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Soliton interactions and degenerate soliton complexes for the focusing nonlinear Schrödinger equation with nonzero background

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Abstract. We characterize soliton interactions in focusing media described by the nonlinear Schrödinger equation in the presenze of a nonzero background field, including the cases of bound states (degenerate soliton trains) and interactions between solitons and Akhmediev breathers. We first characterize bound states, which, as in the case of zero background, are obtained when several solitons travel with the same velocity. We then turn to the case when the soliton velocities are distinct, and we compute the long-time asymptotic behavior of soliton interactions by calculating the position shift for each soliton as $t \to \pm\infty$. We also identify conditions that give rise to large position shifts. Moreover, we characterize the asymptotic phase of the nonzero background in each sector of the xt-plane that is separated by individual solitons or breathers, and we show that the asymptotic phase can be easily determined from whether the region is on the left or on the right of a soliton or an Akhmediev breather.

1 Introduction

Self-focusing media exhibit a variety of interesting phenomena, including modulational instability, supercontinuum generation, critical focusing and collapse, recurrence, chaotic phenomena, integrable turbulence, and rogue waves. As such, they continue to be the subject of intense study (*e.g.*, see [1–20] and references therein). The most commonly used model to study self-focusing media is the nonlinear Schrödinger (NLS) equation, which describes modulations of weakly nonlinear dispersive wave trains in many different physical contexts, such as deep water waves, optical fibers, plasmas and attracting Bose-Einstein condensates [21–26].

The focusing NLS equation is of course a completely integrable infinite-dimensional Hamiltonian system. As such, it possesses a rich family of exact soliton solutions, and it can be studied via the inverse scattering transform (IST) [21,27–29]. The most widely known exact and explicit solutions are the N-soliton generalization of sech-shaped solitons [27], which are generated from a purely discrete spectrum in the associated scattering problem. Soliton interactions were first studied in [27], where the interaction-induced position and phase shifts as $t \to \pm\infty$ were also calculated. Those results were then generalized and extended in a number of later works. Recent studies also considered interactions among so-called "degenerate" solutions [30] (*i.e.*, soliton complexes in which two or more of the solitons have the same velocity) and so-called "multi-pole" solutions [27,31,32] (*i.e.*, soliton solutions corresponding to discrete eigenvalues of higher order). The IST and the study of the corresponding soliton interactions have also been generalized to coupled NLS systems [29,33,34].

The above works studied solutions that are spatially localized (namely, solutions that tend to zero at infinity), *i.e.*, solutions with zero boundary conditions (ZBC). On the other hand, various NLS systems with nonzero boundary conditions (NZBC), *i.e.*, in which the solution tends to nonzero constants at infinity, are also relevant in many applications, and have received considerable interest in recent years (*e.g.*, see [35–47] and references therein).

NLS systems with NZBC possess a larger variety of exact soliton solutions than with ZBC. This is already true in the defocusing case [48–52], and it is even more true for the scalar focusing NLS equation with NZBC. The most general

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one-solutions of the scalar focusing NLS equation with NZBC are the so-called Tajiri-Watanabe solitons [53], which are breather-like traveling solutions. As special cases, these solutions reduce to the Kuznetsov-Ma solitons [54, 55], periodic solutions (Akhmediev breathers) [56] and rational (or rogue-wave) solutions such as the Peregrine soliton [57]. The IST for the focusing NLS equation with NZBC, a spectral classification of all these solutions, and an expression for general N-soliton solutions were recently presented in [38]. Note that, even though Akhmediev breathers and rational solutions are outside the class of solutions studied in [38], they can be obtained as a limit.

Soliton interactions for the scalar defocusing NLS equation with NZBC were studied in [48]. Vector generalizations of the IST in the focusing and defocusing cases with NZBC was then presented in [35, 36, 40, 43] and vector soliton interactions in the defocusing case with NZBC were recently studied in [37, 41, 42]. With one exception [44], however, soliton interactions for the focusing NLS equation with NZBC have not been well studied. The purpose of this work is to address this deficiency and characterize the soliton interactions of the focusing NLS equation with NZBC, including the special case of bound states and cases in which one of the solitons is an Akhmediev breather.

The outline of this work is the following. In sect. 2 we present the formalism for describing the exact N-soliton solutions of the NLS equation with NZBC which is then used in the rest of the work to study soliton interactions. We also study in detail the properties of one-soliton solutions, including Akhmediev breathers. In sect. 3 we discuss the case of bound states, *i.e.*, solutions in which all N solitons have the same velocity. In sect. 4 we study soliton interactions in which none of the solitons is an Akhmediev breather. For simplicity, we limit ourselves to the case of 2-soliton solutions. In sect. 5 we study soliton interactions in which one of the solitons is an Akhmediev breather. Finally, sect. 6 offers some concluding remarks. A few technical results are confined to the appendix.

2 Multi-soliton solutions with NZBC

2.1 NLS equation and solution formulae

We begin by writing the focusing NLS equation in the form

$$iq_t + q_{xx} + 2(|q|^2 - q_o^2)q = 0, \tag{1}$$

where subscripts x and t denote partial differentiation and $q_o > 0$ is the nonzero background amplitude. Namely, we consider solutions with boundary conditions of the form $q(x,t) \to q_\pm$ as $x \to \pm\infty$ with $|q_\pm| = q_o$. The extra term $2q_o^2 q$, which can be removed by the trivial gauge transformation $q(x,t) \mapsto q(x,t)\,\mathrm{e}^{2iq_o^2 t}$, was added to eq. (1) so that the values $q_\pm$ are independent of time. In the context of nonlinear optics, the roles of the spatial and temporal variables are reversed.

Recall that the NLS equation (1) possesses a scaling invariance, namely, if $q(x,t)$ is a solution of eq. (1), then $\tilde{q}(x,t) = aq(ax, a^2 t)$ is also a solution for all real values of a. The NLS equation (1) also admits a phase invariance, *i.e.*, if $q(x,t)$ is a solution, so is $q'(x,t) = \mathrm{e}^{i\phi} q(x,t)$ for all $\phi \in \mathbb{R}$. Thus, without loss of generality, we take the boundary condition as

$$q_- = 1, \qquad x \to -\infty, \tag{2}$$

in the rest of this work. In other words, we take $q_o = |q_\pm| = 1$.

A general expression for the exact N-soliton solution of the focusing NLS equation with NZBC was derived in [38]. In particular, it was shown that the N-soliton solution is uniquely identified in terms of $2N$ complex parameters: the N distinct discrete eigenvalues $\zeta_1, \ldots, \zeta_N$ (where $\zeta = k + \lambda$ is the uniformization variable, $\lambda = (k^2 + q_o^2)^{1/2}$ and k is the scattering parameter, see appendix A.1 for details) and the corresponding norming constants $C_1, \ldots, C_N$. We parameterize them, respectively, as

$$\zeta_j = iz_j \mathrm{e}^{i\alpha_j}, \qquad C_j = \mathrm{e}^{\xi_j + i\phi_j}, \tag{3}$$

for $j = 1, \ldots, N$, with $z_j \geqslant 1$, $-\pi/2 < \alpha_j < \pi/2$, $\xi_j \in \mathbb{R}$ and $0 \leqslant \phi_j < 2\pi$. Note in particular the conditions $|\zeta_j| \geqslant 1$ and $\operatorname{Im}\zeta_j \geqslant 0$, which are a consequence of the definition of the uniformization variable and the parametrization of the discrete eigenvalues within the context of the IST. In total, once amplitude and phase for the nonzero background are specificed, an N-soliton solution is completely determined by $4N$ real parameters (*e.g.*, as given by (3)), just like in the case of ZBC. We will discuss how these parameters relate to the physical properties of solitons in later sections.

In appendix A.2 we show that the exact N-soliton solutions of the focusing NLS equation (1) with NZBC (2) can be written as

$$q(x,t) = q_- \mathrm{e}^{i\alpha_s} \det \begin{pmatrix} (I - 1_{2N} L^*) Z^* & -\mathit{\Gamma} L \\ \mathit{\Gamma}^* L^* & Z^{-1} \end{pmatrix} \Big/ \det \begin{pmatrix} I & -\mathit{\Gamma} L \\ \mathit{\Gamma}^* L^* & I \end{pmatrix}, \tag{4}$$

where $\alpha_s = -4\sum_{j=1}^{N}\alpha_j$, the asterisk denotes complex conjugate, I is the $2N\times 2N$ identity matrix, $1_{2N} = \mathbf{1}\otimes\mathbf{1}$, with $\mathbf{1} = (1,\dots,1)^T$ and $\otimes$ denoting the outer product, and

$$L(x,t) = \mathrm{diag}(l_1(x,t),\dots,l_{2N}(x,t)), \tag{5a}$$

$$Z = \mathrm{diag}(1/\zeta_1,\dots,1/\zeta_{2N}), \qquad \mathit{\Gamma} = (\gamma_{m,n})_{2N\times 2N}, \tag{5b}$$

with $\gamma_{j,j'} = 1/(\zeta_j^* - \zeta_{j'})$ and

$$l_j(x,t) = C_j \mathrm{e}^{-2i\theta(x,t,\zeta_j)} = \mathrm{e}^{\chi_j(x,t)+is_j(x,t)}, \tag{5c}$$

$$\chi_j(x,t) = c_{-,j}\cos\alpha_j\, x + d_{+,j}\sin 2\alpha_j\, t + \xi_j, \tag{5d}$$

$$s_j(x,t) = c_{+,j}\sin\alpha_j\, x - d_{-,j}\cos 2\alpha_j\, t + \phi_j, \tag{5e}$$

$$c_{\pm,j} = z_j \pm 1/z_j, \qquad d_{\pm,j} = z_j^2 \pm 1/z_j^2, \tag{5f}$$

with $\theta(x,t,\zeta) = \lambda(\zeta)[x - 2k(\zeta)t]$, for all $j,j' = 1,\dots,2N$ and symmetries

$$\chi_{j+N} = \chi_j - 2\ln z_j, \qquad s_{j+N} = -s_j + 2\alpha_j, \tag{6}$$

where the constants z_j, α_j, ξ_j and ϕ_j for $j = 1,2,\dots,N$ are still given by eq. (3). As shown in appendix A.2, for $j = N+1,\dots,2N$, similar quantities are given by the symmetric expressions

$$z_j = -1/z_{j-N}, \qquad \alpha_j = \alpha_{j-N}, \tag{7a}$$

$$\xi_j = \xi_{j-N} - 2\ln z_{j-N}, \qquad \phi_j = -\phi_{j-N} + 2\alpha_{j-N} + \pi. \tag{7b}$$

Importantly, eq. (4) holds even when some of the z_j are equal to 1. In other words, the formula includes all three types of solitons of the focusing NLS equation with NZBC: "stationary" Kuznetsov-Ma breathers ($z_j > 1$ and $\alpha_j = 0$), traveling Tajiri-Watanabe breathers ($z_j > 1$ and $\alpha_j \neq 0$), and periodic Akhmediev breathers ($z_j = 1$ and $\alpha_j \neq 0$). Therefore, eq. (4) is a convenient way to analyze the behavior of two-soliton solutions of the focusing NLS equation (1).

It is important to note that: i) the xt-dependence of the solution in eq. (4) comes only from matrices $L(x,t)$ and $L^*(x,t)$, while all other components are constants; ii) these matrices L and L^* are multiplied on the right to each block, so each column of the matrices in the determinants has the same exponential function; iii) each exponential function corresponds to only one of the discrete eigenvalues; iv) $\mathit{\Gamma}$ is a square Cauchy matrix; v) both determinants in eq. (4) are $4N\times 4N$, whereas in [38] the two determinants have different sizes. Later on, one will see that all these new properties of solution (4) will be useful in the calculation of long-time asymptotics.

We should clarify that, for ordinary solitons, the phrase "long-time asymptotic behavior" is used to mean literally the asymptotics of the solution as $t\to\infty$, $x = Vt$ and V constant. The meaning will be slightly different in the analysis of Akhmediev breathers, however. This is because the long-time behavior of Akhmediev breathers is trivial, because these solutions decay exponentially to the nonzero background as $t\to\pm\infty$, *i.e.*, these solutions are localized in time. Thus, in this case it will be more instructive to consider the asymptotics as $x\to\pm\infty$ instead.

2.2 Characterization of one-soliton solutions

We first briefly review the main features of one-soliton solutions, *i.e.*, $N = 1$, to set up notation and terminology that will be used to discuss soliton interactions.

In the special case of a single discrete eigenvalue ζ_1, eq. (4) yields

$$q(x,t) = q_-\mathrm{e}^{-2i\alpha_1}\frac{\cosh(\tilde\chi - 2i\alpha_1) + d_{+,1}\kappa_s + id_{-,1}\kappa_c}{\cosh\tilde\chi + 2\kappa_s}\,, \tag{8}$$

with $q\to q_- = 1$ as $x\to-\infty$, and

$$\tilde\chi = \chi_1 - \ln(2c_{o,1}^2 E_1),$$

$$\kappa_s = E_1[z_1^2\sin(s_1 - 2\alpha_1) - \sin s_1], \qquad E_1 = \cos\alpha_1/(c_{+,1}c_{o,1})$$

$$\kappa_c = E_1[z_1^2\cos(s_1 - 2\alpha_1) - \cos s_1], \qquad c_{o,1} = |1 - \mathrm{e}^{2i\alpha_1}z_1^2|,$$

and all other quantities are defined in eq. (5). Again, in the special case $z_1 = 1$ and $\alpha_1 \neq 0$, eq. (8) yields the expression for Akhmediev breathers. Two examples are shown in fig. 1.

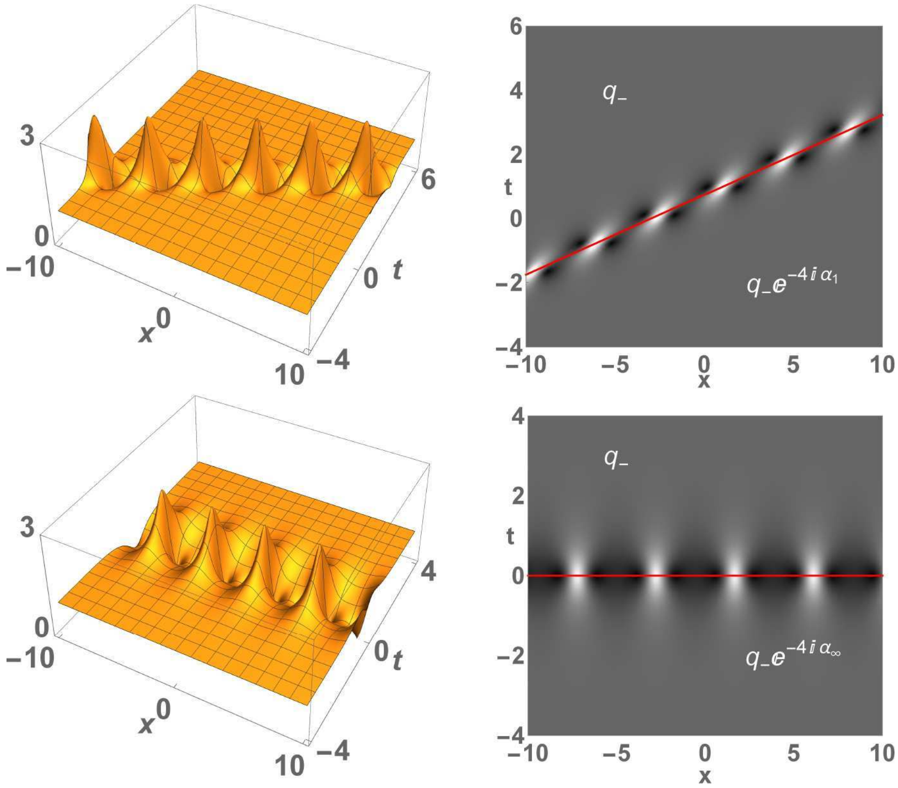

Fig. 1. 3D plot (left) and density plot (right) of the amplitude of a one-soliton solution (top row) and an Akhmediev breather (bottom row). Red lines: the soliton center, given by $\tilde{\chi} = 0$. (In the density plots, black correspond $q(x,t) = 0$, white to $|q(x,t)| = 2.5$, and gray to intermediate values.) Top: $\alpha_1 = -\pi/4$, $z_1 = 2$ and $C_1 = \mathrm{e}^4$. Bottom: $\alpha_1 = -\pi/4$, $z_1 = 1$ and $C_1 = 1$.

First, we introduce some notation that will be useful to treat solutions with solitons and Akhmediev breathers in the same context. In the following sections, we reserve the label $q_j(x,t)$ with a numerical index j to denote one-soliton solutions obtained from eq. (8) with a discrete eigenvalue labeled $\zeta_j = iz_j\,\mathrm{e}^{i\alpha_j}$ with $z_j > 1$, and we use the notation $q_{\pm\infty}(x,t)$ to denote Akhmediev breathers, *i.e.*, a soliton solution corresponding to a discrete eigenvalue ζ_j with $z_j = 1$ and $-\pi/2 < \alpha_j < 0$ or $0 < \alpha_j < \pi/2$, respectively. The subscripts $\pm\infty$ identifies the velocities of Akhmediev breathers, and will be explained below. Consequently, we also use $c_{o,\pm\infty}$, $E_{\pm\infty}$, $\alpha_{\pm\infty}$ and $\xi_{\pm\infty}$ to denote the values of $c_{o,j}$, E_j, α_j and ξ_j with $z_j = 1$ in the case of Akhmediev breathers.

Let us discuss the features of the solution (8). It is easy to see that, when $z_1 > 1$, the solution (8) is localized along the line $\tilde{\chi}(x,t) = 0$. In other words, the "mass" of the soliton is concentrated along the line

$$x = -[d_{+,1}\sin 2\alpha_1\, t + \xi_1 - \ln(2c_{o,1}^2 E_1)]/(c_{-,1}\cos\alpha_1). \tag{9}$$

Notice that $\cos\alpha_1 \neq 0$ and $c_{-,1} \neq 0$ when $|\alpha_1| < \pi/2$ and $z_1 > 1$, so the denominator of the above equation is alway nonzero. Therefore, we can introduce the initial spatial displacement $\bar{\xi}_1$ and soliton velocity V_1 as

$$\bar{\xi}_1 = [\ln(2c_{o,1}^2 E_1) - \xi_1]/(c_{-,1}\cos\alpha_1), \tag{10a}$$

$$V_1 = -2\,d_{+,1}\sin\alpha_1/c_{-,1}. \tag{10b}$$

Figure 2 shows how the soliton velocity V depends on the position of the discrete eigenvalue on the spectral plane. These considerations do not apply to Akhmediev breathers, however, since in this case $c_{-,1} = 0$. On the other hand, the mass of the soliton is still concentrated along $\tilde{\chi}(x,t) = 0$. We then formally define the velocity of an Akhmediev breathers $q_{\pm\infty}(x,t)$ as the limit of eq. (10b), which yields the velocity $V_{\pm\infty} = \pm\infty$, respectively. Hence, this is why we use the subscripts $\pm\infty$ with the Akhmediev breathers. Moreover, from eq. (10a) we define the initial "temporal" displacement for Akhmediev breathers as

$$\bar{\xi}_{\pm\infty} = (\ln|\sin(2\alpha_{\pm\infty})| - \xi_{\pm\infty})/(2\sin 2\alpha_{\pm\infty}), \tag{11}$$

recalling that $c_{o,\pm\infty}$, $E_{\pm\infty}$, $\alpha_{\pm\infty}$ and $\xi_{\pm\infty}$ denote the values of $c_{o,1}$, E_1, α_1 and ξ_1 with $z_1 = 1$.

The final concept we need is the "orientation" of a solution, which will allows us to divide the xt-plane into two half planes, respectively to the left and to the right of the soliton or the breather. For solitons, we say that the soliton is

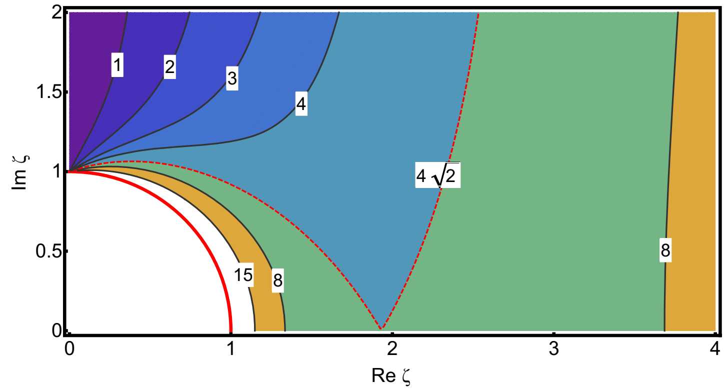

Fig. 2. Contour plot of soliton velocity V from eq. (10b) in the spectral ζ-plane.

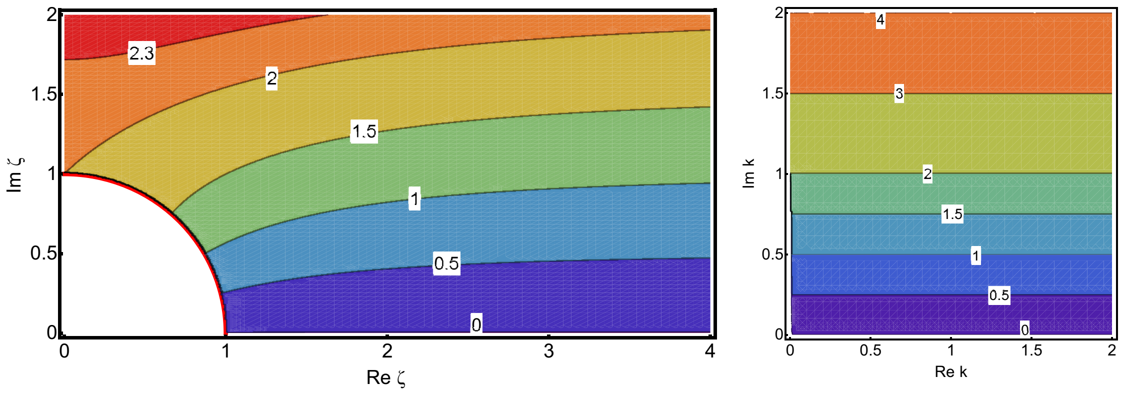

Fig. 3. Contour plot of the amplitude of the envelope of a soliton (8) defined in eq. (13) in the spectral ζ-plane (left) or k-plane (right). The amplitude is obtained by solving eq. (13) numerically. Please see appendix A.1 for the relation between ζ and k.

oriented along the direction of increasing t. For Akhmediev breathers, we say that the orientation of $q_\infty(x,t)$ coincides with that of the positive x-axis and that of $q_{-\infty}(x,t)$ with that of the negative x-axis.

In appendix A.3 we compute the long-time asymptotics of one-soliton solutions and Akhmediev breathers, and we show that these solutions separate the xt-plane into two half planes, divided by the line $\tilde{\chi}(x,t) = 0$. To the left of the soliton or Akhmediev breather,

$$q(x,t) = q_- + o(1), \quad |t| \to \infty, \tag{12a}$$

whereas, to its right,

$$q(x,t) = q_- \mathrm{e}^{-4i\alpha_o} + o(1), \quad |t| \to \infty, \tag{12b}$$

with $\alpha_o = \alpha_1$ or $\alpha_o = \alpha_{\pm\infty}$ for solitons or Akhmediev breathers, respectively.

Notice that for a stationary (Kuznetsov-Ma) soliton ($\alpha_1 = 0$), there is no phase shift. Of course, the fact that the left portion of the background is always unchanged is not surprising. This is because we are fixing the boundary condition $q(x,t) \to q_-$ as $x \to -\infty$. The situation would be reversed if one fixed the boundary condition as $x \to \infty$. What is always true is that each soliton or Akhmediev breather introduces an asymptotic phase shift. In the following sections we use all of the above results to study multi-soliton solutions.

One can define two quantitative measures to characterize the envelope of the one-soliton solutions (8): the amplitude compared to the nonzero background and the width at its half height. The amplitude of the envelope can be defined as

$$q_{\text{amplitude},1} = \max_{x,t} |q_1(x,t)| - q_o, \tag{13}$$

whereas the width of the envelope at its half height does not have a simply formula. Figures 3 and 4 show the numerical results of the amplitude (13) and the width as a function of the discrete eigenvalue ζ_1 against all possible norming constants C_1. One can see from the figures that, all contour lines in the ζ-plane become asymptotically horizontal, as $\operatorname{Re}\zeta_1 \to \infty$. In fact, fig. 3(right) strongly suggests that the amplitude of the envelope of a soliton can be simply expressed in terms of the discrete eigenvalue as

$$q_{\text{amplitude},1} = 2\,\operatorname{Im}(k_1) = \operatorname{Im}(\zeta_1 - 1/\zeta_1),$$

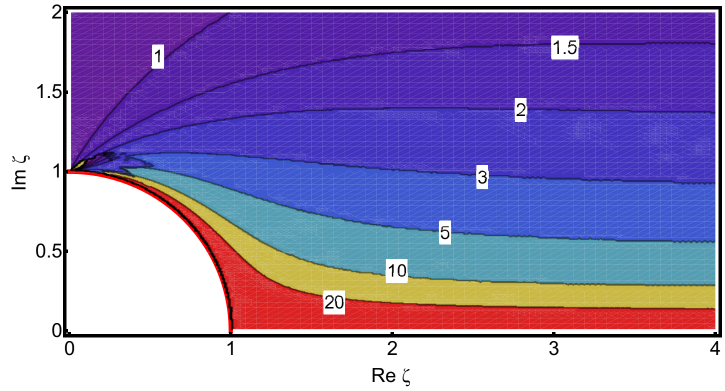

Fig. 4. Contour plot of the width of the envelope of the one-soliton solution (8) at the half height of the amplitude in the spectral ζ-plane. Similarly to fig. 3, the plot is generated by finding the width numerically. Notice how all contour lines converge as the discrete eigenvalue approaches the branch point iq_o. (Some numerical artifacts near the point iq_o are also visible.)

where k_1 is the discrete eigenvalue in the k-plane and ζ_1 is the discrete eigenvalue in the ζ-plane. Please see appendix A.1 for details on the spectral parameters.

3 Bound states and degenerate soliton complexes

The simplest generalization of one-soliton solutions is a so-called bound state, *i.e.*, a degenerate soliton complex, in which two or more solitons travel with the same speed. These solutions are obtained from the N-soliton solution (4) when the discrete eigenvalues ζ_j are such that $V_j = V$ for all $1 \leqslant j \leqslant N$, with V_j defined by eq. (10b) as before. (That is, all discrete eigenvalues lie on the same level curve in fig. 2.) Solving for α_j from eq. (10b), we then have $\alpha_j = -\arcsin[Vc_{-,j}/(2d_{+,j})]$. Therefore, a bound state of N solitons is described by eqs. (4) and (5), but with

$$\chi_j = c_{-,j}\sqrt{4\,d_{+,j}^2 - V^2c_{-,j}^2}(x - Vt)/(2\,d_{+,j}) + \xi_j, \tag{14a}$$

$$s_j = -d_{-,j}[d_{+,j}Vx - (2\,d_{+,j}^2 - V^2c_{-,j}^2)t]/(2\,d_{+,j}^2) + \varphi_j. \tag{14b}$$

An important feature of a bound state, as for a one-soliton solution, is the location of its main “mass”. To characterize this, as in the case of ZBC [30] one can define the center of mass (CoM) as a suitable combination of moments. However, since solutions do not decay as $x \to \pm\infty$ in our case, one must modify the integrand by subtracting the background. We therefore define the CoM as

$$CoM(t) = \int_{\mathbb{R}} x\left(|q|^2 - 1\right)\mathrm{d}x \Big/ \int_{\mathbb{R}} \left(|q|^2 - 1\right)\mathrm{d}x. \tag{15}$$

(Of course different modifications of the integrands are possible. However, eq. (15) is consistent with the definition in the case of ZBG, and it is sufficient for our purposes.)

The equation $x = CoM(t)$ defines a straight line in the xt-plane that accurately mirrors the position of the degenerate soliton train. On the other hand, eq. (15) cannot be easily evaluated. Therefore, in appendix A.4 we give an alternative definition for the soliton center $\bar{x}(t)$, which is based on matching the decay of $q(\bar{x} + \Delta x, t)$ to the background as $\Delta x \to \pm\infty$. This definition is convenient because it bypasses the internal complexity of the soliton complex, and allows us to obtain an analytical expression for the center $\bar{x}(t)$ of an N-soliton train with velocity V. The center $\tilde{x}(t)$ is given by

$$\bar{x}(t) = Vt + \frac{\sum_{j=1}^{N}(\ln z_j - \xi_j) - \ln\sqrt{\det \Gamma}}{\sum_{j=1}^{N} c_{-,j}\cos\alpha_j}\,, \tag{16}$$

where $\cos\alpha_j = (4\,d_{+,j}^2 - V^2c_{-,j}^2)^{1/2}/(2\,d_{+,j})$ and

$$\det \Gamma = \frac{\prod_{m=2}^{2N}\prod_{n=1}^{m-1}|\zeta_n - \zeta_m|^2}{\prod_{m,n=1}^{N}|1 + \zeta_m\zeta_n|^2|\zeta_m^* - \zeta_n|^2}\prod_{n=1}^{N}|\zeta_n|^4. \tag{17}$$

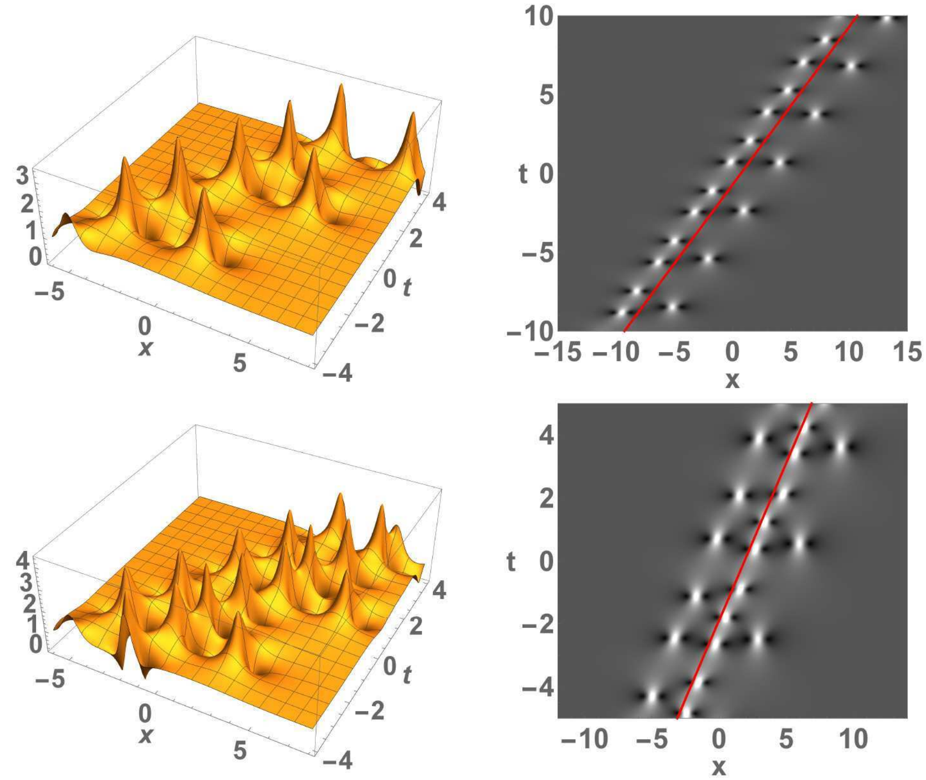

Fig. 5. Similarly to fig. 1 but for a degenerate two-soliton complex (top) and a degenerate three-soliton complex (bottom). Red lines denote the center $\bar{x}(t)$ from eq. (16). Top: Two discrete eigenvalues: $\zeta_1 = 2i\mathrm{e}^{-i\arcsin(3/17)}$ and $\zeta_2 = 3i\mathrm{e}^{-i\arcsin(15/97)}/2$ with velocity $V = 1$, and norming constants: $C_1 = \mathrm{e}^2$ and $C_2 = 1$. Bottom: Three discrete eigenvalues: $\zeta_1 = 2i\mathrm{e}^{-i\arcsin(3/17)}$, $\zeta_2 = 3i\mathrm{e}^{-i\arcsin(15/97)}/2$ and $\zeta_3 = 5i\mathrm{e}^{-i\arcsin(105/641)}/2$ with $V = 1$, and norming constants: $C_1 = \mathrm{e}^2$, $C_2 = C_3 = 1$.

(Notice that $\det\Gamma > 0$ so that $\ln\sqrt{\det\Gamma}$ is always real.) One can also check by calculating $CoM(t)$ numerically from eq. (15), that $\bar{x}(t)$ and $CoM(t)$ indeed coincide. When $N = 1$, eq. (16) reduces to the expression for the center of a single soliton, namely, $\bar{x}(t) = Vt + \bar{\xi}$, where $\bar{\xi}$ is the initial displacement from eq. (10). When $V = 0$ (*i.e.*, in the case of a train of Kuznetsov-Ma solitons), $\bar{x}(t)$ is independent of t. On the other hand, when $V \neq 0$ eq. (16) yields the center of a degenerate train of Tajiri-Watanabe solitons. The center of mass for two degenerate soliton trains is shown in fig. 5.

4 Soliton-soliton interactions

We now discuss soliton interactions in the framework of the exact two-soliton solutions, excluding for now the case of Akhmediev breathers. In other words, we take $z_1 > 1$ and $z_2 > 1$. Without loss of generality, we label the discrete eigenvalues so that $V_1 < V_2$. An example of a two-soliton interaction is shown in fig. 6.

In appendix A.5 we calculate the long-time asymptotic behavior of these solutions and show that the two-soliton solutions with NZBC can be decomposed as

$$q(x,t) = q_- q_1^{\pm}(x,t) q_2^{\pm}(x,t) + o(1), \quad t \to \pm\infty, \tag{18}$$

for x/t bounded, and where

$$q_1^- = \mathrm{e}^{-4i\alpha_2}\tilde{q}_1, \qquad q_1^+ = q_1, \qquad q_2^- = q_2, \qquad q_2^+ = \mathrm{e}^{-4i\alpha_1}\tilde{q}_2.$$

Each component $\tilde{q}_j$ above is a one-soliton solution (8) with $q_- = 1$ and with $\tilde{l}_j$ instead of l_j,

$$\tilde{l}_j = \zeta_2^*\zeta_4^*/[\zeta_2\zeta_4(\beta_{j,2}^*\beta_{j,4}^*)^2]l_j, \quad j = 1,3, \tag{19a}$$

$$\tilde{l}_k = \zeta_1^*\zeta_3^*/[\zeta_1\zeta_3(\beta_{k,1}^*\beta_{k,3}^*)^2]l_k, \quad k = 2,4, \tag{19b}$$

where $\beta_{m,n}$ is defined in eq. (A.19).

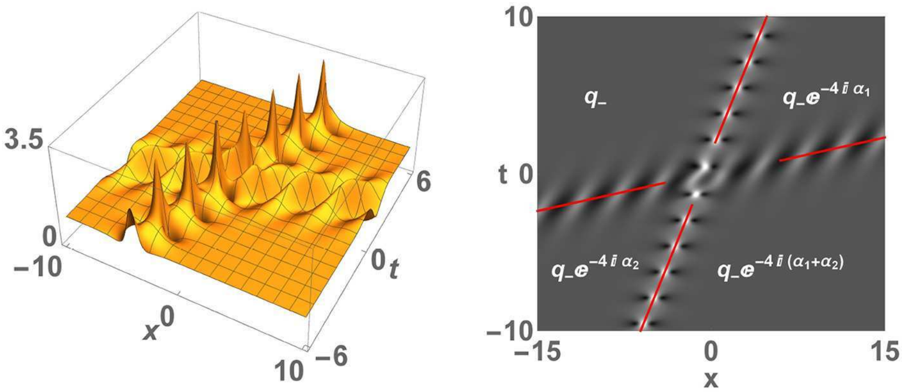

Fig. 6. Similarly to fig. 1 but for a two-soliton solution showing soliton-soliton interactions. Red lines identify the asymptotic displacements $\bar{\xi}_j^{\pm}$ before and after the interaction, as obtained from eq. (20). The two discrete eigenvalues are $\zeta_1 = 2i\mathrm{e}^{-i\pi/32}$ and $\zeta_2 = 3i\mathrm{e}^{-2i\pi/5}/2$, and norming constants are $C_1 = \mathrm{e}^2$ and $C_2 = 1$.

The asymptotic displacement $\bar{\xi}_j^{\pm}$ of each soliton $q_j^{\pm}$ can then be easily computed from using eq. (10), and is given by

$$\bar{\xi}_1^+ = \bar{\xi}_1, \qquad \bar{\xi}_1^- = \bar{\xi}_1 - (\ln \xi_o)/(c_{-,1}\cos\alpha_1), \tag{20a}$$

$$\bar{\xi}_2^+ = \bar{\xi}_2 - (\ln \xi_o)/(c_{-,2}\cos\alpha_2), \qquad \bar{\xi}_2^- = \bar{\xi}_2, \tag{20b}$$

with

$$\xi_o = \frac{d_+ - 2(2\cos(\alpha_1-\alpha_2) + c_{-,1}c_{-,2})\cos(\alpha_1-\alpha_2)}{d_+ - 2(2\cos(\alpha_1+\alpha_2) - c_{-,1}c_{-,2})\cos(\alpha_1+\alpha_2)} \tag{21}$$

and $d_+ = d_{+,1} + d_{+,2}$. The interaction-induced position shift $\Delta\bar{\xi}_j$ of each soliton is thus

$$\Delta\bar{\xi}_j = \bar{\xi}_j^+ - \bar{\xi}_j^- = (-1)^{j-1}(\ln\xi_o)/(c_{-,j}\cos\alpha_j). \tag{22}$$

This position shift was also recently obtained in [44]. From eqs. (21) and (22), the asymptotic position shift only depends on the two discrete eigenvalues but not on the two norming constants. As a result, the asymptotic shift does not depend on the initial soliton displacement (10a). It is only determined by the soliton amplitudes, velocities and soliton widths implicitly through the discrete eigenvalues.

The two solitons separate the xt-plane into four regions. The solution $q(x,t)$ has amplitude 1 asymptotically in each region, but assumes a different phase in each. Similarly to the case of one-soliton solutions, the phase in regions to the left of a soliton is unaffected by that soliton, whereas the phase in regions to the right of a soliton is changed by $-4i\alpha_j$.

5 Soliton-breather interactions

We now extend the above results to cases in which one of the solitons is an Akhmediev breather. We first take $z_2 = 1$ and $\alpha_2 = \alpha_\infty < 0$, corresponding to discrete eigenvalues $\zeta_1 = iz_1\mathrm{e}^{i\alpha_1}$ and $\zeta_2 = i\mathrm{e}^{i\alpha_\infty}$ (plus symmetric counterparts). Thus, we have an Akhmediev breather q_∞ with $V_2 = \infty$, and the relation $V_1 < V_2$ still holds.

In appendix A.6 we calculate the long-time asymptotics of such a two-soliton solution, and we show that

$$q(x,t) = q_1^{\pm}(x,t) + o(1), \quad t \to \pm\infty, \tag{23a}$$

$$q(x,t) = q_\infty^{\pm}(x,t) + o(1), \quad x \to \pm\infty, \tag{23b}$$

where

$$q_1^- = \mathrm{e}^{-4i\alpha_2}\tilde{q}_1, \qquad q_1^+ = q_1, \qquad q_\infty^- = q_\infty, \qquad q_\infty^+ = \mathrm{e}^{-4i\alpha_1}\tilde{q}_\infty.$$

with $\tilde{q}_1$ and $\tilde{q}_\infty$ given by eq. (8) with $\tilde{l}_j$ instead of l_j, given by eq. (19) as before (but where now $\zeta_2 = i\mathrm{e}^{i\alpha_\infty}$).

The asymptotic soliton displacement $\bar{\xi}_j^{\pm}$ are computed in a similar way as before, and are given by

$$\bar{\xi}_1^+ = \bar{\xi}_1, \qquad \bar{\xi}_1^- = \bar{\xi}_1 - (\ln\xi_o)/(c_{-,1}\cos\alpha_1), \tag{24}$$

$$\bar{\xi}_\infty^+ = \bar{\xi}_\infty - \ln\xi_o, \qquad \bar{\xi}_\infty^- = \bar{\xi}_\infty, \tag{25}$$

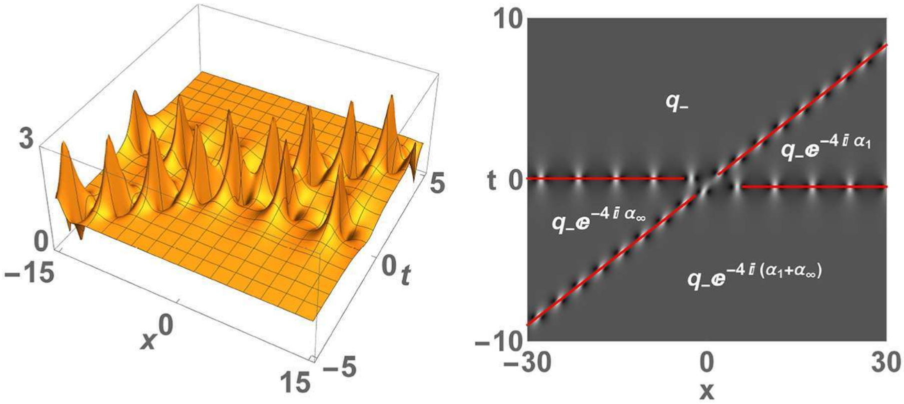

Fig. 7. Similarly to fig. 6 but for a soliton-breather interaction. Parameters are: $\zeta_1 = 2i\mathrm{e}^{-i\arccos(4/17)/2}$, $\zeta_\infty = i\mathrm{e}^{-i\pi/6}$ and $C_1 = C_\infty = 1$.

where now

$$\xi_o = \frac{d_{+,1} - 2\cos[2(\alpha_1 - \alpha_\infty)]}{d_{+,1} - 2\cos[2(\alpha_1 + \alpha_\infty)]}\,. \tag{26}$$

So the asymptotic shifts $\Delta\bar{\xi}_j$ of the soliton and the Akhmediev breather are given by

$$\Delta\bar{\xi}_1 = \bar{\xi}_1^+ - \bar{\xi}_1^- = (\ln\xi_o)/(c_{-,1}\cos\alpha_1), \tag{27a}$$

$$\Delta\bar{\xi}_\infty = \bar{\xi}_\infty^+ - \bar{\xi}_\infty^- = -\ln\xi_o. \tag{27b}$$

Similarly to sect. 4, the asymptotic shifts depends on the discrete eigenvalues not on the norming constants. Therefore, they are only affected by the soliton/breather amplitudes, velocities and widths, but not by the initial spatial displacement (10a) of the soliton or the temporal displacement (11) of the breather.

As before, the soliton and the Akhmediev breather separate the xt-plane into four portions, in which the solution is asymptotically equal to $q_o = 1$ in amplitude but with four different phases. And as before, regions to the left of the soliton (or breather) are unaffected by it, whereas for regions to the right of one soliton (breather) the phase changes by $-4i\alpha_j$. (Recall the definition of orientation for Akhmediev breathers in sect. 2.) One can obtain the interactions between a soliton and an Akhmediev breather with $0 < \alpha_2 < \pi/2$ in a similar way. For brevity, we omit the details. An example of a soliton-breather interaction is shown in fig. 7.

Note that the expression in eq. (26) is significantly simpler than that in eq. (21). In fact, it is easy to show that ξ_o achieves its extremal values when

$$\cos 2\alpha_1 = 2\cos(2\alpha_\infty)/d_{+,1}. \tag{28}$$

In general eq. (28) admits two solutions with opposite sign. Both values yield the largest interaction-induced shifts (in opposite directions). Also, it is obvious that $\alpha_1 = 0$ (a stationary soliton) implies $\ln\xi_o = 0$, *i.e.*, no positions shift occurs. The quantities $\ln\xi_o$ as a function of z_1 and α_1 is shown in fig. 8(left). One example corresponding to a large shift is shown in fig. 8(right).

6 Conclusions

We have presented a study of soliton interactions in focusing media with a nonzero background using the integrable cubic focusing NLS equation with the exact N-soliton solutions. In particular, we discussed degenerate N-soliton trains (*i.e.*, bound states) and we characterized soliton-soliton and soliton-breather interactions. We computed analytically the long-time asymptotics, and we characterized the asymptotic phase of each region defined by the interaction as well as the interaction-induced position shifts. Finally, we identified criteria that result in large position shifts.

It is important to recall that solutions of the focusing NLS equation (1) with NZBC are affected by modulational instability (MI), *i.e.*, the nonzero background is unstable to long-wavelength perturbations. Therefore, when solving such equations numerically, round-off errors grow exponentially. All the soliton solutions considered in this work are exact. Thus, the long-time asymptotic results in this work are not affected by numerical errors. Indeed, the only numerical approximation made in this work is the evaluation of the exact solutions for plotting purposes. Also recall that MI is an instability arising from the continuous spectrum of the scattering problem [16]. Thus, the results of this

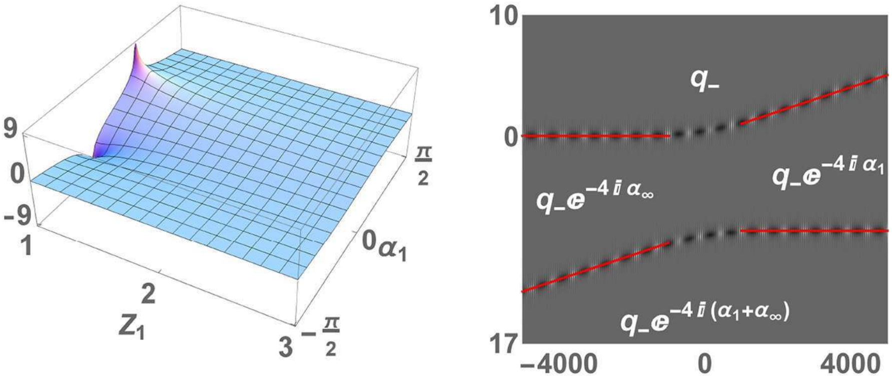

Fig. 8. Left: the asymptotic shift $\Delta\bar{\xi}_\infty$ of the Akhmediev breather (26) with $\alpha_\infty = -\pi/6$, as a function of z_1 and α_1. Right: density plot of a soliton-breather interaction resulting in a large position shift. Here, $z_1 = 1.001$, $\alpha_\infty = -\pi/6$, $C_1 = C_\infty = 1$ and the negative value of α_1 from eq. (28), which result the largest shift. (Note the large spatial window because of the large value of V_1 from eq. (10).)

work can be considered as an idealized situation for the long-time asymptotics in focusing media, in which background radiation is exactly zero. A natural question is then whether the results of this work can be observed in numerical simulations or experiments. In this respect, however, it should be noted that, even though MI plays an important role in numerical simulations or experiments, it was shown in refs. [14,15] that there exists an intermediate time range for which the solutions are not destroyed by the exponentially growing numerical error or background noise. Consequently, one is still able to observe the long-time asymptotics numerically. In fact, recent works have reported the experimental observation of both soliton structures and the nonlinear stage of MI in focusing media with NZBC [18,19]. Finally, because the NLS equation is an universal model appearing in many fields of nonlinear physics, the results in this work apply to all those physical scenarios. For these reasons we expect that, despite MI, our results can be observed in a variety of physical settings.

The results of the present work lay the foundation to study more general scenarios, such as N-soliton interactions with $N > 2$, breather-breather interactions, and even interactions between bound states. We believe that the generalization to N-soliton interactions is straightforward. On the other hand, the study of breather-breather interactions will be more challenging, as well as the study of multi-soliton interactions in which two or more of the soliton velocities coincide.

Another interesting problem will be to study soliton interactions with the simultaneous presence of solitons, Akhmediev breathers and Peregrine solitons. A slightly different formulation of the inverse scattering transform, such as the one in [20], which includes all types of soliton solutions, is needed in order to do so. Finally, a challenging but physically important problem will be the study of interactions between solitons and radiation, generalizing the results of [13,14]. In particular, we note that the soliton-radiation interactions in the focusing case with ZBC, in the defocusing case and in other systems has already been considered [58–62].

Another interesting question is whether there is an analogue with NZBC of the phase-sensitive interactions of two nearby sech pulses in the NLS equation with ZBC. Recall that, in that case, the in-phase and out-of-phase configurations actually correspond to very different spectral portraits. Namely, the in-phase configuration leads to a degenerate two-soliton solution (in the sense of ref. [30]), whereas the out-of phase configuration leads to a solution with a double pole in the spectral problem (as first studied in [27]). Solutions of the focusing NLS equation with NZBC arising from double poles in the scattering problem were recently studied in [17]. We are unaware, however, of whether an analogue of such in-phase and out-of-phase interactions exists in the case of the focusing NLS equation with NZBC.

We plan to study some of the above questions in the near future.

We thank Barbara Prinari for many interesting discussions. This work was partially supported by the National Science Foundation under grant numbers DMS-1614623 and DMS-1615524.

Appendix A.

Here we provide further details about some of the results in the main text.

Appendix A.1. Lax pair and uniformization variable

The NLS equation (1) is the zero-curvature condition

$$X_t - T_x + [X, T] = 0, \tag{A.1}$$

i.e., the compatibility condition $\phi_{xt} = \phi_{tx}$ of the matrix Lax pair

$$\phi_x = X\phi, \qquad \phi_t = T\phi, \tag{A.2}$$

with

$$X = ik\sigma_3 + Q, \qquad T = -i(2k^2 + q_o^2 - |q|^2 - Q_x)\,\sigma_3 - 2kQ, \tag{A.3}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$ is the third Pauli matrix, and

$$Q(x,t) = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}. \tag{A.4}$$

The first half of the Lax pair (A.2) is referred to as the scattering problem and $q(x,t)$ as the potential. The continuous spectrum of the scattering problem is comprised by all values of $k \in \mathbb{C}$ such that $\lambda = (k^2 + q_o^2)^{1/2}$ is real, *i.e.*, $\Sigma = \mathbb{R} \cup i[-q_o, q_o]$. One can deal with the branched nature of λ by introducing a two-sheeted Riemann surface obtained by gluing the two copies of the complex k-plane in which λ takes on the two signs of the complex square root. The surface is then mapped back to the complex plane by introducing the uniformization variable $\zeta = k + \lambda$ similarly to refs. [39,63]. In this way, the first quadrant of the k-plane is mapped to the exterior of the circle $|z| = q_o$ in the first quadrant of the z-plane, shown in fig. 2 (see [38] for details).

Appendix A.2. Alternative soliton solution formulae

In this appendix we show how to rewrite the N-soliton solution formula from [38] to eq. (4). For the sake of the calculations, we start with an N-soliton solution with $q_- = -i$. Recall the total number of N discrete eigenvalues and the corresponding norming constants are parameterized by eq. (3). The IST requires additional auxiliary quantities [38], which can be written as follows:

$$\zeta_{j+N} = -1/\zeta_j^*, \qquad C_{j+N} = C_j^*/(\zeta_j^*)^2. \tag{A.5}$$

A detailed discussion on the symmetries between quantities in eq. (3) and the auxiliary quantities can be found in [38]. As a result, using a similar parametrization for ζ_{j+N} and C_{j+N} as in eq. (3), one obtains eq. (7).

It was shown in [38] that N-soliton solutions of eq. (1) with $q(x,t) \to q_- = -i$ as $x \to -\infty$ can be written as

$$q(x,t) = -i \det M^{\mathrm{aug}} / \det M, \tag{A.6a}$$

with $M = I + A$,

$$A = (A_{n,k})_{2N\times 2N}, \qquad M^{\mathrm{aug}} = \begin{pmatrix} 1 & \mathbf{Y}^T \\ \mathbf{B} & M \end{pmatrix}, \tag{A.6b}$$

$$Y = (l_1^*, \dots, l_{2N}^*)^T, \qquad \mathbf{B} = (B_1, \dots, B_{2N})^T, \tag{A.6c}$$

$$A_{n,k} = \sum_{j=1}^{2N} \gamma_{n,j}\, \gamma_{j,k}^*\, l_j\, l_k^*, \qquad B_n = 1 + \sum_{j=1}^{2N} l_j \gamma_{n,j}/\zeta_j, \tag{A.6d}$$

where the superscript T denotes matrix transpose. It will be convenient to define $\mathbf{1} = (1, \dots, 1)^T$ and $\boldsymbol{\zeta} = (1/\zeta_1, \dots, 1/\zeta_{2N})^T$. It is easy to check that $A(x,t) = \Gamma L(x,t) \Gamma^* L^*(x,t)$ where matrices Γ and L are defined in eq. (5). We consider the denominator in eq. (A.6) first:

$$\det M = \det(I + \Gamma L \Gamma^* L^*) = \det \begin{pmatrix} I & -\Gamma L \\ \Gamma^* L^* & I \end{pmatrix}. \tag{A.7}$$

Next, we consider the numerator in eq. (A.6):

$$\det M^{\mathrm{aug}} = \det(M - \mathbf{B}\mathbf{Y}^T). \tag{A.8}$$

One should regard the term $\mathbf{B}\mathbf{Y}^T$ as a matrix product between a $1 \times 2N$ matrix $\mathbf{B}$ and a $2N \times 1$ matrix $\mathbf{Y}^T$. As a result, the quantity $\mathbf{B}\mathbf{Y}^T$ is a $2N \times 2N$ matrix. We next compute this quantity $\mathbf{B}\mathbf{Y}^T$ explicitly. From eq. (A.6), we have $\mathbf{B} = \mathbf{1} + \Gamma L\boldsymbol{\zeta}$, implying $\mathbf{B}\mathbf{Y}^T = \mathbf{1}\mathbf{Y}^T + \Gamma L\boldsymbol{\zeta}\mathbf{Y}^T$. Again, one should regard the two quantities $\mathbf{1}\mathbf{Y}^T$ and $\boldsymbol{\zeta}\mathbf{Y}^T$ as matrix products. Explicitly, one obtains $\mathbf{1}\mathbf{Y}^T = 1_{2N}L^*$ and $\boldsymbol{\zeta}\mathbf{Y}^T = Z\,1_{2N}L^*$, where again $1_{2N} = \mathbf{1}\otimes\mathbf{1}$. Thus, recalling eq. (A.7), one can simplify eq. (A.8) to

$$\det M^{\mathrm{aug}} = \det[I - 1_{2N}L^* + \Gamma L(\Gamma^* - Z\,1_{2N})L^*]. \tag{A.9}$$

It is easy to verify that $\Gamma^* - Z\,1_{2N} = Z\Gamma^*(Z^*)^{-1}$, where Z is defined in eq. (5). Notice that both L and Z are diagonal, so $(Z^*)^{-1}L^* = L^*(Z^*)^{-1}$. Then eq. (A.9) can be further simplified to

$$\begin{aligned}\det M^{\mathrm{aug}} &= \det\left[I - 1_{2N}L^* + \Gamma LZ\Gamma^*(Z^*)^{-1}L^*\right] \qquad \text{(A.10)}\\ &= \det\begin{pmatrix} I - 1_{2N}L^* & -\Gamma LZ \\ \Gamma^*L^*(Z^*)^{-1} & I\end{pmatrix}\\ &= \prod_{n=1}^{2N}\frac{\zeta_n^*}{\zeta_n}\det\begin{pmatrix}(I - 1_{2N}L^*)Z^* & -\Gamma L \\ \Gamma^*L^* & Z^{-1}\end{pmatrix}. \qquad \text{(A.11)}\end{aligned}$$

Finally, combining eqs. (A.6), (A.7) and (A.11), we obtain the following formula for the N-soliton solution

$$q(x,t) = -i\mathrm{e}^{i\alpha_s}\det\begin{pmatrix}(I - 1_{2N}L^*)Z^* & -\Gamma L \\ \Gamma^*L^* & Z^{-1}\end{pmatrix}\Big/\det\begin{pmatrix} I & -\Gamma L \\ \Gamma^*L^* & I\end{pmatrix}, \tag{A.12}$$

where $\alpha_s = \arg(\prod_{n=1}^{2N}\zeta_n^*/\zeta_n) = -4\sum_{n=1}^{N}\alpha_n$. This solution has the asymptotic behavior $q(x,t) \to q_- = -i$ as $x \to -\infty$. However in the main text we take solutions with $q_- = 1$. Using the phase invariance of the NLS equation, we multiply the above solution by i, and we obtain eq. (4).

Appendix A.3. Asymptotics of one-soliton solutions

Consider a one-soliton solution generated by a discrete eigenvalue ζ_1, with corresponding velocity V_1. (In the special case of an Akhmediev breather, $|\zeta_1| = q_o$ and $V_1 = \pm\infty$.) Taking $V \neq V_1$, we analyze the asymptotic behavior of the solution as $t \to \pm\infty$ along the line $x = Vt + y$ with y an arbitrary constant. Substituting $x = Vt + y$ into eq. (8), it is easy to see from eqs. (5) and (6), that, in the limit $t \to \pm\infty$, $|l_1|$ and $|l_2|$ either both tend to infinity or both decay to zero. In particular, for $j = 1, 2$: i) if $V < V_1$, $|l_j|$ decays as $t \to \infty$ and grows as $t \to -\infty$; ii) if $V > V_1$, $|l_j|$ grows as $t \to \infty$ and decays as $t \to -\infty$. The situation is similar for Akhmediev breathers (*i.e.*, when $|\zeta_1| = q_o$, implying $V_1 = \pm\infty$.) Namely: i) for $q_\infty(x,t)$, $|l_j|$ grows as $t \to \infty$ and decays as $t \to -\infty$; ii) for $q_{-\infty}(x,t)$, the opposite is true. Instead of considering various situations in the limit $t \to \pm\infty$, we then only need to consider two cases: $|l_j|$ grows or decays.

When both l_1 and l_2 tend to zero, L and L^* also tend to the zero matrix. Then, in this limit the solution becomes

$$q(x,t) = q_-\mathrm{e}^{-4i\alpha_o}\det\begin{pmatrix} Z_1^* & 0 \\ 0 & Z_1^{-1}\end{pmatrix}\Big/\det\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} + o(1) = q_- + o(1),$$

where $\alpha_o = \alpha_1$ or $\alpha_o = \alpha_{\pm\infty}$ for solitons or Akhmediev breathers, respectively. The same definition for α_o holds in the rest of appendix. (Recall that $Z_1 = \mathrm{diag}(1/\zeta_1, 1/\zeta_2)$.)

When both $|l_1|$ and $|l_2|$ tend to infinity, we factor out both determinants in the numerator and denominator by $\det[\mathrm{diag}(l_1^*, l_2^*, l_1, l_2)]$ from the right. (This operation leaves the solution invariant.) In the corresponding limit the solution then becomes

$$q(x,t) = q_-\mathrm{e}^{-4i\alpha_o}\det\begin{pmatrix} -1_2Z_1^* & -\Gamma \\ \Gamma^* & 0\end{pmatrix}\Big/\begin{pmatrix} 0 & -\Gamma \\ \Gamma^* & 0\end{pmatrix} + o(1) = q_-\mathrm{e}^{-4i\alpha_o} + o(1). \tag{A.13}$$

Combining these two expressions with all the cases discussed above, finally we obtain eqs. (12).

Appendix A.4. Center of mass of a degenerate soliton train

From eqs. (A.7) and (A.11), one can write the N-soliton solution as

$$q(x,t) = q_-\frac{\det[I - 1_{2N}L^* + \Gamma LZ\Gamma^*(Z^*)^{-1}L^*]}{\det(I + \Gamma L\Gamma^*L^*)}\,. \tag{A.14}$$

As before, the xt-dependence in eq. (A.14) arises solely from the matrix L defined in eq. (5a) with χ_j and s_j from eq. (14) together with the symmetry (6). As discussed in sect. 3, we define the soliton center $\bar{x}(t)$ by matching the asymptotic behavior of $q(x,t)$ with $x = \bar{x}(t) + \Delta x$ as $\Delta x \to \pm\infty$.

From the boundary conditions $q \to q_\pm$ as $x \to \pm\infty$ with $|q_\pm| = 1$, we know that $q(\bar{x} + \Delta x, t) \to q_\pm$ as $\Delta x \to \pm\infty$. Consequently, the asymptotic behavior of $q(\bar{x} + \Delta x, t)$ as $\Delta x \to \pm\infty$ can only differ by a constant phase, and the numerator and denominator of eq. (A.14) must have the same leading-order asymptotics in modulus. As a result, we next only focus on the denominator and ignore the complex phase difference. The asymptotic behavior of the denominator of $q(\bar{x}(t) + \Delta x, t)$ as $\Delta x \to \pm\infty$ in modulus describes how the solution decays to the background.

We first examine the asymptotics as $\Delta x \to \infty$. In this limit every component of L grows exponentially. As a result, as $\Delta x \to \infty$ one can expand the denominator of eq. (A.14) with $x = \bar{x}(t) + \Delta x$ as

$$\det(I + \Gamma L \Gamma^* L^*) = \det(\Gamma L \Gamma^* L^*)[1 + o(1)] = \exp\left\{4\sum_{j=1}^{N}[\chi_j(x,t) - \ln z_j]\right\}(\det \Gamma)^2[1 + o(1)], \tag{A.15}$$

where we used the fact that $\Gamma^* = -\Gamma^T$ implies $\det \Gamma^* = \det \Gamma$ (since the size of the matrix is $2N \times 2N$).

We next look at the asymptotics as $\Delta x \to -\infty$. Every component of $L(x,t)$ decays exponentially in this limit, or in other words, every component of $L^{-1}(x,t)$ grows exponentially. To expand the denominator as before, we rewrite the solution from eq. (A.14) as

$$q(x,t) = q_{\mathrm{num}}/q_{\mathrm{denom}}, \tag{A.16}$$

where $q_{\mathrm{denom}} = \det[I + (\Gamma L \Gamma^* L^*)^{-1}]$ and

$$q_{\mathrm{num}} = \det\left[I - 1_{2N}(\Gamma L Z \Gamma^* (Z^*)^{-1})^{-1} + (\Gamma L Z \Gamma^* (Z^*)^{-1} L^*)^{-1}\right].$$

We again focus on q_{denom} because q_{num} behaves similarly. As $\Delta x \to -\infty$, the denominator is

$$\det[I + (\Gamma L \Gamma^* L^*)^{-1}] = \det(\Gamma L \Gamma^* L^*)^{-1}[1 + o(1)] = \exp\left\{-4\sum_{j=1}^{N}[\chi_j(x,t) - \ln z_j]\right\}(\det \Gamma)^{-2}[1 + o(1)]. \tag{A.17}$$

Since by definition that $\bar{x}$ is the "center" of this degenerate soliton complex, one expects that the asymptotic behavior of $q(\bar{x} + \Delta x, t)$ as $\Delta x \to \pm\infty$ is similar. In other words, one expects that the solution decays to the background in a similar fashion as $\Delta x \to \pm\infty$. Imposing that the leading-order terms from eqs. (A.15) and (A.17) match, one obtains

$$\exp\left[8\sum_{j=1}^{N}[\chi_j(\bar{x},t) - \ln z_j]\right](\det \Gamma)^4 = 1. \tag{A.18}$$

Solving the above equation for $\bar{x}(t)$, one obtains eq. (16). Note that Γ is a Cauchy matrix, and its determinant can be computed from [64].

Appendix A.5. Soliton-soliton interactions

We study two-soliton interactions (excluding the case of Akhmediev breathers for now) by looking at the asymptotic behavior of the solution as $t \to \pm\infty$, similarly to appendix A.3. We substitute $x = Vt + y$ into the solution. For simplicity, we introduce shorthand notation $l_{j,j+2}$ to represent l_j and l_{j+2} for $j = 1, 2$. It is easy to see that, depending on the value of V and the limits $t \to \pm\infty$, there are five cases: i) if $V < V_1$, all l_j's decay/grow as $t \to \pm\infty$, respectively; ii) if $V = V_1$, $l_{1,3}$ is always bounded whereas, $l_{2,4}$ decay/grow as $t \to \pm\infty$, respectively; iii) if $V_1 < V < V_2$, $l_{1,3}$ grow/decay as $t \to \pm\infty$, respectively, while $l_{2,4}$ do the opposite; iv) if $V = V_2$, $l_{2,4}$ are bounded, but $l_{1,3}$ grow/decay as $t \to \pm\infty$, respectively; v) if $V_2 < V$, l_j grow/decay as $t \to \pm\infty$, respectively.

Similarly to the analysis of one-soliton solutions, instead of considering the above five cases directly we consider the following alternatives: 1) all l_j decay; 2) all l_j grow; 3) $l_{1,3}$ decay and $l_{2,4}$ grow; 4) $l_{1,3}$ are bounded and $l_{2,4}$ decay; 5) $l_{1,3}$ are bounded and $l_{2,4}$ grow. Cases i)–v) above can then be pieced together from these five alternatives plus others that are treated similarly.

1) If all l_j decay, then L and L^* tend to the zero matrix. Then the solution (4) reduces to q_- in this limit.
2) If all l_j grow to infinity, we multiply the numerator and denominator of eq. (4) by the inverse of $\mathrm{diag}(l_1^*, \ldots, l_4^*, l_1, \ldots, l_4)$ from the right, which as before leaves the solution invariant. Similarly to eq. (A.13), we then obtain

$$q(x,t) = q_- \mathrm{e}^{-4i(\alpha_1+\alpha_2)} + o(1)$$

in the corresponding limit.

3) If $l_{1,3}$ decay and $l_{2,4}$ grow, we multiply numerator and denominator of eq. (4) by the inverse of $\mathrm{diag}(1, l_2^*, 1, l_4^*, 1, l_2, 1, l_4)$ from the right. The quotient becomes

$$\det\begin{pmatrix} (\bar{L}_o - 1_4 L_o)Z^* & -\Gamma L_o \\ \Gamma^* L_o & Z^{-1}\bar{L}_o \end{pmatrix} \Big/ \det\begin{pmatrix} \bar{L}_o & -\Gamma L_o \\ \Gamma^* L_o & \bar{L}_o \end{pmatrix} + o(1),$$

where $L_o = \mathrm{diag}(0,1,0,1)$ and $\bar{L}_o = \mathrm{diag}(1,0,1,0)$. After straightforward calculations, we then obtain

$$q(x,t) = q_- \mathrm{e}^{-4i\alpha_2} + o(1)$$

in the corresponding limit.

4) If $l_{1,3}$ are bounded and $l_{2,4}$ decay, the quotient in eq. (4) becomes

$$\det\begin{pmatrix} (I - 1_4 L_o^*)Z^* & -\Gamma L_o \\ \Gamma^* L_o^* & Z^{-1} \end{pmatrix} \Big/ \det\begin{pmatrix} I & -\Gamma L_o \\ \Gamma^* L_o^* & I \end{pmatrix} + o(1),$$

where $L_o = \mathrm{diag}(l_1, 0, l_3, 0)$. Next, we perform the same row/column operations to both determinants simultaneously, so that the quotient (*i.e.*, the solution of the NLS equation) remains the same. Taking into account the structure of the two determinants, we expand them in terms of their even-numbered columns, which yields an overall factor of $\zeta_2\zeta_4/(\zeta_2^*\zeta_4^*)$ that cancels out. The remainder has the same form as the one-soliton solution (8) with $j = 1$ and $q_- = 1$. Thus,

$$q(x,t) = q_1(x,t) + o(1)$$

in the corresponding limit.

5) If $l_{1,3}$ are bounded and $l_{2,4}$ grow, we multiply the two determinants by the inverse of $\mathrm{diag}(1, l_2^*, 1, l_4^*, 1, l_2, 1, l_4)$ from the right. The quotient becomes

$$\det\begin{pmatrix} (\bar{L}_o - 1_4 L_o^*)Z^* & -\Gamma L_o \\ \Gamma^* L_o^* & Z^{-1}\bar{L}_o \end{pmatrix} \Big/ \det\begin{pmatrix} \bar{L}_o & -\Gamma L_o \\ \Gamma^* L_o & \bar{L}_o \end{pmatrix} + o(1),$$

where $L_o = \mathrm{diag}(l_1, 1, l_3, 1)$ and $\bar{L}_o = \mathrm{diag}(1,0,1,0)$. We again perform the same row/column operations to both determinants simultaneously. These operations are: i) use the fourth row to cancel all other elements in the last column and then expand according to the same column; ii) use the second row to cancel all other elements in the sixth column and then expand according to the same column; iii) use the fourth column to eliminate all other elements in the last row and then expand according to the same row; iv) use the second column to eliminate all other elements in the fourth row and then expand according to the same row. Next, we multiply/divide the jth row/column of each determinant by a_j, respectively, where

$$a_1 = \zeta_2^*\zeta_4^*\beta_{1,2}\beta_{1,4}/(\zeta_2\zeta_4), \qquad a_2 = \zeta_2^*\zeta_4^*\beta_{3,2}\beta_{3,4}/(\zeta_2\zeta_4),$$
$$a_3 = \beta_{1,2}^*\beta_{1,4}^*, \qquad a_4 = \beta_{3,2}^*\beta_{3,4}^*$$

and

$$\beta_{m,n} = (\zeta_m^* - \zeta_n)/(\zeta_m^* - \zeta_n^*). \tag{A.19}$$

Finally, introducing the quantities

$$\tilde{l}_1 = \zeta_2^*\zeta_4^*/[\zeta_2\zeta_4(\beta_{1,2}^*\beta_{1,4}^*)^2]l_1, \qquad \tilde{l}_3 = \zeta_2^*\zeta_4^*/[\zeta_2\zeta_4(\beta_{3,2}^*\beta_{3,4}^*)^2]l_3,$$

we can rewrite the solution $q(x,t)$ as

$$q(x,t) = \mathrm{e}^{-4i\alpha_2}\tilde{q}_1(x,t) + o(1),$$

where $\tilde{q}_1(x,t)$ has the same form as the solution (8), but with $\tilde{l}_j$ instead of l_j and $q_- = 1$. It can be easily verified that $\tilde{l}_1$ and $\tilde{l}_3$ satisfy the symmetry (6). In other words, $\tilde{q}_1(x,t)$ is indeed a one-soliton solution (8).

Other cases can be computed similarly. Finally, by combining all the asymptotic results in the five cases i)–v), we obtain the results in sect. 4.

Appendix A.6. Soliton-breather interactions

We study soliton-breather interactions by substituting $x = Vt + y$ into the solution (4). Recall that we took $z_2 = 1$ and $\alpha < 0$, so the soliton generated from the second discrete eigenvalue ζ_2 is an Akhmediev breather with $V_2 = \infty$. Notice that since the Akhmediev breather is localized in time, $l_{2,4}$ always grow/decay as $t \to \pm\infty$ for any real value of V. Thus, in order to observe this breather asymptotically, one needs to consider the limits $x \to \pm\infty$ at fixed t. Thus, depending on the value of V and the particular limit considered ($t \to \pm\infty$ or $x \to \pm\infty$), there are several cases: i) if $V < V_1$, l_j decay/grow as $t \to \pm\infty$, respectively; ii) if $V = V_1$, $l_{1,3}$ are always bounded, whereas $l_{2,4}$ decay/grow as $t \to \pm\infty$, respectively; iii) if $V_1 < V < V_2$, $l_{1,3}$ grow/decay as $t \to \pm\infty$, respectively, whereas $l_{2,4}$ do the opposite; iv) $l_{1,3}$ always grows/decays as $x \to \pm\infty$, while $l_{2,4}$ are bounded.

Similarly to what we did when considering soliton-soliton interactions, instead of looking at the above four cases, we next discuss the following cases: 1) all l_j decay; 2) all l_j grow; 3) $l_{1,3}$ decay and $l_{2,4}$ grow; 4) $l_{1,3}$ are bounded and $l_{2,4}$ decay; 5) $l_{1,3}$ are bounded and $l_{2,4}$ grow.

Expressing cases i)–iv) above in terms of the five alternatives, we see that these cases are exactly the same as the ones in the calculations of soliton-soliton interactions. Thus, by simply repeating all calculations and relating all results to the cases i)–iv), one can obtain the results in sect. 5. (The only complication is that case iv) corresponds to the limits $x \to \pm\infty$ instead of $t \to \pm\infty$, so it is necessary to separate results into two groups: $t \to \pm\infty$ and $x \to \pm\infty$.)

One can derive corresponding results for soliton-breather interactions when the Akhmediev breather is $q_{-\infty}(x,t)$ instead of $q_{\infty}(x,t)$ using similar arguments.

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