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Initial-boundary-value problems for discrete linear evolution equations

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We present a transform method for solving initial-boundary-value problems (IBVPs) for linear semidiscrete (differential-difference) and fully discrete (difference-difference) evolution equations. The method is the discrete analogue of the one recently proposed by A. S. Fokas to solve IBVPs for evolution linear partial differential equations. We show that any discrete linear evolution equation can be written as the compatibility condition of a discrete Lax pair, namely, an overdetermined linear system of equations containing a spectral parameter. As in the continuum case, the method employs the simultaneous spectral analysis of both parts of the Lax pair, the symmetries of the evolution equation and a relation, called the global algebraic relation, that couples all known and unknown boundary values. The method applies for differential-difference equations in one lattice variable as well as for multi-dimensional and fully discrete evolution equations. We demonstrate the method by discussing explicitly several examples.

Keywords: initial-boundary-value problems; discrete linear evolution equations.

1. Introduction

Initial-boundary-value problems (IBVPs) are of interest both theoretically and in applications. In particular, the solution of IBVPs for integrable non-linear partial differential equations (PDEs) has been an ongoing problem for over 30 years. Several approaches have been proposed for solving IBVPs for integrable non-linear PDEs on semiinfinite spatial domains (e.g., see Ablowitz & Segur, 1975; Bikbaev & Tarasov, 1991; Biondini & Hwang, 2009; Degasperis *et al.*, 2001, 2002; Khabibullin, 1991; Sabatier, 2006; Skylanin, 1987; Tarasov, 1991 and references therein). In particular, a transform method was recently developed by Fokas and collaborators (see Fokas, 1997, 2000; Fokas & Gelfand, 1994; Fokas *et al.*, 2005 and references therein). The method uses three key ingredients: (i) simultaneous spectral analysis of the Lax pair of the PDE in question, (ii) the global algebraic relation that couples all known and unknown boundary values and (iii) the symmetries of the associated dispersion relation. Interestingly, the method also provides a new and powerful approach to solve IBVPs for 'linear' PDEs in one and several space dimensions (see Fokas, 2002, 2005; Fokas & Pelloni, 1998, 2001; Treharne & Fokas, 2004 and references therein). At the same time, it is generally accepted that discrete problems are often more difficult and than continuum ones and also in some sense more fundamental (e.g., see Ablowitz, 1977; Ablowitz *et al.*, 2000; Ablowitz & Ladik, 1975, 1976; Ablowitz *et al.*, 2003; Biondini & Hwang, 2008; Flaschka, 1974a,b; Habibullin, 1995; Hirota *et al.*, 1988a,b; Maruno & Biondini, 2004; Ragnisco & Santini, 1990; Toda, 1975). The purpose of this work is to show that an approach similar to the one mentioned above for linear PDEs can also be used to solve IBVPs for a general class of discrete linear evolution equations (DLEEs). The method is quite general, and it works for many IBVPs for which Fourier or Laplace methods are not applicable. Even when such methods can be used, the present method has several advantages, in that it provides a representation of the solution which is convenient for both asymptotic analysis and numerical evaluation.

The work is organized as follows. First, we describe the general method for semidiscrete (i.e. differential-difference) evolution equations in 1 + 1 dimensions, namely in one discrete lattice variable and one continuous time variable. We then solve explicitly several examples to illustrate the method. Next, we discuss the extensions of the method to systems of equations, higher-order problems and forced equations, and we present the extensions of the method to linear semidiscrete evolution equation in two lattice variables and to fully discrete (difference-difference) evolution equations. Finally, we conclude this work with some final remarks.

2. Differential-difference equations in one lattice variable

Consider an arbitrary linear discrete evolution equation in one lattice variable, namely

$$\mathrm{i}\dot{q}_n = \omega(\mathrm{e}^{\partial})q_n, \tag{2.1}$$

for a sequence of functions $\{q_n(t)\}_{n\in\mathbb{N}}$ with $q_n\colon \mathbb{R} \to \mathbb{C}$, where e^{∂} is the shift operator (namely $\mathrm{e}^{\partial}q_n = q_{n+1}$), and the dot denotes differentiation with respect to time ($\dot{f} = \mathrm{d}f/\mathrm{d}t$), and $\omega(z)$ is an arbitrary discrete dispersion relation, namely

$$\omega(z) = \sum_{j=-J_1}^{J_2} \omega_j z^j, \tag{2.2}$$

where J_1 and J_2 are arbitrary non-negative integers. Equation (2.1) is the discrete analogue of a linear evolution PDE. Indeed, when $n \in \mathbb{Z}$, (2.1) admits the solution $q_n(t) = z^n\,\mathrm{e}^{-\mathrm{i}\omega(z)t}$, which is the analogue of the plane-wave solutions $\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}$ for linear PDEs. Note that, in order for the IBVP for (2.1) to be well posed on $(n,t) \in \mathbb{N} \times \mathbb{R}_0^+$, one must assign not only an initial condition $q_n(0)$, $\forall\, n \in \mathbb{N}$ but also J_1 boundary conditions (BCs) $q_{-J_1+1}(t), \ldots, q_0(t)$. Indeed, these conditions are necessary and sufficient to ensure that (2.1) can be evaluated $\forall\, n \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}_0^+$. Below we first present the Lax pair formulation of (2.1) and we derive a formal expression for the solution. We then discuss the issue of the unknown boundary data and the symmetries of the equation. Finally, we combine those results to obtain the solution of the IBVP. Section 4 will illustrate the method with various examples.

2.1 *Lax pair and compatibility form*

Equation (2.1) can be written via a discrete Lax pair, i.e. as the compatibility relation of the overdetermined linear system

$$\Phi_{n+1} - z\Phi_n = q_n, \quad \dot{\Phi}_n + \mathrm{i}\omega(z)\Phi_n = X_n, \tag{2.3}$$

where $X_n(z,t)$ is given by

$$X_n(z,t) = -\mathrm{i}\left[\frac{\omega(z)-\omega(\xi)}{z-\xi}\right]_{\xi=\mathrm{e}^{\partial}} q_n(t). \tag{2.4}$$

That is, requiring that $\partial_t(\Phi_{n+1}) = \mathrm{e}^{\partial}(\dot{\Phi}_n)$ implies that $q_n(t)$ satisfies (2.1). Equation (2.3) is a generalization of the Lax pair for the discrete linear Schrdinger (DLS) equation obtained in Biondini &

Hwang (2008) by taking the linear limit of the Lax pair for the Ablowitz–Ladik system. Note that the difference $\omega(z) - \omega(s)$ is always divisible by $z - s$. Thus, the Laurent series of $X_n(z,t)$ as a function of ξ always truncates. In fact,

$$X_n(z,t) = \mathrm{i} \sum_{j=-J_1}^{J_2-1} b_j(z) q_{n+j}(t), \tag{2.5a}$$

where

$$b_j(z) = -\sum_{m=j+1}^{J_2} \omega_m z^{m-j-1}, \quad j = 0, \ldots, J_2 - 1, \tag{2.5b}$$

and

$$b_j(z) = \sum_{m=-j}^{J_1} \omega_{-m} z^{-m-j-1}, \quad j = -J_1, \ldots, -1. \tag{2.5c}$$

The solution of the IBVP can be obtained by performing spectral analysis of the Lax pair. (Indeed, this was the method used in Biondini & Hwang, 2008 because it can be non-linearized.) For linear problems, however, a simplified approach is possible. For this purpose, it is useful to rewrite the Lax pair (2.3) by introducing $\Psi_n(z,t) = z^{-n}\,\mathrm{e}^{\mathrm{i}\omega(z)t}\Phi_n(z,t)$. The modified eigenfunction $\Psi_n(z,t)$ satisfies a simpler Lax pair in which the homogeneous part is trivial:

$$\Psi_{n+1} = z^{-n}\,\mathrm{e}^{\mathrm{i}\omega(z)t} q_n, \quad \dot{\Psi}_n = z^{-n}\,\mathrm{e}^{\mathrm{i}\omega(z)t}(zX_n - z^2 X_{n-1}). \tag{2.6}$$

The compatibility condition of (2.6), which also yields (2.1), can be written as

$$\partial_t(z^{-n}\,\mathrm{e}^{\mathrm{i}\omega(z)t} q_n) = z^{-n}\,\mathrm{e}^{\mathrm{i}\omega(z)t}(X_{n+1} - zX_n).$$

The above condition can be written more conveniently as

$$\partial_t(z^{-n}\,\mathrm{e}^{\mathrm{i}\omega(z)t} q_n) = \Delta(z^{-n+1}\,\mathrm{e}^{\mathrm{i}\omega(z)t} X_n), \tag{2.7}$$

where $\Delta Q_n = Q_{n+1} - Q_n$ is the finite-difference operator. Equation (2.7) is the discrete analogue of the closure condition for a differential 1-form that arises in the continuum case (see Fokas, 2002) and provides the starting point for the solution of the IBVP.

2.2 *Global relation and reconstruction formula*

We now obtain an expression for the solution of (2.1). We introduce the spectral transforms of the initial condition and BC as

$$\hat{q}(z,t) = \sum_{n=1}^{\infty} q_n(t)/z^n, \quad \hat{g}_n(z,t) = \int_0^t \mathrm{e}^{\mathrm{i}\omega(z)t'} q_n(t')\,\mathrm{d}t', \tag{2.8a}$$

defined, respectively, for all $|z| \geqslant 1$ and for all $z \neq 0$, together with

$$\hat{X}_1(z,t) = z\int_0^t \mathrm{e}^{\mathrm{i}\omega(z)t'} X_1(z,t')\,\mathrm{d}t'. \tag{2.8b}$$

(Throughout this work, primes will not denote differentiation.) Henceforth, we require that $q_n(0) \in l^1(\mathbb{N})$ (the space of absolutely summable sequences). This ensures that $\hat{q}(z,t)$ is bounded $\forall\, z \in \mathbb{C}$ with $|z| \geqslant 1$ and is analytic for $|z| > 1$. Similarly, we require that the BCs are continuous functions of t, which ensures that $\hat{X}_1(z,t)$ is analytic everywhere in the punctured complex plane $\mathbb{C}^{(*)}$, and is bounded $\forall\, z \in \bar{D}$, where $D = \{z \in \mathbb{C}\colon \operatorname{Im}\omega(z) > 0\}$. (Throughout this work, we will use the notation $R^{(*)} = R - \{0\}$. As usual, the overbar denotes closure.) In what follows it will be convenient to decompose $D = D_+ \cup D_-$, where $D_\pm$ denotes the portions of D inside and outside the unit disk:

$$D_\pm = \{z \in \mathbb{C}\colon |z| \lessgtr 1 \wedge \operatorname{Im}\omega(z) > 0\}.$$

We now sum (2.7) from $n = 1$ to ∞, obtaining, for $|z| \geqslant 1$,

$$\partial_t(\mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(z,t)) = \sum_{n=1}^{\infty} \Delta(z^{-n+1}\,\mathrm{e}^{\mathrm{i}\omega(z)t} X_n) = -\mathrm{e}^{\mathrm{i}\omega(z)t} z X_1(z,t). \tag{2.9}$$

Integrating (2.9) from $t' = 0$ to $t' = t$ we then get, for $|z| \geqslant 1$,

$$\mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(z,t) = \hat{q}(z,0) - \hat{X}_1(z,t). \tag{2.10}$$

Equation (2.10) is the global algebraic relation, which combines all known and unknown initial and boundary data.

The inverse transform of $\hat{q}(z,t)$ is obtained by noting that the $q_n(t)$ are the Laurent coefficients of $\hat{q}(z,t)$, implying simply

$$q_n(t) = \frac{1}{2\pi\,\mathrm{i}} \int_{|z|=1} z^{n-1}\hat{q}(z,t)\mathrm{d}z, \quad \forall\, n \in \mathbb{N}.$$

Use of (2.10) then yields, $\forall\, n \in \mathbb{N}$ and$\forall\, t \in \mathbb{R}_0^+$,

$$q_n(t) = \frac{1}{2\pi\,\mathrm{i}} \int_{|z|=1} z^{n-1}\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{q}(z,0)\mathrm{d}z - \frac{1}{2\pi\,\mathrm{i}} \int_{|z|=1} z^{n-1}\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{X}_1(z,t)\mathrm{d}z. \tag{2.11}$$

Equation (2.11) allows one to obtain the solution of the IBVP in terms of the spectral data. Indeed, one can easily verify that the function $q_n(t)$ defined by the right-hand side of (2.11) solves the DLEE and satisfies the initial condition and the BC. The right-hand side of (2.11), however, involves both known and unknown boundary data via $\hat{X}_1(z,t)$, which depends on $q_{-J_1+1}(t), \ldots, q_{J_2}(t)$ via their spectral transforms (cf. (2.5a) and (2.8b)). Since only $q_{-J_1+1}(t), \ldots, q_0(t)$ are assigned as BCs, $q_1(t), \ldots, q_{J_2}(t)$ must be considered as unknowns. Thus, in order for the expression (2.12) to provide an effective solution of the IBVP, we must be able to express $\hat{X}_1(z,t)$ only in terms of known ones.

As we show below, the elimination of the unknown boundary data is made possible by using both the global relation and the symmetries of the differential-difference evolution (2.1). A key part of the method, however, is the use of contour deformation to move the integration contour for the second integral in (2.11) away from the unit circle. The integrand in the last term of (2.11) is analytic $\forall\, z \neq 0$ and continuous and bounded for $z \in \bar{D}$. Moreover, $\hat{q}(z,t) \to q_0(t)$ as $z \to \infty$ and $\hat{g}_n(z,t) \to 0$ as $z \to 0$ and $z \to \infty$ in D. Thus, we can deform that integration contour from

$|z| = 1$ to $z \in \partial D_+$, obtaining the reconstruction formula:

$$q_n(t) = \frac{1}{2\pi\mathrm{i}} \int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{q}(z,0)\mathrm{d}z - \frac{1}{2\pi\mathrm{i}} \int_{\partial D_+} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{X}_1(z,t)\mathrm{d}z,$$

$\forall\, n \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}_0^+$. We next show that, when $z \in D_+$, it is indeed possible to eliminate the unknown boundary data. When this is done, (2.12) then provides the solution of the IBVP in terms of the spectral functions.

2.3 *Symmetries*

The spectral functions $\hat{g}_n(z,t)$ (and with them $\hat{X}_1(z,t)$) are invariant under any transformation that leaves the dispersion relation (2.2) invariant; i.e. they are invariant under any map $z \mapsto \xi(z)$ such that $\omega(\xi(z)) = \omega(z)$. (Note that, under any such transformation, we have $\xi(D_\pm) = D_\pm$.) The equation $\omega(z) = \omega(\xi(z))$ has $J_1 + J_2 - 1$ non-trivial roots, of course, in addition to the trivial one $\xi = z$. Using these symmetries in the global relation will allow us to eliminate the unknown boundary data. To do so, however, one needs to identify which of the $J_1 + J_2 - 1$ non-trivial roots are useful for this purpose. In general, it is not possible to express these roots in closed form except in the simplest cases. One must therefore look at the asymptotic behaviour of these roots as $z \to 0$ and $z \to \infty$. (Note that $z = 0$ and $z = \infty$ are both images of $k = \infty$ under $z = \mathrm{e}^{\mathrm{i}kh}$, where h is the lattice spacing.)

As $z \to \infty$, we have $\omega(z) \sim \omega_{J_2} z^{J_2}$, and as $z \to 0$, it is $\omega(z) \sim \omega_{-J_1} z^{-J_1}$. Thus, as $z \to \infty$, D_- is asymptotically equivalent to $S^{(\infty)}$, and as $z \to 0$, D_+ is asymptotically equivalent to $S^{(0)}$, where

$$S^{(\infty)} = \bigcup_{j=0}^{J_2-1} S_j^{(\infty)}, \qquad S^{(0)} = \bigcup_{j=-J_1+1}^{0} S_j^{(0)},$$

where, for $j = 0, \ldots, J_2 - 1$ and for $j = -J_1 + 1, \ldots, 0$, respectively, it is

$$S_j^{(\infty)} = \{z \in \mathbb{C}\colon 2j\pi/J_2 - \arg\omega_{J_2}/J_2 < \arg z < (2j{+}1)\pi/J_2 - \arg\omega_{J_2}/J_2\},$$

$$S_j^{(0)} = \{z \in \mathbb{C}\colon (2j{-}1)\pi/J_1 + \arg\omega_{-J_1}/J_1 < \arg z < 2j\pi/J_1 + \arg\omega_{-J_1}/J_1\}.$$

To study the asymptotic behaviour of the symmetries, note that

$$\omega(\xi) - \omega(z) = \sum_{j=-J_1}^{J_2} \omega_j(\xi^j - z^j) = (\xi - z)Q(\xi, z)/\xi^{J_1} z^{J_1},$$

where, owing to (2.5a),

$$Q(\xi, z) = -z^{J_1} \sum_{j=0}^{J_1+J_2-1} b_{j-J_1}(z)\xi^j, \qquad (2.12)$$

and the coefficients $b_j(z)$ are as in (2.5c). Thus, $Q(\xi, z)$ is a polynomial of degree $J_1 + J_2 - 1$ in ξ. Its $J_1 + J_2 - 1$ roots, which we denote by $\xi_{-J_1+1}(z), \ldots, \xi_{J_2-1}(z)$, yield the non-trivial roots of the equation $\omega(\xi) = \omega(z)$. In Section 3, we compute the asymptotic behaviour of these non-trivial roots via a singular perturbation expansion. In particular, we show that, as $z \to 0$, the $J_1 + J_2 - 1$ non-trivial roots behave as follows:

$$\xi_n(z) \sim \begin{cases} e^{2\pi i n/J_1} z, & n = -J_1+1, \ldots, -1, \\ (\omega_{-J_1}/\omega_{J_2})^{1/J_2}\, e^{2\pi i n/J_2} z^{-J_1/J_2}, & n = 0, \ldots, J_2 - 1. \end{cases} \tag{2.13}$$

In particular, (2.13) implies that each of $\xi_0(z), \ldots, \xi_{J_2-1}(z)$ maps one of the J_1 sectors of $S_j^{(0)}$ (and therefore D_+) onto one of the J_2 sectors of $S_j^{(\infty)}$ (and therefore D_-). These roots are precisely those needed to eliminate the unknown boundary data. Using similar arguments, one can also show that, as $z \to \infty$, these roots behave as follows:

$$\xi_{\sigma_n}(z) \sim \begin{cases} (\omega_{-J_1}/\omega_{J_2})^{1/J_1}\, e^{2\pi i n/J_1} z^{-J_2/J_1}, & n = -J_1+1, \ldots, 0, \\ e^{2\pi i n/J_2} z, & n = 1, \ldots, J_2 - 1, \end{cases}$$

where $\sigma = (\sigma_1, \ldots, \sigma_{J_1+J_2-1})$ is a permutation of $-J_1+1, \ldots, J_2-1$. The behaviour of the roots as $z \to \infty$ and that as $z \to 0$ could of course be connected if desired using matched asymptotic expansions. This, however, is not necessary for our purposes.

2.4 *Elimination of the unknown boundary data*

The solution in (2.12) depends on $\hat{X}_1(z,t)$, which involves the J_2 unknown functions $q_1(t), \ldots, q_{J_2}(t)$ via these spectral transforms. Applying the transformations $z \to \xi_j(z)$, with $j = 0, \ldots, J_2 - 1$, in the discrete global relation (2.10), we obtain, $\forall\, z \in \bar{D}_+^*$, the J_2 algebraic equations:

$$\hat{X}_1(\xi_j(z), t) + e^{i\omega(z)t}\hat{q}(\xi_j(z), t) = \hat{q}(\xi_j(z), 0), \tag{2.14}$$

i.e. $\forall\, z \in \bar{D}_+^*$ and for $n = 0, \ldots, J_2 - 1$,

$$i\xi_j(z) \sum_{n=-J_1+1}^{J_2} b_n(\xi_j(z))\hat{g}_n(z,t) + e^{i\omega(z)t}\hat{q}(\xi_j(z), t) = \hat{q}(\xi_j(z), 0).$$

These can be regarded as a linear system of J_2 equations for the J_2 unknowns $\hat{g}_1(z,t), \ldots, \hat{g}_{J_2}(z,t)$. In fact, they are precisely these equations that allow us to solve for these unknown boundary data in terms of the given BCs $\hat{g}_{-J_1+1}(z,t), \ldots, \hat{g}_0(z,t)$. (Or, we can solve for any other combination of J_2 unknown boundary data with any J_1 given BCs.) Indeed, the determinant of the coefficient matrix M of the system (2.14) is

$$\det M = (\omega_{J_2})^{J_2} \prod_{0 \leqslant n < n' \leqslant J_2 - 1} (\xi_n(z) - \xi_{n'}(z)),$$

which is always non-zero as long as the roots $\xi_j(z)$ are distinct. Here, we assume that this condition is satisfied $\forall\, z \in \bar{D}_+$. (This condition is always satisfied in the limit $z \to 0$.) By substituting $\hat{g}_1(z,t), \ldots, \hat{g}_{J_2}(z,t)$ into (2.12), one then finally obtains the solution of IBVP (2.1) only in terms of known initial-boundary data.

A careful reader will obviously note that the term $e^{i\omega(z)t}\hat{q}(\xi_n(z), t)$ appearing in the left-hand side of (2.14) is (apart from the change $z \to \xi_n(z)$) just the transform of the solution we are trying to recover. Note, however, that for all $n \in \mathbb{N}$ this term gives zero contribution to the reconstruction formula (2.12) since term $z^{n-1}\,\mathrm{e}^{\mathrm{i}\omega(z)(t-t')}\hat{q}(\xi_n(z),t')$ is analytic and bounded in D_+ and therefore its integral over ∂D_+ is zero. This is exactly the same as to what happens for the method in the continuum limit (Fokas, 2002).

3. Asymptotic behaviour of the symmetries

Here, we briefly show how to obtain the asymptotic behaviour of the roots $\xi_j(z)$ of $\omega(\xi(z)) = \omega(z)$. As $z \to 0$, collecting the lowest powers of z in $Q(\xi, z)$, we obtain

$$Q(\xi,z) \sim \sum_{j=J_1}^{J_1+J_2-1} \omega_{j-J_1+1} z^{J_1}\xi^j - \sum_{j=0}^{J_1-1} \omega_{-J_1} z^{J_1-j+1}\xi^j. \tag{3.1}$$

We therefore look for the values of $\xi(z)$ that make the right-hand side of (3.1) zero. Two possible situations arise:

(i) If $z^{J_1-2}\xi \sim z^{J_1-1}$, it is $\xi = \mathrm{O}(z)$ as $z \to 0$. This is a consistent assumption because z^{J_1-1}, $z^{J_1-2}\xi, \ldots, z\xi^{J_1-2}, \xi^{J_1-1}$ are all $\mathrm{O}(z^{J_1-1})$, i.e. all these terms are the highest order terms in (3.1), and other terms $z^{J_1}\xi^{J_1}, z^{J_1}\xi^{J_1+1}, \ldots, z^{J_1}\xi^{J_1+J_2-2}, z^{J_1}\xi^{J_1+J_2-1}$ are negligible compared with $\mathrm{O}(z^{J_1-1})$. Letting $\xi(z) = zk$, for some non-zero constant k, and substituting $\xi(z)$ into (3.1) gives

$$\omega_{-J_1} z^{J_1-1} \sum_{j=0}^{J_1-1} k^j = z^{J_1} \sum_{j=1}^{J_2} \omega_j z^{J_1+j-1} k^{J_1+j-1},$$

or, equivalently,

$$\sum_{j=0}^{J_1-1} k^j = \sum_{j=1}^{J_2} (\omega_j/\omega_{-J_1}) z^{J_1+j} k^{J_1+j-1}. \tag{3.2}$$

As $z \to 0$, the right side of (3.2) goes to zero. Thus, we need $k^{J_1-1} + k^{J_1-2} + \cdots + k + 1 = 0$. We therefore have $J_1 - 1$ non-trivial roots $k_n = \mathrm{e}^{2\pi\mathrm{i}n/J_1}$ for $n = -J_1+1, \ldots, -1$. Thus, $\xi_n(z) \sim \mathrm{e}^{2\pi\mathrm{i}n/J_1} z$ for $n = -J_1+1, \ldots, -1$.

(ii) If $z^{J_1}\xi^{J_1+J_2-1} \sim \xi^{J_1-1}$, it is $\xi = \mathrm{O}(z^{-J_1/J_2})$ as $z \to 0$. This is also a consistent assumption because the other terms in the equation, namely, $z^{J_1}\xi^{J_1+J_2-2}, \ldots, z^{J_1}\xi^{J_1+1}, z\xi^{J_1-2}, \ldots, z^{J_1-1}$ are negligible compared with $z^{J_1}\xi^{J_1+J_2-1}$ and ξ^{J_1-1}. Letting $\xi(z) = z^{-J_2/J_1}k$, for some non-zero constant k, and substituting into (3.1), we get

$$\sum_{j=1}^{J_2} \omega_j z^{-J_1(J_1-J_2+j-1)/J_2} k^{J_1+j-1} - \omega_{-J_1}\left(\sum_{j=1}^{J_1} z^{J_1(J_2-j+1)/J_2-j} k^{j-1}\right) = 0.$$

As $z \to 0$, the leading order yields, after simplifications, $k^{J_1-1}(\omega_{J_2}k^{j_2} - \omega_{-J_1}) = 0$. It is then clear that we have J_2 non-trivial roots $k_n = (\omega_{-J_1}/\omega_{J_2})^{1/J_2}\,\mathrm{e}^{2\pi\mathrm{i}n/J_2}$ for $n = 0, \ldots, J_2-1$. Hence,

$$\xi_n(z) \sim (\omega_{-J_1}/\omega_{J_2})^{1/J_2}\,\mathrm{e}^{2\pi\mathrm{i}n/J_2} z^{-J_1/J_2}, \quad n = 0, \ldots, J_2-1.$$

Summarizing, as $z \to 0$, the $J_1 + J_2 - 1$ non-trivial roots behave as in (2.13). The asymptotic behaviour of the roots as $z \to \infty$ can be obtained in a similar way. A similar approach can also be used for multi-dimensional and fully discrete evolution equations.

4. Examples

We now illustrate the method by discussing various examples that are discretizations of physically significant PDEs. For simplicity, we set the lattice spacing constant h to 1 whenever this can be done without loss of generality by rescaling dependent and/or independent variables.

4.1 *Discrete one-directional wave equations*

We start by considering two different semidiscretizations of the one-directional wave equation. (Of course, both these could also be solved using more traditional methods. This will not be the case for many of the other examples, however.)

Consider first the forward-difference DLEE

$$\dot{q}_n = (q_{n+1} - q_n)/h.$$

We take $h = 1$, as mentioned earlier. The dispersion relation is $\omega(z) = \mathrm{i}(z-1)$ and it is $J_1 = 0$ and $J_2 = 1$, implying $X_n(z,t) = q_n(t)$ (including $\hat{X}_1(z,t) = \hat{g}_1(z,t)$). The domain D is the union of $D_+ = \emptyset$ and $D_- = \{z \in \mathbb{C}\colon \operatorname{Re} z > 1\}$ (see Fig. 1). The global relation yields, for $|z| \geqslant 1$,

$$\mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(z,t) + \hat{g}_1(z,t) = \hat{q}(z,0).$$

Hence, we have

$$q_n(t) = \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{q}(z,0)\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{g}_1(z,t)\mathrm{d}z. \tag{4.1}$$

Since $J_1 + J_2 = 1$, the equation $\omega(\xi) = \omega(z)$ has only one root, i.e. the trivial one $\xi = z$. Since $D_+ = \emptyset$, however, $z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{g}_1(z,t)$ is analytic $\forall\, z \neq 0$ and bounded for all $|z| \leqslant 1$, so the second integral in (4.1) vanishes $\forall\, n \in \mathbb{N}$. We therefore obtain the solution simply as

$$q_n(t) = \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{q}(z,0)\mathrm{d}t.$$

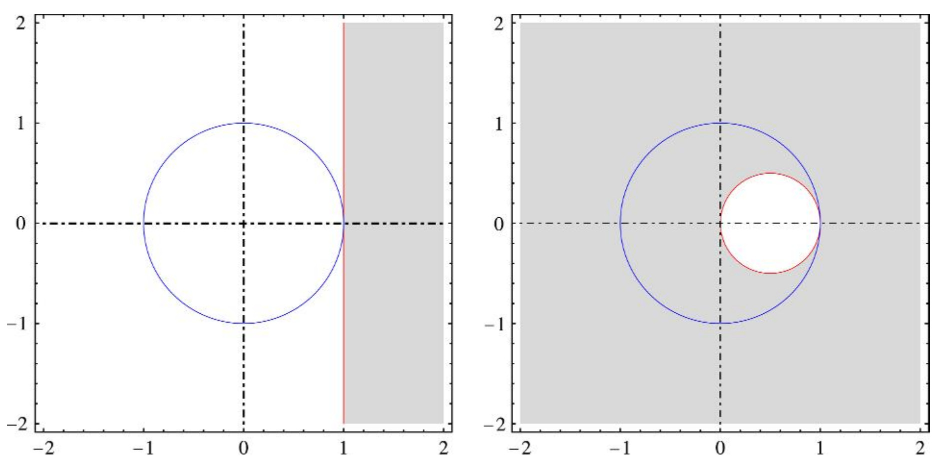

FIG. 1. The dispersion relation $\omega(z)$ for the discrete one-directional wave equation in the complex z-plane. Left: The forward-difference DLEE. Right: The backward-difference DLEE. Here, and in all subsequent figures, the shaded regions show the domains $D_\pm$ where $\operatorname{Im}\omega(z) > 0$.

That is, no BC is needed in this case, as expected.

Consider now the backward-difference DLEE

$$\dot{q}_n = (q_n - q_{n-1})/h.$$

Again, let $h = 1$. In this case, the dispersion relation is $\omega(z) = \mathrm{i}(1 - 1/z)$ and it is $J_1 = 1$ and $J_2 = 0$. We have $X_n(z,t) = q_{n-1}(t)/z$ and $\hat{X}_1(z,t) = \hat{g}_0(z,t)$. The domain D is now the union of $D_+ = \{z \in \mathbb{C}\colon |z| < 1 \wedge |z - 1/2| > 1/2\}$ and $D_- = \{z \in \mathbb{C}\colon |z| > 1\}$ (see Fig. 1). The global relation (2.10) is, for $|z| \geqslant 1$,

$$\mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(z,t) + \hat{g}_0(z,t) = \hat{q}(z,0)$$

from which we obtain the solution as

$$q_n(t) = \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{q}(z,0)\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{g}_0(z,t)\mathrm{d}z.$$

The boundary of D_+ includes all the unit circle, so it is not necessary to use contour deformation. Since $J_1 + J_2 = 1$ as before, however, the equation $\omega(\xi) = \omega(z)$ has no non-trivial root. Hence, in this case no elimination is possible, with the result that as expected, we need one BC, $q_0(t)$.

4.2 *DLS equation*

A discrete analogue of the linear Schrdinger equation $\mathrm{i}q_t + q_{xx} = 0$ is

$$\mathrm{i}\dot{q}_n + (q_{n+1} - 2q_n + q_{n-1})/h^2 = 0. \tag{4.2}$$

Again, let $h = 1$. Here, the dispersion relation is $\omega(z) = 2 - (z + 1/z)$, implying $J_1 = J_2 = 1$ and $X_n = \mathrm{i}(q_n - q_{n-1}/z)$. Thus, we have

$$\hat{X}_1(z,t) = \mathrm{i}(z\hat{g}_1(z,t) - \hat{g}_0(z,t)) \tag{4.3}$$

for $|z| \geqslant 1$, which contains the unknown boundary datum $q_1(t)$ via its spectral transform. The domains $D_\pm$ are simply $D_\pm = \{z \in \mathbb{C}\colon |z| \lessgtr 1 \wedge \operatorname{Im} z \gtrless 0\}$ (see Fig. 2). The global relation is

$$\mathrm{i}[z\hat{g}_1(z,t) - \hat{g}_0(z,t)] + \mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(z,t) = \hat{q}(z,0), \quad \forall\, z \in \bar{D}_-. \tag{4.4}$$

The elimination of the unknown boundary data is simple because $\omega(\xi) = \omega(z)$ is a quadratic equation, whose only non-trivial root is $\xi = 1/z$, and (4.4) with $z \to 1/z$ gives, $\forall\, z \in \bar{D}_+^*$,

$$\mathrm{i}[\hat{g}_1(z,t)/z - \hat{g}_0(z,t)] + \mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(1/z,t) = \hat{q}(1/z,0). \tag{4.5}$$

We then solve for $\hat{g}_1(z,t)$ to get, $\forall\, z \in \bar{D}_+^*$,

$$\hat{g}_1(z,t) = z[\hat{g}_0(z,t) + \mathrm{i}(\mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(1/z,t) - \hat{q}(1/z,0))].$$

We therefore obtain the following expression for the solution: $\forall\, n \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}_0^+$,

$$q_n(t) = \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{q}(z,0)\,\mathrm{d}z + \frac{1}{2\pi}\int_{\partial D_+} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\left[\mathrm{i}z^2\hat{q}(1/z,0) - (z^2 - 1)\hat{g}_0(z,t)\right]\mathrm{d}z. \tag{4.6}$$

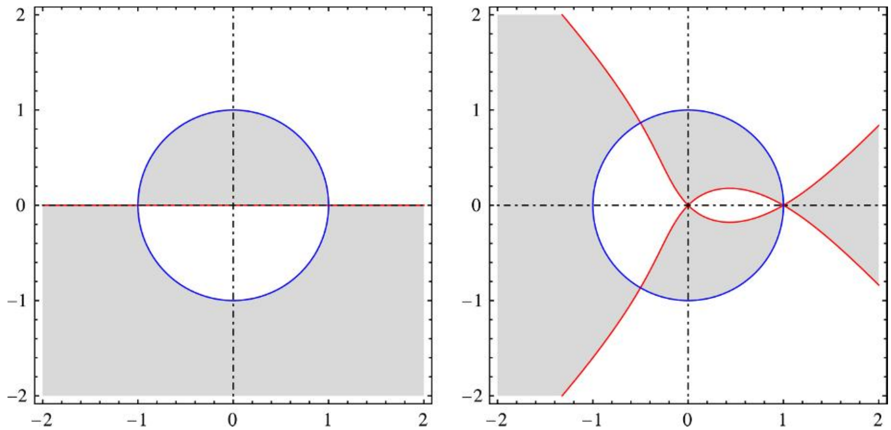

FIG. 2. Left: The dispersion relation $\omega(z)$ for the DLS (4.2) in the complex z-plane. Right: The dispersion relation $\omega(z)$ for the discrete linear Korteweg-de Vries (4.10) in the complex z-plane. The shaded regions show the domains $D_\pm$ where $\operatorname{Im}\omega(z) > 0$.

The above solution could also be obtained by using Fourier sine series. Unlike Fourier sine/cosine series, however, the present method applies to any discrete evolution equation. Moreover, the method can also deal with other kind of BCs just as effectively, as we show next. Consider (4.2) with BCs

$$\alpha q_1(t) + q_0(t) = b(t), \tag{4.7}$$

with $b(t)$ given, and $\alpha \in \mathbb{C}$ an arbitrary constant. Such kinds of BCs, which are the discrete analogue of Robin BCs for PDEs, cannot be treated using sine/cosine series. The present method, however, works equally well; one just needs to solve the global relation for a different unknown. Indeed, $\forall\, n \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}_0^+$, the solution of this IBVP is given by (cf. Biondini & Hwang, 2008)

$$\begin{aligned} q_n(t) = & \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{q}(z,0)\,\mathrm{d}z \\ & - \frac{1}{2\pi\mathrm{i}}\int_{\partial D_+} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\frac{\hat{G}(z,t)}{1/z-\alpha}\mathrm{d}z - \nu_\alpha \alpha^{1-n}\,\mathrm{e}^{-\mathrm{i}\omega(\alpha)t}\hat{G}(1/\alpha,t), \end{aligned} \tag{4.8}$$

where

$$\hat{G}(z,t) = \mathrm{i}(2\alpha - z - 1/z)\hat{b}(z,t) - \mathrm{i}(z-\alpha)\hat{q}(1/z,0) \tag{4.9}$$

and where $\nu_\alpha = 1$ if $\alpha \in D_-$, $\nu_\alpha = 1/2$ if $\alpha \in \partial D_-$ and $\nu_\alpha = 0$ otherwise, and where the integral along ∂D_+ is to be taken in the principal value sense when $\alpha \in \partial D_-$. As before, one can easily verify that the expression in (4.8) indeed solves (4.2) and satisfies the initial condition and the BC (4.7). One can verify that, in the limit $\alpha \to \infty$ with $b(t)/\alpha = b'(t)$ finite, the solution (4.6) of the IBVP with 'Dirichlet-type' BCs is recovered.

4.3 *Discrete linear Korteweg-de Vries equations*

A discrete analogue of the linear Korteweg-de Vries equation $q_t = q_{xxx}$ is given by

$$\dot{q}_n = (q_{n+2} - 2q_{n+1} + 2q_{n-1} - q_{n-2})/h^3. \tag{4.10}$$

Note that the IBVP for (4.10) cannot be solved with Fourier methods. Again, we set $h = 1$. The dispersion relation $\omega(z) = \mathrm{i}(z^2 - 2z + 2/z - 1/z^2)$, implying $J_1 = J_2 = 2$ and

$$X_n(z,t) = -(q_{n+1} + (z-2)q_n + (1/z^2 - 2/z)q_{n-1} + q_{n-2}/z).$$

The domains $D_\pm$ are now significantly more complicated (see Fig. 2). However, as $z \to 0$, D_+ is asymptotically equivalent to

$$S^{(0)} = \{z \in \mathbb{C}\colon (\pi/4 < \arg z < 3\pi/4) \cup (5\pi/4 < \arg z < 7\pi/4)\}$$

and as $z \to \infty$, D_- is asymptotically equivalent to

$$S^{(\infty)} = \{z \in \mathbb{C}\colon (-\pi/4 < \arg z < \pi/4) \cup (3\pi/4 < \arg z < 5\pi/4)\}.$$

Then we have that, for $|z| \geqslant 1$,

$$\hat{X}_1(z,t) = -z\hat{g}_2(z,t) - z(z-2)\hat{g}_1(z,t) - (1/z-2)\hat{g}_0(z,t) - \hat{g}_{-1}(z,t). \tag{4.11}$$

Inserting the above into (2.10), we get

$$-z\hat{g}_2(z,t) - z(z-2)\hat{g}_1(z,t) - (1/z-2)\hat{g}_0(z,t) - \hat{g}_{-1}(z,t) + \mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(z,t) = \hat{q}(z,0), \quad \forall z \in \bar{D}_-. \tag{4.12}$$

The use of the symmetries of the discrete linear Korteweg-de Vries to eliminate the unknown boundary data is more complicated than in the previous examples since $\omega(\xi) = \omega(z)$ yields a quartic equation. Note $[\omega(\xi) - \omega(z)]/(\xi - z) = 0$ is equivalent to $z^2\xi^3 + z^2(z-2)\xi^2 + (1-2z)\xi + z = 0$. Let ξ_{-1}, ξ_0 and ξ_1 be the non-trivial roots of (4.3) in our case. One can show that, as $z \to 0$,

$$\xi_{-1} = \mathrm{i}/z + (1-\mathrm{i}) - \mathrm{i}z + \mathrm{O}(z^2), \quad \xi_0 = -z + \mathrm{O}(z^2), \quad \xi_1 = -\mathrm{i}/z + (1+\mathrm{i}) + \mathrm{i}z + \mathrm{O}(z^2),$$

while, as $z \to \infty$,

$$\xi_{-1} = -z + 2 + \mathrm{O}(1/z^2), \quad \xi_0 = -\mathrm{i}/z + \mathrm{O}(1/z^2), \quad \xi_1 = \mathrm{i}/z + \mathrm{O}(1/z^2).$$

The functions $\hat{g}_n(z,t)$ are invariant under the transformations $z \to \xi_j(z)$ for $j = -1, 0, 1$. Moreover, $z \in D_-$ implies $\xi_0(z), \xi_1(z) \in D_+$ and viceversa. Then, substituting $z \to \xi_0$ and $z \to \xi_1$ in (4.12) and solving for $\hat{g}_1(z,t)$ and $\hat{g}_2(z,t)$, we obtain, $\forall z \in \bar{D}_+^*$

$$\begin{aligned}\hat{g}_{1,\mathrm{eff}}(z,t) = \{&\xi_0^2\xi_1\hat{q}(\xi_1,0) - \xi_0\xi_1^2\hat{q}(\xi_0,0) + [\xi_0^2\xi_1 - \xi_0\xi_1^2]\hat{g}_{-1}(z,t) \\ &+ [\xi_0^2 - \xi_1^2 - 2\xi_0^2\xi_1 + 2\xi_0\xi_1^2]\hat{g}_0(z,t)\}/[\xi_0^2\xi_1^2(\xi_1 - \xi_0)],\end{aligned} \tag{4.13a}$$

$$\begin{aligned}\hat{g}_{2,\mathrm{eff}}(z,t) = \{&\xi_1^2(\xi_1 - 2)[\xi_0\hat{q}(\xi_0,0) + \xi_0\hat{g}_{-1}(z,t) + (1-2\xi_0)\hat{g}_0(z,t)] \\ &- \xi_0^2(\xi_0 - 2)[\xi_1\hat{q}(\xi_1,0) + \xi_1\hat{g}_{-1}(z,t) + (1-2\xi_1)\hat{g}_0(z,t)]\}/[\xi_0^2\xi_1^2(\xi_1 - \xi_0)],\end{aligned} \tag{4.13b}$$

where the z-dependence of ξ_0 and ξ_1 was omitted for brevity. As before, terms containing $\hat{q}(\xi_j,t)$ give no contribution to the solution and have been neglected. The solution of (4.10) is then given by (2.12) with $\hat{X}_1(z,t)$ given by (4.12) with $\hat{g}_1(z,t)$ and $\hat{g}_2(z,t)$ replaced by (4.13).

Consider now the following alternative discretization of the linear Korteweg-de Vries equation:

$$\dot{q}_n = (q_{n+1} - 3q_n + 3q_{n-1} - q_{n-2})/h^3. \tag{4.14}$$

Note that in this case, the truncation error is dissipative rather than dispersive. (The right-hand side of (4.14) is asymptotic to $q_{xxx} - hq_{xxxx}/2 + \mathrm{O}(h^2)$ as $h \to 0$, as opposed to $q_{xxx} + h^2 q_{xxxxx}/4 + \mathrm{O}(h^3)$ for (4.10).) Set $h = 1$ as before. The dispersion relation is $\omega(z) = \mathrm{i}(z - 3 + 3/z - 1/z^2)$, implying $J_1 = 2$ and $J_2 = 1$. Also,

$$X_n = q_n - (3/z + 1/z^2)q_{n-1} + 1/zq_{n-2}.$$

We have, for $|z| \geqslant 1$,

$$\hat{X}_1(z,t) = z\hat{g}_1(z,t) - (3 + 1/z)\hat{g}_0(z,t) + \hat{g}_{-1}(z,t), \tag{4.15}$$

which contains one unknown datum $q_1(t)$. The domain $D_\pm$ shown in Fig. 3 are somewhat complicated. As $z \to 0$, however, D_+ is asymptotically equivalent to

$$S^{(0)} = \{z \in \mathbb{C}\colon (\pi/4 < \arg z < 3\pi/4) \cup (5\pi/4 < \arg z < 7\pi/4)\},$$

and as $z \to \infty$, D_- is asymptotically equivalent to

$$S^{(\infty)} = \{z \in \mathbb{C}\colon -\pi/2 < \arg z < \pi/2\}.$$

Owing to (4.15), the global relation (2.10) is, $\forall z \in \bar{D}_-$,

$$z\hat{g}_1(z,t) - (3 + 1/z)\hat{g}_0(z,t) + \hat{g}_{-1}(z,t) + \mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(z,t) = \hat{q}(z,0), \quad \forall z \in \bar{D}_-. \tag{4.16}$$

We now use the symmetries of (4.14). The equation $[\omega(\xi) - \omega(z)]/(\xi - z) = 0$ yields

$$z^2\xi^2 - (3z + 1)\xi + z = 0. \tag{4.17}$$

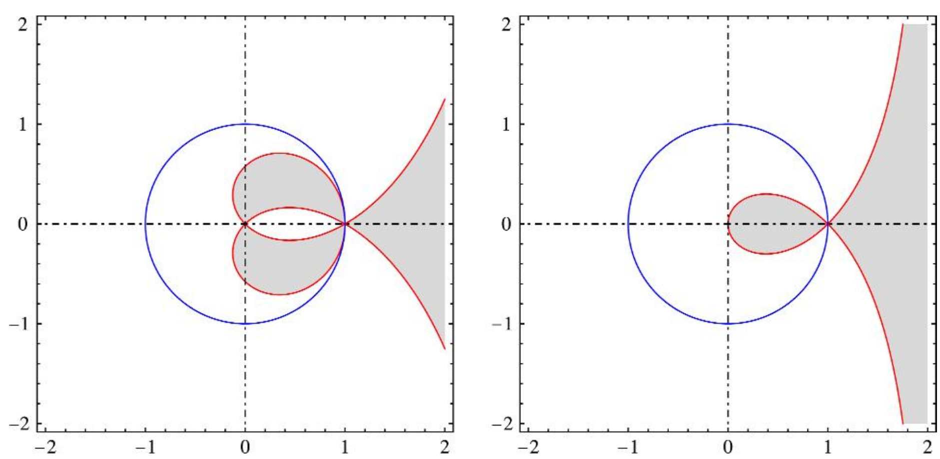

FIG. 3. Left: The dispersion relation $\omega(z)$ for (4.14) in the complex z-plane. Right: the dispersion relation $\omega(z)$ for (4.18) with $c = 0$. As before, the shaded regions show the domains $D_\pm$ where $\operatorname{Im}\omega(z) > 0$.

Let ξ_{-1} and ξ_0 be the roots of (4.17). As $z \to 0$, it is

$$\xi_{-1} = z - 3z^2 + 9z^3 + \mathrm{O}(z^4), \quad \xi_0 = 1/z^2 + 3/z - z + 3z^2 - 9z^3 + \mathrm{O}(z^4),$$

while, as $z \to \infty$, it is

$$\xi_{-1} = -\mathrm{i}/\sqrt{z} + 3/(2z) + 9\mathrm{i}/\big(8z^{3/2}\big) + 1/\big(2z^2\big) + \mathrm{O}\big(1/z^3\big),$$

$$\xi_0 = \mathrm{i}/\sqrt{z} + 3/(2z) - 9\mathrm{i}/\big(8z^{-3/2}\big) + 1/\big(2z^2\big) + \mathrm{O}\big(1/z^3\big).$$

Moreover, $z \in D_-$ implies $\xi_0 \in D_+$. Then (4.16) with $z \to \xi_0$ yields, $\forall\, z \in \bar{D}_+^*$,

$$\xi_0 \hat{g}_1(z,t) - (3 + 1/\xi_0)\hat{g}_0(z,t) + \hat{g}_{-1}(z,t) + \mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(\xi_0,t) = \hat{q}(\xi_0,0).$$

We can then solve for $\hat{g}_1(z,t)$, $\forall\, z \in \bar{D}_+^*$,

$$\hat{g}_1(z,t) = [(3 + 1/\xi_0)\hat{g}_0(z,t) - \hat{g}_{-1}(z,t) - \mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(\xi_0,t)]/\xi_0,$$

where as before the z-dependence of ξ_0 was omitted. As before, $\hat{q}(\xi_j,t)$ gives no contribution to the solution, which is therefore given by (2.12) with $\hat{X}_1(z,t)$ replaced by

$$\hat{X}_{1,\mathrm{eff}}(z,t) = [z(3 + 1/\xi_0)/\xi_0 - (3 + 1/z)]\hat{g}_0(z,t) - (z/\xi_0 - 1)\hat{g}_{-1}(z,t) + \hat{q}(\xi_0,0)/\xi_0.$$

4.4 *A discrete convection–diffusion equation*

Consider the semidiscrete equation

$$\dot{q}_n = c(q_{n+1} - q_{n-1})/h + (q_{n+1} - 2q_n + q_{n-1})/h^2, \tag{4.18}$$

with $c \in \mathbb{R}$ being the group speed in the continuum limit. Again, we take $h = 1$, which can be done without loss of generality by rescaling the time variable and redefining the constant c. The dispersion relation is $\omega(z) = \mathrm{i}[(1+c)z - 2 + (1-c)/z]$, implying $J_1 = J_2 = 1$ and

$$X_n = -\mathrm{i}[(1+c)q_n - (1-c)q_{n-1}/z].$$

Then we obtain, for $|z| \geqslant 1$,

$$\hat{X}_1(z,t) = -\mathrm{i}[(1+c)z\hat{g}_1(z,t) - (1-c)\hat{g}_0(z,t)], \tag{4.19}$$

which contains the unknown datum $q_1(t)$. The domains $D_\pm$ for $c = 0$ are shown in Fig. 3 and their boundary for some values of c in Fig. 4. As $z \to 0$, D_+ is asymptotically equivalent to

$$S^{(0)} = \begin{cases} \{z \in \mathbb{C}\colon -\pi/2 < \arg z < \pi/2\}, & c < 1, \\ \{z \in \mathbb{C}\colon \pi/2 < \arg z < 3\pi/2\}, & c > 1. \end{cases}$$

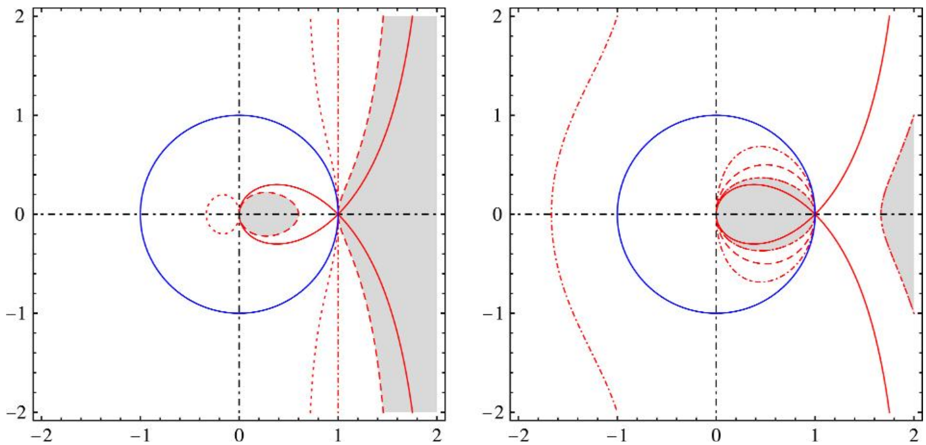

FIG. 4. The boundaries of the regions $D_\pm$ for (4.18) for various values of c. Left: $c = 0$ (solid), $c = 1/4$ (dot-dashed), $c = 1$ (dashed) and $c = 2$ (dotted). Right: $c = 0$ (solid), $c = -1/4$ (dotted), $c = -1$ (dashed) and $c = -4$ (dot-dashed). The shaded regions show the domains $D_\pm$ for $c = 1/4$ (left) and $c = -1/4$ (right).

As $z \to \infty$, the domain D_- is asymptotically equivalent to

$$S^{(\infty)} = \begin{cases} \{z \in \mathbb{C}\colon \pi/2 < \arg z < 3\pi/2\}, & c < -1, \\ \{z \in \mathbb{C}\colon -\pi/2 < \arg z < \pi/2\}, & c > -1. \end{cases}$$

As $c \to \infty$, it is $D_\pm = \{z \in \mathbb{C}\colon |z| \lesseqgtr 1 \wedge \mathrm{Re}\, z \lesseqgtr 0\}$ (where the upper/lower inequalities in the right-hand side go with the upper/lower sign in the left-hand side). As $c \to -\infty$, $D_\pm = \{z \in \mathbb{C}\colon |z| \lesseqgtr 1 \wedge \mathrm{Re}\, z \gtreqless 0\}$. Note that $c = \pm 1$ are special cases since the domains $D_\pm$ change character at these two points (see Fig. 4).

Inserting (4.19) into (2.10), gives, $\forall\, z \in \bar{D}_-$, the global relation as

$$-\mathrm{i}[(1+c)z\hat{g}_1(z,t) - (1-c)\hat{g}_0(z,t)] + \mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(z,t) = \hat{q}(z,0). \tag{4.20}$$

The elimination of the unknown boundary datum here is simple, since $\omega(\xi) = \omega(z)$ is a quadratic equation, whose only one non-trivial root is $\xi(z) = \nu_c/z$, where $\nu_c = (1-c)/(1+c)$ for $c \neq -1$. Using the same arguments as before, (4.20) with $z \to \nu_c/z$ yields, $\forall\, z \in \bar{D}_+^*$,

$$-\mathrm{i}[(1-c)\hat{g}_1(z,t)/z - (1-c)\hat{g}_0(z,t)] + \mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(\nu_c/z,t) = \hat{q}(\nu_c/z,0).\hat{q}$$

Then after some algebra, we obtain the solution of (4.18), $\forall\, n \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}_0^+$,

$$q_n(t) = \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{q}(z,0)\mathrm{d}z + \frac{1}{2\pi}\int_{\partial D_+} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\{\mathrm{i}z^2\hat{q}(\nu_c/z,0)/\nu_c + [(1+c)z^2 - (1-c)]\hat{g}_0(z,t)\}\mathrm{d}z.$$

Hence, only one BC is needed at $n = 0$ for $c \neq \pm 1$, i.e. $q_0(t)$.

Things change if $c = \pm 1$. For $c = 1$, we have $\hat{X}_1(z,t) = -2\mathrm{i}z\hat{g}_1(z,t)$ for $|z| \geqslant 1$. The domain D is the union of $D_+ = \emptyset$ and $D_- = \{z \in \mathbb{C}\colon \operatorname{Re} z > 1\}$. The global relation is, for $|z| > 1$,

$$\mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(z,t) + 2\mathrm{i}z\hat{g}_1(z,t) = \hat{q}(z,0).$$

But the term $z^{n-1}\,\mathrm{e}^{\mathrm{i}\omega(z)t}\hat{g}_1(z,t)$ is analytic $\forall\, z \neq 0$ and bounded for $|z| \leqslant 1$. Using (2.12), the solution of (4.18) is then

$$q_n(t) = \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{q}(z,0)\mathrm{d}z.$$

Thus, no BC is needed for $c = 1$.

When $c = -1$ instead for $|z| \geqslant 1$, we get $\hat{X}_1(z,t) = 2\mathrm{i}\hat{g}_0(z,t)$. The domain D is the union of $D_+ = \{z \in \mathbb{C}\colon |z - 1/2| < 1/2\}$ and $D_- = \emptyset$. The global relation (2.10) yields, for $|z| > 1$,

$$\mathrm{e}^{\mathrm{i}\omega(z)t}\hat{q}(z,t) + 2\mathrm{i}\hat{g}_0(z,t) = \hat{q}(z,0).$$

The equation $\omega(\xi) = \omega(z)$ has no non-trivial root, hence, no elimination is possible here. Thus, we obtain the solution as

$$q_n(t) = \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{q}(z,0)\mathrm{d}z - \frac{1}{\pi}\int_{\partial D_+} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\hat{g}_0(z,t)\mathrm{d}z,$$

by deforming the integration contour from $|z| = 1$ to $z \in \partial D_+$. Thus, in this case, we need one BC, $q_0(t)$.

5. Systems of equations, higher-order equations and forced problems

The method presented in Section 2 can be extended in a straightforward way to solve more general kinds of IBVPs, as we show next.

5.1 *Systems of DLEEs*

Consider the linear system of semidiscrete evolution equations

$$\mathrm{i}\dot{\mathbf{q}}_n = \Omega(\mathrm{e}^{\partial})\mathbf{q}_n, \tag{5.1}$$

where $\mathbf{q}_n = (q_n^{(1)}, \ldots, q_n^{(M)})^t$ is an M-component vector and $\Omega(z)$ is an $M \times M$ matrix. One can easily verify that a Lax pair for (5.1) is given by

$$\mathbf{v}_{n+1} - z\mathbf{v}_n = \mathbf{q}_n, \quad \dot{\mathbf{v}}_n + \mathrm{i}\Omega(z)\mathbf{v}_n = \mathbf{X}_n, \tag{5.2}$$

where

$$\mathbf{X}_n(z,t) = -\mathrm{i}\left.\frac{\Omega(z) - \Omega(s)}{z - s}\right|_{s=\mathrm{e}^{\partial}} \mathbf{q}_n(t),$$

and as in the scalar case, $\mathbf{X}_n$ has a finite-principal part. The compatibility of (5.2) can also be written as

$$\partial_t(z^{-n}\,\mathrm{e}^{\mathrm{i}\Omega(z)t}\mathbf{q}_n) = \Delta(z^{-n+1}\,\mathrm{e}^{\mathrm{i}\Omega(z)t}\mathbf{X}_n).$$

Following similar steps as in the scalar case, one then obtains

$$\mathbf{q}_n = \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} z^{n-1}\, \mathrm{e}^{-\mathrm{i}\Omega(z)t} \hat{\mathbf{q}}(z,0)\mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} z^{n-1}\, \mathrm{e}^{-\mathrm{i}\Omega(z)t} \hat{\mathbf{X}}_1(z,t)\mathrm{d}z, \tag{5.3}$$

where

$$\hat{\mathbf{q}}(z,t) = \sum_{n=1}^{\infty} \mathbf{q}_n(t)/z^n, \quad \hat{\mathbf{X}}_1(z,t) = z\int_0^t \mathrm{e}^{\mathrm{i}\Omega(z)t'} \mathbf{X}_1(z,t')\mathrm{d}t',$$

respectively, for $|z| \geqslant 1$ and for all $z \neq 0$. The global algebraic relation is

$$\mathrm{e}^{\mathrm{i}\Omega(z)t} \hat{\mathbf{q}}(z,t) = \hat{\mathbf{q}}(z,0) - \hat{\mathbf{X}}_1(z,t).$$

For simplicity, we consider the case of a simple matrix $\Omega(z)$. Cases with non-trivial Jordan blocks can be treated similarly. Let $\omega_1(z),\ldots,\omega_M(z)$ and $\mathbf{v}_1(z),\ldots,\mathbf{v}_M(z)$ are the eigenvalues of $\Omega(z)$ and the corresponding eigenvectors. That is, let $\Omega(z) = V\Lambda V^{-1}$, where $V(z) = (\mathbf{v}_1,\ldots,\mathbf{v}_M)$ and

$$\Lambda(z) = \mathrm{diag}(\omega_1,\ldots,\omega_M) = \Lambda_1 + \cdots + \Lambda_M,$$

with $\Lambda_m(z) = \mathrm{diag}(0,\ldots,0,\omega_m,0,\ldots,0)$. We use the spectral decomposition of $\Omega(z)$ to write

$$\Omega(z) = V\Lambda_1 W^\dagger + \cdots + V\Lambda_M W^\dagger = \omega_1 \mathbf{v}_1 \mathbf{w}_1^\dagger + \cdots + \omega_M \mathbf{v}_M \mathbf{w}_M^\dagger,$$

where $W = (V^{-1})^\dagger = (\mathbf{w}_1(z),\ldots,\mathbf{w}_M(z))$, and the dagger denotes conjugate transpose. Hence,

$$\mathrm{e}^{-\mathrm{i}A(z)t} = \mathrm{e}^{-\mathrm{i}\omega_1(z)t}\mathbf{v}_1\mathbf{w}_1^\dagger + \cdots + \mathrm{e}^{-\mathrm{i}\omega_M(z)t}\mathbf{v}_M\mathbf{w}_M^\dagger.$$

Now define the domains $D_\pm^{(m)} = \{z \in \mathbb{C}\colon |z| \gtrless 1 \wedge \mathrm{Im}\,\omega_m(z) > 0\}, m = 1,\ldots,M$. Note that the roots $\omega_1(z),\ldots,\omega_M(z)$ may have branch cuts. If that is the case, one must carefully integrate around these branch cuts. We discuss one such case later. If, instead, there are no branch cuts, we can use contour deformation to move the integration contour in the second integral of (5.3) from $|z| = 1$ to $z \in \partial D_+^{(1)} \cup,\ldots,\cup \partial D_+^{(M)}$, obtaining $\forall\, n \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}_0^+$,

$$\mathbf{q}_n(t) = \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} z^{n-1}\, \mathrm{e}^{-\mathrm{i}A(z)t} \hat{\mathbf{q}}(z,0)\mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \sum_{m=1}^{M} \int_{\partial D_+^{(m)}} z^{n-1}\, \mathrm{e}^{-\mathrm{i}\omega_m(z)t} \mathbf{v}_m \mathbf{w}_m^\dagger \hat{\mathbf{X}}_1(z,t)\mathrm{d}z. \tag{5.4}$$

Equation (5.4) is the analogue of the reconstruction formula (2.12) of the scalar case. One can now use the symmetries of the roots $\omega_1(z),\ldots,\omega_M(z)$ to eliminate the unknown boundary data, following similar steps as in the scalar case.

5.2 *Higher-order problems*

Now, consider higher-order equations of the type systems

$$\frac{\mathrm{d}^M q_n}{\mathrm{d}t^M} = \sum_{m=0}^{M-1} P_m(\mathrm{e}^{\partial}) \frac{\mathrm{d}^m q_n}{\mathrm{d}t^m}, \tag{5.5}$$

where the $P_m(z)$ are rational functions. We can convert this into a matrix system of first-order (in time) DLEEs. More precisely, introducing the vector dependent variable $\mathbf{q}_n = (q_n, \ldots, \mathrm{d}^M q_n/\mathrm{d}t^M)^\top$, where the superscript $\top$ denotes matrix transpose, (5.5) can be written in the form of (5.1), with

$$\Omega(z) = \mathrm{i}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ P_0(z) & P_1(z) & P_2(z) & \cdots & P_{M-1}(z) \end{pmatrix}.$$

One can then use the methods presented for systems of DLEEs.

5.3 *Forced problems*

Suppose $q_n(t)$ satisfies the forced version of (2.1), i.e.

$$\mathrm{i}\dot{q}_n(t) - \omega(\mathrm{e}^{\partial})q_n(t) = h_n(t), \tag{5.6}$$

$\forall\, n \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}_0^+$, where $q_{-J_1+1}(t), \ldots, q_0(t)$ are given boundary data, and $h_n(t)$ is a sequence of functions with sufficient smoothness.

The solution of this problem can be reduced to the solution of (2.1). Indeed using spectral transforms, it is relatively easy to show that a particular solution of (5.6) is given by

$$H_n(t) = -\frac{1}{2\pi}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z)t}\int_0^t \mathrm{e}^{\mathrm{i}\omega(z)t'}\sum_{m=1}^{\infty} h_m(t')/z^m\,\mathrm{d}t'\,\mathrm{d}z,$$

$\forall\, n \in \mathbb{Z}$. Since $H_n(0) \equiv 0$, $\forall\, n \in \mathbb{N}$, the solution of the IBVP defined by (5.6) is then given by

$$q_n(t) = \tilde{q}_n(t) + H_n(t),$$

where $\tilde{q}_n(t)$ satisfies the homogeneous equation (2.1) with the given initial condition and BC.

5.4 *Example*

We illustrate the above results by solving the IBVP for the discrete wave equation

$$\ddot{q}_n = q_{n+1} - 2q_n + q_{n-1}. \tag{5.7}$$

Let $\mathbf{q}_n = (q_n, \dot{q}_n)^\top$. Then (5.7) becomes (5.1), with

$$\Omega(z) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i}(z+1/z-2) & 0 \end{pmatrix}, \quad \mathbf{X}_n = \begin{pmatrix} 0 \\ q_n - q_{n-1}/2 \end{pmatrix}.$$

The eigenvalues of $\Omega(z)$ are $\omega_\pm(z) = \pm\sqrt{2-(z+1/z)}$, and the corresponding eigenvectors are $\mathbf{v}_\pm(z) = (1, -\mathrm{i}\omega_\pm)^\top$. So the spectral decomposition of $\Omega(z)$ is $\Omega(z) = V\Lambda V^{-1}$, where $\Lambda(z) = \mathrm{diag}(\omega_+, \omega_-)$, $V(z) = (\mathbf{v}_+, \mathbf{v}_-)$ and $V^{-1} = (\mathbf{w}_+, \mathbf{w}_-)^\dagger$, where $\mathbf{w}_\pm(z) = (1, \mathrm{i}/\omega_\pm)^\top/2$. Now, introduce the projection operators

$$P_\pm(z) = \mathbf{v}_\pm\mathbf{w}_\pm^\dagger(z) = \begin{pmatrix} 1/2 & \mathrm{i}/(2\omega_\pm) \\ -\mathrm{i}\omega_\pm/2 & 1/2 \end{pmatrix}.$$

Then, we have $\mathrm{e}^{-\mathrm{i}A(z)t} = \mathrm{e}^{-\mathrm{i}\omega_+(z)t}P_+(z) + \mathrm{e}^{-\mathrm{i}\omega_-(z)t}P_-(z)$. Inserting this into the reconstruction formula, one obtains

$$\begin{aligned}\mathbf{q}_n(t) &= \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}\,\mathrm{e}^{-\mathrm{i}A(z)t}\hat{\mathbf{q}}(z,0)\mathrm{d}z\\ &\quad - \frac{1}{2\pi\mathrm{i}}\left[\int_{\partial D_+^{(+)}} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega_+(z)t}P_+(z)\hat{\mathbf{X}}_1(z,t)\mathrm{d}z + \int_{\partial D_+^{(-)}} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega_-(z)t}P_-(z)\hat{\mathbf{X}}_1(z,t)\mathrm{d}z\right].\end{aligned} \tag{5.8}$$

The deformation of the integrals to obtain (5.8) is not trivial, however, as we discuss next.

It is $\mathrm{Im}\,\omega^{(\sigma)}(z) > 0$ for $z \in D^{(\sigma)} = D_+^{(\sigma)} \cup D_-^{(\sigma)}$ and $\sigma = \pm$, where the domains are $D_\pm^{(+)} = \{z \in \mathbb{C}\colon |z| \lessgtr 1 \wedge \mathrm{Im}\,z \gtrless 0\}$ and $D_\pm^{(-)} = \{z \in \mathbb{C}\colon |z| \lessgtr 1 \wedge \mathrm{Im}\,z \lessgtr 0\}$. Note also that $\omega_\pm(z)$ have branch points at $z = 0, 1, \infty$. Taking the branch cut of the square root to be along negative real values of its argument, the branch cuts of $\omega_\pm(z)$ are along $(0,1)\cup(1,\infty)$. Since the first of these branch cuts is inside the unit circle $|z| = 1$ in (5.3), we can decompose the integration contour as $\partial D_+^{(+)} \cup \partial D_+^{(-)} \cup C$, where $C = C^- \cup (-C^+)$ and $C^\pm = (\pm\mathrm{i}\varepsilon, 1 \pm \mathrm{i}\varepsilon)$ and $\partial D_+^{(\pm)}$ are above/below the branch cut, respectively (see Fig. 5).

Using the symmetry of $\omega_\pm(z)$, i.e. $\omega_\pm(x+\mathrm{i}y) = \omega_\mp(x-\mathrm{i}y)$, we have

$$\int_{C^\pm} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega_+(z)t}P_+(z)\hat{\mathbf{X}}_1(z,t)\mathrm{d}z = \int_{C^\mp} z^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega_-(z)t}P_-(z)\hat{\mathbf{X}}_1(z,t)\mathrm{d}z.$$

Thus,

$$\int_C z^{n-1}\mathrm{e}^{-\mathrm{i}\omega_+(z)t}P_+(z)\hat{\mathbf{X}}_1(z,t)\mathrm{d}z + \int_C z^{n-1}\mathrm{e}^{-\mathrm{i}\omega_-(z)t}P_-(z)\hat{\mathbf{X}}_1(z,t)\mathrm{d}z = 0\,.$$

That is, the sum of the integrals around the branch cuts is zero. Thus, we obtain with (5.8), as anticipated.

We now discuss the elimination of the unknown boundary data. Let

$$\hat{g}_n^\pm(z,t) = \int_0^t \mathrm{e}^{\mathrm{i}\omega_\pm(z)t'}q_n(t')\mathrm{d}t'.$$

We have, for $|z| \geqslant 1$,

$$\hat{\mathbf{X}}_1(z,t) = \begin{pmatrix} \mathrm{i}[z\hat{g}_1^+(z,t) - \hat{g}_0^+(z,t)]/[2\omega_+(z)] + \mathrm{i}[z\hat{g}_1^-(z,t) - \hat{g}_0^-(z,t)]/[2\omega_-(z)] \\ [z\hat{g}_1^+(z,t) - \hat{g}_0^+(z,t) + z\hat{g}_1^-(z,t) - \hat{g}_0^-(z,t)]/2 \end{pmatrix},$$

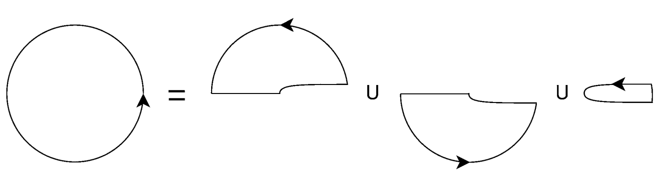

FIG. 5. Integration contour for (5.7).

which contains the unknown boundary datum $q_1(t)$. The global relation for (5.7) is, $\forall z \in \bar{D}_-^{(1)} \cup \bar{D}_-^{(2)}$,

$$\hat{\mathbf{X}}_1(z,t) + \mathrm{e}^{\mathrm{i}A(z)t}\hat{\mathbf{q}}(z,t) = \hat{\mathbf{q}}(z,0)\,. \tag{5.9}$$

The elimination of the unknown boundary datum is not difficult because $\omega_\pm(\xi) = \omega_\pm(z)$ is a quadratic equation whose only non-trivial root is $\xi = 1/z$. Replacing $z \to 1/z$ in (5.9) and solving for $\hat{g}_1^+(z,t)$ and $\hat{g}_1^-(z,t)$ we then obtain the solution via (5.8) with $\hat{\mathbf{X}}_1(z,t)$ replaced by

$$\hat{\mathbf{X}}_{1,\mathrm{eff}}(z,t) = \begin{pmatrix} \hat{q}(1/z,0) + \mathrm{i}(z^2-1)[\hat{g}_0^+(z,t)/\omega_+(z) + \hat{g}_0^-(z,t)/\omega_-(z)]/2 \\ z^2\hat{q}(1/z,0) + (z^2-1)[\hat{g}_0^+(z,t) + \hat{g}_0^-(z,t)]/2 \end{pmatrix}.$$

Thus, the only BC needed is $q_0(t)$.

6. Differential-difference equations in two lattice variables

We now show how the method presented in Section 2 can be extended to solve IBVPs for linear separable differential-difference evolution equations for a double sequence of functions $\{q_{m,n}(t)\}_{m,n\in\mathbb{N}}$. Consider a multi-dimensional analogue of (2.1) in the form

$$\mathrm{i}\dot{q}_{m,n} = \omega(\mathrm{e}^{\partial_m}, \mathrm{e}^{\partial_n})q_{m,n}, \tag{6.1}$$

where $\mathrm{e}^{\partial_m}q_{m,n} = q_{m+1,n}$ and $\mathrm{e}^{\partial_n}q_{m,n} = q_{m,n+1}$ and $\omega(z_1,z_2)$ is an arbitrary discrete dispersion relation. In particular, we will restrict our attention to the class of so-called 'separable' equations for which the dispersion relation can be written as the sum

$$\omega(z_1,z_2) = \omega_1(z_1) + \omega_2(z_2) = \sum_{m=-M_1}^{M_2} \omega_{1,m} z_1^m + \sum_{n=-N_1}^{N_2} \omega_{2,n} z_2^n. \tag{6.2}$$

This class includes many physically significant examples. An identical restriction exists for the method in the continuum case (Fokas, 2002).

6.1 *The general method*

Similar to Section 2, we can write (6.1) in the discrete version of a divergence equation as

$$\partial_t[z_1^{-m}z_2^{-n}\,\mathrm{e}^{\mathrm{i}\omega(z_1,z_2)t}q_{m,n}] = \Delta_m(z_1^{-m+1}z_2^{-n}\,\mathrm{e}^{\mathrm{i}\omega(z_1,z_2)t}X_{m,n}^{(1)}) + \Delta_n(z_1^{-m}z_2^{-n+1}\,\mathrm{e}^{\mathrm{i}\omega(z_1,z_2)t}X_{m,n}^{(2)}), \tag{6.3}$$

where $\Delta_m Q_m = Q_{m+1} - Q_m$ and $\Delta_n Q_n = Q_{n+1} - Q_n$ are the difference operators, and where

$$X_{m,n}^{(1)}(z_1,z_2,t) = -\mathrm{i}\left[\frac{\omega_1(z_1)-\omega_1(s_1)}{z_1-s_1}\right]_{s_1=\mathrm{e}^{\partial_m}} q_{m,n}(t), \tag{6.4a}$$

$$X_{m,n}^{(2)}(z_1,z_2,t) = -\mathrm{i}\left[\frac{\omega_2(z_2)-\omega_2(s_2)}{z_2-s_2}\right]_{s_2=\mathrm{e}^{\partial_n}} q_{m,n}(t). \tag{6.4b}$$

Let $\hat{q}(z_1,z_2,t)$, $\hat{q}_{m,n}(z_1,z_2,t)$, $\hat{X}_1^{(1)}(z_1,z_2,t)$ and $\hat{X}_1^{(2)}(z_1,z_2,t)$ be the z-transforms of the initial condition and BC, respectively. That is,

$$\hat{q}(z_1,z_2,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} z_1^{-m}z_2^{-n}q_{m,n}(t), \tag{6.5a}$$

$$\hat{q}_{m,n}(z_1, z_2, t) = \int_0^t e^{i\omega(z_1,z_2)t'} q_{m,n}(t')dt' \tag{6.5b}$$

$$\hat{X}_1^{(1)}(z_1, z_2, t) = \int_0^t e^{i\omega(z_1,z_2)t'} z_1 X_{1,n}^{(1)}(z_1, z_2, t')dt', \tag{6.5c}$$

$$\hat{X}_1^{(2)}(z_1, z_2, t) = \int_0^t e^{i\omega(z_1,z_2)t'} z_2 X_{m,1}^{(2)}(z_1, z_2, t')dt'. \tag{6.5d}$$

For $(|z_1| > 1) \wedge (|z_2| > 1)$, summing (6.3) from $m = 1$ to ∞ and $n = 1$ to ∞, it is

$$\partial_t[e^{i\omega(z_1,z_2)t}\hat{q}(z_1, z_2, t)] + \sum_{n=1}^{\infty} e^{i\omega(z_1,z_2)t} z_1 X_{1,n}^{(1)}(z_1, z_2, t) + \sum_{m=1}^{\infty} e^{i\omega(z_1,z_2)t} z_2 X_{m,1}^{(2)}(z_1, z_2, t) = 0. \tag{6.6}$$

Again, note that if $q_{m,n}(0) \in l^1(\mathbb{N}\times\mathbb{N})$, then $\hat{q}(z_1, z_2, t)$ is defined $\forall\,(z_1, z_2) \in \mathbb{C}\times\mathbb{C}$ with $|z_1|, |z_2| \geqslant 1$ and is analytic for $|z_1|, |z_2| > 1$, while $\hat{X}_1^{(1)}(z_1, z_2, t)$ and $\hat{X}_1^{(2)}(z_1, z_2, t)$ are defined $\forall\, z_1 \in \bar{D}^{(1)}$, $\forall\, z_2 \in \bar{D}^{(2)}$, and are analytic $\forall\, z_1 \in D^{(1)}$, $\forall\, z_2 \in D^{(2)}$, where $D^{(1)} = \{z_1 \in \mathbb{C}\colon \operatorname{Im}\omega_1(z_1) > 0\}$ and $D^{(2)} = \{z_2 \in \mathbb{C}\colon \operatorname{Im}\omega_2(z_2) > 0\}$.

As in Section 2, we decompose $D^{(j)} = D_+^{(j)} \cup D_-^{(j)}$ for $j = 1, 2$, where

$$D_\pm^{(j)} = \{z_j \in \mathbb{C}\colon |z_j| \lesseqgtr 1 \wedge \operatorname{Im}\omega_j(z_j) > 0\}.$$

Now integrate (6.6) from $t' = 0$ to $t' = t$ to get, for $(|z_1| > 1) \wedge (|z_2| > 1)$,

$$e^{i\omega(z_1,z_2)t}\hat{q}(z_1, z_2, t) = \hat{q}(z_1, z_2, 0) - \hat{X}_1^{(1)}(z_1, z_2, t) - \hat{X}_1^{(2)}(z_1, z_2, t)\,. \tag{6.7}$$

Equation (6.7) is the discrete global relation in two lattice variables.

Since $q_{m,n}(t)$ are the Laurent coefficients of $\hat{q}(z_1, z_2, t)$, the inverse transform of $\hat{q}(z_1, z_2, t)$ is:

$$q_{m,n}(t) = \frac{1}{(2\pi i)^2}\int_{|z_1|=1}\int_{|z_2|=1} z_1^{m-1} z_2^{n-1}\hat{q}(z_1, z_2, t)dz_2\, dz_1, \quad \forall\, m, n \in \mathbb{N}.$$

Then (6.5) provides, $\forall\, m, n \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}_0^+$,

$$\begin{aligned} q_{m,n}(t) = {} & \frac{1}{(2\pi i)^2}\int_{|z_1|=1}\int_{|z_2|=1} z_1^{m-1} z_2^{n-1}\, e^{-i\omega(z_1,z_2)t}\hat{q}(z_1, z_2, 0)dz_1\, dz_1 \\ & - \frac{1}{(2\pi i)^2}\int_{|z_1|=1}\int_{|z_2|=1} z_1^{m-1} z_2^{n-1}\, e^{-i\omega(z_1,z_2)t}[\hat{X}_1^{(1)}(z_1, z_2, t) + \hat{X}_1^{(2)}(z_1, z_2, t)]dz_1\, dz_1. \end{aligned} \tag{6.8}$$

As in Section 2, we now use contour deformation to move the integration contour for the second integral in (6.8) away from the unit circle. The integrand in the last term of (6.8) is analytic $\forall\, z_1, z_2 \neq 0$ and continuous and bounded for $z_1 \in \bar{D}^{(1)}$ and $z_2 \in \bar{D}^{(2)}$. Thus, we can deform that integration contour from $|z_1| = 1$ to $z_1 \in \partial D_+^{(1)}$ and $|z_2| = 1$ to $z_2 \in \partial D_+^{(2)}$ obtaining, $\forall\, m, n \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}_0^+$, the reconstruction formula for the solution of the IBVP:

$$\begin{aligned}q_{m,n}(t) = {} & \frac{1}{(2\pi\,\mathrm{i})^2}\int_{|z_1|=1}\int_{|z_2|=1} z_1^{m-1} z_2^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z_1,z_2)t}\hat{q}(z_1,z_2,0)\mathrm{d}z_2\,\mathrm{d}z_1 \\ & - \frac{1}{(2\pi\,\mathrm{i})^2}\int_{\partial D_+^{(1)}}\int_{\partial D_+^{(2)}} z_1^{m-1} z_2^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z_1,z_2)t}[\hat{X}_1^{(1)}(z_1,z_2,t)+\hat{X}_1^{(2)}(z_1,z_2,t)]\mathrm{d}z_2\,\mathrm{d}z_1 \\ & - \frac{1}{(2\pi\,\mathrm{i})^2}\int_{\partial D_+^{(1)}}\int_{|z_2|=1} z_1^{m-1} z_2^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z_1,z_2)t}\hat{X}_1^{(1)}(z_1,z_2,t)\mathrm{d}z_2\,\mathrm{d}z_1 \\ & - \frac{1}{(2\pi\,\mathrm{i})^2}\int_{|z_1|=1}\int_{\partial D_+^{(2)}} z_1^{m-1} z_2^{n-1}\,\mathrm{e}^{-\mathrm{i}\omega(z_1,z_2)t}\hat{X}_1^{(2)}(z_1,z_2,t)\mathrm{d}z_2\,\mathrm{d}z_1. \end{aligned} \tag{6.9}$$

Of course (6.9) depends on the unknown data $\{q_{m,n}(t)\}_{n=1,\ldots,N_2;m=1,\ldots,M_2}$ via their spectral transforms $\{\hat{q}_{m,n}(z_1,z_2,t)\}_{n=1,\ldots,N_2;m=1,\ldots,M_2}$ appearing in $\hat{X}_1^{(1)}(z_1,z_2,t)$ and $\hat{X}_1^{(2)}(z_1,z_2,t)$. As before, we must express these unknown boundary values in terms of known quantities.

The spectral functions $\hat{X}_1^{(1)}(z_1,z_2,t)$ and $\hat{X}_1^{(2)}(z_1,z_2,t)$ are invariant under any transformation that leaves the dispersion relation (6.2) invariant; i.e. any map $z_1 \mapsto \xi^{(1)}(z_1)$ and $z_2 \mapsto \xi^{(2)}(z_2)$ such that $\omega_1(\xi^{(1)}) = \omega_1(z_1)$ and $\omega_2(\xi^{(2)}) = \omega_2(z_2)$. Note that, as a result, we have $\xi^{(1)}(D_\pm^{(1)}) \subseteq D_\pm^{(1)}$ and $\eta^{(2)}(D_\pm)^{(2)} \subseteq D_\pm^{(2)}$.

The equation $\omega_1(z_1) = \omega_1(\xi^{(1)})$ has M_1+M_2-1 non-trivial roots in addition to the trivial one $\xi^{(1)} = z_1$, and $\omega_2(z_2) = \omega_2(\xi^{(2)})$ has N_1+N_2-1 non-trivial roots in addition to the trivial one $\xi^{(2)} = z_2$. Using these symmetries in the global relation allow us to eliminate the unknown boundary data. To do so, however, one must identify which of the $(M_1+M_2-1)(N_1+N_2-1)$ non-trivial roots are useful for this purpose. As in Section 2, it is not possible to find these roots in closed form except in the simplest cases. As before, we then look at the asymptotic behaviour of these roots as $z_1 \to 0$, $z_2 \to 0$, $z_1 \to \infty$ and $z_2 \to \infty$. As $z_1 \to \infty$, we have $\omega_1(z_1) \sim \omega_{1,M_2} z_1^{M_2}$. Similarly, as $z_1 \to 0$, we have $\omega_1(z_1) \sim \omega_{1,-M_1} z_1^{-M_1}$. Thus, as $z_1 \to \infty$ and $z_1 \to 0$, the domain $D_-^{(1)}$ and $D_+^{(1)}$ are, respectively, asymptotically equivalent to

$$S^{(1,\infty)} = \bigcup_{m=0}^{M_2-1} S_m^{(1,\infty)}, \quad S^{(1,0)} = \bigcup_{m=-M_1+1}^{0} S_m^{(1,0)},$$

where, for $m = 0,\ldots,M_2-1$,

$$S_m^{(1,\infty)} = \{z_1 \in \mathbb{C}\colon 2m\pi/M_2 - \arg\omega_{1,M_2}/M_2 < \arg z_1 < (2m+1)\pi/M_2 - \arg\omega_{1,M_2}/M_2\},$$

while for $m = -M_1+1,\ldots,0$,

$$S_m^{(1,0)} = \{z_1 \in \mathbb{C}\colon (2m-1)\pi/M_1 + \arg\omega_{1,-M_1}/M_1 < \arg z_1 < 2m\pi/M_1 + \arg\omega_{1,-M_1}/M_1\}.$$

Similarly, as $z_2 \to \infty$, we have $\omega_2(z_2) \sim \omega_{2,N_2} z_2^{N_2}$ and as $z_2 \to 0$, we have $\omega_2(z_2) \sim \omega_{2,-N_1} z_2^{-N_1}$. Thus, as $z_2 \to \infty$ and $z_2 \to 0$, the domain $D_-^{(2)}$ and $D_+^{(2)}$ are, respectively, asymptotically equivalent to domains

$$S^{(2,\infty)} = \bigcup_{n=0}^{N_2-1} S_n^{(2,\infty)}, \quad S^{(2,0)} = \bigcup_{n=-N_1+1}^{0} S_n^{(2,0)},$$

where, for $n = 0, \ldots, N_2 - 1$,

$$S_n^{(2,\infty)} = \{z_2 \in \mathbb{C}\colon 2n\pi/N_2 - \arg\omega_{2,N_2}/N_2 < \arg z_2 < (2n+1)\pi/N_2 - \arg\omega_{2,N_2}/N_2\},$$

while for $n = -N_1 + 1, \ldots, 0$,

$$S_n^{(2,0)} = \{z_2 \in \mathbb{C}\colon (2n-1)\pi/N_1 + \arg\omega_{2,-N_1}/N_1 < \arg z_2 < 2n\pi/N_1 + \arg\omega_{2,-N_1}/N_1\}.$$

The asymptotic behaviour of the $(M_1 + M_2 - 1)(N_1 + N_2 - 1)$ non-trivial roots of the equations $\omega_1(\xi^{(1)}) = \omega_1(z_1)$ and $\omega_2(\xi^{(2)}) = \omega_2(z_2)$ can be found via a singular perturbation expansion as before. Namely, using similar arguments to Section 2, we find that as $z_1 \to 0$, the $M_1 + M_2 - 1$ non-trivial roots of $\omega_1(\xi^{(1)}(z_1)) = \omega_1(z_1)$ become

$$\xi_m^{(1)}(z_1) \sim \begin{cases} \mathrm{e}^{2\pi\mathrm{i}m/M_1} z_1, & m = -M_1 + 1, \ldots, -1, \\ (\omega_{1,-M_1}/\omega_{1,M_2})^{1/M_2}\, \mathrm{e}^{2\pi\mathrm{i}m/M_2} z_1^{-M_1/M_2}, & m = 0, \ldots, M_2 - 1; \end{cases}$$

as $z_2 \to 0$, the $N_1 + N_2 - 1$ non-trivial roots of $\omega_2(\xi^{(2)}(z_2)) = \omega_2(z_2)$ are

$$\xi_n^{(2)}(z_2) \sim \begin{cases} \mathrm{e}^{2\pi\mathrm{i}n/N_1} z_2, & n = -N_1 + 1, \ldots, -1, \\ (\omega_{2,-N_1}/\omega_{2,N_2})^{1/N_2}\, \mathrm{e}^{2\pi\mathrm{i}n/N_2} z_2^{-N_1/N_2}, & n = 0, \ldots, N_2 - 1. \end{cases}$$

Thus, using $\xi_0^{(1)}(z_1), \ldots, \xi_{M_2-1}^{(1)}(z_1)$, each of the M_1 sectors in $S_m^{(1,0)}$ is mapped onto one of the M_2 sectors of $S_m^{(1,\infty)}$, and using $\xi_0^{(2)}(z_2), \ldots, \xi_{N_2-1}^{(2)}(z_2)$, each of the N_1 sectors in $S_n^{(2,0)}$ is mapped onto one of the N_2 sectors of $S_n^{(2,\infty)}$. We can then perform the substitutions $z_1 \to \xi_m^{(1)}(z_1)$ and $z_2 \to \xi_n^{(2)}(z_2)$ in the global relation for $m = 0, \ldots, M_2 - 1$, $n = 0, \ldots, N_2 - 1$. Applying these transformations in the discrete global relation (6.7), we then get $M_2 N_2$ algebraic equations

$$\hat{X}_1^{(1)}(z_1, z_2, t) + \hat{X}_1^{(2)}(z_1, z_2, t) + \mathrm{e}^{\mathrm{i}\omega(z_1,z_2)t}\hat{q}(\xi_m^{(1)}(z_1), \xi_n^{(2)}(z_2), t) = \hat{q}(\xi_m^{(1)}(z_1), \xi_n^{(2)}(z_2), 0) \quad (6.10)$$

for $m = 0, \ldots, M_2 - 1$; $n = 0, \ldots, N_2 - 1$. These are precisely the equation that allow us to solve for the unknown boundary data $\{\hat{q}_{m,n}(z_1, z_2, t)\}_{n=1,\ldots,N_2}^{m=1,\ldots,M_2}$ with $\{\hat{q}_{m,n}(z_1, z_2, t)\}_{n=-N_1+1,\ldots,0}^{m=-M_1+1,\ldots,0}$ given, then we can get the solution of (6.1) with given boundary data. As in Section 2, the left-hand side of (6.10) contains $\mathrm{e}^{\mathrm{i}\omega(z_1,z_2)t}\hat{q}(\xi_m^{(1)}(z_1), \xi_n^{(2)}(z_2), t)$, which is (apart from the change $z_1 \to \xi_m^{(1)}(z_1)$ and $z_2 \to \xi_n^{(2)}(z_2)$) just the transform of solution we are trying to recover. As before, however, this term gives zero contribution to the reconstruction formula (6.9). This is because the term $z_1^{m-1} z_2^{n-1}\, \mathrm{e}^{\mathrm{i}\omega(z_1,z_2)(t-t')} \times \hat{q}(\xi_m^{(1)}(z_1), \xi_n^{(2)}(z_2), t')$ is analytic and bounded in $D_+^{(1)} \cap D_+^{(2)}$, and therefore its integral over $\partial D_+^{(1)} \cup \partial D_+^{(2)}$ is zero.

EXAMPLE. Consider the diffusive–dispersive DLEE

$$\dot{q}_{m,n} = b(q_{m+1,n} - 2q_{m,n} + q_{m-1,n})/h^2 + (q_{m,n+1} - 3q_{m,n} + 3q_{m,n-1} - q_{m,n-2})/h^3,$$

where $b \in \mathbb{R}^+$ and the same lattice spacing in m and n was taken. As before, we take $h = 1$. Here, $M_1 = M_2 = 1$, $N_1 = 2$ and $N_2 = 1$. The dispersion relation is $\omega(z_1, z_2) = \omega_1(z_1) + \omega_2(z_2)$ with

$\omega_1(z_1) = \mathrm{i}b(z_1 - 2 + 1/z_1)$ and $\omega_2(z_2) = \mathrm{i}(z_2 - 3 + 3/z_2 - 1/z_2^2)$. Note that $\omega_1(z_1)$ equals b times the dispersion relation of (4.2), and $\omega_2(z_2)$ is same as the dispersion relation of (4.14). We obtain

$$X_{m,n}^{(1)} = b(q_{m,n} - q_{m-1,n}/z), \quad X_{m,n}^{(2)} = q_{m,n} - (3/z_2 - 1/z_2^2)q_{m,n-1} + 1/z_2 q_{m,n-2}.$$

The domains are

$$D_\pm^{(1)} = \{z_1 \in \mathbb{C}\colon |z_1| \lessgtr 1 \wedge \operatorname{Im} z_1 \gtrless 0\}, \quad D_\pm^{(2)} = \{z_2 \in \mathbb{C}\colon |z_2| \lessgtr 1 \wedge \operatorname{Im} \omega_2(z_2) > 0\},$$

with $D_\pm^{(2)}$ coinciding with the domain shown in Fig. 2 (right). The solution is given by (6.9) with

$$\hat{X}_1^{(1)}(z_1, z_2, t) = b[z_1 \hat{q}_{1,n}(z_1, z_2, t) - \hat{q}_{0,n}(z_1, z_2, t)], \tag{6.11a}$$

$$\hat{X}_1^{(2)}(z_1, z_2, t) = z_2 \hat{q}_{m,1}(z_1, z_2, t) - (3 - 1/z_2)\hat{q}_{m,0}(z_1, z_2, t) + \hat{q}_{m,-1}(z_1, z_2, t). \tag{6.11b}$$

Substituting (6.11) into (6.7). The global relation is, for $\forall z_1 \in \bar{D}_-^{(1)}$ and $\forall z_2 \in \bar{D}_-^{(2)}$,

$$\begin{aligned} b[z_1 \hat{q}_{1,n}(z_1, z_2, t) - \hat{q}_{0,n}(z_1, z_2, t)] + z_2 \hat{q}_{m,1}(z_1, z_2, t) - (3 - 1/z_2)\hat{q}_{m,0}(z_1, z_2, t) \\ + \hat{q}_{m,-1}(z_1, z_2, t) + \mathrm{e}^{\mathrm{i}\omega(z_1,z_2)t}\hat{q}(z_1, z_2, t) = \hat{q}(z_1, z_2, 0). \end{aligned}$$

We then eliminate the unknown boundary data by transformation $z_1 \to 1/z_1$ as for (4.2) and transformation $z_2 \to \xi_0(z_2)$ as for (4.14). We obtain, $\forall z_1 \in \bar{D}_-^{*(1)}, \forall z_2 \in \bar{D}_-^{*(2)}$,

$$\begin{aligned} b[1/z_1 \hat{q}_{1,n}(z_1, z_2, t) - \hat{q}_{0,n}(z_1, z_2, t)] + z_2 \hat{q}_{m,1}(z_1, z_2, t) - (3 - 1/z_2)\hat{q}_{m,0}(z_1, z_2, t) \\ + \hat{q}_{m,-1}(z_1, z_2, t) + \mathrm{e}^{\mathrm{i}\omega(z_1,z_2)t}\hat{q}(1/z_1, z_2, t) = \hat{q}(1/z_1, z_2, 0), \\ b[z_1 \hat{q}_{1,n}(z_1, z_2, t) - \hat{q}_{0,n}(z_1, z_2, t)] + \xi_0(z_2)\hat{q}_{m,1}(z_1, z_2, t) - (3 - 1/\xi_0(z_2))\hat{q}_{m,0}(z_1, z_2, t) \\ + \hat{q}_{m,-1}(z_1, z_2, t) + \mathrm{e}^{\mathrm{i}\omega(z_1,z_2)t}\hat{q}(z_1, \xi_0(z_2), t) = \hat{q}(z_1, \xi_0(z_2), 0), \end{aligned}$$

we can then solve for $\hat{q}_{m,1}(z_1, z_2, t)$ and $\hat{q}_{1,n}(z_1, z_2, t)$ with $\hat{q}_{m,0}(z_1, z_2, t)$, $\hat{q}_{m,-1}(z_1, z_2, t)$ and $\hat{q}_{0,n}(z_1, z_2, t)$ and substitute in (6.11) to obtain the reconstruction formula.

7. Fully discrete evolution equations

We now show how the method can be extended to solve IBVPs for a general class of fully DLEEs of the type $\mathrm{i}(q_n^{m+1} - q_n^m)/\Delta t = \omega(\mathrm{e}^{\partial_n})q_n^m$, which are the fully discrete analogue of (2.1). Equivalently, we write these equations as

$$q_n^{m+1} = W(\mathrm{e}^{\partial_n})q_n^m, \tag{7.1}$$

where $W(z) = 1 - \mathrm{i}\Delta t\omega(z)$ is an arbitrary fully discrete dispersion relation:

$$W(z) = \sum_{j=-J_1}^{J_2} c_j z^j, \tag{7.2}$$

Equation (7.1) admits the solution $q_n^m = z^n W^m$. It should then be clear that the role of the condition $\mathrm{Im}\,\omega(z) \gtrless 0$ will now be played by the condition $|W(z)| \gtrless 1$.

7.1 *The general method*

Equation (7.1) can be written as the compatibility relation of a fully discrete Lax pair, i.e. an overdetermined linear system

$$\Phi_{n+1}^m - z\Phi_n^m = q_n^m, \quad \Phi_n^{m+1} - W\Phi_n^m = X_n^m, \tag{7.3}$$

where $X_n^m(z)$ is given by the explicit formula

$$X_n^m(z) = \left[\frac{W(z) - W(s)}{z - s}\right]_{s=\mathrm{e}^{\partial_n}} q_n^m.$$

If $\Psi_n^m(z) = z^{-n}W^{-m}\Phi_n^m(z)$, then Ψ_n^m satisfies the modified fully discrete Lax pair:

$$\Psi_{n+1}^m = z^{-n}W^{-m+1}q_n^m, \quad \Psi_n^{m+1} = z^{-n+1}W^{-m}X_n^m. \tag{7.4}$$

Using Ψ_n^m, the compatibility of (7.4) (namely, the condition $\Delta_m(\Psi_{n+1}^m) = \Delta_n(\Psi_n^{m+1})$) can be written as $q_n^{m+1} - Wq_n^m = X_{n+1}^m - zX_n^m$ or equivalently as:

$$\Delta_m(z^{-n}W^{-m+1}q_n^m) = \Delta_n(z^{-n+1}W^{-m}X_n^m), \tag{7.5}$$

where $\Delta_m Q_m = Q_{m+1} - Q_m$ and $\Delta_n Q_n = Q_{n+1} - Q_n$ are the finite-difference operators.

Let $\hat{q}^m(z)$, $\hat{g}_n^m(z)$ and $\hat{X}_1^m(z)$ be the z-transforms of the initial condition and BC, respectively:

$$\hat{q}^m(z) = \sum_{n=1}^{\infty} z^{-n}q_n^m, \quad \hat{g}_n^m(z) = \sum_{m'=0}^{m} W^{-m'}q_n^{m'}, \quad \hat{X}_1^m(z) = \sum_{m'=0}^{m} z\,W^{-m'}X_1^{m'}. \tag{7.6}$$

Summing (7.5) from $n = 1$ to ∞, we obtain, for $|z| \geqslant 1$,

$$\Delta_m(W^{-m+1}\hat{q}^m(z)) = -zW^{-m}X_1^m(z). \tag{7.7}$$

Suppose $q_n^0 \in l^1(\mathbb{N})$. Then, $\hat{q}^m(z)$ is defined $\forall\, z \in \mathbb{C}$ with $|z| \geqslant 1$ and is analytic for $|z| > 1$, while $\hat{X}_1^m(z)$ is defined $\forall\, z \in \bar{D}$ and is analytic $\forall\, z \in D$, where $D = \{z \in \mathbb{C}\colon |W(z)| > 1\}$. Similar to the semidiscrete case, we decompose D as $D = D_+ \cup D_-$, where $D_\pm = \{z \in \mathbb{C}\colon |z| \lessgtr 1 \wedge |W(z)| > 1\}$. Summing (7.7) from $m' = 0$ to m, we get, for $|z| \geqslant 1$,

$$\hat{q}^m(z) = W^{m-1}(z)[W(z)\hat{q}^0(z) - \hat{X}_1^m(z)]. \tag{7.8}$$

Equation (7.8) is the fully discrete version of the global relation, which contains all known and unknown initial-boundary data. Again, the inverse transform of $\hat{q}^m(z)$ is obtained by noting that the q_n^m are the

Laurent coefficients of $\hat{q}^m(z)$. Using (7.6) then yields, $\forall\, n \in \mathbb{N}, \forall\, m \in \mathbb{N}_0$,

$$q_n^m = \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} z^{n-1} W^m(z) \hat{q}^0(z) \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} z^{n-1} W^{m-1}(z) \hat{X}_1^m(z) \mathrm{d}z. \tag{7.9}$$

Again, we use contour deformation to move the integration contour for the second integral in (7.9) away from the unit circle. Since $z^n W^{m-1} \hat{X}_1^m(z)$ is analytic $\forall\, z \neq 0$ and bounded and continuous for $z \in \bar{D}$, we can deform that integration contour from $|z| = 1$ to $z \in \partial D_+$ to obtain the reconstruction formula, $\forall\, n \in \mathbb{N}, \forall\, m \in \mathbb{N}_0$,

$$q_n^m = \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} z^{n-1} W^m(z) \hat{q}^0(z) \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{\partial D_+} z^{n-1} W^{m-1}(z) \hat{X}_1^m(z) \mathrm{d}z. \tag{7.10}$$

As in the semidiscrete case, the solution in (7.10) depends on the unknown data $q_1^m, \ldots, q_{J_2}^m$ via their spectral transforms $\hat{q}_1^m(z), \ldots, \hat{q}_{J_2}^m(z)$ appearing in $\hat{X}_1^m(z)$. Thus, in order for the method to yield an effective solution, one must be able to express these unknown boundary values in terms of known quantities. In fact, an immediate consequences of the method is that it allows one to verify that, to make the IBVP (7.1) well posed on the naturals, one needs to assign exactly J_1 BCs at $n = 0$. As before, the elimination of the unknown boundary data can be accomplished using the global relation (7.8) together with the symmetries of the equation. The spectral function $\hat{X}_1^m(z)$ is invariant under any transformation that leaves the dispersion relation (7.2) invariant; i.e. any map $z \mapsto \zeta(z)$ such that $W(\zeta(z)) = W(z)$. Note that, as a result, we have $\zeta(D_\pm) \subseteq D_\pm$. The equation $W(z) = W(\zeta(z))$ has $J_1 + J_2 - 1$ non-trivial roots in addition to the trivial root $\zeta = z$. Using these symmetries in the global relation will allow us to eliminate the unknown boundary data. In order to do so, however, one needs to identify which of the $J_1 + J_2 - 1$ non-trivial roots are useful for this purpose. As before, we look at the asymptotic behaviour of these roots as $z \to 0$ and $z \to \infty$. As $z \to \infty$, we have $W(z) \sim c_{J_2} z^{J_2}$. Similarly, as $z \to 0$, we have $W(z) \sim c_{-J_1} z^{-J_1}$. Thus, as $z \to \infty$ and as $z \to 0$, the domains D_- and D_+ are, respectively, asymptotically equivalent to the domains

$$S^{(\infty)} = \bigcup_{j=0}^{J_2-1} S_j^{(\infty)}, \qquad S^{(0)} = \bigcup_{j=-J_1+1}^{0} S_j^{(0)},$$

where, for $j = 0, \ldots, J_2 - 1$,

$$S_j^{(\infty)} = \{z \in \mathbb{C}\colon 2j\pi/J_2 < \arg z < (2j+1)\pi/J_2\}.$$

while, for $j = -J_1 + 1, \ldots, 0$,

$$S_j^{(0)} = \{z \in \mathbb{C}\colon (2j-1)\pi/J_1 < \arg z < 2j\pi/J_1\}.$$

Again, the asymptotic behaviour of the $J_1 + J_2 - 1$ non-trivial roots of the equation $W(\zeta) = W(z)$ can be found by a singular perturbation expansion. Using similar arguments as in Section 2, we have that the $J_1 + J_2 - 1$ non-trivial roots for equation $W(\zeta(z)) = W(z)$ as $z \to 0$

$$\zeta_n(z) \sim \begin{cases} \mathrm{e}^{2\pi \mathrm{i} n/J_1} z, & n = -J_1 + 1, \ldots, -1, \\ (c_{-J_1}/c_{J_2})^{1/J_1}\, \mathrm{e}^{2\pi \mathrm{i} n/J_2} z^{-J_1/J_2}, & n = 0, \ldots, J_2 - 1. \end{cases}$$

From the above calculations we now see that, using $\zeta_0(z), \ldots, \zeta_{J_2-1}(z)$, each of the J_1 sectors in $S_j^{(0)}$ is mapped onto one of the J_2 sectors of $S_j^{(\infty)}$. Applying transformations $z \to \zeta_n(z)$, $n = 0, \ldots, J_2 - 1$ in the discrete global relation (7.8), we then obtain J_1 algebraic equations:

$$\hat{X}_1^m(z) + W^{-m+1}(z)\,\hat{q}^m(\zeta_n(z)) = W(z)\hat{q}^0(\zeta_n(z)) \tag{7.11}$$

for $n = -J_1 + 1, \ldots, 0$. These are precisely the equations that allow us to solve for the unknown boundary data $\hat{q}_1^m(z), \ldots, \hat{q}_{J_2}^m(z)$ with $\hat{q}_{-J_1+1}^m(z), \ldots, \hat{q}_0^m(z)$ given, and then substitute them in (7.10), we gain the solution of (7.1) with given boundary data. Again, the left-hand side of (7.11) contains the unknown term $W^{-m+1}(z)\hat{q}^m(\zeta_n(z))$. As before, however, this term gives zero contribution to the reconstruction formula (7.10) thanks to analyticity.

EXAMPLE. Consider the fully discrete convection–diffusion equation

$$(q_n^{n+1} - q_n^m)/\Delta t = c(q_{n+1}^m - q_{n-1}^m)/h + (q_{n+1}^m - 2q_n^m + q_{n-1}^m)/h^2, \tag{7.12}$$

with $c \in \mathbb{R}$. Letting $\Delta t = h = 1$, we have

$$q_n^{m+1} = (1+c)q_{n+1}^m - q_n^m + (1-c)q_{n-1}^m.$$

The fully discrete dispersion relation $W(z) = (1+c)z - 1 + (1-c)/z$, implying $J_1 = J_2 = 1$ and $X_n^m(z) = (1+c)q_n^m - (1-c)q_{n-1}^m/z$. The domains $D_\pm = \{z \in \mathbb{C}\colon |z| \lessgtr 1 \wedge |W(z)| > 1\}$. We obtain, for $|z| \geqslant 1$,

$$\hat{X}_1^m(z) = (1+c)z\hat{g}_1^m(z) - (1-c)\hat{g}_0^m(z), \tag{7.13}$$

which contains the unknown datum q_1^m. The domains $D_\pm$ and their boundary for some values of c are shown in Fig. 6. For $c \neq 1$, as $z \to 0$, D_+ is asymptotically equivalent to $S^{(0)} = \{z \in \mathbb{C}\colon 0 < \arg z < 2\pi\}$, while as $z \to \infty$, the domain D_- is asymptotically equivalent to $S^{(\infty)} = \{z \in \mathbb{C}\colon 0 < \arg z < 2\pi\}$. Note that the values $c = \pm 1$ are special cases since $D_\pm$ change character at these two points (see Fig. 6).

Inserting (7.13) into (7.8), we have, $\forall\, z \in \bar{D}_-$,

$$W^{m-1}[(1+c)z\hat{g}_1^m(z) - (1-c)\hat{g}_0^m(z)] + \hat{q}^m(z) = W^m\hat{q}^0(z). \tag{7.14}$$

Taking $z \to \nu_c/z$ in (7.14) (where $\nu_c = (1-c)/(1+c)$ for $c \neq \pm 1$), we then get, $\forall\, z \in \bar{D}_+^*$,

$$W^{m-1}[(1-c)\hat{g}_1^m(z)/z - (1-c)\hat{g}_0^m(z)] + \hat{q}^m(\nu_c/z) = W^m\hat{q}^0(\nu_c/z).$$

After straightforward calculations, for $c \neq \pm 1$, we then obtain the solution of the IBVP as

$$q_n^m = \frac{1}{2\pi\,\mathrm{i}}\int_{|z|=1} z^{n-1}W^m\hat{q}^0(z)\mathrm{d}z$$
$$- \frac{1}{2\pi\,\mathrm{i}}\int_{\partial D_+} z^{n-1}W^{m-1}\{[(1+c)z^2 - (1-c)]\hat{g}_0^M(z) + z^2W\hat{q}^0((2-c)/[z(2+c)])/\nu_c\}\mathrm{d}z.$$

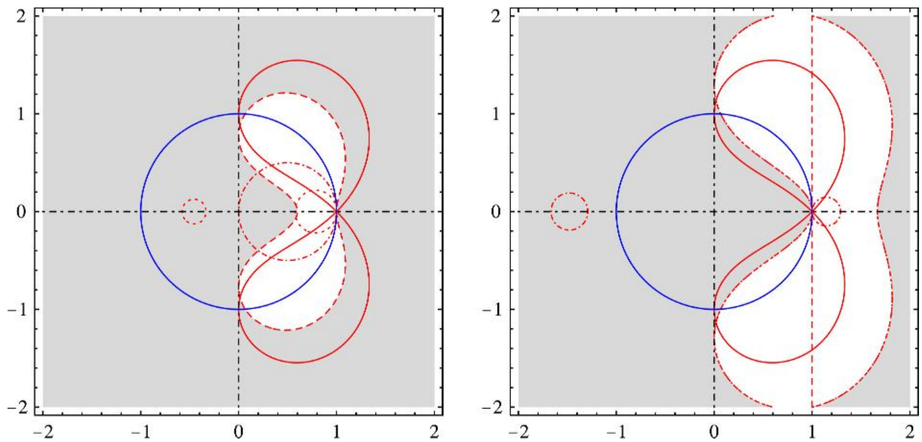

FIG. 6. The boundaries of the regions $D_\pm$ for (7.12) for various values of c. Left: $c = 0$ (solid), $c = 1/4$ (dashed), $c = 1$ (dot-dashed) and $c = 2$ (dotted). Right: $c = 0$ (solid), $c = -1/4$ (dotted), $c = -1$ (dashed) and $c = -4$ (dot-dashed). The shaded regions show the domains $D_\pm$ for $c = 1/4$ (left) and $c = -1/4$ (right).

Therefore, only one BC is needed at $n = 0$ for $c \neq \pm 1$, i.e. q_0^m. For $c = \pm 1$, we can use similar methods as in (4.18) to find the solution of IBVPs (7.12), as we show next.

When $c = 1$, we have $\hat{X}_1^m(z) = 2z\hat{g}_1^m(z)$ for $|z| \geqslant 1$. The domain D is the union of $D_+ = \{z \in \mathbb{C}\colon |z| < 1 \wedge |z - 1/2| > 1/2\}$ and $D_- = \{z \in \mathbb{C}\colon |z| > 1\}$. The global relation is, for $|z| > 1$,

$$2W^{m-1}z\hat{g}_1^m(z) + \hat{q}^m(z) = W^m\hat{q}^0(z).$$

But the term $z^{n-1}W^{m-1}(z)\hat{g}_1^m(z)$ is analytic $\forall\, z \neq 0$ and bounded for $|z| \leqslant 1$. By (7.10), the solution is

$$q_n^m = \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}W^m\hat{q}^0(z)\mathrm{d}z.$$

When $c = -1$, we get $\hat{X}_1^m(z) = -2\hat{g}_0^m(z)$. The domain D is the union of $D_+ = \{z \in \mathbb{C}\colon |z| < 1\}$ and $D_- = \{z \in \mathbb{C}\colon \operatorname{Re} z < 1 \wedge |z| > 1\}$. The global relation (7.8) yields, for $|z| > 1$,

$$-2W^{m-1}\hat{g}_0^m(z) + \hat{q}^m(z) = W^m\hat{q}^0(z).$$

Now $W(\zeta) = W(z)$ has nontrivial root, so no elimination is possible. The solution is thus

$$q_n^m = \frac{1}{2\pi\mathrm{i}}\int_{|z|=1} z^{n-1}W^m\hat{q}^0(z)\mathrm{d}z + \frac{1}{\pi\mathrm{i}}\int_{\partial D_+} z^{n-1}W^{m-1}\hat{g}_0^m(z)\mathrm{d}z.$$

Summarizing, no BC is needed when $c = 1$, and the BC q_0^m is needed when $c = -1$.

8. Concluding remarks

We have presented a method to solve IBVPs for DLEEs. The method, which is quite general but simple to implement, yields an integral representation of the solution of the IBVP. It also provides an easy way to check the number of BCs that are needed at the lattice boundary in order for the IBVP to be well posed. The method also applies for forced equations, DLEEs that are higher order in time, systems of DLEEs, fully discrete evolution equations and DLEEs with more than one lattice variable. As such, it works for many IBVPs that cannot be treated with Fourier sine/cosine series and/or Laplace transforms. In the previous sections, we pointed out several cases that cannot be treated with Fourier methods. As for Laplace transform methods, they are ineffective for IBVPs for (2 + 1)-dimensional equations since the application of Laplace transforms in this case yields a boundary-value problem for a partial difference equation on the same ‘spatial’ domain as the original IBVP. Moreover, Laplace transform methods are not applicable to IBVPs for fully discrete (difference-difference) equations. Even when a Laplace transform approach can be used, the present method has several advantages compared to it, since the use of Laplace transforms: (i) leads to complicated expressions involving terms $z^{\lambda(s)}\mathrm{e}^{-st}$, where $\lambda(s)$ is the solution of the ‘implicit’ equation $s + \mathrm{i}\omega(\lambda) = 0$, as opposed to expressions of the type $z^n \mathrm{e}^{-\mathrm{i}\omega(z)t}$, where $\omega(z)$ is explicit, in the present method; (ii) requires t going to infinity, which is unnatural for an evolution equation. Finally, unlike Fourier or Laplace methods, the present method can also be non-linearized to solve IBVPs for integrable non-linear differential-difference evolution equations, as demonstrated in Biondini & Hwang (2008).

Finally, let us briefly comment on the relation between our method and the Wiener–Hopf (WH) method. WH problems typically arise in elliptic problems, for regular domains, and when the BCs change type (e.g., see Lawrie & Abrahams, 2007; Noble, 1988). The problems treated in our work are of evolution type. Nonetheless, a relationship between the WH method and our method does exist. As discussed in Fokas (2008), for IBVPs for PDEs in simple domains the global algebraic relation and the equations obtained using the symmetries of the problem provide a generalization of the WH technique. The same is true for the discrete evolution equations that are the subject of our work. Moreover, it is well known that the application of the WH technique is *ad hoc* and problem dependent; again, see Lawrie & Abrahams (2007) and references therein. In contrast, our method is essentially algorithmic: the analyticity properties of the relevant functions in the spectral domain are determined by construction. In contrast, one would have to use an *ad hoc* approach on a case-by-case basis to formulate a WH problem with equivalent properties. So, in this context, one can view our method as an effectivization and a generalization of the WH method for the kinds of IBVPs considered here.

The integral representation of the solution obtained by the present method is the practical implementation of the Ehrenpreis principle (e.g., see Ehrenpreis, 1970; Henkin, 1990; Palamodov, 1970). As such, it is especially convenient in order to compute the long-term asymptotics of the solution using the steepest descent method. Also, since the integrals in the reconstruction formula are uniformly convergent, even when they cannot be calculated exactly they provide a convenient way to evaluate the solution numerically. We therefore believe that this method will also prove to be a useful comparison test for finite-difference discretizations of IBVPs for linear PDEs.

We showed in detail how the elimination of the unknown boundary data works for a semiinfinite range of integers. The same techniques can be used to solve IBVPs on finite ranges of integers. Indeed, using similar arguments as the ones in Section 2, it is easy to show that for the IBVP on the finite domain $0 \leqslant n \leqslant N$, one also needs to assign exactly J_2 BCs at $n = N$.

While the main steps of the method are similar to the continuum case, its implementation presents some significant differences. One such difference arises in the elimination of the unknown boundary data, where instead of the asymptotic behaviour of the dispersion relation $\omega(k)$ at the single point $k = \infty$ in the continuum case, one needs the asymptotic behaviour of $\omega(z)$ as $z \to \infty$ and as $z \to 0$. This difference is understood intuitively by recalling that $z = \mathrm{e}^{\mathrm{i}kh}$, and therefore there are two limiting points corresponding to $k = \infty$, depending on whether $\operatorname{Im} k \gtrless 0$. Perhaps more importantly, even when the DLEE has a continuum limit as $h \to 0$, the number of BCs to be assigned in the discrete case is determined by the specific finite-difference stencil considered, and it does not coincide in general with the number of BCs needed in the continuum case. The unknown boundary data in the continuum case are the spatial derivatives at the origin, and their number depends on the order and sign of the highest spatial derivative in the PDE (which also determines its characteristics). In the discrete case, the unknown boundary data are the first J_2 values of the solution inside the lattice. Therefore, even when the discrete dispersion relation is a finite-difference approximation of a continuous one, the number of unknown boundary values is determined by the order of accuracy of the finite-difference stencil not by the order of derivative that it represents.

REFERENCES

ABLOWITZ, M. J. (1977) Nonlinear evolution equations—continuous and discrete. *SIAM Rev.*, **19**, 663–684.

ABLOWITZ, M. J., HALBURD, R. G. & HERBST, B. (2000) On the extension of the Painlevé property to difference equations. *Nonlinearity*, **13**, 889–905.

ABLOWITZ, M. J. & LADIK, J. F. (1975) Nonlinear differential-difference equations. *J. Math. Phys.*, **16**, 598.

ABLOWITZ, M. J. & LADIK, J. F. (1976) Nonlinear differential-difference equations and Fourier analysis. *J. Math. Phys.*, **17**, 1011.

ABLOWITZ, M. J., PRINARI, B. & TRUBATCH, A. D. (2003) *Discrete and Continuous Nonlinear Schrdinger Systems*. Cambridge: Cambridge University Press.

ABLOWITZ, M. J. & SEGUR, H. (1975) The inverse scattering transform: semi-infinite interval. *J. Math. Phys.*, **16**, 1054–1056.

BIKBAEV, R. F. & TARASOV, V. O. (1991) Initial-boundary value problem for the nonlinear Schrdinger equation. *J. Phys. A: Math. Gen.*, **24**, 2507–2516.

BIONDINI, G. & HWANG, G. (2008) Initial-boundary value problems for differential-difference evolution equations: discrete linear and integrable nonlinear Schrdinger equations. *Inverse Probl.*, **24**, 065011: 1–43.

BIONDINI, G. & HWANG, G. (2009) Solitons, boundary value problems and a nonlinear method of images. *J. Phys. A*, **42**, 205207:1–17.

DEGASPERIS, A., MANAKOV, S. V. & SANTINI, P. M. (2001) On the initial-boundary value problems for soliton equations. *JETP Lett.*, **74**, 481–485.

DEGASPERIS, A., MANAKOV, S. V. & SANTINI, P. M. (2002) Initial-boundary value problems for linear and soliton PDEs. *Theor. Math. Phys.*, **133**, 1475–1489.

EHRENPREIS, L. (1970) *Fourier Analysis in Several Complex Variables*. New York: Wiley.

FLASCHKA, H. (1974a) On the Toda lattice. 1. Existence of integrals. *Phys. Rev. B*, **9**, 1924–1925.

FLASCHKA, H. (1974b) On the Toda lattice. 2. Inverse-scattering solution. *Prog. Theor. Phys.*, **51**, 703.

FOKAS, A. S. (1997) A unified transform method for solving linear and certain nonlinear PDEs. *Proc. R. Soc. A*, **453**, 1411–1443.

FOKAS, A. S. (2000) On the integrability of certain linear and nonlinear partial differential equations. *J. Math. Phys.*, **41**, 4188–4237.

FOKAS, A. S. (2002) A new transform method for evolution partial differential equations. *IMA J. Appl. Math.*, **67**, 559–590.

FOKAS, A. S. (2005) A transform method for linear evolution PDEs on a finite interval. *IMA J. Appl. Math.*, **70**, 564–587.

FOKAS, A. S. (2008) *A Unified Approach to Boundary Value Problems*. Philadelphia, PA: SIAM.

FOKAS, A. S. & GELFAND, I. M. (1994) Integrability of linear and nonlinear evolution equations and associated Fourier transforms. *Lett. Math. Phys.*, **32**, 189–210.

FOKAS, A. S., ITS, A. R. & SUNG, L.-Y. (2005) The nonlinear Schrdinger equation on the half-line. *Nonlinearity*, **18**, 1771–1822.

FOKAS, A. S. & PELLONI, B. (1998) The solution of certain initial boundary-value problems for the linearized Korteweg-de Vries equations. *Proc. R. Soc. A*, **454**, 645–657.

FOKAS, A. S. & PELLONI, B. (2001) Two-point boundary value problems for linear evolution equations. *Math. Proc. Camb. Philol. Soc.*, **131**, 521–543.

HABIBULLIN, I. (1995) Boundary conditions for integrable chains. *Phys. Lett. A*, **207**, 263–268.

HENKIN, G. (1990) Method of integral representations in complex analysis. *Encyclopedia of Mathematical Sciences. Several Complex Variables*, vol. 3. Berlin, Germany: Springer.

HIROTA, R., ITO, M. & KAKO, F. (1988a) Two-dimensional Toda lattice equations. *Prog. Theor. Phys. Suppl.*, **94**, 42–58.

HIROTA, R., OHTA, Y. & SATSUMA, J. (1988b) Wronskian structures of solutions for soliton equations. *Prog. Theor. Phys. Suppl.*, **94**, 59–72.

KHABIBULLIN, I. T. (1991) Integrable initial-boundary-value problems. *Theor. Math. Phys.*, **28**, 28–36.

LAWRIE, J. B. & ABRAHAMS, I. D. (2007) A brief historical perspective on the Wiener-Hopf technique. *J. Eng. Math.*, **59**, 351–358.

MARUNO, K.-I. & BIONDINI, G. (2004) Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete version. *J. Phys. A*, **37**, 11819–11839.

NOBLE, B. (1988) *Methods Based on the Wiener-Hopf Technique*. New York: American Mathematical Society.

PALAMODOV, V. P. (1970) *Linear Differential Operators with Constant Coefficients*. Berlin, Germany: Springer.

RAGNISCO, O. & SANTINI, P. M. (1990) A unified algebraic approach to integral and discrete evolution equations. *Inverse Probl.*, **6**, 441–452.

SABATIER, P. C. (2006) Generalized inverse scattering transform applied to linear partial differential equations. *Inverse Probl.*, **22**, 209–228.

SKYLANIN, E. K. (1987) Boundary conditions for integrable equations. *Funct. Anal. Appl.*, **21**, 86–87.

TARASOV, V. O. (1991) The integrable initial-boundary value problem on a semiline: nonlinear Schrdinger equation and sine-Gordon equation. *Inverse Probl.*, **7**, 435–449.

TODA, M. (1975) Studies of a nonlinear lattice. *Phys. Rep.*, **18**, 1–123.

TREHARNE, P. A. & FOKAS, A. S. (2004) Boundary value problems for systems of linear evolution equations. *IMA J. Appl. Math.*, **69**, 539–555.