* order), the pivot columns of $A'P^{-1}$ can be identified with its last $M-N$ columns which correspond to the nonpivot columns of $AP^{-1}=[I_N, G]$, and vice versa. The same correspondence between pivot and nonpivot columns also holds for A and A' because the columns of both matrices are permuted by the same matrix P^{-1}. This proves the first part of the lemma.

To establish that A' is irreducible, note first from Definition 2.2 that the permutation of columns preserves irreducibility of a matrix. Since A is irreducible, Definition 2.2 implies that all rows or columns of G and G^T are nonzero. Therefore the matrix $A'P^{-1}=[-G^T, I_{M-N}]$, and hence A', are both irreducible. □

Note that A' is *not* in the canonical RREF, but can be set in RREF by a GL $(N,\mathbb{R})$ transformation. Next, we define the matrix B which is also of rank $M-N$ and irreducible like A', and whose columns are obtained from A' as

$$B[m] = (-1)^m A'[m], \quad m = 1, \ldots, M. \tag{C3}$$

Then using Eqs. (C2) and (C3), the minors of A can be expressed in terms of the complementary minors of B via (see, e.g., Ref. 6, p. 21)

$$A(l_1, \ldots, l_N) = (-1)^\sigma \det(P) B(m_1, \ldots, m_{M-N}), \tag{C4}$$

where $\sigma = M(M+1)/2 + N(N+1)/2$, and where the indices $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_{M-N}$ are the complement of $1 \leqslant l_1 \leqslant l_2 \leqslant \cdots l_N$ in $\{1, 2, \ldots, M\}$. The matrix B plays the role of a coefficient matrix for the dual tau function as given by the following lemma.

Lemma C.4 (Duality): If $\tau_{N,M}(x,y,t)$ is the tau function associated with an irreducible $N \times M$ coefficient matrix A, then the matrix B defined via Eq. (C3) *generates a tau function $\tau_{M-N,M}(x,y,t)$ that is dual to $\tau_{N,M}(x,y,t)$.*

Proof: Without loss of generality we choose the tau function $\tau_{N,M}(x,y,t)$ associated with the given equivalence class of solutions such that $\theta_{m,0}=0$ for all $m=1, \ldots, M$ in Eq. (2.2). Then, using Eq. (C4) we can express the tau function as

$$\tau_{N,M}(-x,-y,-t) = (-1)^\sigma \det(P) e^{-\theta_{1,\ldots,M}} \tau'(x,y,t), \tag{C5a}$$

where

$$\tau'(x,y,t) = \sum_{1 \leqslant m_1 < m_2 < \cdots < m_{M-N} \leqslant M} V(l_1, \ldots, l_N) B(m_1, \ldots, m_{M-N}) e^{\theta_{m_1,\ldots,m_{M-N}}}, \tag{C5b}$$

with $V(l_1, \ldots, l_N)$ denoting the Van der Monde determinant as in Eq. (2.2) and where the sum is now taken over the complementary indices $m_1, \ldots, m_{M-N}$ instead of $l_1, \ldots, l_N$. [The number of terms in the sum remains the same since $\binom{M}{N} = \binom{M}{M-N}$.] It is clear from Eq. (1.2) that both $\tau_{N,M}(-x,-y,-t)$ and $\tau'(x,y,t)$ in Eq. (C5a) generate the same solution $u(x,y,t)$ of KPII although $\tau'(x,y,t)$ itself is *not* a tau function. Note that all the nonzero minors of B have the same sign, which is determined by the sign of $(-1)^\sigma \det(P) > 0$. Thus, by replacing each Van der Monde coefficient $V(l_1, \ldots, l_N)$ by $V(m_1, \ldots, m_{M-N})$ in Eq. (C5b), it is possible to obtain from $\tau'(x,y,t)$, a new tau function $\tau_{M-N,M}(x,y,t)$ associated with the irreducible coefficient matrix B. Since both $\tau'(x,y,t)$ and $\tau_{M-N,M}(x,y,t)$ are sign-definite sums of the *same* exponential phase combinations, they generate solutions that are in the same equivalence class. Therefore, the tau function $\tau_{M-N,M}(x,y,t)$ constructed via the above prescription is dual to the tau function $\tau_{N,M}(x,y,t)$. This yields the desired result. □

By applying Lemma C.4, it is easy to show that part (i) of Theorem 3.6 implies part (ii) and vice versa. For example, by applying part (i) of Theorem 3.6 to the tau function $\tau_{M-N,M}(x,y,t)$ in Lemma C.4 one can conclude that as $y\to\infty$, $\tau_{M-N,M}(x,y,t)$ generate a solution with exactly $M-N$ line solitons, identified by the *pivot* indices $g_1,\ldots,g_{M-N}$ of the associated coefficient matrix B. One should however note that since the ordering of the pivot and nonpivot columns of B is reversed with respect to that of A, if $[i,j]$ with $i<j$ labels an asymptotic line soliton generated by $\tau_{M-N,M}(x,y,t)$ as $y\to\infty$, then j is the pivot index, not i. The solution generated by $\tau_{M-N,M}(x,y,t)$ is in the same equivalence class as $u(-x,-y,-t)$ because $\tau_{M-N,M}(x,y,t)$ is dual to $\tau_{M,N}(x,y,t)$. Consequently, as $y\to\infty$, $u(-x,-y,-t)$ has $M-N$ asymptotic line solitons labeled by exactly the same indices $g_1,\ldots,g_{M-N}$. Then as $y\to-\infty$, it follows that the solution $u(x,y,t)$ generated by $\tau_{N,M}(x,y,t)$ also has $M-N$ asymptotic line solitons. Furthermore, these line solitons are labeled by the same indices $g_1,\ldots,g_{M-N}$ which are the nonpivot indices of the coefficient matrix A of the tau function $\tau_{N,M}(x,y,t)$. This proves part (ii) of Theorem 3.6. Similarly, one could also prove part (i) of the Theorem using part (ii) and Lemma C.4.

Another consequence of Lemma C.4 is that the dominant pairs of phase combinations for the asymptotic line solitons of $\tau_{M-N,N}(x,y,t)$ as $y\to\infty$ are the complement of those for the asymptotic line solitons of the dual tau function $\tau_{N,M}(x,y,t)$ as $y\to-\infty$. Thus, if the dominant pair of phase combinations for $\tau_{M-N,M}(x,y,t)$ as $y\to\infty$ along the line $L_{i,j}$ is given by $\theta_{i,m_2,\ldots,m_{M-N}}$ and $\theta_{j,m_2,\ldots,m_{M-N}}$, the dominant phase combinations for $\tau_{N,M}(x,y,t)$ as $y\to-\infty$ along $L_{i,j}$ are $\theta_{i,l_2,\ldots,l_N}$ and $\theta_{j,l_2,\ldots,l_N}$, where the index set $\{l_2,\ldots,l_N\}$ is the complement of $\{i,j,m_2,\ldots,m_{M-N}\}$ in $\{1,\ldots,M\}$.

A particularly interesting subclass of (N_-,N_+)-soliton solutions is obtained by requiring the solutions $u(x,y,t)$ and $u(-x,-y,-t)$ to be in the same equivalence class. Thus, this class of solutions is generated tau functions which can be regarded as "self-dual." The corresponding solutions are the elastic N-soliton solutions of KPII, for which the amplitudes and directions of the N incoming line solitons coincide with those of the N outgoing line solitons. Hence, the set of incoming line solitons and the set of outgoing line solitons are both labeled by the same index pairs $\{[i_n,j_n]\}_{n=1}^{N}$. Clearly, in this case we have $N_+=N_-=N$ and $M=2N$. Some properties of the elastic N-soliton solution have been studied in Ref. 12, and we will discuss several other properties in a future presentation. Here we only mention one result which is a direct consequence of Theorem 3.6 and the above discussions:

Corollary C.5: A necessary condition for a set of index pairs $\{[i_n,j_n]\}_{n=1}^{N}$ *to describe an elastic N-soliton solution is that the indices* $i_1,\ldots,i_N$ *and* $j_1,\ldots,j_N$ *form a disjoint partition of the integers* $1,\ldots,2N$.

Proof: From part (i) of Theorem 3.6, the indices $i_1,\ldots,i_N$ for the N asymptotic line solitons as $y\to\infty$ label the pivot columns of A, and from part (ii) of Theorem 3.6, the indices $j_1,\ldots,j_N$ for the N asymptotic line solitons as $y\to-\infty$ label the nonpivot columns of A. In order for the N asymptotic line solitons as $y\to-\infty$ to be the same as those as $y\to\infty$, however, the index pairs $[i_n,j_n]$ must obviously be the same as $y\to\pm\infty$ for all $n=1,\ldots,N$. But the sets of pivot and nonpivot indices of any matrix are of course disjoint; hence the desired result. □

Note however that the condition in Corollary C.5 is necessary but *not* sufficient to describe an elastic N-soliton solution. It is indeed possible to have N-soliton solutions where the index pairs labeling the asymptotic line solitons as $y\to\infty$ and as $y\to-\infty$ form two different disjoint partition of integers $\{1,2,\ldots,2N\}$. Such N-soliton solutions are *not* elastic. See, for example, the 2-soliton solution in Fig. 4(e).

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