Long-Time Asymptotics for the Focusing Nonlinear Schrödinger Equation with Nonzero Boundary Conditions at Infinity and Asymptotic Stage of Modulational Instability

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Abstract

We characterize the long-time asymptotic behavior of the focusing nonlinear Schrödinger (NLS) equation on the line with symmetric, nonzero boundary conditions at infinity by using a variant of the recently developed inverse scattering transform (IST) for such problems and by employing the nonlinear steepest-descent method of Deift and Zhou for oscillatory Riemann-Hilbert problems. First, we formulate the IST over a single sheet of the complex plane without introducing the uniformization variable that was used by Biondini and Kovačič in 2014. The solution of the focusing NLS equation with nonzero boundary conditions is thereby associated with a matrix Riemann-Hilbert problem whose jumps grow exponentially with time for certain portions of the continuous spectrum. This growth is the signature of the well-known modulational instability within the context of the IST. We then eliminate this growth by performing suitable deformations of the Riemann-Hilbert problem in the complex spectral plane. The results demonstrate that the solution of the focusing NLS equation with nonzero boundary conditions remains bounded at all times. Moreover, we show that, asymptotically in time, the xt-plane decomposes into two types of regions: a left far-field region and a right far-field region, where the solution equals the condition at infinity to leading order up to a phase shift, and a central region in which the asymptotic behavior is described by slowly modulated periodic oscillations. Finally, we show how, in the latter region, the modulus of the leading-order solution, initially obtained as a ratio of Jacobi theta functions, can be reduced to the well-known elliptic solutions of the focusing NLS equation. These results provide the first characterization of the long-time behavior of generic perturbations of a constant background in a modulationally unstable medium. © 2017 Wiley Periodicals, Inc.

1 Introduction

The present work is devoted to the study of the long-time asymptotic behavior of the one-dimensional focusing nonlinear Schrödinger (NLS) equation formulated

Communications on Pure and Applied Mathematics, Vol. LXX, 2300–2365 (2017)

on the line with symmetric, nonzero boundary conditions at infinity:

$$iq_t + q_{xx} + 2\big(|q|^2 - q_o^2\big)q = 0, \tag{1.1}$$

with $q = q(x,t)$ complex-valued, $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, $q(x,0)$ given, and with

$$\lim_{x\to\pm\infty} q(x,0) = q_\pm. \tag{1.2}$$

Hereafter, $q_\pm$ are complex constants with $|q_\pm| = q_o > 0$, and

$$q(x,0) - q_\pm \in L^{1,1}(\mathbb{R}^\pm), \tag{1.3}$$

where

$$L^{1,1}(\mathbb{R}^\pm) = \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \int_{\mathbb{R}^\pm} (1+|x|)|f(x)|dx < \infty \right\}. \tag{1.4}$$

The NLS equation is an important model in applied mathematics and theoretical physics due to both its surprisingly rich mathematical structure and its physical significance and broad applicability to a number of different areas, ranging from nonlinear optics to water waves and from plasmas to Bose-Einstein condensates. In particular, recall that the NLS equation is a universal model for the evolution of almost monochromatic waves in a weakly nonlinear dispersive medium [9, 20]. Specifically, the equation was derived in the context of nonlinear optics [25,26,71], and in the framework of fluid mechanics for water waves of small amplitude over infinite depth [76] and finite depth [10, 49]. A rigorous justification of the model in the latter case was also given in [29]. The equation has also been suggested as a model for rogue waves [24, 68]. Further references on the physical aspect of the NLS equation can be found in [63, 70]. Moreover, the NLS equation is also one of the prototypical infinite-dimensional, completely integrable systems. A Lax pair for the equation was derived in [81], where it was also shown that the equation admits an infinite number of conservation laws as well as exact N-soliton solutions for arbitrary N.

The initial value problem for the NLS equation has been studied extensively during the past five decades and a plethora of results are available. Assuming sufficient smoothness of the initial data, Zakharov and Shabat [81] developed in 1972 the inverse scattering transform (IST) for the solution of the NLS initial value problem on the line for initial conditions with sufficiently rapid decay at infinity. In 1974 Ablowitz, Kaup, Newell, and Segur [3] generalized those results by introducing the so-called AKNS system, which contains the NLS equation as a special case. The IST for first-order systems was also developed rigorously by Beals and Coifman in 1984 [6]. The periodic problem for NLS was studied by Its and Kotlyarov in 1976 [51], while the half-line and the finite interval problems were analyzed via an appropriate extension of IST for bounded domains by Fokas and others [41, 42]. Problems with *nonzero* boundary conditions (NZBC) at infinity of the kind (1.2) have also been studied. The IST for the *defocusing* NLS equation on the line with nonzero boundary conditions at infinity was developed soon after the case of zero boundary conditions [38, 82]. The IST for the *focusing* NLS equation on the line with nonzero boundary conditions at infinity was recently developed by Kovačič and the first author [12]. (Partial results were contained in [44, 59, 65].) From a different point of view, sharp well-posedness of the NLS equation on the line with initial data in Sobolev spaces H^s for any $s \geq 0$ was proved by Bourgain [15] (see also [16]). Well-posedness of the NLS equation on the half-line with data in Sobolev spaces was established independently and via different approaches by Holmer [50]; Bona, Sun, and Zhang [14]; and Fokas, Himonas, and the second author [40]. Further functional-analytic results for NLS can be found in Craig, Kappeler, and Strauss [28]; Cazenave [22]; Cazenave and Weissler [23]; Kenig, Ponce, and Vega, [58]; Ghidaglia and Saut [46]; Linares and Ponce [64]; Carroll and Bu [21]; Kato [57]; Ginibre and Velo [47]; Tsutsumi [73]; and the references therein.

The motivation for the present work stems from its intriguing connection with the physical phenomenon of modulational instability (MI, also known as Benjamin-Feir instability in the context of deep water waves [8]), namely, the instability of a constant background with respect to long-wavelength perturbations. Modulational instability is one of the most ubiquitous phenomena in nonlinear science (e.g., see [80] and the references therein). In many cases, the dynamics of systems affected by modulational instability is governed by the one-dimensional focusing NLS equation. Hence, the initial (i.e., linear) stage of modulational instability can be studied by linearizing the focusing NLS equation around the constant background. It is then easily seen that all Fourier modes below a certain threshold are unstable, and the corresponding perturbations grow exponentially. However, the linearization ceases to be valid as soon as perturbations become comparable with the background. A natural question is therefore what happens beyond this linear stage. Surprisingly, despite some early work [59–62], this question, which is referred to as the characterization of the *nonlinear stage of modulational instability*, has remained essentially open for about 50 years.

Since the focusing NLS equation is a completely integrable system, a natural conjecture (by analogy with the case of localized initial conditions) is that modulational instability is mediated by solitons [45, 78]. On the other hand, Fagerstrom and the first author [11] employed the recently developed IST for the focusing NLS equation with nonzero boundary conditions at infinity in order to study modulational instability by computing the spectrum of the scattering problem for simple classes of perturbations of a constant background. In particular, it was shown in [11] that there exist classes of perturbations for which no solitons are present. Therefore, since all generic perturbations of the constant background are linearly unstable, *solitons cannot be the mechanism that mediates modulational instability*, contradicting the conjecture of [78]. Instead, in [11] the instability mechanism was identified within the context of IST. More precisely, it was shown that the instability emerges from certain portions of the continuous spectrum of the scattering problem associated with the focusing NLS equation. At the same time, [11] did not offer any insight about the actual nonlinear dynamics of the solutions of the focusing NLS equation past the linear stage of modulational instability.

The aim of the present work is to address this challenge and characterize the nonlinear stage of modulational instability. We do so by computing the long-time asymptotic behavior of the solution of the focusing NLS equation. Recall that formal but ingenious results for the long-time asymptotics of the NLS equation with zero boundary conditions at infinity were obtained in 1976 by Segur and Ablowitz [69] and Zakharov and Manakov [79]. In 1993, Deift and Zhou [34] introduced a method for the *rigorous* asymptotic analysis of oscillatory Riemann-Hilbert problems like those arising in the inverse problem for the solution of integrable evolution equations via IST. The Deift-Zhou method can be regarded as the nonlinear analogue of the classical steepest-descent method used in the analysis of the long-time asymptotics for linear evolution equations. The method relies on appropriate factorizations of the jump matrices of the Riemann-Hilbert problem and suitable deformations of the associated jump contours in order to extract the leading-order asymptotic behavior as well as obtain rigorous estimates for the corrections.

The Deift-Zhou method was first used to compute the long-time asymptotics of the modified KdV equation [34], and was subsequently extended and employed in numerous works, including the study of the long-time asymptotics of the KdV equation [32], of the defocusing NLS equation [35], and of the Toda lattice [55], all with decaying data at infinity, as well as the analysis of the zero dispersion (semiclassical) limit of the KdV equation [33] and the focusing NLS equation [67, 72]. We also note that the Deift-Zhou method has found useful applications in the theory of orthogonal polynomials [30, 31]. Recent results concerning the NLS equation were also presented in the works of Buckingham and Venakides [18]; Boutet de Monvel, Kotlyarov, and Shepelsky [17]; Jenkins and McLaughlin [53]; and Jenkins [52]. On the other hand, none of those works studied the problem considered in the present work, namely, the long-time asymptotics of the solution of the focusing NLS equation (1.1) with *generic* initial conditions satisfying (1.2) and (1.3). Our results are summarized by the following theorems.

THEOREM 1.1 (Plane wave region). *Let* $q(x,0)$ *satisfy* (1.2), (1.3), *and* (3.1) *with* $\epsilon = q_o$ *and be such that no discrete spectrum is present. For* $x < -4\sqrt{2}q_o t$, *the long-time asymptotic behavior of the solution of the focusing NLS equation* (1.1) *is given by*

$$q(x,t) = e^{2ig_\infty} q_- + O\big(t^{-\frac{1}{2}}\big), \quad t \to \infty, \tag{1.5}$$

where the real quantity g_∞ *is defined by equation* (5.47) *and depends only on the initial datum* $q(x,0)$ *via the reflection coefficient* (2.28) *and on the ratio* x/t *via the stationary point* k_1 *defined by equation* (3.9).

For $x > 4\sqrt{2}q_o t$, *the leading-order asymptotic behavior of the solution* q *is given by a formula analogous to formula* (1.5), *with* q_- *replaced by* q_+ *and with*

k_1 in the definition of g_∞ replaced by the stationary point k_2, also defined by equation (3.9)*.*

THEOREM 1.2 (Modulated elliptic wave region). *Let $q(x,0)$ satisfy the hypotheses of Theorem* 1.1*. For $-4\sqrt{2}q_o t < x < 0$, the long-time asymptotic behavior of the solution of the focusing NLS equation* (1.1) *is given by*

$$\begin{aligned} q(x,t) = {} & \frac{q_o(q_o+\alpha_{\mathrm{im}})}{\overline{q}_-} \\ & \cdot \frac{\Theta\Big(\frac{\sqrt{q_o\alpha_{\mathrm{im}}}}{mK(m)}(x-2\alpha_{\mathrm{re}}t) - X_o + 2v_\infty - \frac{1}{2}\Big)\Theta\big(\frac{1}{2}\big)}{\Theta\Big(\frac{\sqrt{q_o\alpha_{\mathrm{im}}}}{mK(m)}(x-2\alpha_{\mathrm{re}}t) - X_o - \frac{1}{2}\Big)\Theta\big(2v_\infty - \frac{1}{2}\big)} e^{2i(g_\infty - G_\infty t)} \\ & + O\big(t^{-\frac{1}{2}}\big), \quad t \to \infty, \end{aligned} \tag{1.6}$$

where Θ is defined in terms of the third Jacobi theta function as

$$\Theta(k) = \theta_3(\pi k, e^{i\pi\tau}), \quad \tau = \frac{iK(\sqrt{1-m^2})}{K(m)}, \tag{1.7}$$

and $K(m)$ is the complete elliptic integral of the first kind with elliptic modulus m. The real quantities m, α_{re}, and α_{im} are independent of the initial datum and are given by the unique solutions of the algebraic system of equations

$$\frac{x}{2t} = 2\alpha_{\mathrm{re}} + \frac{q_o^2 - \alpha_{\mathrm{im}}^2}{\alpha_{\mathrm{re}}}, \quad m^2 = \frac{4q_o\alpha_{\mathrm{im}}}{\alpha_{\mathrm{re}}^2 + (q_o+\alpha_{\mathrm{im}})^2}, \tag{1.8a}$$

$$\big[\alpha_{\mathrm{re}}^2 + (q_o - \alpha_{\mathrm{im}})^2\big]K(m) = \big(\alpha_{\mathrm{re}}^2 - \alpha_{\mathrm{im}}^2 + q_o^2\big)E(m), \tag{1.8b}$$

$E(m)$ being the complete elliptic integral of the second kind. The real quantity G_∞ and the complex quantity v_∞ are also independent of the initial conditions and are given, respectively, by equations (5.41) *and* (5.77)*. The real quantities g_∞ and X_o are defined, respectively, by equation* (5.47) *and by*

$$X_o = \frac{1}{2\pi}\left[\omega - i\ln\left(\frac{q_-}{q_o}\right)\right] + \frac{1}{4}, \tag{1.9}$$

where the real quantity ω is given by equation (5.44)*, and they depend only on the initial datum $q(x,0)$ via the reflection coefficient* (2.28) *and on the ratio x/t.*

For $0 < x < 4\sqrt{2}q_o t$, the leading-order asymptotic behavior of the solution q is given by a formula analogous to formula (1.6)*, with $\overline{q}_-$ replaced by $\overline{q}_+$ and the remaining quantities modified accordingly.*

THEOREM 1.3 (Elliptic representation). *The modulus of the leading-order asymptotic solution* (1.6) *in the modulated elliptic wave region can be expressed in terms of the Jacobi elliptic function* sn *via the formula*

$$\begin{aligned} |q_{\mathrm{asymp}}(x,t)|^2 = {} & (q_o+\alpha_{\mathrm{im}})^2 \\ & - 4q_o\alpha_{\mathrm{im}}\,\mathrm{sn}^2\bigg(\frac{2\sqrt{q_o\alpha_{\mathrm{im}}}}{m}(x-2\alpha_{\mathrm{re}}t) - 2K(m)X_o, m\bigg), \end{aligned} \tag{1.10}$$

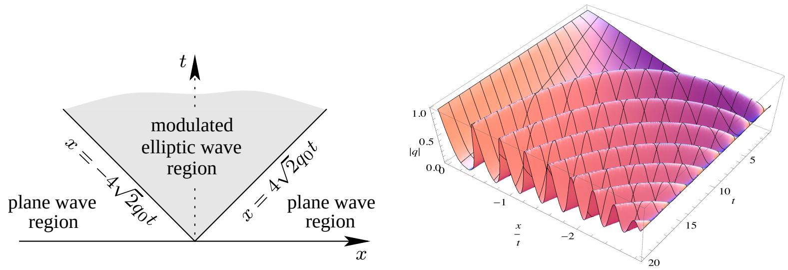

FIGURE 1.1. Left: The plane wave and modulated elliptic wave regions of the xt-plane. Right: The leading-order modulus $|q(x,t)|$ in the modulated elliptic wave region, as given by equation (1.10), for $q_o = \frac{1}{2}$ and $(x/t, t) \in [-2\sqrt{2}, 0] \times [1, 20]$.

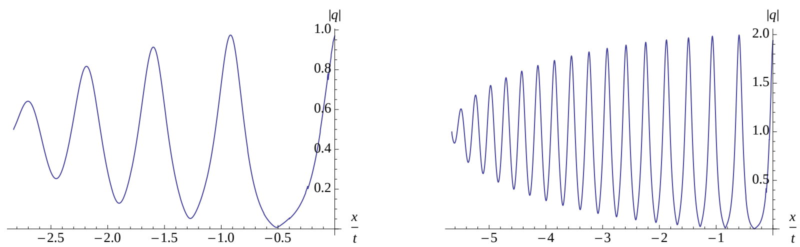

FIGURE 1.2. The leading-order modulus $|q(x,t)|$ in the modulated elliptic wave region, as given by equation (1.10), as a function of $x/t \in [-4\sqrt{2}q_o, 0]$. Left: For $q_o = \frac{1}{2}$ and $t = 10$. Right: For $q_o = 1$ and $t = 10$.

where the real quantities α_{re}, α_{im}, $K(m)$, m, *and* X_o *are defined as in Theorem* 1.2.

Plots of $|q(x,t)|$ against x/t and t in the modulated elliptic wave region, obtained using the elliptic representation (1.10), are provided in Figures 1.1 and 1.2.

Remark 1.4. Together, Theorems 1.1 and 1.2 demonstrate that the solution of the focusing NLS equation with nonzero boundary conditions remains bounded at all times.

Remark 1.5. The modulated elliptic wave solution in equation (1.10) in the special case in which the phase shift X_o is absent had been formally obtained by Kamchatnov (see [54, p. 234, eq. (5.29)]) using Whitham's modulation theory. The motion of the branch points for said solution was also formally derived in [36] using similar methods. No connection with initial conditions was provided in either of those works, however. In contrast, Theorems 1.1 and 1.2 rigorously establish the validity of the solution (1.6) as the long-time asymptotic state of *generic* initial conditions satisfying the hypotheses of said theorems.

Remark 1.6. The leading-order asymptotic solution (1.5) in the plane wave region depends on the initial condition $q(x,0)$ only through the phase shift g_∞, which is defined in terms of the reflection coefficient $r(k)$ by equation (4.25). Similarly, the modulated elliptic wave solution (1.6) depends on $q(x,0)$ only via the phase shift g_∞ and the slowly varying offset X_o, defined, respectively, by equations (4.25) and (1.9). Conversely, the envelope of the modulated elliptic wave solution is completely described by the parameters α_{re} and α_{im}, which are determined by (1.8) and are independent of the initial datum. Hence, the results of this work show that a large class of localized perturbations of the constant background in modulationally unstable media on the infinite line exhibits the same long-time asymptotic behavior as described by equations (1.5) and (1.6). Therefore, *the asymptotic stage of modulational instability is universal.*

The rest of this paper is organized as follows. In Section 2 we provide an alternative formulation of the IST for the initial value problem for the focusing NLS equation (1.1) with nonzero boundary conditions, without making use of the uniformization variable employed in [12]. In Section 3 we give an outline of the derivation of the long-time asymptotics of this problem, and we show that the xt-plane decomposes into two regions, namely a plane wave region and a modulated elliptic wave region. Sections 4 and 5 establish Theorems 1.1 and 1.2, respectively, while Section 6 provides the proof of Theorem 1.3. Section 7 contains some final remarks—including further comments on the issues mentioned in Remarks 1.4–1.6 above—a discussion of the physical significance of the results, and a few open problems. Finally, a rigorous estimation of the leading-order error is provided in the Appendix.

2 Inverse Scattering Transform with Nonzero Boundary Conditions

In this section we formulate the IST for the initial value probelm for the NLS equation (1.1) with nonzero boundary conditions (1.1), which we then use in the subsequent sections to compute the long-time asymptotic behavior of the solutions. In [12], the IST relied on introducing a two-sheeted Riemann surface and a suitable uniformization variable, following the approach of [38]. Here, instead, we work directly in the spectral plane, which is more advantageous for our purposes.

The focusing NLS equation is typically written in the form

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{2.1}$$

in which case the corresponding nonzero boundary conditions are

$$\lim_{x\to\pm\infty} u(x,t) = q_\pm e^{2iq_o^2 t}. \tag{2.2}$$

Equation (2.1) is trivially converted into (1.1) via the transformation

$$u(x,t) = q(x,t)\, e^{2iq_o^2 t}. \tag{2.3}$$

On the other hand, the advantage of equation (1.1) over equation (2.1) for our purposes is that, for the former case, the boundary conditions (1.2) at infinity are constant, while in the case of (2.3) they depend on t. That is, (1.1), (1.2), and (1.3) imply that $\lim_{x\to\pm\infty} q(x,t) = q_\pm$ for all $t > 0$.

Recall that equation (1.1) admits the Lax pair representation

$$\Psi_x = X\Psi, \quad X = ik\sigma_3 + Q, \quad k \in \mathbb{C}, \tag{2.4a}$$

$$\Psi_t = T\Psi, \quad T = -2ik^2\sigma_3 + i\sigma_3\big(Q_x - Q^2 - q_o^2 I\big) - 2kQ \tag{2.4b}$$

(namely, equation (1.1) is the compatibility condition $X_t - T_x + [X,T] = 0$ of equation (2.4)), where $\Psi(x,t,k)$ is a 2×2 matrix-valued function and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\overline{q}(x,t) & 0 \end{pmatrix}, \tag{2.5}$$

with the overbar denoting complex conjugation as usual.

Direct problem. Let $X_\pm = \lim_{x\to\pm\infty} X(x,t,k)$ and $T_\pm = \lim_{x\to\pm\infty} T(x,t,k)$. It is straightforward to see that the eigenvector matrix of $X_\pm$ can be written as

$$E_\pm(k) = \begin{pmatrix} 1 & \frac{i(\lambda-k)}{\overline{q}_\pm} \\ \frac{i(\lambda-k)}{q_\pm} & 1 \end{pmatrix}, \tag{2.6}$$

while the associated eigenvalues $\pm i\lambda$ are defined by the complex square root

$$\lambda(k) = \big(k^2 + q_o^2\big)^{\frac{1}{2}}. \tag{2.7}$$

Specifically, introducing the branch cut

$$B = i[-q_o, q_o], \tag{2.8}$$

we take $\lambda(k)$ to be the unique, single-valued function defined by equation (2.7) with a jump discontinuity across B, identified with its limiting value from the right along B so that

$$\lambda(k) = \begin{cases} \sqrt{k^2 + q_o^2}, & k \in \mathbb{R}^+ \cup B, \\ -\sqrt{k^2 + q_o^2}, & k \in \mathbb{R}^-, \end{cases} \tag{2.9}$$

where the square root sign denotes the principal branch of the real square root.

Motivated by the above remarks, and observing that $T_\pm = -2kX_\pm$, we seek simultaneous solutions $\Psi_\pm$ of the Lax pair (2.4) such that

$$\Psi_\pm(x,t,k) = E_\pm(k)\, e^{i\vartheta(x,t,k)\sigma_3}(1 + o(1)), \quad x \to \pm\infty, \tag{2.10}$$

where we define

$$\vartheta(x,t,k) = \lambda(x - 2kt). \tag{2.11}$$

It is straightforward to see from equations (2.9) and (2.11) that the exponentials in the normalization (2.10) of $\Psi_\pm$ remain bounded as $x \to \pm\infty$ provided that $k \in \Sigma$, with $\Sigma = \mathbb{R} \cup B$. It is convenient to also introduce the matrix-valued functions $\mu_\pm$ defined by

$$\mu_\pm(x,t,k) = \Psi_\pm(x,t,k)e^{-i\vartheta(x,t,k)\sigma_3}. \tag{2.12}$$

The normalization (2.10) of $\Psi_\pm$ implies

$$\mu_\pm(x,t,k) = E_\pm(k) + o(1), \quad x \to \pm\infty,\ k \in \Sigma. \tag{2.13}$$

Using standard methods, we then obtain linear integral equations of Volterra type for $\mu_\pm$:

$$\begin{aligned}\mu_-(x,t,k) = E_-(k) &+ \int_{-\infty}^{x} E_-(k)e^{i\lambda(x-y)\sigma_3}E_-^{-1}(k)\\ &\cdot \Delta Q_-(y,t)\mu_-(y,t,k)e^{-i\lambda(x-y)\sigma_3}\,dy,\end{aligned} \tag{2.14a}$$

$$\begin{aligned}\mu_+(x,t,k) = E_+(k) &- \int_{x}^{\infty} E_+(k)e^{i\lambda(x-y)\sigma_3}E_+^{-1}(k)\\ &\cdot \Delta Q_+(y,t)\mu_+(y,t,k)e^{-i\lambda(x-y)\sigma_3}\,dy,\end{aligned} \tag{2.14b}$$

where $\Delta Q_\pm(x,t) = Q(x,t) - Q_\pm$ and $Q_\pm = \lim_{x\to\pm\infty} Q(x,t)$.

The analysis of the Neumann series for the integral equations (2.14) (see [12] for more details) allows one to prove existence and uniqueness of the eigenfunctions $\mu_\pm$ for all $k \in \Sigma$ provided that $(q(x,t)-q_\pm) \in L^{1,1}(\mathbb{R}_x^\pm)$. Moreover, denoting by $\mu_{\pm1}$ and $\mu_{\pm2}$ the first and second column of $\mu_\pm$, respectively, one can conclude that μ_{+1} and μ_{-2} (and, respectively, μ_{-1} and μ_{+2}) can be analytically continued as functions of k in $\mathbb{C}^+ \setminus B^+$ (and, respectively, $\mathbb{C}^- \setminus B^-$), where

$$\mathbb{C}^\pm = \{k \in \mathbb{C} : \Im(k) \gtrless 0\}, \quad B^\pm = B \cap \mathbb{C}^\pm. \tag{2.15}$$

Consequently, definition (2.12) implies that Ψ_{-1} and Ψ_{+2} are analytic for $k \in \mathbb{C}^- \setminus B^-$, while Ψ_{+1} and Ψ_{-2} are analytic for $k \in \mathbb{C}^+ \setminus B^+$.

Scattering matrix. Since X and T are traceless, by Abel's theorem the determinants of $\Psi_\pm$ are independent of x and t. Thus, using the asymptotic conditions (2.10) we have

$$\det \Psi_\pm(x,t,k) = \lim_{x\to\pm\infty} \det \Psi_\pm(x,t,k) = \frac{2\lambda}{\lambda + k} \doteq d(k). \tag{2.16}$$

The definition (2.7) of λ implies that d is nonzero and nonsingular for all $k \in \Sigma^*$, where

$$\Sigma^* = \Sigma \setminus \{\pm i q_o\}. \tag{2.17}$$

Therefore, the matrices $\Psi_\pm$ are fundamental matrix solutions of both parts of the system (2.4) for all $k \in \Sigma^*$. It then follows that there exists a matrix $S(k)$ independent of x and t, hereafter called the scattering matrix, such that

$$\Psi_-(x,t,k) = \Psi_+(x,t,k)S(k), \quad k \in \Sigma^*. \tag{2.18}$$

Note that S is unimodular as

$$\det S(k) = \frac{\det \Psi_-(x,t,k)}{\det \Psi_+(x,t,k)} \equiv 1. \tag{2.19}$$

Moreover, we will see that the diagonal entries of $S(k)$ can be analytically continued off Σ^*.

Symmetries. The identity

$$\overline{X(q,\overline{q},\overline{k})} = -\sigma_* X(q,\overline{q},k)\sigma_*$$

implies that

$$\overline{\Psi_\pm(x,t,\overline{k})} = -\sigma_* \Psi_\pm(x,t,k)\sigma_*$$

for $k \in \Sigma^*$, which in turns yields the symmetry condition

$$\overline{S(\overline{k})} = -\sigma_* S(k)\sigma_*, \quad k \in \Sigma^*, \;\; \sigma_* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.20}$$

In fact, by letting $s_{11}(k) = a(k)$ and $s_{21}(k) = b(k)$, the scattering matrix S takes the form

$$S(k) = \begin{pmatrix} a(k) & -\overline{b}(k) \\ b(k) & \overline{a}(k) \end{pmatrix}, \quad k \in \Sigma^*, \tag{2.21}$$

where $\overline{a}(k) = \overline{a(\overline{k})}$ and $\overline{b}(k) = \overline{b(\overline{k})}$ denote the Schwartz conjugates. The determinant condition (2.19) then becomes

$$a(k)\overline{a}(k) + b(k)\overline{b}(k) = 1, \quad k \in \Sigma^*. \tag{2.22}$$

Finally, equations (2.18) and (2.21) imply

$$a(k) = (1/d(k))\,\mathrm{wr}(\Psi_{-1}(x,t,k), \Psi_{+2}(x,t,k)), \quad k \in \Sigma^*, \tag{2.23a}$$

$$\overline{a}(k) = (1/d(k))\,\mathrm{wr}(\Psi_{+1}(x,t,k), \Psi_{-2}(x,t,k)), \quad k \in \Sigma^*, \tag{2.23b}$$

$$b(k) = (1/d(k))\,\mathrm{wr}(\Psi_{+1}(x,t,k), \Psi_{-1}(x,t,k)), \quad k \in \Sigma^*, \tag{2.23c}$$

$$\overline{b}(k) = (1/d(k))\,\mathrm{wr}(\Psi_{+2}(x,t,k), \Psi_{-2}(x,t,k)), \quad k \in \Sigma^*, \tag{2.23d}$$

where wr denotes the Wronskian determinant. Thus, recalling that $d(k)$ in equation (2.16) is analytic for $k \notin B$, we infer that the function $a(k)$ is analytic in $\mathbb{C}^- \setminus B^-$. Similarly, the Schwartz conjugate $\overline{a}(k)$ is analytic in $\mathbb{C}^+ \setminus B^+$. On the other hand, the function $b(k)$ cannot be analytically continued away from Σ^* in general.

Jumps of the eigenfunctions and scattering data across the branch cut. The jump discontinuity of λ across the branch cut B induces a corresponding jump for the eigenfunctions and scattering data. (This is one of the points in which the present formulation of the IST differs most significantly from that in [12].) Since λ has been taken to be continuous from the right of B, the same is true for the analytic eigenfunctions and scattering data. That is, taking B to be oriented upwards,

$$\begin{aligned} \mu_{-1}^{-}(k) &= \lim_{\varepsilon\to 0^+} \mu_{-1}(k+\varepsilon) = \mu_{-1}(k), \\ \mu_{+2}^{-}(k) &= \lim_{\varepsilon\to 0^+} \mu_{+2}(k+\varepsilon) = \mu_{+2}(k), \end{aligned} \qquad k \in B^-,$$

with similar relations for the function a defined by (2.23a) and the eigenfunctions analytic in $\mathbb{C}^+ \setminus B^+$.

Noting that the pairs $\{\Psi_{+1}, \Psi_{+2}\}$ and $\{\Psi_{-1}, \Psi_{-2}\}$ are both fundamental sets of solutions of equation (2.4a), we find that the jumps of $\mu_\pm$ across B are given by

$$\begin{aligned} \mu_{-1}^{+}(x,t,k) &= \frac{\lambda+k}{iq_-}\,\mu_{-2}(x,t,k), \\ \mu_{+2}^{+}(x,t,k) &= \frac{\lambda+k}{i\overline{q}_+}\,\mu_{+1}(x,t,k), \end{aligned} \qquad k \in B^-, \tag{2.24a}$$

$$\begin{aligned} \mu_{-2}^{+}(x,t,k) &= \frac{\lambda+k}{i\overline{q}_-}\,\mu_{-1}(x,t,k), \\ \mu_{+1}^{+}(x,t,k) &= \frac{\lambda+k}{iq_+}\,\mu_{+2}(x,t,k), \end{aligned} \qquad k \in B^+. \tag{2.24b}$$

It then follows that the function a satisfies the following condition across B^+:

$$\overline{a}^{+}(k) = (q_-/q_+)\,a(k), \quad k \in B^+. \tag{2.25}$$

Inverse problem. Exploiting the analyticity properties of $\mu_{\pm 1,2}$ in the spectral plane, we proceed to define the sectionally analytic matrix-valued function

$$M(x,t,k) = \begin{cases} \left(\frac{\mu_{+1}}{\overline{a}d}, \mu_{-2}\right) = \left(\frac{\Psi_{+1}}{\overline{a}d}, \Psi_{-2}\right)e^{-i\vartheta\sigma_3}, & k \in \mathbb{C}^+ \setminus B^+, \\ \left(\mu_{-1}, \frac{\mu_{+2}}{ad}\right) = \left(\Psi_{-1}, \frac{\Psi_{+2}}{ad}\right)e^{-i\vartheta\sigma_3}, & k \in \mathbb{C}^- \setminus B^-, \end{cases} \tag{2.26}$$

where the dependence on the variables x, t, k on the right-hand side has been suppressed for brevity. Note the presence of d in definition (2.26) (as compared to [12] and other standard formulations of the inverse problem), which implies that M is unimodular, i.e.,

$$\det M(x,t,k) \equiv 1, \quad k \in \mathbb{C} \setminus B. \tag{2.27}$$

As usual, the inverse problem is formulated in terms of an appropriate Riemann-Hilbert problem. To define this Riemann-Hilbert problem, one needs an appropriate jump condition for M. We first compute the jump of M across $k \in \mathbb{R}$. Denoting by $M^\pm$ the limits of M as $\Im(k) \to 0^\pm$, recalling that ϑ, a, and d are continuous across $\mathbb{R}$, and using the scattering relation (2.18) and the determinant condition

(2.22), we obtain the jump condition

$$M^+(x,t,k) = M^-(x,t,k)\begin{pmatrix} \frac{1}{d(k)}[1 + r(k)\overline{r}(k)] & \overline{r}(k)e^{2i\vartheta(x,t,k)} \\ r(k)e^{-2i\vartheta(x,t,k)} & d(k) \end{pmatrix}, \quad k \in \mathbb{R},$$

where the real k-axis is oriented from left to right as usual, ϑ is defined by equation (2.11), and the reflection coefficient $r(k)$ with Schwartz conjugate $\overline{r}(k)$ is defined by

$$r(k) = -\frac{b(k)}{\overline{a}(k)}. \tag{2.28}$$

Next, we note that the jump of M across the branch cut B is also affected by the discontinuities of μ_{-1} and μ_{+2} across B^- and of μ_{+1} and μ_{-2} across B^+. Hence, starting from the definition (2.26) of M and employing equations (2.24) and (2.25) in combination with straightforward algebraic computations, we obtain the jump conditions

$$M^+(x,t,k) = \\ M^-(x,t,k)\begin{pmatrix} -\frac{\lambda-k}{iq_-}\,\overline{r}(k)e^{2i\vartheta(x,t,k)} & \frac{2\lambda}{i\overline{q}_-} \\ \frac{\overline{q}_-}{2i\lambda}[1 + r(k)\overline{r}(k)] & -\frac{\lambda+k}{i\overline{q}_-}\,r(k)e^{-2i\vartheta(x,t,k)} \end{pmatrix}, \quad k \in B^+,$$

$$M^+(x,t,k) = \\ M^-(x,t,k)\begin{pmatrix} \frac{\lambda+k}{iq_-}\,\overline{r}(k)e^{2i\vartheta(x,t,k)} & \frac{q_-}{2i\lambda}[1 + r(k)\overline{r}(k)] \\ \frac{2\lambda}{iq_-} & \frac{\lambda-k}{i\overline{q}_-}\,r(k)e^{-2i\vartheta(x,t,k)} \end{pmatrix}, \quad k \in B^-,$$

with B oriented upwards as before, and with ϑ and r as above.

Finally, to complete the formulation of the Riemann-Hilbert problem one must specify a normalization condition and, if a discrete spectrum is present, appropriate residue conditions. The latter will not be necessary in our case, since in what follows we will assume that no discrete spectrum is present. Using the integral equations (2.14) for $\mu_\pm$ and the relationship (2.26) between M and $\mu_\pm$, it can be shown (see [12]) that the function M admits the large-k asymptotic expansion

$$M(x,t,k) = I + \frac{M_1(x,t)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty. \tag{2.29}$$

Combining equations (2.4a), (2.26), and (2.29), one can also recover the solution of the focusing NLS equation (1.1) in the form

$$q(x,t) = -2i(M_1(x,t))_{12}. \tag{2.30}$$

Based on the above discussion, the function M satisfies the following Riemann-Hilbert problem.

Riemann-Hilbert problem 2.1. Suppose that $a(k) \neq 0$ for all $k \in \mathbb{C}^- \cup \Sigma$ so that no discrete spectrum is present. Determine a sectionally analytic matrix-valued function $M(k) = M(x,t,k)$ in $\mathbb{C} \setminus \Sigma$ satisfying the jump conditions

$$M^+(k) = M^-(k)V_1, \qquad k \in \mathbb{R}, \tag{2.31a}$$

$$M^+(k) = M^-(k)V_2, \qquad k \in B^+, \tag{2.31b}$$

$$M^+(k) = M^-(k)V_3, \qquad k \in B^-, \tag{2.31c}$$

and the normalization condition

$$M(k) = I + O(1/k), \quad k \to \infty, \tag{2.31d}$$

where

$$V_1(x,t,k) = \begin{pmatrix} \frac{1}{d(k)}[1 + r(k)\overline{r}(k)] & \overline{r}(k)e^{2i\theta(\xi,k)t} \\ r(k)e^{-2i\theta(\xi,k)t} & d(k) \end{pmatrix}, \tag{2.32a}$$

$$V_2(x,t,k) = \begin{pmatrix} -\frac{\lambda-k}{iq_-}\overline{r}(k)e^{2i\theta(\xi,k)t} & \frac{2\lambda}{i\overline{q}_-} \\ \frac{\overline{q}_-}{2i\lambda}[1 + r(k)\overline{r}(k)] & -\frac{\lambda+k}{i\overline{q}_-}r(k)e^{-2i\theta(\xi,k)t} \end{pmatrix}, \tag{2.32b}$$

$$V_3(x,t,k) = \begin{pmatrix} \frac{\lambda+k}{iq_-}\overline{r}(k)e^{2i\theta(\xi,k)t} & \frac{q_-}{2i\lambda}[1 + r(k)\overline{r}(k)] \\ \frac{2\lambda}{iq_-} & \frac{\lambda-k}{i\overline{q}_-}r(k)e^{-2i\theta(\xi,k)t} \end{pmatrix}, \tag{2.32c}$$

the reflection coefficient r is defined by equation (2.28), the similarity variable is

$$\xi = x/t, \tag{2.33}$$

and the function $\theta(\xi,k) = \vartheta(x,t,k)/t$ is given by

$$\theta(\xi,k) = \lambda(\xi - 2k). \tag{2.34}$$

3 Long-Time Asymptotics: Preliminaries

We now compute the long-time asymptotic behavior of the solution q of the focusing NLS equation, as obtained by equation (2.30), by analyzing the Riemann-Hilbert Problem 2.1 via the Deift-Zhou nonlinear steepest descent for oscillatory Riemann-Hilbert problems [34]. Recall that, in general, the Deift-Zhou method is based on deforming the jump contours of the Riemann-Hilbert problem to contours in the complex k-plane across which the relevant jumps have a well-defined limit as $t \to \infty$. In our case, it turns out that, after the appropriate deformations have been performed, the majority of the jumps across the deformed contours tend to the identity matrix, while those that yield the leading-order contribution to the asymptotics tend to constant matrices.

Analyticity in a neighborhood of the continuous spectrum. Hereafter, in order to be able to deform the contours away from the continuous spectrum Σ, we place an additional restriction on the initial datum $q(x,0)$ that ensures that the reflection coefficient r can be analytically extended off the continuous spectrum.

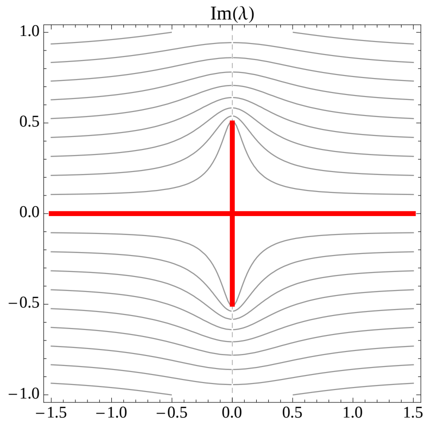

FIGURE 3.1. Contour plot of $\Im(\lambda)$ for $q_o = \frac{1}{2}$.

LEMMA 3.1 (Analyticity of the reflection coefficient). *Suppose that there exists a constant* $\epsilon > 0$ *such that*

$$e^{\pm\epsilon x}(q(x,0) - q_\pm) \in L^1(\mathbb{R}^\pm), \tag{3.1}$$

where $q(x,0)$ *and* $q_\pm$ *are the initial and boundary conditions, respectively, of the focusing NLS equation* (1.1)*. Then, all the eigenfunctions and the spectral functions* a *and* b *are analytic in the region*

$$\Sigma_\epsilon = \{k \in \mathbb{C} \setminus B : |\mathrm{Im}(\lambda)| < \epsilon\},$$

where λ *is defined by equation* (2.7)*. The same conclusion follows for the reflection coefficient* r *defined by equation* (2.28) *as long as* $a(k) \neq 0$ *for all* $k \in \Sigma_\epsilon$*.*

The region Σ_ϵ in Lemma 3.1 is the analogue of what is known as the Bargmann strip in the case of zero conditions at infinity [48]. Lemma 3.1 can be established by employing a Neumann series for the integral equations (2.14) that define $\mu_\pm$, using similar arguments as in [12]. It is important to note that the contour lines of $\Im(\lambda)$ do not intersect the branch cut B; e.g., see Figure 3.1. Hence, the existence of any $\epsilon > 0$ ensures that r is analytic in a neighborhood of B (except possibly at the branch points), which is the domain of analyticity required for the deformations in the Deift-Zhou method.

When the hypothesis of Lemma 3.1 is satisfied, the jump condition (2.25) also holds across B^-. In addition, (2.23) and (2.24) yield a similar jump condition for b, namely, $b^+(k) = -(\overline{q}_+/q_-)\overline{b}(k)$ for all $k \in B$. Hence, we obtain the jump condition for the reflection coefficient r across B, namely,

$$r^+(k) = -(\overline{q}_-/q_-)\overline{r}(k), \quad k \in B. \tag{3.2}$$

An example of an initial condition satisfying the hypothesis of Lemma 3.1 is the following boxlike initial datum considered in [11]:

$$q(x,0) = \begin{cases} q_o, & |x| > L, \\ \beta e^{i\chi}, & |x| < L, \end{cases} \tag{3.3}$$

with $\beta > 0$, $L > 0$, and $\chi \in \mathbb{R}$, which gives rise to the reflection coefficient

$$r(k) = \frac{e^{2i\lambda L}[(\beta\cos\chi - q_o)k - i\beta\lambda\sin\chi]}{\lambda\sqrt{k^2+\beta^2}\cot(2L\sqrt{k^2+\beta^2}) - i(k^2 + q_o\beta\cos\chi)}. \tag{3.4}$$

The sign structure of $\Re(i\theta)$. The choice of deformations of the Deift-Zhou method depends crucially on the sign structure of the quantity $\Re(i\theta)$, which is involved in all three jump matrices (2.32) of the Riemann-Hilbert problem 2.1. From the definition (2.34) of θ we have

$$\Re(i\theta) = -\lambda_{\mathrm{im}}(\xi - 2k_{\mathrm{re}}) + 2\lambda_{\mathrm{re}}k_{\mathrm{im}}. \tag{3.5}$$

The above expression simplifies significantly in the special cases $|k_{\mathrm{im}}| \ll 1$ and $|k_{\mathrm{im}}| \gg 1$. When $|k_{\mathrm{im}}| \gg 1$, recalling that $\lambda \sim k$ as $k \to \infty$, we have

$$\Re(i\theta) = (4k_{\mathrm{re}} - \xi)k_{\mathrm{im}} + O(1/k_{\mathrm{im}}), \quad k_{\mathrm{im}} \to \pm\infty. \tag{3.6}$$

Thus, as $k_{\mathrm{im}} \to \pm\infty$, the sign of $\Re(i\theta)$ is determined by that of $k_{\mathrm{re}} - \xi/4$. When $0 < k_{\mathrm{im}} \ll 1$, on the other hand, the situation is more complicated. The definition of λ implies

$$\begin{aligned} \lambda &= \mathrm{sign}(k_{\mathrm{re}})\sqrt{k_{\mathrm{re}}^2 + q_o^2} \\ &\quad \cdot \left[1 - \frac{q_o^2}{2(k_{\mathrm{re}}^2+q_o^2)^2}\,k_{\mathrm{im}}^2 + \frac{i\,k_{\mathrm{re}}}{k_{\mathrm{re}}^2+q_o^2}\,k_{\mathrm{im}} + O(k_{\mathrm{im}}^3)\right]. \end{aligned} \tag{3.7}$$

Hence,

$$\Re(i\theta) = \frac{\mathrm{sign}(k_{\mathrm{re}})}{\sqrt{k_{\mathrm{re}}^2+q_o^2}}\big(4k_{\mathrm{re}}^2 - \xi\,k_{\mathrm{re}} + 2q_o^2\big)k_{\mathrm{im}} + O(k_{\mathrm{im}}^3), \quad k_{\mathrm{im}} \to 0^+. \tag{3.8}$$

For $|\xi| < 4\sqrt{2}q_o$ the quadratic expression in the leading-order term of (3.8) is always positive, while for $|\xi| \geqslant 4\sqrt{2}q_o$ it has real roots k_1, k_2 equal to

$$k_{1,2} = \frac{1}{8}\Big(\xi \pm \sqrt{\big(\xi - 4\sqrt{2}q_o\big)\big(\xi + 4\sqrt{2}q_o\big)}\Big), \tag{3.9}$$

where we take $k_1 < k_2$. A similar expansion is obtained when $k_{\mathrm{im}} \to 0^-$, leading to the same roots as in equation (3.9). The overall sign structure of $\Re(i\theta)$ in the complex k-plane is illustrated in Figure 3.2.

Remark 3.2. Importantly, the points k_1 and k_2 are the stationary points of θ. In the following sections we will show that the sectors of the xt-plane where $|\xi| > 4\sqrt{2}q_o$ correspond to *plane wave regions*, whereas the sectors where $|\xi| < 4\sqrt{2}q_o$ correspond to *modulated elliptic wave regions*.

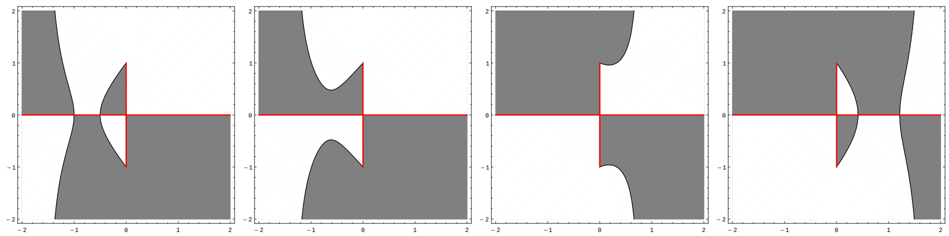

FIGURE 3.2. The sign structure of $\Re(i\theta)$ in the complex k-plane for various values of ξ and $q_o = 1$: (a) $\xi = -6$, corresponding to $x < -4\sqrt{2}q_o t$; (b) $\xi = -5.2$, corresponding to $-4\sqrt{2}q_o t < x < 0$; (c) $\xi = 3$, corresponding to $0 < x < 4\sqrt{2}q_o t$; (d) $\xi = 6.5$, corresponding to $x > 4\sqrt{2}q_o t$. In the gray regions $\Re(i\theta) < 0$, whereas in the white regions $\Re(i\theta) > 0$. Cases (a) and (d) correspond to plane wave regions, while cases (b) and (c) correspond to modulated elliptic wave regions. The points $k_{1,2}$ in cases (a) and (d) correspond with the sign changes of $\Re(i\theta)$ along the real axis away from $k = 0$.

4 The Plane Wave Region: Proof of Theorem 1.1

In this section we prove Theorem 1.1; i.e., we compute the leading-order long-time asymptotic behavior of the solution of the focusing NLS equation (1.1) in the plane wave region $|x| > 4\sqrt{2}q_o t$. As mentioned earlier, we do so by performing appropriate deformations of the Riemann-Hilbert problem 2.1 similar to those employed in [18].

All three jump matrices $V_1^{(0)} = V_1$, $V_2^{(0)} = V_2$, and $V_3^{(0)} = V_3$ of the Riemann-Hilbert problem 2.1 contain both $e^{i\theta t}$ and $e^{-i\theta t}$. Recall also that $\theta(\xi, k) \in \mathbb{R}$ for all $k \in \Sigma$. The first step required in order to take the limit $t \to \infty$ is to express these jumps as products of matrices that involve only one of the two aforementioned exponentials, and with the matrices ordered in such a way that they remain bounded when the contour is deformed away from Σ.

Preliminary factorizations: $x < -4\sqrt{2}q_o t$. We will show that, in this case, the factorizations of $V_1^{(0)}$ convenient for our purposes are

$$V_1^{(0)} = \begin{cases} V_2^{(1)} V_0^{(1)} V_1^{(1)}, & k_{\mathrm{re}} < k_1, \\ V_4^{(1)} V_3^{(1)}, & k_{\mathrm{re}} > k_1, \end{cases} \tag{4.1}$$

where

$$V_0^{(1)} = \begin{pmatrix} 1 + r\bar{r} & 0 \\ 0 & \frac{1}{1+r\bar{r}} \end{pmatrix}, \tag{4.2a}$$

$$V_1^{(1)} = \begin{pmatrix} d^{-1/2} & \frac{d^{1/2}\bar{r}e^{2i\theta t}}{1+r\bar{r}} \\ 0 & d^{1/2} \end{pmatrix}, \quad V_2^{(1)} = \begin{pmatrix} d^{-1/2} & 0 \\ \frac{d^{1/2}re^{-2i\theta t}}{1+r\bar{r}} & d^{1/2} \end{pmatrix}, \tag{4.2b}$$

$$(4.2c)\quad V_3^{(1)} = \begin{pmatrix} d^{-1/2} & 0 \\ d^{-1/2} r e^{-2i\theta t} & d^{1/2} \end{pmatrix}, \quad V_4^{(1)} = \begin{pmatrix} d^{-1/2} & d^{-1/2}\overline{r} e^{2i\theta t} \\ 0 & d^{1/2} \end{pmatrix}.$$

Moreover, since $k_{1,2} < 0$ in this region (cf. Figure 3.2a), the branch cut B lies to the right of the point k_1. Thus, the appropriate factorizations for the jumps $V_2^{(0)}$ and $V_3^{(0)}$ will turn out to be

$$(4.3a)\qquad V_2^{(0)} = \big(V_{3-}^{(1)}\big)^{-1} V_B V_{3+}^{(1)}, \quad k \in B^+,$$

$$(4.3b)\qquad V_3^{(0)} = V_{4-}^{(1)} V_B \big(V_{4+}^{(1)}\big)^{-1}, \quad k \in B^-,$$

where $V_{3,4\pm}^{(1)}$ denote the left- and right-sided limits of $V_{3,4}^{(1)}$ and V_B is the *constant* matrix

$$(4.4)\qquad V_B = \begin{pmatrix} 0 & q_-/iq_o \\ \overline{q}_-/iq_o & 0 \end{pmatrix}.$$

Remark 4.1 (k_1 versus k_2). In the case of $x < -4\sqrt{2}q_o t$, the point k_2 is not significant concerning the choice of factorization of the jump $V_1^{(0)}$. This is because, as can be seen from Figure 3.2a, the change of sign at k_2 affects only a finite region of the k-plane, which can therefore be bypassed by suitable deformations of the jump contours. The opposite is of course true for $x > 4\sqrt{2}q_o t$.

Remark 4.2 ($x < -4\sqrt{2}q_o t$ versus $x > 4\sqrt{2}q_o t$). For $x < -4\sqrt{2}q_o t$, using the factorizations (4.1)–(4.3) and performing suitable deformations in the complex k-plane, we will eventually be able to reduce the jump across the branch cut B to the *constant* matrix V_B defined above. This reduction is crucial, since it implies that the Riemann-Hilbert problem that yields the leading-order asymptotic behavior of the focusing NLS solution can be solved explicitly. For $x > 4\sqrt{2}q_o t$, however, the sign structure of $\Re(i\theta)$ is such that the desired reduction cannot be accomplished using the above factorizations. In that case, it is then necessary to first perform a rescaling of the original Riemann-Hilbert problem, as discussed next.

Preliminary factorizations: $x > 4\sqrt{2}q_o t$. In this case we first rescale the Riemann-Hilbert problem 2.1 as follows. Let

$$(4.5)\qquad \begin{aligned} \widetilde{M}(x,t,k) &= \begin{cases} M(x,t,k)\overline{A}(k), & k \in \mathbb{C}^+ \setminus \Sigma^+, \\ M(x,t,k)A^{-1}(k), & k \in \mathbb{C}^- \setminus \Sigma^-, \end{cases} \\ A(k) &= \begin{pmatrix} a(k) & 0 \\ 0 & a(k)^{-1} \end{pmatrix}. \end{aligned}$$

Then, $\widetilde{M}(k) = \widetilde{M}(x,t,k)$ is analytic in $\mathbb{C} \setminus \Sigma$ and satisfies the jump conditions

$$\widetilde{M}^+(k) = \widetilde{M}^-(k)\widetilde{V}_1^{(0)}, \quad k \in \mathbb{R}, \tag{4.6a}$$

$$\widetilde{M}^+(k) = M^-(k)\widetilde{V}_2^{(0)}, \quad k \in B^+, \tag{4.6b}$$

$$\widetilde{M}^+(k) = \widetilde{M}^-(k)\widetilde{V}_3^{(0)}, \quad k \in B^-, \tag{4.6c}$$

and the normalization condition

$$\widetilde{M}(k) = I + O(1/k), \quad k \to \infty, \tag{4.6d}$$

where

$$\widetilde{V}_1^{(0)} = A V_1^{(0)} \overline{A}, \qquad \widetilde{V}_2^{(0)} = (\overline{A^-})^{-1} V_2^{(0)} \overline{A^+}, \qquad \widetilde{V}_3^{(0)} = A^- V_3^{(0)} (A^+)^{-1}. \tag{4.7}$$

The advantage of considering $\widetilde{M}$ instead of M for $x > 4\sqrt{2}q_o t$ is that, in contrast to the jumps $V_1^{(0)}$, $V_2^{(0)}$, and $V_3^{(0)}$, the jumps $\widetilde{V}_1^{(0)}$, $\widetilde{V}_2^{(0)}$, and $\widetilde{V}_3^{(0)}$ can be factorized in a way that eventually leads to a Riemann-Hilbert problem with a *constant* jump across B, just like in the case $x < -4\sqrt{2}q_o t$. Indeed, we may now use the factorizations

$$\widetilde{V}_1^{(0)} = \begin{cases} \widetilde{V}_2^{(1)} \widetilde{V}_0^{(1)} \widetilde{V}_1^{(1)}, & k_{\mathrm{re}} > k_2, \\ \widetilde{V}_4^{(1)} \widetilde{V}_3^{(1)}, & k_{\mathrm{re}} < k_2, \end{cases} \tag{4.8}$$

where for $\rho(k) = -b(k)/a(k)$ we define

$$\widetilde{V}_0^{(1)} = \begin{pmatrix} \frac{1}{1+\rho\overline{\rho}} & 0 \\ 0 & 1+\rho\overline{\rho} \end{pmatrix}, \tag{4.9a}$$

$$\widetilde{V}_1^{(1)} = \begin{pmatrix} d^{-1/2} & 0 \\ \frac{d^{-1/2}\rho e^{-2i\theta t}}{1+\rho\overline{\rho}} & d^{1/2} \end{pmatrix}, \quad \widetilde{V}_2^{(1)} = \begin{pmatrix} d^{-1/2} & \frac{d^{-1/2}\overline{\rho} e^{2i\theta t}}{1+\rho\overline{\rho}} \\ 0 & d^{1/2} \end{pmatrix}, \tag{4.9b}$$

$$\widetilde{V}_3^{(1)} = \begin{pmatrix} d^{-1/2} & d^{1/2}\overline{\rho} e^{2i\theta t} \\ 0 & d^{1/2} \end{pmatrix}, \quad \widetilde{V}_4^{(1)} = \begin{pmatrix} d^{-1/2} & 0 \\ d^{1/2}\rho e^{-2i\theta t} & d^{1/2} \end{pmatrix}. \tag{4.9c}$$

Also, since the branch cut B lies to the left of k_2 (see Figure 3.2d), the appropriate factorizations for $\widetilde{V}_2^{(0)}$ and $\widetilde{V}_3^{(0)}$ are

$$\widetilde{V}_2^{(0)} = \big(\widetilde{V}_{3-}^{(1)}\big)^{-1} \widetilde{V}_B \widetilde{V}_{3+}^{(1)}, \quad k \in B^+, \tag{4.10a}$$

$$\widetilde{V}_3^{(0)} = \widetilde{V}_{4-}^{(1)} \widetilde{V}_B \big(\widetilde{V}_{4+}^{(1)}\big)^{-1}, \quad k \in B^-, \tag{4.10b}$$

where the *constant* matrix $\widetilde{V}_B$ is defined by

$$\widetilde{V}_B = \begin{pmatrix} 0 & q_+/iq_o \\ \overline{q}_+/iq_o & 0 \end{pmatrix}. \tag{4.10c}$$

Remark 4.3. The factorizations (4.8) and (4.10) are completely analogous to the factorizations (4.1) and (4.3). Moreover, it is straightforward to see that

$$(M_1(x,t))_{12} = (\widetilde{M}_1(x,t))_{12}$$

since $a(k) = 1 + O(1/k^2)$ as $k \to \infty$ (e.g., see [12]). Hence, the rescaling does not affect the reconstruction formula (2.30) of the potential. As a result, once the rescaling (4.5) has been performed, the steps required for the implementation of the Deift-Zhou method in the two cases $x < -4\sqrt{2}q_o t$ and $x > 4\sqrt{2}q_o t$ are identical. Therefore, we hereafter limit ourselves to discussing the case $x < -4\sqrt{2}q_o t$ for brevity.

First deformation. Define $M^{(1)} = M^{(1)}(x,t,k)$ in terms of the solution $M^{(0)} = M$ of the Riemann-Hilbert problem 2.1 according to Figure 4.1, with the matrices $V_0^{(1)}, \dots, V_B$ given by equations (4.2) and (4.4). Then $M^{(1)}$ is analytic in $\mathbb{C} \setminus (B \cup L_0 \cup L_1 \cup L_2 \cup \bigcup_{j=1}^4 L_{3,j} \cup L_{4,j})$ and satisfies the jump conditions

$$M^{(1)+}(k) = M^{(1)-}(k)V_B, \qquad k \in B, \tag{4.11a}$$

$$M^{(1)+}(k) = M^{(1)-}(k)V_j^{(1)}, \qquad k \in L_j,\ j = 0,1,2, \tag{4.11b}$$

$$M^{(1)+}(k) = M^{(1)-}(k)V_3^{(1)}, \qquad k \in L_{3,1} \cup L_{3,2}, \tag{4.11c}$$

$$M^{(1)+}(k) = M^{(1)-}(k)\big(V_3^{(1)}\big)^{-1}, \quad k \in L_{3,3} \cup L_{3,4}, \tag{4.11d}$$

$$M^{(1)+}(k) = M^{(1)-}(k)V_4^{(1)}, \qquad k \in L_{4,1} \cup L_{4,3}, \tag{4.11e}$$

$$M^{(1)+}(k) = M^{(1)-}(k)\big(V_4^{(1)}\big)^{-1}, \quad k \in L_{4,2} \cup L_{4,4}, \tag{4.11f}$$

with the jump contours L_j as in Figure 4.1, and the normalization condition

$$M^{(1)}(k) = I + O(1/k), \quad k \to \infty. \tag{4.11g}$$

Note that the jump across (k_1, ∞) has been eliminated as a result of the transformation. Furthermore, note that all the jump contours of Figure 4.1 apart from $(-\infty, k_1)$ and the branch cut B can be deformed to a single contour that intersects with the continuous spectrum Σ only at k_1, as shown in Figure 4.2. Near the branch point iq_o in particular, we first deform the contours $L_{3,2}$ and $L_{3,3}$ as shown in Figure 4.3 and then further deform the contours $\bigcup_{j=1}^4 L_{3,j}$ of Figure 4.1 to the contour L_3 of Figure 4.2, which does not go through the origin. The point $-iq_o$ is handled in an analogous way, so that the contours $\bigcup_{j=1}^4 L_{4,j}$ of Figure 4.1 are deformed to the contour L_4 of Figure 4.2.

Second deformation. The purpose of this deformation is to eliminate the jump across $(-\infty, k_1)$. This goal is accomplished by introducing an auxiliary scalar function $\delta = \delta(k)$ that is analytic in $\mathbb{C} \setminus (-\infty, k_1)$ and satisfies the jump condition

$$\delta^+(k) = \delta^-(k)[1 + r(k)\overline{r}(k)], \quad k \in (-\infty, k_1), \tag{4.12a}$$

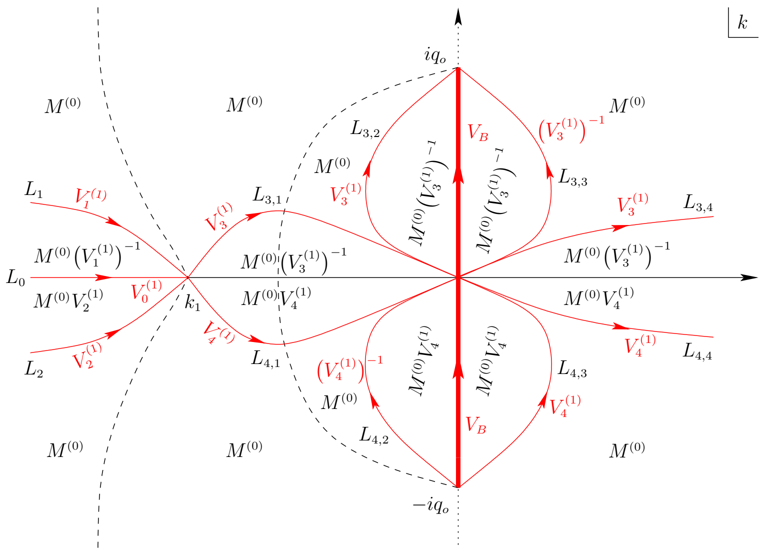

FIGURE 4.1. The first deformation in the plane wave region.

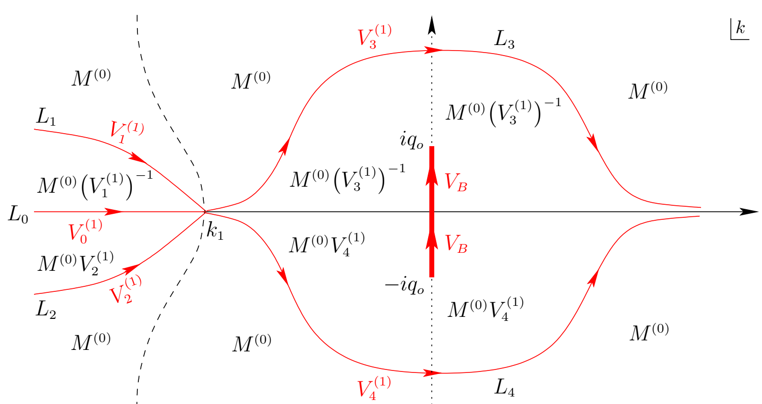

FIGURE 4.2. Final form of the Riemann-Hilbert problem for $M^{(1)}$ in the plane wave region.

and the normalization condition

$$\delta(k) = 1 + O(1/k), \quad k \to \infty. \tag{4.12b}$$

The solution of this scalar Riemann-Hilbert problem is obtained in explicit form via the Plemelj formulae [43] as

$$\delta(k) = \exp\left\{\frac{1}{2i\pi}\int_{-\infty}^{k_1} \frac{\ln[1 + r(\nu)\overline{r}(\nu)]}{\nu - k}\, d\nu\right\}, \quad k \notin (-\infty, k_1). \tag{4.13}$$

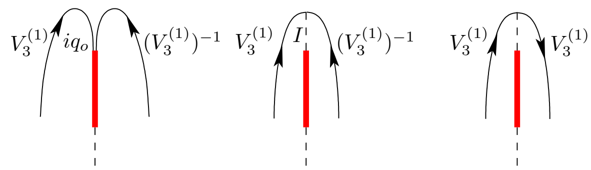

FIGURE 4.3. The sequence of deformations around the branch point iq_o.

Then, the function $M^{(2)}$ defined by

$$M^{(2)}(x,t,k) = M^{(1)}(x,t,k)\delta(k)^{-\sigma_3} \tag{4.14}$$

does not have a discontinuity across $(-\infty, k_1)$ since

$$M^{(2)+}(k) = M^{(2)-}(k)(\delta^-)^{\sigma_3} V_0^{(1)} \big(\delta^+\big)^{-\sigma_3} = M^{(2)-}(k)$$

for all $k \in (-\infty, k_1)$. Moreover, the jumps off the real k-axis become

$$V_B^{(2)} = \begin{pmatrix} 0 & \frac{q_-}{iq_o}\delta^2 \\ \frac{\bar{q}_-}{iq_o}\delta^{-2} & 0 \end{pmatrix}, \quad V_1^{(2)} = \begin{pmatrix} d^{-1/2} & \frac{d^{1/2}\bar{r}e^{2i\theta t}}{1+r\bar{r}}\delta^2 \\ 0 & d^{1/2} \end{pmatrix}, \tag{4.15a}$$

$$V_2^{(2)} = \begin{pmatrix} d^{-1/2} & 0 \\ \frac{d^{1/2}re^{-2i\theta t}}{1+r\bar{r}}\delta^{-2} & d^{1/2} \end{pmatrix}, \quad V_3^{(2)} = \begin{pmatrix} d^{-1/2} & 0 \\ d^{-1/2}\, re^{-2i\theta t}\,\delta^{-2} & d^{1/2} \end{pmatrix}, \tag{4.15b}$$

$$V_4^{(2)} = \begin{pmatrix} d^{-1/2} & d^{-1/2}\,\bar{r}e^{2i\theta t}\,\delta^2 \\ 0 & d^{1/2} \end{pmatrix}, \tag{4.15c}$$

and, since δ is analytic away from $(-\infty, k_1)$, no additional jumps are introduced. Overall, the function $M^{(2)}$ is analytic in $\mathbb{C} \setminus (\bigcup_{j=1}^4 L_j \cup B)$ and satisfies the jump conditions

$$M^{(2)+}(k) = M^{(2)-}(k)V_B^{(2)}, \quad k \in B, \tag{4.16a}$$

$$M^{(2)+}(k) = M^{(2)-}(k)V_j^{(2)}, \quad k \in L_j,\ j = 1,2,3,4, \tag{4.16b}$$

with the jump contours L_j as shown in Figure 4.4, and the normalization condition

$$M^{(2)}(k) = I + O(1/k), \quad k \to \infty. \tag{4.16c}$$

Third deformation. The next step consists in removing the function d from the jump matrices (4.15) so that the jumps along the contours L_j eventually tend to the identity as $t \to \infty$. This is accomplished by switching from $M^{(2)}$ to $M^{(3)}$ according to Figure 4.5. It then follows that

$$M^{(3)+}(k) = M^{(3)-}(k)V_j^{(2)} d^{\frac{\sigma_3}{2}}, \quad k \in L_j,\ j = 1,3,$$

$$M^{(3)+}(k) = M^{(3)-}(k) d^{\frac{\sigma_3}{2}} V_j^{(2)}, \quad k \in L_j,\ j = 2,4.$$

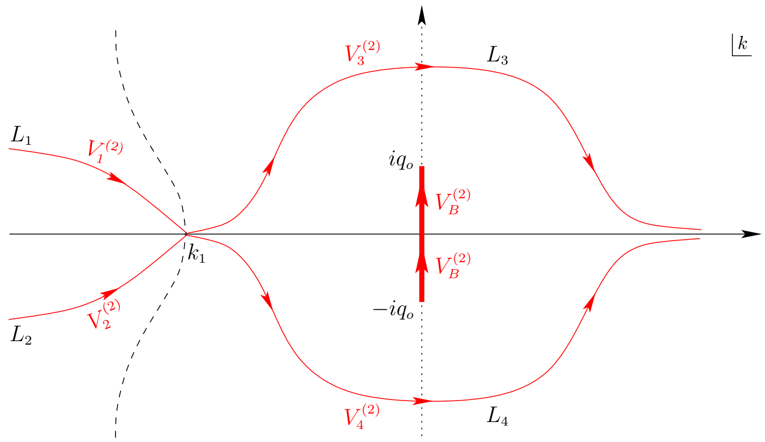

FIGURE 4.4. The jumps of $M^{(2)}$ in the plane wave region.

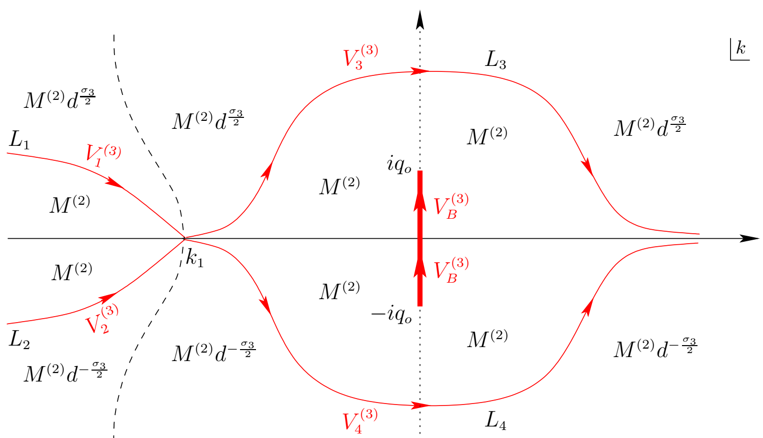

FIGURE 4.5. The third deformation in the plane wave region.

Furthermore, the jump $V_B^{(2)}$ of $M^{(2)}$ across B, which does not involve d, remains the same for $M^{(3)}$. Overall, $M^{(3)}$ is analytic in $\mathbb{C} \setminus (\bigcup_{j=1}^4 L_j \cup B)$ and satisfies the jump conditions

(4.17a) $$M^{(3)+}(k) = M^{(3)-}(k)V_B^{(3)}, \quad k \in B,$$

(4.17b) $$M^{(3)+}(k) = M^{(3)-}(k)V_j^{(3)}, \quad k \in L_j,\ j = 1, 2, 3, 4,$$

and the normalization condition

(4.17c) $$M^{(3)}(k) = I + O(1/k), \qquad k \to \infty,$$

where the jump contours L_1, L_2, L_3, L_4 are shown in Figure 4.5 and

$$V_B^{(3)} = V_B^{(2)}, \quad V_1^{(3)} = \begin{pmatrix} 1 & \frac{\overline{r}e^{2i\theta t}}{1+r\overline{r}}\,\delta^2 \\ 0 & 1 \end{pmatrix}, \quad V_2^{(3)} = \begin{pmatrix} 1 & 0 \\ \frac{re^{-2i\theta t}}{1+r\overline{r}}\,\delta^{-2} & 1 \end{pmatrix}, \tag{4.18a}$$

$$V_3^{(3)} = \begin{pmatrix} 1 & 0 \\ re^{-2i\theta t}\,\delta^{-2} & 1 \end{pmatrix}, \quad V_4^{(3)} = \begin{pmatrix} 1 & \overline{r}e^{2i\theta t}\,\delta^2 \\ 0 & 1 \end{pmatrix}. \tag{4.18b}$$

Fourth deformation. Next, we eliminate the auxiliary function δ from the jump $V_B^{(3)}$ across the branch cut B—and hence turn this jump into a constant—by employing the so-called g-function mechanism [32, 33]. More precisely, we let

$$M^{(4)}(x,t,k) = M^{(3)}(x,t,k)e^{ig(k)\sigma_3}, \tag{4.19}$$

where the yet-to-be-determined scalar function g is analytic in $\mathbb{C} \setminus B$ and satisfies the discontinuity condition

$$e^{i(g^+(k)+g^-(k))} = \delta(k)^2, \quad k \in B. \tag{4.20}$$

It is straightforward to check that if condition (4.20) holds, then the jump of $M^{(4)}$ across B is constant and equal to

$$V_B^{(4)} = V_B = \begin{pmatrix} 0 & q_-/iq_o \\ \overline{q}_-/iq_o & 0 \end{pmatrix}, \quad k \in B. \tag{4.21a}$$

Moreover, since g is analytic away from B, no new jumps are introduced. The remaining jump matrices in equation (4.18) are changed into

$$V_1^{(4)} = \begin{pmatrix} 1 & \frac{\overline{r}e^{2i(\theta t-g)}}{1+r\overline{r}}\,\delta^2 \\ 0 & 1 \end{pmatrix}, \quad V_2^{(4)} = \begin{pmatrix} 1 & 0 \\ \frac{re^{-2i(\theta t-g)}}{1+r\overline{r}}\,\delta^{-2} & 1 \end{pmatrix}, \tag{4.21b}$$

$$V_3^{(4)} = \begin{pmatrix} 1 & 0 \\ re^{-2i(\theta t-g)}\delta^{-2} & 1 \end{pmatrix}, \quad V_4^{(4)} = \begin{pmatrix} 1 & \overline{r}e^{2i(\theta t-g)}\delta^2 \\ 0 & 1 \end{pmatrix}, \tag{4.21c}$$

The function g can be determined explicitly as follows. Dividing condition (4.20) by λ and using equation (4.13) for δ, we deduce that the scalar function $g/\lambda = (g/\lambda)(k)$ is analytic in $\mathbb{C} \setminus B$ and satisfies the jump condition

$$\left(\frac{g}{\lambda}\right)^+(k) - \left(\frac{g}{\lambda}\right)^-(k) = \frac{1}{\pi\lambda}\int_{-\infty}^{k_1} \frac{\ln[1+r(\nu)\overline{r}(\nu)]}{\nu - k}\,d\nu, \quad k \in B, \tag{4.22a}$$

and the normalization condition

$$\frac{g}{\lambda}(k) = O(1/k), \quad k \to \infty. \tag{4.22b}$$

Plemelj's formulae then yield g in the explicit form

$$g(k) = \frac{\lambda(k)}{2i\pi^2}\int_{\zeta\in B} \frac{1}{\lambda(\zeta)(\zeta-k)}\int_{-\infty}^{k_1} \frac{\ln[1+r(\nu)\overline{r}(\nu)]}{\nu-\zeta}\,d\nu\,d\zeta, \quad k \notin B. \tag{4.23}$$

In summary, the function $M^{(4)}$ defined by equation (4.19), with g given by equation (4.23), satisfies the following Riemann-Hilbert problem.

Riemann-Hilbert problem 4.1 (Final problem in the plane wave region). Determine a sectionally analytic matrix-valued function $M^{(4)}(k) = M^{(4)}(x,t,k)$ in $\mathbb{C} \setminus (\bigcup_{j=1}^4 L_j \cup B)$ satisfying the jump conditions

$$M^{(4)+}(k) = M^{(4)-}(k)V_B, \qquad k \in B, \tag{4.24a}$$

$$M^{(4)+}(k) = M^{(4)-}(k)V_j^{(4)}, \qquad k \in L_j,\ j = 1,2,3,4, \tag{4.24b}$$

and the normalization condition

$$M^{(4)}(k) = [I + O(1/k)]e^{ig_\infty \sigma_3}, \quad k \to \infty, \tag{4.24c}$$

where the jump contours L_j are shown in Figure 4.5, the jump matrices are given by equations (4.21), and the real constant g_∞ is given by

$$g_\infty = -\frac{1}{2i\pi^2}\int_{\zeta \in B} \frac{1}{\lambda(\zeta)} \int_{-\infty}^{k_1} \frac{\ln[1 + r(\nu)\overline{r}(\nu)]}{\nu - \zeta}\, d\nu\, d\zeta. \tag{4.25}$$

Decomposition of the final Riemann-Hilbert problem and limit $t \to \infty$. Starting from formula (2.30) for the solution q of the focusing NLS equation (1.1) in terms of $M^{(0)}$ and applying the four successive deformations that lead to $M^{(4)}$, we obtain

$$q(x,t) = -2i\left(M_1^{(4)}(x,t)\right)_{12} e^{ig_\infty}, \tag{4.26}$$

where $M_1^{(4)}$ is the $O(1/k)$ coefficient of the large-k expansion of $M^{(4)}$:

$$M^{(4)}(x,t,k) = e^{ig_\infty \sigma_3} + \frac{M_1^{(4)}(x,t)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty. \tag{4.27}$$

In order to be able to take the limit $t \to \infty$ in formula (4.26), a suitable decomposition of $M^{(4)}$ is first required into an asymptotic problem, which will yield the leading-order contribution to the solution of the NLS equation, and an error problem, which will yield the leading-order error.

It is evident from the structure of the jump matrices $V_1^{(4)}$, $V_2^{(4)}$, $V_3^{(4)}$, and $V_4^{(4)}$ in equations (4.21b) and (4.21c) and the sign structure of $\Re(i\theta)$ (see Figure 3.2a) that k_1 is the only point at which the above matrices fail to tend uniformly to I as $t \to \infty$. Hence, a neighborhood of the point k_1—in addition, of course, to the branch cut B—is the only region expected to yield the leading-order contribution to the long-time asymptotics of $M^{(4)}$, while the remaining contours are expected to contribute only in the error. This reasoning motivates the following decomposition of the Riemann-Hilbert problem 4.1.

Let $D_{k_1}^{\varepsilon}$ be a disk of radius ε centered at k_1, with ε sufficiently small so that $D_{k_1}^{\varepsilon} \cap B = \varnothing$. Then, write

$$M^{(4)} = M^{\text{err}} M^{\text{asymp}}, \quad M^{\text{asymp}} = \begin{cases} M^B, & k \in \mathbb{C} \setminus D_{k_1}^{\varepsilon}, \\ M^D, & k \in D_{k_1}^{\varepsilon}, \end{cases} \tag{4.28}$$

where:

- the function $M^B(k) = M^B(x,t,k)$ is analytic in $\mathbb{C} \setminus B$ and satisfies the jump condition

$$M^{B+}(k) = M^{B-}(k)\, V_B, \quad k \in B, \tag{4.29a}$$

and the normalization condition

$$M^B(k) = [I + O(1/k)]e^{ig_\infty \sigma_3}, \quad k \to \infty; \tag{4.29b}$$

- the function M^D is analytic in $D_{k_1}^\varepsilon \setminus L_j$, $j = 1,2,3,4$, with jumps

$$M^{D+}(k) = M^{D-}(k) V_j^{(4)}, \quad k \in \widehat{L}_j = L_j \cap D_{k_1}^\varepsilon, \ j = 1,2,3,4; \tag{4.30}$$

- the function $M^{\mathrm{err}}(k) = M^{\mathrm{err}}(x,t,k)$ is analytic in $\mathbb{C} \setminus (\bigcup_{j=1}^4 \check{L}_j \cup \partial D_{k_1}^\varepsilon)$ and satisfies the jump condition

$$M^{\mathrm{err}+}(k) = M^{\mathrm{err}-}(k) V^{\mathrm{err}}, \quad k \in \bigcup_{j=1}^4 \check{L}_j \cup \partial D_{k_1}^\varepsilon, \ \check{L}_j = L_j \setminus \overline{D_{k_1}^\varepsilon}, \tag{4.31a}$$

and the normalization condition

$$M^{\mathrm{err}}(k) = I + O(1/k), \quad k \to \infty, \tag{4.31b}$$

with the jump V^{err} defined by

$$V^{\mathrm{err}} = \begin{cases} M^B V_j^{(4)} (M^B)^{-1}, & k \in \check{L}_j, \\ M^{\mathrm{asymp}-} (V_D^{\mathrm{asymp}})^{-1} (M^{\mathrm{asymp}-})^{-1}, & k \in \partial D_{k_1}^\varepsilon, \end{cases} \tag{4.32}$$

where V_D^{asymp} is the jump of M^{asymp} across the circle $\partial D_{k_1}^\varepsilon$, which is yet unknown.

Under the decomposition (4.28) of $M^{(4)}$, formula (4.26) becomes

$$q(x,t) = -2i \big(M_1^B(x,t) e^{ig_\infty} + M_1^{\mathrm{err}}(x,t) \big)_{12}. \tag{4.33}$$

Moreover, thanks to the fact that the jump matrix V_B is constant, M^B is actually given by the explicit formula

$$M^B(k) = e^{ig_\infty \sigma_3} \begin{pmatrix} \frac{1}{2}(\Lambda + \Lambda^{-1}) & -\frac{q_o}{2\bar{q}_-}(\Lambda - \Lambda^{-1}) \\ \frac{q_o}{2q_-} \overline{(\Lambda - \Lambda^{-1})} & \frac{1}{2} \overline{(\Lambda + \Lambda^{-1})} \end{pmatrix}, \tag{4.34}$$

where the function Λ is defined by

$$\Lambda(k) = \left(\frac{k - iq_o}{k + iq_o} \right)^{\frac{1}{4}}. \tag{4.35}$$

In addition, in the Appendix we show that M_1^{err} admits the following estimate:

$$|M_1^{\mathrm{err}}(x,t)| = O\big(t^{-\frac{1}{2}}\big), \quad t \to \infty. \tag{4.36}$$

Thus, returning to equation (4.33) we conclude that the long-time asymptotic behavior of the solution of the focusing NLS equation (1.1) in the plane wave region is given to leading order by

$$q(x,t) = -2i\big(M_1^B(x,t)\big)_{12} e^{ig_\infty} + O\big(t^{-\frac{1}{2}}\big), \quad t \to \infty,$$

or, explicitly,

$$q(x,t) = e^{2ig_\infty} q_- + O\big(t^{-\frac{1}{2}}\big), \quad t \to \infty. \tag{4.37}$$

Note that definitions (3.9) and (4.25) of k_1 and g_∞ imply that $\lim_{\xi \to -\infty} g_\infty = 0$. Hence, in consistency with the infinity condition (1.2), in the plane wave region as $t \to \infty$ we have $q(x,t) \longrightarrow q_-$ in the limit $x \to -\infty$, $\xi \to -\infty$.

The proof of Theorem 1.1 is complete.

5 The Modulated Elliptic Wave Region: Proof of Theorem 1.2

In this section we prove Theorem 1.2; i.e., we compute the leading-order long-time asymptotic behavior of the solution of the focusing NLS equation (1.1) in the modulated elliptic wave region $|x| < 4\sqrt{2}q_o t$.

Recall that the sign structure of $\Re(i\theta)$ in this region was discussed in Section 4 and is depicted in Figures 3.2b and 3.2c. As in the plane wave region, we consider only the case $-4\sqrt{2}q_o t < x < 0$ since the case $0 < x < 4\sqrt{2}q_o t$ is entirely analogous after suitably reformulating the Riemann-Hilbert problem 2.1 (see discussion below Remark 4.2).

Remark 5.1. The main difference between the plane wave and the modulated elliptic wave regions is the absence of real stationary points in the latter case. As a result, the curves identifying the sign changes of $\Re(i\theta)$ in Figure 3.2b as $k \to \infty$ connect directly to the branch cut B. This implies that it is not possible anymore to use the previous factorizations and deformations to lift the contours off the real k-axis in such a way that the corresponding jump matrices remain bounded as $t \to \infty$. In other words, in order to connect the negative real k-axis to the white region in the upper half-plane, one cannot avoid passing through the gray region, in which $\Re(i\theta)$ has the "wrong" sign. To circumvent this problem, similarly to [18] we will introduce an appropriate, artificial change-of-factorization point $k_o \in \mathbb{R}^-$, which will be determined as part of the problem. Importantly, one can also show that the choice of k_o has no effect on the solution of the problem.

First, second, and third deformations. The first deformation, which is defined in Figure 5.1, is the same as the first deformation in the plane wave region except that the change of factorization now occurs at the point k_o instead of the point k_1. The second and third deformations are also the same as the corresponding deformations in the plane wave region and lead to the function $M^{(3)}(k) = M^{(3)}(x,t,k)$, which is analytic in $\mathbb{C} \setminus (\bigcup_{j=1}^4 L_j \cup B)$ and satisfies the jump conditions

$$M^{(3)+}(k) = M^{(3)-}(k) V_B^{(3)}, \quad k \in B, \tag{5.1a}$$

$$M^{(3)+}(k) = M^{(3)-}(k) V_j^{(3)}, \quad k \in L_j,\ j = 1, 2, 3, 4, \tag{5.1b}$$

and the normalization condition

$$M^{(3)}(k) = I + O(1/k), \qquad k \to \infty, \tag{5.1c}$$

with the jump contours L_j shown in Figure 5.2 and the jump matrices $V_j^{(3)}$ given by equations (4.18) after changing the definition of the function δ to

$$\delta(k) = \exp\left\{\frac{1}{2i\pi}\int_{-\infty}^{k_o} \frac{\ln\left[1 + r(\nu)\overline{r}(\nu)\right]}{\nu - k}\, d\nu\right\}, \quad k \notin (-\infty, k_o). \tag{5.2}$$

Remark 5.2. Importantly, the contour L_3 now intersects the curve $\Re(i\theta) = 0$ in the second quadrant at a point α, as depicted in Figure 5.2, unlike what happens in the plane wave region. This intersection point will be determined in terms of k_o in due course. Note also that the analyticity condition (3.1) with $\epsilon = q_o$ ensures that the reflection coefficient $r(k)$ admits a unique analytic continuation in the neighborhood of the entire contours L_3 and L_4.

Fourth deformation: Elimination of the exponential growth. So far we have managed to deform the jump contour across $\mathbb{R}$ to contours in the complex k-plane as in the plane wave region. We now encounter a new phenomenon that is not

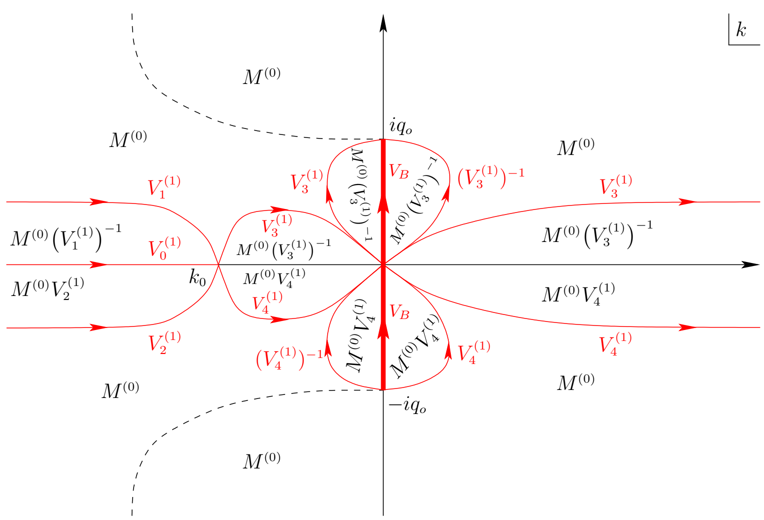

FIGURE 5.1. The first deformation in the modulated elliptic wave region.

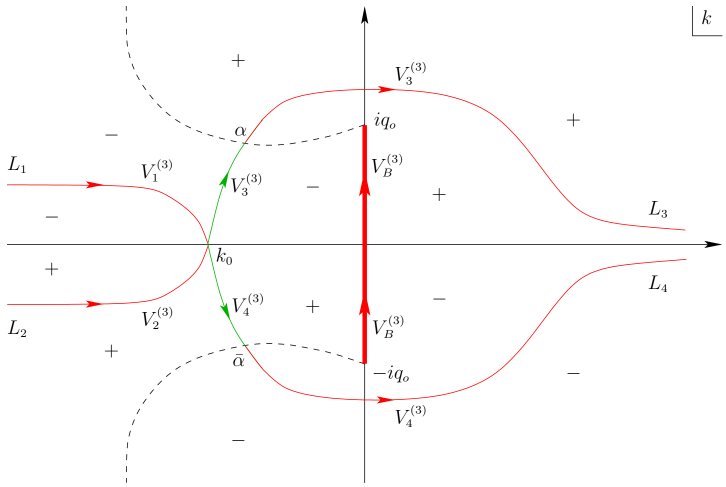

FIGURE 5.2. The jumps of $M^{(3)}$ in the modulated elliptic wave region. The jump matrices $V_3^{(3)}$ and $V_4^{(3)}$ grow exponentially as a function of t on the green-colored contours.

present in the plane wave region; however, the jumps $V_3^{(3)}$ and $V_4^{(3)}$ grow exponentially with t along the green-colored segments of the deformed contours shown in Figure 5.2.

To address this issue, we employ the following new factorizations for these jumps:

$$V_3^{(3)} = V_5^{(3)} V_7^{(3)} V_5^{(3)}, \quad V_4^{(3)} = V_6^{(3)} V_8^{(3)} V_6^{(3)}, \tag{5.3}$$

where

$$V_5^{(3)} = \begin{pmatrix} 1 & \frac{\delta^2}{r}\, e^{2i\theta t} \\ 0 & 1 \end{pmatrix}, \quad V_6^{(3)} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\overline{r}\delta^2}\, e^{-2i\theta t} & 1 \end{pmatrix}, \tag{5.4a}$$

$$V_7^{(3)} = \begin{pmatrix} 0 & -\frac{\delta^2}{r}\, e^{2i\theta t} \\ \frac{r}{\delta^2}\, e^{-2i\theta t} & 0 \end{pmatrix}, \quad V_8^{(3)} = \begin{pmatrix} 0 & \overline{r}\delta^2\, e^{2i\theta t} \\ -\frac{1}{\overline{r}\delta^2}\, e^{-2i\theta t} & 0 \end{pmatrix}. \tag{5.4b}$$

These factorizations allow for the deformation of the green contours of Figure 5.2 into the green contours shown in Figure 5.3, where for $j = 1, \ldots, 8$ we denote by $V_j^{(3)}$ the jump associated with the contour L_j.

Observe that the jumps $V_5^{(3)}$ and $V_6^{(3)}$ are now bounded along the contours L_5 and L_6, respectively. On the other hand, the 21-entry of the jump $V_7^{(3)}$ and the 12-entry of the jump $V_8^{(3)}$ are still unbounded along the contours L_7 and L_8, respectively. To resolve this issue, we employ once again the g-function mechanism

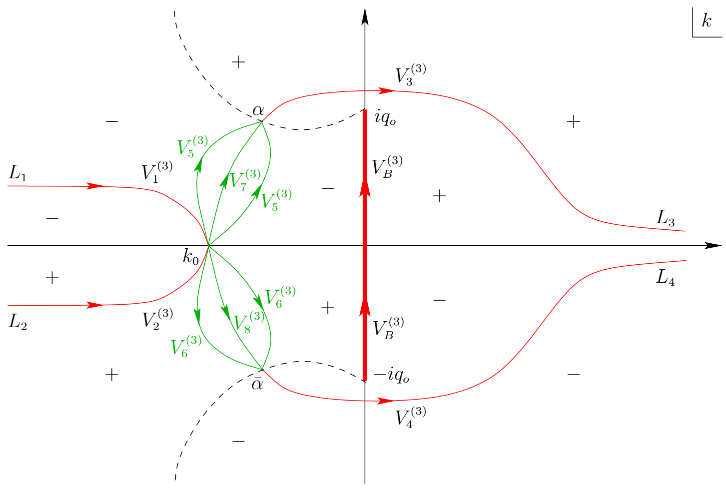

FIGURE 5.3. The fourth deformation in the modulated elliptic wave region.

as in the previous section (cf. transformation (4.19)). This time, however, we need to do so through a t-*dependent* exponential, namely, by letting

$$M^{(4)}(x,t,k) = M^{(3)}(x,t,k)e^{-iG(\xi,\alpha,k_o,k)t\sigma_3} \tag{5.5}$$

for a function $G(\xi,\alpha,k_o,k)$ that is required to be analytic in $\mathbb{C}\setminus(B\cup L_7\cup L_8)$. In fact, instead of working with G it will be more convenient to consider the function h defined by

$$h(\xi,\alpha,k_o,k) = \theta(\xi,k) + G(\xi,\alpha,k_o,k). \tag{5.6}$$

From the above definition, we infer that h must be analytic in $\mathbb{C}\setminus(B\cup L_7\cup L_8)$ and has jump discontinuities across B and $\widetilde{B}$. Moreover, according to transformation (5.5) the jumps of $M^{(4)}$ read

$$\begin{gathered}
V_B^{(4)} = \begin{pmatrix} 0 & \frac{q_-\delta^2}{iq_o}\, e^{i(h^++h^-)t} \\ \frac{\bar q_-}{iq_o\delta^2}\, e^{-i(h^++h^-)t} & 0 \end{pmatrix}, \quad V_1^{(4)} = \begin{pmatrix} 1 & \frac{\bar r\delta^2}{1+r\bar r}\, e^{2iht} \\ 0 & 1 \end{pmatrix}, \\
V_2^{(4)} = \begin{pmatrix} 1 & 0 \\ \frac{r}{(1+r\bar r)\delta^2}\, e^{-2iht} & 1 \end{pmatrix}, \quad V_3^{(4)} = \begin{pmatrix} 1 & 0 \\ \frac{r}{\delta^2}\, e^{-2iht} & 1 \end{pmatrix}, \\
V_4^{(4)} = \begin{pmatrix} 1 & \bar r\delta^2\, e^{2iht} \\ 0 & 1 \end{pmatrix}, \quad V_5^{(4)} = \begin{pmatrix} 1 & \frac{\delta^2}{r}\, e^{2iht} \\ 0 & 1 \end{pmatrix}, \quad V_6^{(4)} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\bar r\delta^2}\, e^{-2iht} & 1 \end{pmatrix}, \\
V_7^{(4)} = \begin{pmatrix} 0 & -\frac{\delta^2}{r}\, e^{i(h^++h^-)t} \\ \frac{r}{\delta^2}\, e^{-i(h^++h^-)t} & 0 \end{pmatrix}, \\
V_8^{(4)} = \begin{pmatrix} 0 & \bar r\delta^2\, e^{i(h^++h^-)t} \\ -\frac{1}{\bar r\delta^2}\, e^{-i(h^++h^-)t} & 0 \end{pmatrix}.
\end{gathered} \tag{5.7}$$

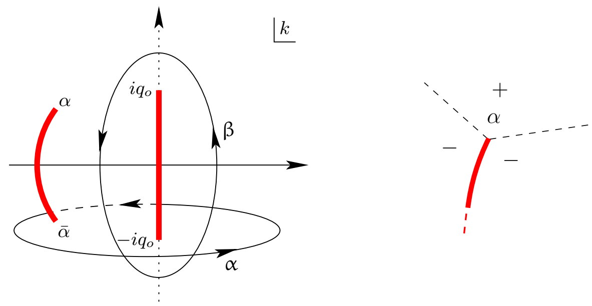

FIGURE 5.4. Left: The basis $\{\alpha, \beta\}$ of cycles of the genus-1 Riemann surface Σ. Right: The desired sign structure of $\Re(ih)$ in a neighborhood of α (cf. Figure 5.3).

The task that we turn to next is to determine k_o, $\alpha = \alpha_{\text{re}} + i\alpha_{\text{im}}$, and h so that all these jump matrices remain bounded as $t \to \infty$.

The definition of h. We begin by introducing the upwardly oriented branch cut $\widetilde{B} = L_7 \cup (-L_8)$, and we define the single-valued function γ with branch cuts B and $\widetilde{B}$ by

$$\gamma(k) = \left[\left(k^2 + q_o^2\right)(k-\alpha)(k-\overline{\alpha})\right]^{\frac{1}{2}}, \tag{5.8}$$

where we identify γ with its right-sided limit along B and $\widetilde{B}$; i.e., we set $\gamma(k) = \gamma^-(k) = -\gamma^+(k)$ for $k \in B \cup \widetilde{B}$, so that $\gamma(k) \sim k^2$ as $k \to \infty$. Then γ gives rise to a genus-1 Riemann surface Σ with sheets Σ_1, Σ_2 and a basis $\{\alpha, \beta\}$ of cycles defined as follows: the β-cycle is a closed, anticlockwise contour around the branch cut B that remains entirely on the first sheet Σ_1 of the Riemann surface; the α-cycle consists of an anticlockwise contour that starts on the left of $\widetilde{B}$, then approaches B from the right while on the first sheet Σ_1, and finally returns to the starting point via the second sheet Σ_2. These cycles are depicted in Figure 5.4 (left).

Next, similarly to [17], we let h be given by the Abelian integral

$$h(k) = \frac{1}{2}\left(\int_{iq_o}^{k} + \int_{-iq_o}^{k}\right) dh(z), \tag{5.9}$$

where the Abelian differential dh is defined by

$$dh(k) = -4\,\frac{(k-k_o)(k-\alpha)(k-\overline{\alpha})}{\gamma(k)}\,dk. \tag{5.10}$$

The task is therefore to determine the triplet $\{k_o, \alpha_{\text{re}}, \alpha_{\text{im}}\}$ so that: (a) the function h defined by equation (5.9) satisfies a Riemann-Hilbert problem that removes the existing growth from the jumps $V_7^{(4)}$ and $V_8^{(4)}$ of (5.7), and (b) the sign structure of $\Re(ih)$ is such that no new growth is introduced in any of the other jumps of (5.7).

The sign structure of $\Re(ih)$. It is convenient to write the numerator in (5.10) as

$$(k-k_o)(k-\alpha)(k-\overline{\alpha}) = k^3 + c_2k^2 + c_1k + c_0, \tag{5.11}$$

where the real constants c_0, c_1, c_2 are given by

$$c_0 = -k_o\big(\alpha_{\mathrm{re}}^2 + \alpha_{\mathrm{im}}^2\big), \quad c_1 = \alpha_{\mathrm{re}}^2 + 2k_o\alpha_{\mathrm{re}} + \alpha_{\mathrm{im}}^2, \quad c_2 = -(k_o + 2\alpha_{\mathrm{re}}). \tag{5.12}$$

We then have

$$h(k) = -2\left(\int_{iq_o}^{k} + \int_{-iq_o}^{k}\right)\frac{z^3 + c_2z^2 + c_1z + c_0}{\gamma(z)}\,dz. \tag{5.13}$$

We next use this representation to study the sign structure of $\Re(ih)$ for all $k \in \mathbb{C}$. Note that, since h is completely specified in terms of k_o and α by (5.9) and (5.10), determining k_o and α is equivalent to determining c_0, c_1, and c_2.

We start by studying the sign structure of $\Re(ih)$ as $k \to \infty$. Recall that the deformations of the Riemann-Hilbert problem were chosen according to the sign structure of $\Re(i\theta)$. Thus, in the transition from θ to h, this structure should be preserved. We can ensure that this is the case by requiring

$$\Re(ih) = \Re(i\theta) + O(1/k), \quad k \to \infty. \tag{5.14}$$

In this regard, note that since

$$\gamma(k) = k^2\left[1 - \frac{\alpha+\overline{\alpha}}{2k} + \frac{4q_o^2 - (\alpha-\overline{\alpha})^2}{8k^2} + O\left(\frac{1}{k^3}\right)\right], \quad k \to \infty,$$

it follows that

$$\begin{aligned}\frac{dh}{dk} = -4\Bigg[&k + (c_2 + \alpha_{\mathrm{re}}) \\ &+ \frac{c_1 + c_2\alpha_{\mathrm{re}} - \frac{1}{2}\big(q_o^2 + \alpha_{\mathrm{im}}^2\big) + \alpha_{\mathrm{re}}^2}{k} + O\left(\frac{1}{k^2}\right)\Bigg], \quad k \to \infty.\end{aligned} \tag{5.15}$$

On the other hand, from the definition (2.34) of θ we have

$$\theta(k) = -2k^2 + \xi k - q_o^2 + O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{5.16}$$

Hence, to satisfy relation (5.14) we require

$$c_1 = \frac{1}{2}\big(q_o^2 + \alpha_{\mathrm{im}}^2\big) + \frac{\xi}{4}\alpha_{\mathrm{re}}, \quad c_2 = -\frac{\xi}{4} - \alpha_{\mathrm{re}}. \tag{5.17}$$

Then, integrating expansion (5.15) we find

$$h(k) = -2k^2 + \xi k + H_o + O\left(\frac{1}{k}\right), \quad k \to \infty, \tag{5.18}$$

where the constant H_o can be determined by observing that

$$2\left(\int_{iq_o}^{k} + \int_{-iq_o}^{k}\right)\left(z - \frac{\xi}{4}\right)dz = 2k^2 - \xi k + 2q_o^2. \tag{5.19}$$

Indeed, combining equations (5.13) and (5.19) we can express h in the form

$$h(k) = -2\bigg(\int_{iq_o}^{k} + \int_{-iq_o}^{k}\bigg)\bigg[\frac{z^3 + c_2 z^2 + c_1 z + c_0}{\gamma(z)} - \Big(z - \frac{\xi}{4}\Big)\bigg]dz \\ - \big(2k^2 - \xi k + 2q_o^2\big).$$

Hence, using also equation (5.18) we deduce

$$H_o = -2\bigg(\int_{iq_o}^{\infty} + \int_{-iq_o}^{\infty}\bigg)\bigg[\frac{z^3 + c_2 z^2 + c_1 z + c_0}{\gamma(z)} - \Big(z - \frac{\xi}{4}\Big)\bigg]dz \\ - 2q_o^2 \tag{5.20}$$

with c_1 and c_2 given by equation (5.17) and c_0 yet to be determined. Note that H_o is well-defined since the relation

$$\frac{z^3 + c_2 z^2 + c_1 z + c_0}{\gamma(z)} - \Big(z - \frac{\xi}{4}\Big) = O\Big(\frac{1}{z^2}\Big), \quad z \to \infty,$$

ensures that the integrals in (5.20) are convergent. Moreover, it is evident from the contours of integration and the fact that $c_0, c_1, c_2 \in \mathbb{R}$ that $H_o \in \mathbb{R}$. Thus the desired behavior (5.14) of $\Re(ih)$ for large k is achieved. Also, combining (5.12) and (5.17) we have

$$\alpha_{\mathrm{re}} = -k_o + \frac{\xi}{4}, \quad \alpha_{\mathrm{im}} = \sqrt{2k_o^2 - \frac{\xi}{2}k_o + q_o^2}. \tag{5.21}$$

It thus remains to determine k_o. We do so by analyzing the behavior of h near α. In order for the jumps $V_3^{(4)}$, $V_5^{(4)}$, and $V_6^{(4)}$ to be bounded near α, $\Re(ih)$ should have the sign structure shown in Figure 5.4 (right). Letting $\zeta = k - \alpha$, we have

$$\big(k^2 + q_o^2\big)^{\frac{1}{2}} = \big(\alpha^2 + q_o^2\big)^{\frac{1}{2}}\bigg[1 + \frac{\alpha\zeta}{\alpha^2 + q_o^2} + O\big(\zeta^2\big)\bigg], \quad \zeta \to 0, \tag{5.22a}$$

$$[(k-\alpha)(k-\overline{\alpha})]^{\frac{1}{2}} = (\alpha - \overline{\alpha})^{\frac{1}{2}}\bigg[\zeta^{\frac{1}{2}} + \frac{\zeta^{3/2}}{2(\alpha - \overline{\alpha})} + O\big(\zeta^{\frac{5}{2}}\big)\bigg], \quad \zeta \to 0. \tag{5.22b}$$

Hence, in a neighborhood of α, we have

$$\frac{dh}{dk} = -\frac{4(\alpha - \overline{\alpha})^{1/2}(\alpha - k_o)}{\big(\alpha^2 + q_o^2\big)^{1/2}}\bigg\{\zeta^{\frac{1}{2}} + \bigg[\frac{1}{\alpha - k_o} + \frac{1}{2(\alpha - \overline{\alpha})} - \frac{\alpha}{\alpha^2 + q_o^2}\bigg]\zeta^{\frac{3}{2}} \\ + O(\zeta^{\frac{5}{2}})\bigg\}, \quad \zeta \to 0,$$

from which, integrating, we obtain the expansion

$$h(k) = h(\alpha) - \frac{4(\alpha-\overline{\alpha})^{1/2}(\alpha-k_o)}{\left(\alpha^2+q_o^2\right)^{\frac{1}{2}}}\left\{\frac{2}{3}(k-\alpha)^{\frac{3}{2}} + \frac{2}{5}\left[\frac{1}{(\alpha-k_o)} + \frac{1}{2(\alpha-\overline{\alpha})} - \frac{\alpha}{\alpha^2+q_o^2}\right](k-\alpha)^{\frac{5}{2}} + O((k-\alpha)^{\frac{7}{2}})\right\}, \quad k \to \alpha,$$

where

$$h(\alpha) = -2\left(\int_{iq_o}^{\alpha} + \int_{-iq_o}^{\alpha}\right)\frac{z^3+c_2z^2+c_1z+c_0}{\gamma(z)}\,dz. \tag{5.23}$$

On the other hand, in order to obtain the sign structure in Figure 5.4 (right), the leading-order term of the expansion of $\Re(ih)$ near α should be of $O(k-\alpha)^{3/2}$. Thus, we must have

$$\Im h(\alpha) = 0. \tag{5.24}$$

Using contour deformations, we can write $h(\alpha)$ in the form

$$\begin{aligned} h(\alpha) = &-2\left(\int_{iq_o}^{\alpha} + \int_{-iq_o}^{\overline{\alpha}}\right)\frac{z^3+c_2z^2+c_1z+c_0}{\gamma(z)}\,dz \\ &-2\int_{\overline{\alpha}}^{\alpha}\frac{z^3+c_2z^2+c_1z+c_0}{\gamma(z)}\,dz, \end{aligned} \tag{5.25}$$

where, as before, $\gamma(z)$ is taken to be continuous from the right on the branch cut $\widetilde{B} = [\overline{\alpha}, \alpha]$. We then note that the first term on the right-hand side of equation (5.25) is real, while the second term is imaginary. Hence (5.24) is equivalent to the following condition:

$$\int_{\overline{\alpha}}^{\alpha}\frac{z^3+c_2z^2+c_1z+c_0}{\gamma(z)}\,dz = 0. \tag{5.26}$$

It is convenient to reformulate the above condition as follows. The integrals of dh from $\overline{\alpha}$ to $-iq_o$ and from α to iq_o are both equal to half of the integral of dh along the α-cycle. Hence, by analyticity the contour of integration from $\overline{\alpha}$ to α on the right of $\widetilde{B}$ can be deformed to the contour from $-iq_o$ to iq_o on the left of B. Thus, recalling the fact that $\gamma^+ = -\gamma^- = -\gamma$ across B and also equation (5.11), condition (5.26) takes the equivalent forms

$$\int_{-iq_o}^{iq_o}\frac{z^3+c_2z^2+c_1z+c_0}{\gamma(z)}\,dz = \int_{-iq_o}^{iq_o}\sqrt{\frac{(z-\alpha_{\mathrm{re}})^2+\alpha_{\mathrm{im}}^2}{z^2+q_o^2}}(z-k_o)dz = 0. \tag{5.27}$$

Finally, recalling equation (5.21), condition (5.27) yields the following integral equation for k_o:

$$\int_{-iq_o}^{iq_o} \sqrt{\frac{\left(z+k_o-\frac{\xi}{4}\right)^2+2k_o^2-\frac{\xi}{2}k_o+q_o^2}{z^2+q_o^2}}(z-k_o)\,dz=0. \tag{5.28}$$

The solution of equation (5.28) uniquely determines the points α and k_o, and hence the function h, in terms of ξ and q_o.

Remark 5.3. The integral equation (5.28) is trivially satisfied for:

(i) $\xi = -4\sqrt{2}q_o$ (i.e., $x = -4\sqrt{2}q_o t$). In this case $k_o = -q_o/\sqrt{2}$, and $\alpha_{\mathrm{re}} = k_o$ and $\alpha_{\mathrm{im}} = 0$. Note that for $\xi = -4\sqrt{2}q_o$ equation (3.9) implies that $k_1 = k_2 = k_o$. Thus, at the interface between the plane wave and modulated elliptic wave regions, the two asymptotic descriptions are consistent.

(ii) $\xi = 0$ (i.e., either $x = 0$ or in the limit $t \to \infty$). In this case, the point k_o collapses to the origin and the branch points α and $\overline{\alpha}$ collapse to the branch points $\pm iq_o$.

More generally, we can show the following, similarly to [17]:

LEMMA 5.4. *For all $\xi \in (-4\sqrt{2}q_o, 0)$, the integral equation* (5.28) *has a unique solution $k_o = k_o(\xi)$.*

PROOF. The result essentially follows from the implicit function theorem. The change of variables

$$x = -\frac{\xi}{8q_o}, \quad y = \frac{k_o - \frac{\xi}{4}}{q_o}, \tag{5.29}$$

and the parametrization $z = iq_o\zeta$ turn equation (5.28) into

$$f(x,y) = \int_{-1}^{1} \sqrt{\frac{(i\zeta+y)^2+2y(y-2x)+1}{1-\zeta^2}}(i\zeta+2x-y)\,d\zeta = 0. \tag{5.30}$$

Since $\xi \in [-4\sqrt{2}q_o, 0]$ and $k_o \in (\xi/8, 0)$, we consider equation (5.30) only for $(x,y) \in \mathcal{D} = (-\sqrt{2}/2, \sqrt{2}) \times (0, \sqrt{2}/2)$. Since $f(0,0) = 0$ and $f_x(0,0) < 0$, by the implicit function theorem there exists a neighborhood $\mathcal{X} \times \mathcal{Y}$ of $(0,0)$ and a unique function $F : \mathcal{X} \mapsto \mathcal{Y}$ such that $\{(x, F(x)) : x \in \mathcal{X}\} = \{(x,y) \in \mathcal{X} \times \mathcal{Y} : f(x,y) = 0\}$. Hence, there exists a unique solution $y(x)$ of the integral equation (5.30) in a neighborhood of $(0,0)$. Moreover, $f_x(x,y) < 0$ throughout the domain $\mathcal{D}$ with the exception of the point $(\sqrt{2}/2, \sqrt{2}/2)$. Thus, there exists a neighborhood around every point $(x,y) \in \mathcal{D}$ in which there exists a unique function $F(x)$ such that $y = F(x)$ represents the unique solution of equation (5.30), and the only point where uniqueness is violated is $(\sqrt{2}/2, \sqrt{2}/2)$. However, this point is itself a solution of equation (5.30) (see case (i) above for $\xi = -4\sqrt{2}q_o$). Thus, for any $x \in (-\sqrt{2}/2, \sqrt{2})$ there exists a unique $y(x)$ such that $f(x, y(x)) = 0$, and $y(\sqrt{2}/2) = \sqrt{2}/2$. Equivalently, for any $\xi \in [-4\sqrt{2}q_o, 0]$ there exists a unique solution $k_o(\xi)$ of the integral equation (5.28) with $k_o(-4\sqrt{2}q_o) = -(\sqrt{2}/2)q_o$. □

Remark 5.5. We note that

$$\int_{-iq_o}^{iq_o} \frac{dz}{\gamma(z)} = -\frac{2iK(m)}{|\alpha + iq_o|}, \tag{5.31}$$

where $K(m)$ is the complete elliptic integral of the first kind defined by

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{dz}{\sqrt{1 - m^2 \sin^2 z}} \tag{5.32}$$

with elliptic modulus m equal to

$$m = \frac{2\sqrt{q_o \alpha_{\mathrm{im}}}}{|\alpha + iq_o|}. \tag{5.33}$$

Moreover, similar expressions in terms of elliptic integrals can be obtained for the three remaining integrals of condition (5.27) that involve z, z^2, and z^3. Combining all of these expressions as well as equations (5.21) then reduces condition (5.27), and hence the integral equation (5.28), to equation (1.8b).

Global sign structure of $\Re(ih)$. In summary, when the points α and k_o are chosen according to equations (5.21) and (5.28), the function h defined by equation (5.9) eliminates the growth from the matrices $V_7^{(4)}$ and $V_8^{(4)}$, and the quantity $\Re(ih)$ has the correct sign structure both for large k and for k near α and $\overline{\alpha}$. Before formulating the Riemann-Hilbert problem for the function $M^{(4)}$, however, one must check that $\Re(ih)$ has the correct sign structure *for all finite* $k \in \mathbb{C}$. To confirm that this is indeed the case, it remains to verify that $\Re(ih)$ has the correct behavior near $k = 0$. Specifically, we want to show that for any $k_o, \alpha_{\mathrm{re}} < 0$, there exists a neighborhood of the origin in which $\Re(ih)$ has the required sign structure. To see this, using the expansions

$$[(k-\alpha)(k-\overline{\alpha})]^{\frac{1}{2}} = |\alpha|\left[1 - \frac{\alpha_{\mathrm{re}} k}{|\alpha|^2} + \frac{\alpha_{\mathrm{im}}^2}{2|\alpha|^4}k^2 + O(k^3)\right],$$
$$\lambda = \operatorname{sign}(k_{\mathrm{re}})\, q_o \left[1 + \frac{k^2}{2q_o^2} + O(k^4)\right],$$

as $k \to 0$ with $k_{\mathrm{im}} > 0$, we find

$$\frac{dh}{dk} = \frac{4|\alpha|k_o}{\operatorname{sign}(k_{\mathrm{re}})q_o}\left[1 - \left(\frac{\alpha_{\mathrm{re}}}{|\alpha|^2} + \frac{1}{k_o}\right)k + \left(\frac{\alpha_{\mathrm{im}}^2}{2|\alpha|^4} - \frac{1}{2q_o^2} + \frac{\alpha_{\mathrm{re}}}{k_o|\alpha|^2}\right)k^2 + O(k^3)\right],$$

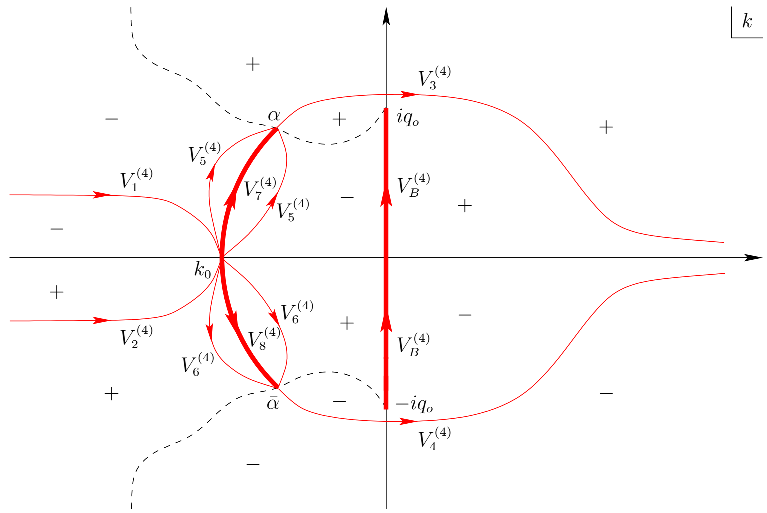

FIGURE 5.5. The sign structure of $\Re(ih)$ for all $k \in \mathbb{C}$.

which implies

$$h(k) = \frac{4|\alpha|k_o}{\operatorname{sign}(k_{\rm re})q_o}\left[k - \frac{1}{2}\left(\frac{\alpha_{\rm re}}{|\alpha|^2} + \frac{1}{k_o}\right)k^2 + O(k^3)\right] + h(0) \tag{5.34}$$

in the same limit. Since by the definition (5.13) of h it follows that $h(0) \in \mathbb{R}$, equation (5.34) yields

$$\Re(ih) = -\frac{4|\alpha|k_o k_{\rm im}}{\operatorname{sign}(k_{\rm re})q_o}\left[1 - \left(\frac{\alpha_{\rm re}}{|\alpha|^2} + \frac{1}{k_o}\right)k_{\rm re}\right] + O(k^3) \tag{5.35}$$

as $k \to 0$ with $k_{\rm im} > 0$. Recalling that $k_o, \alpha_{\rm re} < 0$ we thereby deduce that for small k in the first quadrant we have $\Re(ih) > 0$, while for small k in the second quadrant $\Re(ih) < 0$ provided that $(k_o\alpha_{\rm re} + |\alpha|^2)k_{\rm re} > k_o|\alpha|^2$. The treatment of the lower half-plane is analogous.

Remark 5.6. The analysis of the sign structure of $\Re(ih)$ as $k \to \infty$, for k near α and $\overline{\alpha}$, and as $k \to 0$ implies that it is always possible to deform the contours of Figure 5.3 so that they do not go through regions in which $\Re(ih)$ has the "wrong" sign (i.e., a sign that causes exponential growth). This is because $\Re(ih)$ is a harmonic function away from the branch cuts. If there existed a region of "wrong" sign separating two regions of "correct" sign, then there would be further critical points of $\Re(ih)$ in addition to $\alpha, \overline{\alpha}, k_o$. In turn, this would imply additional critical points for h besides $\alpha, \overline{\alpha}, k_o$. This is impossible, however, since the Abelian differential (5.9) has exactly three zeros, namely $\alpha, \overline{\alpha}, k_o$. Therefore, $\Re(ih)$ has the appropriate sign structure *for all* $k \in \mathbb{C}$, as shown in Figure 5.5.

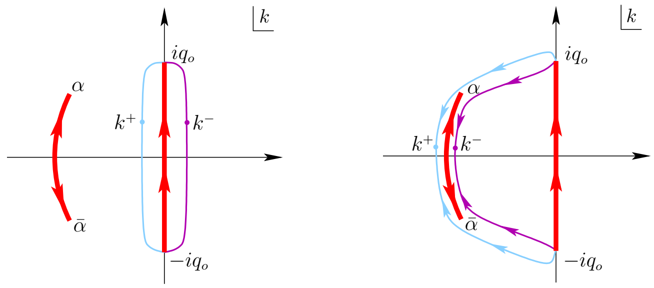

FIGURE 5.6. Left: The computation of the jump of h across B is done by recalling that both Abelian integrals in (5.9) change sign due to the function γ involved in the Abelian differential dh. Right: The computation of the jump of h across $\widetilde{B}$ is done by first deforming the contours of integration of the Abelian integrals in (5.9) as shown in the figure and then by noting that dh changes sign across $\widetilde{B}$ due to the function γ.

The jumps of h. Recall that $h(k) = h(\xi, \alpha, k_o, k)$ is analytic in $\mathbb{C} \setminus (B \cup \widetilde{B})$. We now compute the jumps of h across the branch cuts B and $\widetilde{B}$, which are needed to determine the jump matrices in (5.7). We do so by deforming the contours involved in equation (5.13) as shown in Figures 5.6, obtaining the jump conditions

$$h^+(k) + h^-(k) = 0, \quad k \in B, \tag{5.36a}$$

$$h^+(k) + h^-(k) = \Omega, \quad k \in L_7 \cup L_8, \tag{5.36b}$$

where the real constant Ω is defined by

$$\Omega = -4\left(\int_{iq_o}^{\alpha} + \int_{-iq_o}^{\bar{\alpha}}\right) \frac{(z-k_o)(z-\alpha)(z-\bar{\alpha})}{\gamma(z)}\, dz. \tag{5.37}$$

Moreover, recalling the large-k behavior of h specified by equation (5.18), we infer that h satisfies the normalization condition

$$h(k) = -2k^2 + \xi k + H_o + O(1/k), \quad k \to \infty, \tag{5.38}$$

where, for α and k_o given by equations (5.21) and (5.28), the jump contours L_7 and L_8 are depicted in Figure 5.3, and the real constant H_o is defined by equation (5.20).

The Riemann-Hilbert problem for $M^{(4)}$. By the definition (5.5) of $M^{(4)}$ and the Riemann-Hilbert problem (5.1) for $M^{(3)}$, we infer that $M^{(4)}$ is analytic in $\mathbb{C} \setminus (\bigcup_{j=1}^{8} L_j \cup B)$ and satisfies the jump conditions

$$M^{(4)+}(k) = M^{(4)-}(k) V_B^{(4)}, \quad k \in B, \tag{5.39a}$$

$$M^{(4)+}(k) = M^{(4)-}(k) V_j^{(4)}, \quad k \in L_j,\ j = 1, \ldots, 8, \tag{5.39b}$$

and the normalization condition

$$M^{(4)}(k) = [I + O(1/k)] e^{-iG_\infty t \sigma_3}, \quad k \to \infty, \tag{5.39c}$$

where the jump contours L_j are shown in Figure 5.3, the jump matrices $V_j^{(4)}$ (5.7) simplify thanks to the jumps (5.36) satisfied by h to

$$
\begin{aligned}
&V_B^{(4)} = \begin{pmatrix} 0 & \frac{q_- \delta^2}{i q_o} \\ \frac{\overline{q}_-}{i q_o \delta^2} & 0 \end{pmatrix}, \qquad && V_1^{(4)} = \begin{pmatrix} 1 & \frac{\overline{r}\,\delta^2}{1+r\overline{r}}\, e^{2iht} \\ 0 & 1 \end{pmatrix},\\
&V_2^{(4)} = \begin{pmatrix} 1 & 0 \\ \frac{r}{(1+r\overline{r})\delta^2}\, e^{-2iht} & 1 \end{pmatrix}, && V_3^{(4)} = \begin{pmatrix} 1 & 0 \\ \frac{r}{\delta^2}\, e^{-2iht} & 1 \end{pmatrix},\\
&V_4^{(4)} = \begin{pmatrix} 1 & \overline{r}\delta^2\, e^{2iht} \\ 0 & 1 \end{pmatrix}, && V_5^{(4)} = \begin{pmatrix} 1 & \frac{\delta^2}{r}\, e^{2iht} \\ 0 & 1 \end{pmatrix},\\
&V_6^{(4)} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\overline{r}\delta^2}\, e^{-2iht} & 1 \end{pmatrix}, && V_7^{(4)} = \begin{pmatrix} 0 & -\frac{\delta^2}{r}\, e^{i\Omega t} \\ \frac{r}{\delta^2}\, e^{-i\Omega t} & 0 \end{pmatrix},\\
&V_8^{(4)} = \begin{pmatrix} 0 & \overline{r}\delta^2\, e^{i\Omega t} \\ -\frac{1}{\overline{r}\delta^2}\, e^{-i\Omega t} & 0 \end{pmatrix},
\end{aligned} \tag{5.40}
$$

and, using the expansions (5.16) and (5.18), the real constant G_∞ involved in the normalization condition (5.39c) is equal to

$$
\begin{aligned}
G_\infty &= H_o + q_o^2 \\
&= -2\left(\int_{iq_o}^{\infty} + \int_{-iq_o}^{\infty}\right)\left[\frac{z^3 + c_2 z^2 + c_1 z + c_0}{\gamma(z)} - \left(z - \frac{\xi}{4}\right)\right] dz - q_o^2,
\end{aligned} \tag{5.41}
$$

with the real constant H_o defined by equation (5.20).

Fifth deformation. Our final task is to eliminate the dependence on k from the three jumps $V_B^{(4)}$, $V_7^{(4)}$, and $V_8^{(4)}$ across the branch cuts B and $\widetilde{B}$. This can be achieved with the help of an additional g-function, this time introduced via a t*-independent* exponential, exactly as in the fourth deformation (4.19) for the plane wave region (as opposed to transformation (5.5)). In particular, we let

$$
M^{(5)}(x,t,k) = M^{(4)}(x,t,k) e^{ig(k)\sigma_3}, \tag{5.42}
$$

where the function g is analytic in $\mathbb{C} \setminus (B \cup \widetilde{B})$ with jumps

$$
g^+(k) + g^-(k) = -i \ln(\delta^2), \qquad k \in B, \tag{5.43a}
$$

$$
g^+(k) + g^-(k) = -i \ln(\delta^2/r) + \omega, \quad k \in L_7, \tag{5.43b}
$$

$$
g^+(k) + g^-(k) = -i \ln(\delta^2 \overline{r}) + \omega, \quad k \in L_8, \tag{5.43c}
$$

with the function δ defined by equation (5.2) and the real constant ω given by

$$\omega = i\,\frac{\int_B \frac{\ln[\delta^2(\nu)]}{\gamma(\nu)}\,d\nu + \int_{L_7} \frac{\ln\left[\frac{\delta^2(\nu)}{r(\nu)}\right]}{\gamma(\nu)}\,d\nu - \int_{L_8} \frac{\ln[\delta^2(\nu)\,\overline{r}(\nu)]}{\gamma(\nu)}\,d\nu}{\int_{\widetilde{B}} \frac{d\nu}{\gamma(\nu)}}. \tag{5.44}$$

Proceeding as in the plane wave region, we arrive at a scalar, additive Riemann-Hilbert problem analogous to problem (4.22), which is solved via Plemelj's formulae to yield

$$\begin{aligned} g(k) = \frac{\gamma(k)}{2\pi}\Bigg[\int_B \frac{\ln[\delta^2(\nu)]}{\gamma(\nu)(\nu-k)}\,d\nu + \int_{L_7} \frac{\ln\left[\frac{\delta^2(\nu)}{r(\nu)}\right] + i\omega}{\gamma(\nu)(\nu-k)}\,d\nu \\ - \int_{L_8} \frac{\ln[\delta^2(\nu)\overline{r}(\nu)] + i\omega}{\gamma(\nu)(\nu-k)}\,d\nu\Bigg]. \end{aligned} \tag{5.45}$$

The definition (5.44) of ω ensures that $g(k) = O(1)$ as $k \to \infty$. In particular, we have

$$g(k) = g_\infty + O(1/k), \quad k \to \infty, \tag{5.46}$$

where the real constant g_∞ is given by

$$\begin{aligned} g_\infty = \frac{1}{2\pi}\Bigg[-\int_B \frac{\ln[\delta^2(\nu)]}{\gamma(\nu)}\,\nu\,d\nu - \int_{L_7} \frac{\ln\left[\frac{\delta^2(\nu)}{r(\nu)}\right] + i\omega}{\gamma(\nu)}\,\nu\,d\nu \\ + \int_{L_8} \frac{\ln[\delta^2(\nu)\overline{r}(\nu)] + i\omega}{\gamma(\nu)}\,\nu\,d\nu\Bigg]. \end{aligned} \tag{5.47}$$

Overall, the function $M^{(5)}$ defined by equation (5.42) satisfies the following Riemann-Hilbert problem.

Riemann-Hilbert problem 5.1 (Final problem in the modulated elliptic wave region). Determine a sectionally analytic matrix-valued function $M^{(5)}(k) = M^{(5)}(x,t,k)$ in $\mathbb{C} \setminus (\bigcup_{j=1}^{6} L_j \cup B \cup \widetilde{B})$ satisfying the jump conditions

$$M^{(5)+}(k) = M^{(5)-}(k)V_B, \quad k \in B, \tag{5.48a}$$

$$M^{(5)+}(k) = M^{(5)-}(k)V_{\widetilde{B}}, \quad k \in \widetilde{B}, \tag{5.48b}$$

$$M^{(5)+}(k) = M^{(5)-}(k)V_j^{(5)}, \quad k \in L_j,\ j = 1,\dots,6, \tag{5.48c}$$

and the normalization condition

$$M^{(5)}(k) = [I + O(1/k)]e^{i(g_\infty - G_\infty t)\sigma_3}, \quad k \to \infty, \tag{5.48d}$$

where the jump V_B across the branch cut B is defined by equation (4.4), the jump $V_{\widetilde{B}}$ across the branch cut $\widetilde{B}$ is defined by

$$V_{\widetilde{B}} = \begin{pmatrix} 0 & -e^{i(\Omega t-\omega)} \\ e^{-i(\Omega t-\omega)} & 0 \end{pmatrix}, \tag{5.49}$$

the jumps $V_j^{(5)}$ across the contours L_j shown in Figure 5.3 are given by

$$\begin{aligned} V_1^{(5)} &= \begin{pmatrix} 1 & \frac{\overline{r}\,\delta^2}{1+r\overline{r}}\, e^{2i(ht-g)} \\ 0 & 1 \end{pmatrix}, \quad & V_2^{(5)} &= \begin{pmatrix} 1 & 0 \\ \frac{r}{(1+r\overline{r})\delta^2}\, e^{-2i(ht-g)} & 1 \end{pmatrix}, \\ V_3^{(5)} &= \begin{pmatrix} 1 & 0 \\ \frac{r}{\delta^2}\, e^{-2i(ht-g)} & 1 \end{pmatrix}, \quad & V_4^{(5)} &= \begin{pmatrix} 1 & \overline{r}\delta^2\, e^{2i(ht-g)} \\ 0 & 1 \end{pmatrix}, \\ V_5^{(5)} &= \begin{pmatrix} 1 & \frac{\delta^2}{r}\, e^{2i(ht-g)} \\ 0 & 1 \end{pmatrix}, \quad & V_6^{(5)} &= \begin{pmatrix} 1 & 0 \\ \frac{1}{\overline{r}\delta^2}\, e^{-2i(ht-g)} & 1 \end{pmatrix}, \end{aligned} \tag{5.50}$$

the function δ is defined by equation (5.2), the function h is defined by the Abelian integral (5.9) and satisfies the jumps (5.36), the function g is defined by equation (5.45), and the real constants Ω, G_∞, ω, and g_∞ are given by equations (5.37), (5.41), (5.44), and (5.47) respectively.

Decomposition of $M^{(5)}$. As for the final Riemann-Hilbert problem in the plane wave region, a suitable decomposition of $M^{(5)}$ is now required. Namely, we decompose $M^{(5)}$ in a way that separates the jumps expected to yield the leading-order contribution from the jumps expected to contribute only in the error. In particular, denoting by $D_{k_o}^{\varepsilon}$, D_{α}^{ε}, and $D_{\overline{\alpha}}^{\varepsilon}$ the disks of radius ε centred at k_o, α, and $\overline{\alpha}$, respectively, with ε sufficiently small so that these disks do not intersect with each other or with B, we write

$$M^{(5)} = M^{\text{err}} M^{\text{asymp}} \quad \text{with } M^{\text{asymp}} = \begin{cases} M^B, & k \in \mathbb{C} \setminus (D_{k_o}^{\varepsilon} \cup D_{\alpha}^{\varepsilon} \cup D_{\overline{\alpha}}^{\varepsilon}), \\ M^D, & k \in D_{k_o}^{\varepsilon} \cup D_{\alpha}^{\varepsilon} \cup D_{\overline{\alpha}}^{\varepsilon}, \end{cases} \tag{5.51}$$

where:

- the function M^B is analytic in $\mathbb{C} \setminus (B \cup \widetilde{B})$ and satisfies the jump conditions

$$M^{B+}(k) = M^{B-}(k)\, V_B, \quad k \in B, \tag{5.52a}$$

$$M^{B+}(k) = M^{B-}(k)\, V_{\widetilde{B}}, \quad k \in \widetilde{B}, \tag{5.52b}$$

(see Figure 5.7), and the normalization condition

$$M^B(k) = [I + O(1/k)]e^{i(g_\infty - G_\infty t)\sigma_3}, \quad k \to \infty, \tag{5.52c}$$

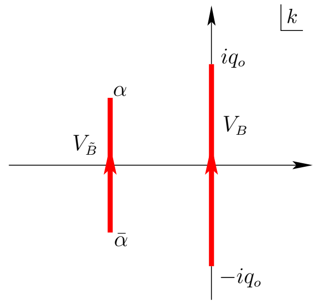

FIGURE 5.7. The jumps of M^B in the modulated elliptic wave region. Note that, since the jump $V_{\widetilde{B}}$ is constant, the contour $\widetilde{B}$ can be deformed to the straight line segment from $\overline{\alpha}$ to α.

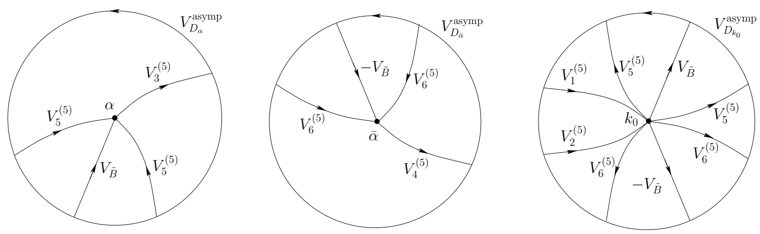

FIGURE 5.8. The jumps of M^D inside the disks $\overline{D_\alpha^\varepsilon}$, $\overline{D_{\overline{\alpha}}^\varepsilon}$, and $\overline{D_{k_o}^\varepsilon}$ in the modulated elliptic wave region.

- the function M^D is analytic in $D_{k_o}^\varepsilon \cup D_\alpha^\varepsilon \cup D_{\overline{\alpha}}^\varepsilon \setminus \bigcup_{j=1}^8 L_j$ with jumps (see Figure 5.8)

$$M^{D+}(k) = M^{D-}(k) V_j^{(5)}, \qquad k \in \widehat{L}_j = L_j \cap \big(D_{k_o}^\varepsilon \cup D_\alpha^\varepsilon \cup D_{\overline{\alpha}}^\varepsilon\big), \; j = 1, \dots, 8, \tag{5.53}$$

- the function M^{err} is analytic in $\mathbb{C} \setminus (\bigcup_{j=1}^6 \check{L}_j \cup \partial D_{k_o}^\varepsilon \cup \partial D_\alpha^\varepsilon \cup \partial D_{\overline{\alpha}}^\varepsilon)$ and satisfies the jump conditions

$$M^{\text{err}+}(k) = M^{\text{err}-}(k) V^{\text{err}}, \quad k \in \bigcup_{j=1}^{6} \check{L}_j \cup \partial D_{k_o}^\varepsilon \cup \partial D_\alpha^\varepsilon \cup \partial D_{\overline{\alpha}}^\varepsilon \tag{5.54a}$$

(see Figure 5.9), and the normalization condition

$$M^{\text{err}}(k) = I + O(1/k), \quad k \to \infty, \tag{5.54b}$$

with $\check{L}_j = L_j \setminus (\overline{D_{k_o}^\varepsilon} \cup \overline{D_\alpha^\varepsilon} \cup \overline{D_{\overline{\alpha}}^\varepsilon})$ and

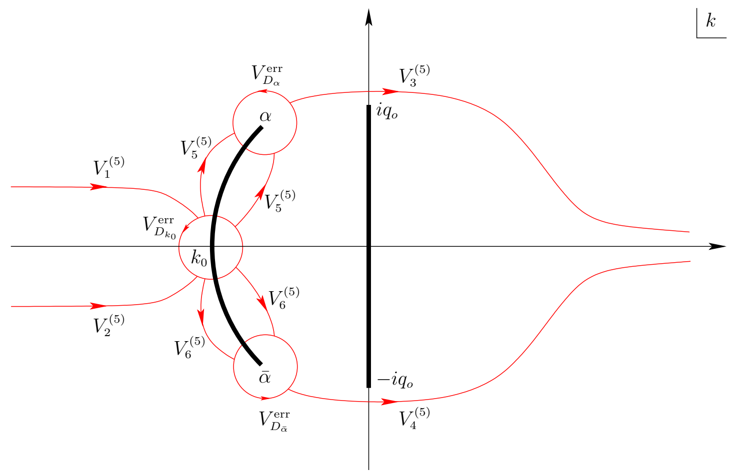

FIGURE 5.9. The jumps of M^{err} in the modulated elliptic wave region.

$$V^{\text{err}} = \begin{cases} M^B V_j^{(5)} (M^B)^{-1}, & k \in \check{L}_j, \\ M^{\text{asymp}-} (V_D^{\text{asymp}})^{-1} (M^{\text{asymp}-})^{-1}, & k \in \partial D_{k_o}^{\varepsilon} \cup \partial D_{\alpha}^{\varepsilon} \cup \partial D_{\bar\alpha}^{\varepsilon}, \end{cases} \tag{5.55}$$

where V_D^{asymp} is the jump of M^{asymp} across the circles $\partial D_{k_o}^{\varepsilon}$, $\partial D_{\alpha}^{\varepsilon}$, and $\partial D_{\bar\alpha}^{\varepsilon}$, which is yet unknown.

Solution of the leading-order asymptotic problem. We now determine M^B, which yields the leading-order contribution to the asymptotics of the solution of the NLS equation as $t \to \infty$. Our approach requires the introduction of appropriate theta functions, similarly to [18]. In particular, recall that the function γ, defined by equation (5.8), gives rise to the Riemann surface Σ with sheets Σ_1, Σ_2 and cycles $\{\alpha, \beta\}$ depicted in Figure 5.4. Then, consider the Abelian differential

$$dw = \frac{C}{\gamma(k)}\, dk, \quad C = \left(\oint_{\beta} \frac{dk}{\gamma(k)} \right)^{-1}, \tag{5.56}$$

which is normalized so that

$$\oint_{\beta} dw = 1 \tag{5.57}$$

and has Riemann period τ defined by

$$\tau = \oint_{\alpha} dw. \tag{5.58}$$

Note that $C \in i\mathbb{R}$. In addition, it can be shown that $\tau \in i\mathbb{R}^+$ (see, for example, [39]). Also, note that the normalization (5.57) and the definition (5.58) of τ imply

$$\int_{-iq_o}^{iq_o} dw = \frac{1}{2}, \quad \int_{\overline{\alpha}}^{-iq_o} dw = \frac{\tau}{2}, \quad k \in \Sigma_1. \tag{5.59}$$

In fact, using the α- and β-cycles of Figure 5.4 in the definition (5.58) we have

$$\tau = \frac{iK(\sqrt{1-m^2})}{K(m)}, \tag{5.60}$$

where $K(m)$ is the complete elliptic integral of the first kind defined by equation (5.32).

Next, we introduce the genus-1 theta function Θ as

$$\Theta(k) = \sum_{\ell \in \mathbb{Z}} e^{2i\pi\ell k + i\pi\ell^2\tau} = \theta_3(\pi k, e^{i\pi\tau}), \tag{5.61}$$

where θ_3 denotes the third Jacobi theta function defined by

$$\theta_3(z, \varrho) = \sum_{\ell \in \mathbb{Z}} e^{2i\ell z} \varrho^{\ell^2}. \tag{5.62}$$

The function θ_3 is analytic for $(z, \varrho) \in \mathbb{C} \times \{\mathbb{C} : |\varrho| < 1\}$. Hence, since $i\tau < 0$, the function Θ is analytic for all $k \in \mathbb{C}$. Moreover, Θ is even and possesses the translation properties

$$\Theta(k+n) = \Theta(k), \quad \Theta(k+n\tau) = e^{-2i\pi nk - i\pi n^2\tau}\Theta(k), \quad n \in \mathbb{Z}. \tag{5.63}$$

We then define the vector-valued function $\mathbf{M}$ as

$$\begin{aligned} \mathbf{M}(k,c) = \Bigg(& \frac{\Theta\big(-\frac{\Omega t}{2\pi} + \frac{\omega}{2\pi} + \frac{i\ln(\frac{\overline{q}_-}{iq_o})}{2\pi} + v(k) + c\big)}{\sqrt{\frac{iq_o}{\overline{q}_-}}\,\Theta(v(k)+c)}, \\ & \frac{\Theta\big(-\frac{\Omega t}{2\pi} + \frac{\omega}{2\pi} + \frac{i\ln(\frac{\overline{q}_-}{iq_o})}{2\pi} - v(k) + c\big)}{\sqrt{\frac{\overline{q}_-}{iq_o}}\,\Theta(-v(k)+c)} \Bigg) \end{aligned} \tag{5.64}$$

where the constants Ω and ω are defined by equations (5.37) and (5.44) as before, the constant c is for now arbitrary and will be determined in due course, and v is the Abelian map

$$v(k) = \int_{iq_o}^{k} dw. \tag{5.65}$$

Note that since v is analytic for all $k \notin B \cup \widetilde{B}$ and Θ is analytic for all $k \in \mathbb{C}$, the only sources of nonanalyticity of the function $\mathbf{M}$ in the k-plane are the branch cuts B and $\widetilde{B}$ and the possible zeros of the functions $\Theta(v(k) \pm c)$.

The function $\mathbf{M}$ has been introduced in order to satisfy the jumps and the normalization condition of the Riemann-Hilbert problem (5.52) for M^B. To compute the jumps of $\mathbf{M}$ note that, using the α- and β-cycles of the Riemann surface Σ, we have

$$v^+(k) + v^-(k) = n, \qquad n \in \mathbb{Z},\ k \in B, \tag{5.66a}$$

$$v^+(k) + v^-(k) = -\tau + n, \quad n \in \mathbb{Z},\ k \in \widetilde{B}. \tag{5.66b}$$

Also, using the translation properties (5.63) of Θ, we find

$$\mathbf{M}^+(k) = \mathbf{M}^-(k) \begin{pmatrix} 0 & -q_-/iq_o \\ \overline{q}_-/iq_o & 0 \end{pmatrix}, \qquad k \in B, \tag{5.67a}$$

$$\mathbf{M}^+(k) = \mathbf{M}^-(k) \begin{pmatrix} 0 & e^{i(\Omega t - \omega)} \\ e^{-i(\Omega t - \omega)} & 0 \end{pmatrix}, \quad k \in \widetilde{B}. \tag{5.67b}$$

The jumps (5.67) differ from those of Riemann-Hilbert problem (5.52) only by a negative sign in the 12-entry. Thus, instead of $\mathbf{M}$ we consider the matrix-valued function N defined by

$$N(k,c) = \frac{1}{2} \begin{pmatrix} \big[p(k) + p^{-1}(k)\big]\mathbf{M}^{(1)}(k,c) & i\big[p(k) - p^{-1}(k)\big]\mathbf{M}^{(2)}(k,c) \\ -i\big[p(k) - p^{-1}(k)\big]\mathbf{M}^{(1)}(k,-c) & \big[p(k) + p^{-1}(k)\big]\mathbf{M}^{(2)}(k,-c) \end{pmatrix}, \tag{5.68}$$

where

$$p(k) = \left[\frac{(k - iq_o)(k - \alpha)}{(k + iq_o)(k - \overline{\alpha})}\right]^{\frac{1}{4}}. \tag{5.69}$$

The function p is analytic away from the branch cuts B and $\widetilde{B}$, is nonzero away from $k = \alpha$ and iq_o, and has fourth-root singularities at $k = \overline{\alpha}$ and $-iq_o$. Moreover, p has the same jump discontinuity across both B and $\widetilde{B}$, namely,

$$p^+(k) = ip^-(k), \quad k \in B \cup \widetilde{B}. \tag{5.70}$$

In addition, p admits the large-k expansion

$$p(k) = 1 - \frac{i(q_o + \alpha_{\mathrm{im}})}{4k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty. \tag{5.71}$$

The jumps (5.67) and (5.70) of $\mathbf{M}$ and p imply that N has the following jumps across B and $\widetilde{B}$:

$$N^+(k) = N^-(k)V_B, \quad k \in B, \tag{5.72a}$$

$$N^+(k) = N^-(k)V_{\widetilde{B}}, \quad k \in \widetilde{B}. \tag{5.72b}$$

Therefore, the function N defined by equation (5.68) has the same jumps as M^B across B and $\widetilde{B}$.

Regarding the analyticity of N away from B and $\widetilde{B}$, we observe that the only possible singularities of N other than the usual branch points $\pm iq_o$, α, and $\overline{\alpha}$ could arise from the functions $\Theta(v(k) \pm c)$ in the denominator of equation (5.64). In this regard, we note that, from the definition (5.61) of Θ in terms of the Jacobi θ_3 function, the zeros of Θ are simple and located at $\frac{1}{2}(1+\tau) + \mathbb{Z} + \tau\mathbb{Z}$. Moreover, the function $p - p^{-1}$ has a unique finite simple zero on the cut complex k-plane given by

$$k_* = \frac{q_o \alpha_{\mathrm{re}}}{q_o + \alpha_{\mathrm{im}}}. \tag{5.73}$$

Then, setting the constant c, which was introduced as an arbitrary constant in the definition (5.64) of $\mathbf{M}$, equal to

$$c = v(k_*) + \frac{1}{2}(1+\tau), \tag{5.74}$$

implies that (i) the function $\Theta(v(k) - c)$ has a unique zero on the first sheet Σ_1 of the Riemann surface Σ, located at the pre-image of k_*, and (ii) the function $\Theta(v(k) + c)$ is nonzero on Σ_1 (see, for example, [39], pp. 290–291). Hence, the choice (5.74) of the constant c ensures that the unique singularity of $\mathbf{M}^{(2)}(k, c)$ and $\mathbf{M}^{(1)}(k, -c)$ on Σ_1 is compensated by the unique zero of $p - p^{-1}$, while $\mathbf{M}^{(1)}(k, c)$ and $\mathbf{M}^{(2)}(k, -c)$ are nonsingular on Σ_1. As a consequence, N is analytic as a function in Σ_1 away from the branch points. Therefore, N is analytic for all $k \in \mathbb{C} \setminus (B \cup \widetilde{B})$.

Finally, we discuss the large-k behavior of N. Combining the definition (5.68) of N and the expansion (5.71) for p we find

$$\lim_{k \to \infty} N(k, c) = N(\infty, c) = \begin{pmatrix} \mathbf{M}^{(1)}(\infty, c) & 0 \\ 0 & \mathbf{M}^{(2)}(\infty, -c) \end{pmatrix}, \tag{5.75}$$

where $\mathbf{M}^{(j)}(\infty, c) = \lim_{k \to \infty} \mathbf{M}^{(j)}(k, c)$ for $j = 1, 2$, and

$$\begin{aligned} \mathbf{M}(\infty, c) = \Bigg(&\frac{\Theta\big(-\frac{\Omega t}{2\pi} + \frac{\omega}{2\pi} + \frac{i \ln(\frac{\overline{q}_-}{iq_o})}{2\pi} + v_\infty + c\big)}{\sqrt{\frac{iq_o}{\overline{q}_-}}\, \Theta(v_\infty + c)}, \\ &\frac{\Theta\big(-\frac{\Omega t}{2\pi} + \frac{\omega}{2\pi} + \frac{i \ln(\frac{\overline{q}_-}{iq_o})}{2\pi} - v_\infty + c\big)}{\sqrt{\frac{\overline{q}_-}{iq_o}}\, \Theta(-v_\infty + c)} \Bigg) \end{aligned} \tag{5.76}$$

with

$$v_\infty = \int_{iq_o}^{\infty} dw. \tag{5.77}$$

In summary, both of the matrix-valued functions M^B and N are analytic away from $B \cup \widetilde{B}$ and satisfy identical jumps across B and $\widetilde{B}$. Thus, we deduce that the unique solution of the Riemann-Hilbert problem (5.52) is

$$M^{B}(k)=e^{i(g_\infty-G_\infty t)\sigma_3}N^{-1}(\infty,c)N(k,c), \tag{5.78}$$

where the exponential term and the constant matrix $N(\infty,c)$ in equation (5.78) are necessary so as to match the normalization of M^B at infinity.

The asymptotic limit $t\to\infty$. Starting from formula (2.30) for the solution q of the focusing NLS equation (1.1) in terms of $M^{(0)}$ and applying the five successive deformations that lead to $M^{(5)}$, we find

$$q(x,t)=-2i\big(M_1^{(5)}(x,t)\big)_{12}e^{i(g_\infty-G_\infty t)}, \tag{5.79}$$

where $M_1^{(5)}$ is the $O(1/k)$ coefficient in the large-k expansion of $M^{(5)}$, i.e.,

$$M^{(5)}(x,t,k)=e^{i(g_\infty-G_\infty t)\sigma_3}+\frac{M_1^{(5)}(x,t)}{k}+O\bigg(\frac{1}{k^2}\bigg),\quad k\to\infty. \tag{5.80}$$

Recalling that for large k the decomposition (5.51) of $M^{(5)}$ involves M^B and M^{err}, we have

$$q(x,t)=-2i\big(M_1^{B}(x,t)e^{i(g_\infty-G_\infty t)}+M_1^{\mathrm{err}}(x,t)\big)_{12}. \tag{5.81}$$

Combining equation (5.78) and the expansion (5.71) for p, we obtain

$$\big(M_1^{B}(x,t)\big)_{12}=\frac{1}{2}(q_o+\alpha_{\mathrm{im}})\frac{\mathbf{M}^{(2)}(\infty,c)}{\mathbf{M}^{(1)}(\infty,c)}\,e^{i(g_\infty-G_\infty t)}. \tag{5.82}$$

Moreover, as in the plane wave region, the term M_1^{err} admits the estimate

$$|M_1^{\mathrm{err}}(x,t)|=O(t^{-\frac{1}{2}}),\quad t\to\infty, \tag{5.83}$$

which can be established by constructing appropriate parametrices near the points $\alpha,\overline{\alpha},k_o$ and then employing the techniques presented in the Appendix. Since the construction of these parametrices is similar to the relevant construction presented in [18], it is omitted here for brevity.

Overall, inserting equation (5.82) and estimate (5.83) in equation (5.81), we conclude that the long-time asymptotic behavior of the solution of the focusing NLS equation (1.1) in the modulated elliptic wave region is given by

$$\begin{aligned} q(x,t)={}&\frac{q_o(q_o+\alpha_{\mathrm{im}})}{\overline{q}_-}\\ &\cdot\frac{\Theta\big(-\frac{\Omega t}{2\pi}+\frac{\omega}{2\pi}+\frac{i\ln(\frac{\overline{q}_-}{iq_o})}{2\pi}-v_\infty+c\big)\Theta(v_\infty+c)}{\Theta\big(-\frac{\Omega t}{2\pi}+\frac{\omega}{2\pi}+\frac{i\ln(\frac{\overline{q}_-}{iq_o})}{2\pi}+v_\infty+c\big)\Theta(-v_\infty+c)}\\ &\cdot e^{2i(g_\infty-G_\infty t)}+O\big(t^{-\frac{1}{2}}\big),\quad t\to\infty, \end{aligned} \tag{5.84}$$

where the constants α_{im}, Ω, G_∞, ω, g_∞, c, and v_∞ are given by equations (5.21), (5.37), (5.41), (5.44), (5.47), (5.74) and (5.77), respectively.

The asymptotic solution (5.84) can actually be expressed in a simpler form. Specifically, similarly to [53], we consider

$$f(k) = 1 - \frac{(k-\alpha)(k-iq_o)}{\gamma(k)} = p(k)[p^{-1}(k) - p(k)] \tag{5.85}$$

as a function on the Riemann surface Σ such that $\gamma(k) \sim k^2$ as $k \to \infty_1$, where ∞_1 denotes the point at infinity on Σ_1. The function f has singularities at $\overline{\alpha}$ and $-iq_o$, and zeros at ∞_1 and at the finite point k_*, which was introduced earlier by equation (5.73) as the unique zero of the function $p - p^{-1}$. Therefore, f is a meromorphic function on Σ with divisor (f) equal to

$$(f) = k_* + \infty_1 - (-iq_o) - \overline{\alpha}. \tag{5.86}$$

Since the divisor (f) of a meromorphic function is principal, by Abel's theorem (e.g., see [7], theorem 2.14) it follows that $v((f)) = 0$, or, equivalently,

$$v(k_*) = v(-iq_o) + v(\overline{\alpha}) - v(\infty_1). \tag{5.87}$$

Recalling the definition of the cycles (see Figure 5.4) and using equations (5.59), we find that $v(-iq_o) = \frac{1}{2} + \mathbb{Z}$ and $v(\overline{\alpha}) = \frac{1}{2} - \frac{\tau}{2} + \mathbb{Z}$ on Σ_1. Hence, $v(k_*) = -\frac{\tau}{2} - v_\infty$ on Σ_1 and then equation (5.74) implies

$$c = \frac{1}{2} - v_\infty + \mathbb{Z} + \tau\mathbb{Z}. \tag{5.88}$$

Therefore, the leading-order asymptotic solution in equation (5.84) simplifies to

$$\begin{aligned} q_{\mathrm{asymp}}(x,t) = {} & \frac{q_o\big(q_o + \alpha_{\mathrm{im}}\big)}{\overline{q}_-} \\ & \cdot \frac{\Theta\big(\frac{1}{2}\big)\Theta\big(\frac{1}{2\pi}\big[\Omega t - \omega - i\ln\big(\frac{\overline{q}_-}{iq_o}\big)\big] + 2v_\infty - \frac{1}{2}\big)}{\Theta\big(2v_\infty - \frac{1}{2}\big)\Theta\big(\frac{1}{2\pi}\big[\Omega t - \omega - i\ln\big(\frac{\overline{q}_-}{iq_o}\big)\big] - \frac{1}{2}\big)} \\ & \cdot e^{2i(g_\infty - G_\infty t)}, \end{aligned} \tag{5.89}$$

where the real constants α_{im}, G_∞, ω, and g_∞ are given by equations (5.21), (5.41), (5.44) and (5.47), respectively, the complex constant v_∞ is defined by equation (5.77), and the real constant Ω is defined in equation (5.37). Finally, note that the constant Ω can be calculated explicitly in terms of elliptic functions. Indeed, using standard results from the theory of Abelian differentials of the second kind, we obtain the expression

$$\Omega = \frac{\pi|\alpha + iq_o|}{K(m)}(\xi - 2\alpha_{\mathrm{re}}), \tag{5.90}$$

where $K(m)$ and m are defined by equations (5.32) and (5.33).

The proof of Theorem 1.2 is complete.

6 Representation via Elliptic Functions: Proof of Theorem 1.3

In this section we prove Theorem 1.3; i.e., we express the modulus of the asymptotic solution (5.89) in the modulated elliptic wave region in terms of elliptic functions.

Recall that the function Θ appearing in equation (5.89) was defined in formula (5.61) in terms of the third Jacobi theta function θ_3, with the period τ given by equation (5.60) and the nome ϱ of θ_3 set to

$$\varrho = e^{i\pi\tau} = e^{-\pi K(\sqrt{1-m^2}/K(m)}, \tag{6.1}$$

where $K(m)$ and m are defined by equations (5.32) and (5.33), respectively. It is also be useful to introduce the three other Jacobi theta functions

$$\theta_1(z,\varrho) = \sum_{\ell\in\mathbb{Z}} (-1)^{\ell-\frac{1}{2}} e^{(2\ell+1)iz} \varrho^{(\ell+\frac{1}{2})^2}, \tag{6.2a}$$

$$\theta_2(z,\varrho) = \sum_{\ell\in\mathbb{Z}} e^{(2\ell+1)iz} \varrho^{(\ell+\frac{1}{2})^2}, \qquad \theta_4(z,\varrho) = \sum_{\ell\in\mathbb{Z}} (-1)^n e^{2i\ell z} \varrho^{\ell^2}. \tag{6.2b}$$

With ϱ given by equation (6.1), the above theta functions are associated with the Jacobi elliptic function sn and the elliptic modulus m via the relations

$$\operatorname{sn}(z,m) = \frac{\theta_3(0)}{\theta_2(0)} \frac{\theta_1(z\theta_3^{-2}(0))}{\theta_4(z\theta_3^{-2}(0))}, \quad m = \frac{\theta_2^2(0)}{\theta_3^2(0)}. \tag{6.3}$$

Hereafter, the nome ϱ will be suppressed from the arguments of the theta functions for brevity. We are now ready to express the asymptotic solution in terms of elliptic functions.

We begin with the constant v_∞ defined by equation (5.77). Note that by the definition of the β-cycle we have

$$2v_\infty - \frac{1}{2} = \int_{-iq_o}^{\infty} dw + \int_{iq_o}^{\infty} dw + \mathbb{Z}; \tag{6.4}$$

hence $2v_\infty - \frac{1}{2}$ is imaginary modulo $\mathbb{Z}$. Moreover, we introduce the real quantities

$$\phi = \frac{1}{2}\left[\Omega t - \omega - i\ln\left(\frac{\overline{q}_-}{iq_o}\right)\right], \quad \psi = -i\pi\left(2v_\infty - \frac{1}{2}\right), \tag{6.5}$$

with Ω given by equation (5.90). Then, we note that the identity $\theta_3(k,\varrho) = \theta_4\big(k+\frac{\pi}{2},\varrho\big)$, together with the evenness of θ_3, implies that $\theta_3(\frac{\pi}{2}) = \theta_3(-\frac{\pi}{2}) = \theta_4(0)$ and $\theta_3(\phi - \frac{\pi}{2}) = \theta_4(\phi)$. Using these relations as well as the relation (5.61) between Θ and θ_3, we can write the leading-order asymptotic solution (5.89) in the form

$$q_{\text{asymp}}(x,t) = \frac{q_o(q_o+\alpha_{\text{im}})}{\overline{q}_-} \frac{\theta_4(0)\theta_3(\phi+i\psi)}{\theta_3(i\psi)\theta_4(\phi)} e^{2i(g_\infty - G_\infty t)}. \tag{6.6}$$

Moreover, recalling that the constants g_∞, G_∞ are real, and employing the addition formula

$$\theta_3(\phi + i\psi)\theta_3(\phi - i\psi)\theta_4^2(0) = \theta_4^2(\phi)\theta_3^2(i\psi) - \theta_1^2(\phi)\theta_2^2(i\psi), \tag{6.7}$$

we have

$$\left|q_{\mathrm{asymp}}(x,t)\right|^2 = (q_o + \alpha_{\mathrm{im}})^2\left[1 - \frac{\theta_1^2(\phi)\theta_2^2(i\psi)}{\theta_4^2(\phi)\theta_3^2(i\psi)}\right]. \tag{6.8}$$

Thus, according to definition (6.3) we have

$$|q_{\mathrm{asymp}}(x,t)|^2 = (q_o + \alpha_{\mathrm{im}})^2\left[1 - m\,\frac{\theta_2^2(i\psi)}{\theta_3^2(i\psi)}\,\mathrm{sn}^2(\theta_3^2(0)\phi, m)\right]. \tag{6.9}$$

It now remains to compute the ratio of the theta functions in equation (6.9) as well as the constant $\theta_3^2(0)$. For the latter, we simply recall the standard formulae [19]

$$\theta_2^2(0) = \frac{2mK(m)}{\pi}, \quad \theta_3^2(0) = \frac{2K(m)}{\pi}, \quad \theta_4^2(0) = \frac{2\sqrt{1-m^2}K(m)}{\pi}, \tag{6.10}$$

where $\theta_2^2(0)$ and $\theta_4^2(0)$ were also included for later use.

The calculation of the ratio $\theta_2^2(i\psi)/\theta_3^2(i\psi)$, on the other hand, is more involved. We first note that, using the identities

$$\theta_1(k,\varrho) = -i e^{ik + \frac{i\pi\tau}{4}}\theta_4\Big(k + \frac{\pi\tau}{2}, \varrho\Big), \tag{6.11a}$$

$$\theta_2(k,\varrho) = \theta_1\Big(k + \frac{\pi}{2}, \varrho\Big), \tag{6.11b}$$

$$\theta_3(k,\varrho) = \theta_4\Big(k + \frac{\pi}{2}, \varrho\Big), \tag{6.11c}$$

we have

$$\frac{\theta_2^2(i\psi)}{\theta_3^2(i\psi)} = \frac{\theta_2^2(2y - \frac{\pi}{2})}{\theta_3^2(2y - \frac{\pi}{2})} = \frac{\theta_1^2(2y)}{\theta_4^2(2y)}, \quad y = \pi v_\infty. \tag{6.12}$$

Subsequently, using the duplication formulae

$$\theta_1(2y)\theta_2(0)\theta_3(0)\theta_4(0) = 2\theta_1(y)\theta_2(y)\theta_3(y)\theta_4(y), \tag{6.13a}$$

$$\theta_4(2y)\theta_4^3(0) = \theta_3^4(y) - \theta_2^4(y), \tag{6.13b}$$

we find

$$\frac{\theta_1^2(2y)}{\theta_4^2(2y)} = \frac{4\theta_4^4(0)}{\theta_2^2(0)\theta_3^2(0)}\frac{\theta_1^2(y)}{\theta_2^2(y)}\frac{\theta_3^2(y)}{\theta_2^2(y)}\frac{\theta_4^2(y)}{\theta_2^2(y)}\left(\left[\frac{\theta_3(y)}{\theta_2(y)}\right]^4 - 1\right)^{-2}. \tag{6.14}$$

The three ratios

$$\frac{\theta_1^2(y)}{\theta_2^2(y)}, \quad \frac{\theta_3^2(y)}{\theta_2^2(y)}, \quad \frac{\theta_4^2(y)}{\theta_2^2(y)} \tag{6.15}$$

appearing in equation (6.14) can be computed in a similar way. Here we present a detailed derivation of the formula for the third ratio and provide the relevant formulae for the first two ratios for brevity.

The idea is to express the desired ratio in terms of a meromorphic function, similarly to [52]. Specifically, to compute the last ratio in equation (6.15), we consider the function

$$f_{42}(k) = e^{i\pi \int_{iq_o}^{k} dw - \frac{i\pi\tau}{4}} \frac{\Theta\left(\int_{\alpha}^{k} dw - \frac{1}{2} - \frac{\tau}{2}\right)}{\Theta\left(\int_{-iq_o}^{k} dw - \frac{1}{2} - \frac{\tau}{2}\right)} \tag{6.16}$$

as a function on the Riemann surface Σ. Noting that on the first sheet Σ_1 we have

$$\int_{\alpha}^{k} dw = \frac{\tau}{2} + \int_{iq_o}^{k} dw + \mathbb{Z}, \qquad \int_{-iq_o}^{k} dw = \frac{1}{2} + \int_{iq_o}^{k} dw + \mathbb{Z}, \tag{6.17}$$

and recalling also the periodicity properties of Θ, we find

$$f_{42}(k) = e^{i\pi \int_{iq_o}^{k} dw - \frac{i\pi\tau}{4}} \frac{\theta_3\left(\pi \int_{iq_o}^{k} dw - \frac{\pi}{2}\right)}{\theta_3\left(\pi \int_{iq_o}^{k} dw - \frac{\pi\tau}{2}\right)}. \tag{6.18}$$

Employing the second and the third of the identities (6.11), we then obtain

$$f_{42}(k) = \frac{\theta_4\left(\pi \int_{iq_o}^{k} dw\right)}{\theta_2\left(\pi \int_{iq_o}^{k} dw\right)}. \tag{6.19}$$

Observe that f_{42} evaluated at $k = \infty_1$ is equal to the ratio $\theta_4(y)/\theta_2(y)$ that we wish to compute.

Furthermore, note that the function f_{42} defined by equation (6.16) is meromorphic on Σ_1, with a simple pole at $k = -iq_o$ and a simple zero at $k = \alpha$. Hence, f may be expressed in the form

$$f_{42}(k) = A_{42}(k) \frac{(k-\alpha)^{1/2}}{(k+iq_o)^{1/2}}, \tag{6.20}$$

where A_{42} is a holomorphic function bounded at ∞_1. Now note that $f_{42}^2(k)$ is also a meromorphic function on the complex k-plane. Hence $A_{42}^2(k)$ must be a constant by Liouville's theorem. Therefore, to determine the function f_{42} (and hence the ratio $\theta_4(y)/\theta_2(y)$) it remains to determine the constant A_{42}.

The computation of A_{42} is done by evaluating the residue of f_{42} at $k = -iq_o$ in two different ways. First, using the representation (6.20) we find

$$\operatorname{Res}[f_{42}(k), -iq_o] = A_{42} \cdot i(iq_o + \alpha)^{\frac{1}{2}}. \tag{6.21}$$

Alternatively, employing the form (6.19) and noting that

$$\theta_2\left(\pi \int_{iq_o}^{k} dw\right) = R(k)(k+iq_o)^{\frac{1}{2}}$$

for some function R such that $R(-iq_o) \neq 0$, we obtain

$$\text{Res}[f_{42}(k), -iq_o] = \frac{\theta_4\big(\pi \int_{iq_o}^{iq_o} dw\big)}{R(-iq_o)} = \frac{\theta_4\big(\frac{\pi}{2}\big)}{R(-iq_o)}. \tag{6.22}$$

Actually, we have

$$\begin{aligned} [R(-iq_o)]^2 &= \frac{\partial}{\partial k}\theta_2^2\left(\pi \int_{iq_o}^{k} dw\right)\bigg|_{k=-iq_o} \\ &= \frac{[2\pi C\, \theta_2'(\frac{\pi}{2})]^2}{-2iq_o(iq_o+\alpha)(iq_o+\overline{\alpha})}, \end{aligned} \tag{6.23}$$

where the imaginary constant C is defined by equation (5.56). Thus, equation (6.22) becomes

$$\text{Res}[f_{42}(k), -iq_o] = \theta_4\big(\tfrac{\pi}{2}\big)\left[-\frac{2iq_o(iq_o+\alpha)(iq_o+\overline{\alpha})}{\big[2\pi C\, \theta_2'\big(\frac{\pi}{2}\big)\big]^2}\right]^{\frac{1}{2}}. \tag{6.24}$$

Matching the expressions (6.21) and (6.24), we deduce

$$A_{42}^2 = \frac{2iq_o(iq_o+\overline{\alpha})\theta_4^2\big(\frac{\pi}{2}\big)}{\big[2\pi C\, \theta_2'\big(\frac{\pi}{2}\big)\big]^2}. \tag{6.25}$$

Finally, evaluating equations (6.19) and (6.20) at $k = \infty_1$ and using the identities (6.11), we obtain

$$\frac{\theta_4^2(y)}{\theta_2^2(y)} = \frac{2iq_o(iq_o+\overline{\alpha})\theta_3^2(0)}{[2\pi C\theta_1'(0)]^2}. \tag{6.26}$$

The constant C defined by equation (5.56) can be expressed in terms of the complete elliptic integral of the first kind $K(m)$ via the formula

$$C = \frac{i|\alpha + iq_o|}{4K(m)}. \tag{6.27}$$

Thus, using also the identity $\theta_1'(0) = \theta_2(0)\theta_3(0)\theta_4(0)$ we can write expression (6.26) in the form

$$\frac{\theta_4^2(y)}{\theta_2^2(y)} = -\frac{2iq_o(iq_o+\overline{\alpha})}{m\sqrt{1-m^2}|\alpha+iq_o|^2}. \tag{6.28}$$

Computations identical to the above yield the following expressions for the other two ratios in (6.15):

$$\frac{\theta_1^2(y)}{\theta_2^2(y)} = -\frac{(iq_o+\alpha)(iq_o+\overline{\alpha})}{\sqrt{1-m^2}|\alpha+iq_o|^2}, \qquad \frac{\theta_3^2(y)}{\theta_2^2(y)} = -\frac{2iq_o(iq_o+\alpha)}{m|\alpha+iq_o|^2}. \tag{6.29}$$

Inserting expressions (6.28) and (6.29) in equation (6.14) we obtain

$$\frac{\theta_1^2(2y)}{\theta_4^2(2y)} = \frac{1}{m}\frac{4q_o\alpha_{\mathrm{im}}}{(q_o+\alpha_{\mathrm{im}})^2}. \tag{6.30}$$

Inserting in turn this expression into equation (6.9) yields expression (1.10) for the modulus of the leading-order solution of the focusing NLS equation (1.1) in the modulated elliptic wave region.

The proof of Theorem 1.3 is complete.

7 Discussion and Final Remarks

We conclude this work with a discussion of the physical implications of our results and with some general remarks about how they fit in a broader context.

(1) We have shown that, for all initial conditions that satisfy the hypothesis of Theorem 1.1, the long-time asymptotics decomposes the xt-plane into two plane wave regions, in each of which the solution is approximately equal to the background value up to a phase, separated by a central region in which the leading-order behavior is given by a slow modulation of the traveling wave solutions of the focusing NLS equation. As already discussed in Section 1, the spatial structure of the asymptotic solution (both in the plane wave regions and in the modulated elliptic wave region) is independent of the initial conditions of the problem, and the initial conditions only determine the slowly varying offset X_o of the elliptic solution (via the reflection coefficient), whereas the envelope of the modulated elliptic wave is independent of it. Thus, the long-time asymptotics of generic localized perturbations of the constant background in modulationally unstable media on the infinite line displays universal behavior. In this sense, *the asymptotic stage of modulational instability is universal.*

(2) The results of this work also show that, even though the jumps along the branch cut B of the original Riemann-Hilbert problem (2.1) grow exponentially with t, the solution of this Riemann-Hilbert problem—and hence the solution of the focusing NLS equation—remains bounded in the limit $t \to \infty$. We note that this is a fairly common result of the analysis of Riemann-Hilbert problems via the Deift-Zhou method, as the factorizations of the jump matrices and the deformations of the jump contours allow one to eliminate all the terms that exhibit the exponential growth in the jump conditions.

(3) We reiterate that a special case of the modulated elliptic wave (without the slowly varying offset parameter X_o and the phase g_∞) had been previously obtained using Whitham theory [54]. The motion of the Riemann invariants had also been previously obtained [36]. Compared to those works, however, the present results represent a significant step forward in that: (i) they establish rigorously the

validity of the modulated elliptic solution as the long-time asymptotic state of the problem; (ii) they establish rigorously that the solution of the focusing NLS equation with nonzero boundary conditions at infinity remains bounded for all times; and (iii) they establish the universality of the modulated elliptic solution as the asymptotic state of a large class of perturbations of the constant background. At the same time, it is remarkable that the particular solution of the genus-1 Whitham system obtained in [36] is relevant for the long-time asymptotics even though the Whitham equations for the focusing NLS equation are elliptic (which implies that the corresponding initial value problem is ill-posed for generic initial data).

(4) The results of this work are relevant in describing the nonlinear evolution of plain waves in modulationally unstable media in all fields where the focusing NLS equation applies, ranging from deep water waves to nonlinear fiber optics, to attractive Bose-Einstein condensates and beyond. A brief summary of some of the main results of this work recently appeared in [13]. Importantly, an extensive set of careful numerical simulations with a variety of initial conditions shows that, for a broad class of localized perturbations of the constant background, the dynamics is indeed characterized by the asymptotic state described in this work. See item (9) below for further discussion.

(5) In the physics literature, modulational instability is often studied in the framework of spatially periodic perturbations of the constant background. The results of this work should therefore be compared to those available in the case of periodic boundary conditions. But the IST machinery used to study the periodic case (namely, the theory of finite-genus solutions [51,66]) is very different from the one in the initial value problem with nonzero boundary conditions used here. (In fact, the question of how the IST of the periodic case reduces to that of the infinite line is an interesting one [74], which does not appear to have a simple answer that can be used effectively to translate results from one setting into the other.) Also, the physics in the two cases is markedly different. For example: (i) In the periodic case, there is an amplitude threshold below which no instability occurs, whereas no such threshold exists on the infinite line. (ii) In the periodic case, radiation cannot escape to infinity, and, therefore, it is doubtful that a long-time asymptotic state exists. Also, sinusoidal excitations are a special case of perturbations with several Fourier components, each contributing with its own amplitude and phase. Such generic perturbations are characterized by their spectral data, and this is precisely the situation studied in this work.

(6) The results of this work can also be compared to the semiclassical limit of the focusing NLS equation with zero boundary conditions [56]. The study of that scenario requires more sophisticated analysis, and the results are also more complicated. (Namely, even though there is a similar bifurcation from plane waves

to modulated genus-2 oscillations in the case of zero boundary conditions, the genus-2 solution also breaks along a certain caustic curve in the xt-plane, with numerical evidence suggesting the generation of regions of higher and higher genus in the dispersionless limit.) Moreover, a fundamental difference between the two physical settings is that numerical simulations of the semiclassical case become more and more sensitive to roundoff error in this limit. (Essentially, the initial value problem becomes ill-posed in this limit.) In contrast, the scenario studied in this work (namely, modulational instability on the line) does not appear to be as sensitive to numerical issues, as demonstrated by the favorable comparison between the asymptotic results and direct numerical simulations of a large class of initial conditions. See also item (9) below.

(7) Of course semiclassical limits and the study of long-time asymptotics are often studied using Whitham theory [75]. As mentioned above, however, the Whitham equations for the focusing NLS equation are elliptic, and therefore the corresponding initial value problem is ill-posed. This is well-known in the case of zero boundary conditions (e.g., see [56]) and remains true in the case of nonzero boundary conditions. Interestingly enough, however, special, real-valued solutions to the Whitham equations also exist in the focusing case, even though the characteristic velocities are complex. Indeed, the modulation equations (1.8) coincide exactly with those obtained in [36, 54]. It should be clear, however, that the IST-related methods used here are the only way to study the nonlinear stage of modulational instability for generic perturbations of the constant background.

(8) The results of this work open up a number of interesting problems, both from a mathematical and a physical point of view. From a physical point of view, an obvious question is whether the asymptotic behavior described in this work is robust. We can reformulate this question in terms of whether the results are stable under perturbations. Let us briefly elaborate on this issue. The fundamental dichotomy in this respect concerns whether the underlying dynamics of the system under study is exactly governed by the NLS equation (1.1). If the answer is yes, the results of this work rigorously establish the long-time asymptotics of the solution for any initial conditions satisfying the hypotheses of Theorems 1.1 and 1.2. On the other hand, if additional terms are present in the governing equations, i.e., if the NLS equation only provides an approximation of the actual behavior, and the exact dynamics is governed by a perturbation of equation (1.1), the situation is different. This is because in many cases perturbed NLS equations give rise to chaotic behavior [1, 2, 4]. Therefore, it is likely that initially small differences between the exact solution of the NLS equation and that of the perturbed system will grow exponentially, and that, as a result, the asymptotic state described in this work will not persist for all times. Such is the case, for example, when one solves the focusing NLS equation with periodic boundary conditions numerically.

Indeed, it is known that such a scenario is characterized by catastrophic roundoff accumulation [2,4].

(9) Nonetheless, we next show that even when one integrates the NLS equation numerically—and therefore the NLS equation is only an approximate description of the exact dynamics due to truncation error—an intermediate time range exists for which the nonlinear stage of modulational instability reaches the asymptotic state described in this work and at the same time catastrophic roundoff has not yet taken hold.

This claim is borne out both by general estimates and by careful numerical simulations. Regarding the former, recall that the growth rate of an unstable Fourier mode $\zeta \in [0, 2\pi]$ in the focusing case is $\gamma(\zeta) = \zeta\sqrt{4q_o^2 - \zeta^2}$ [11]. Therefore $\gamma_{\max} = 2q_o^2$, which is achieved for $\zeta = \sqrt{2}q_o$. A trivial calculation then shows that, working in double precision, the characteristic time needed for roundoff error to grow to $O(1)$ is $\tau_{\text{roundoff}} = 16\ln(10)/\gamma_{\max} = 8\ln(10)/q_o^2$. One can therefore expect that the asymptotic state described in this work will be destroyed by roundoff error after this time.

On the other hand, it is possible to observe the asymptotic state numerically up to $t \simeq \tau_{\text{roundoff}}$. Indeed, we verified that this is the case by performing numerical simulations of the focusing NLS equation (using an eighth-order split-step method) and a variety of initial conditions representing a localized perturbation of the constant background. The numerical results, shown in Figure 7.1, confirm that in all cases the dynamics of the system for $t < \tau_{\text{roundoff}}$ is accurately described by the asymptotic state presented in this work. Remarkably, this is true even when the

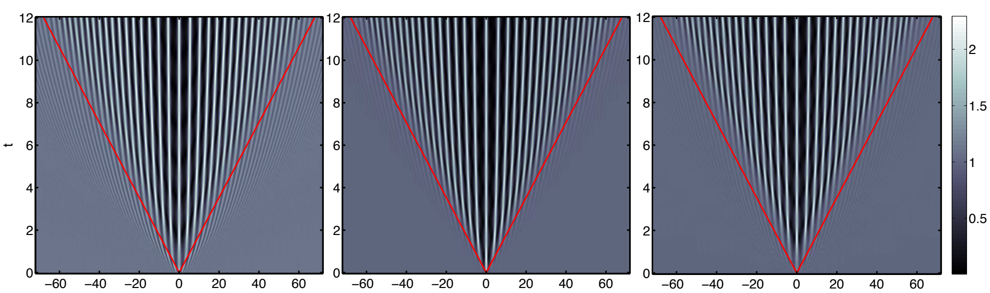

FIGURE 7.1. Numerically computed density plot of $|q(x,t)|$, describing the solution of the focusing NLS equation with various choices of initial condition representing a localized perturbation of a constant background. Left: $q(x,0) = 1 + i\cos(\pi x)$ for $|x| < 1$ and $q(x,0) = 1$ otherwise. Center: $q(x,0) = 1 + ie^{-x^2}\cos(\sqrt{2}x)$. Right: $q(x,0) = 1 + i\,\text{sech}(10x)$. Also shown (red lines) are the boundaries between the plane wave regions and the modulated elliptic wave region. Numerical simulations performed by Sitai Li.

initial conditions are not analytic, in which case the initial value problem for the NLS-Whitham equations has no solution. Note also that no small-norm assumption for the perturbation is required in order for the results of this work to be valid. In fact, we conjecture that the nonlinear structure called "the beard," which was observed to appear seemingly out of nowhere in numerical simulations of the focusing NLS equation [27, 37], is nothing else but the asymptotic behavior state in this work. (For example, compare Figure 7.1 with the structure in the center of the spatial domain in figure 14 of [37].)

(10) From a mathematical point of view, an interesting problem is the generalization of the long-time asymptotics to initial data for which the reflection coefficient is not analytic in a neighborhood of the continuous spectrum. This situation corresponds to the case of initial conditions that tend to the constant background algebraically as $x \to \pm\infty$. One would hope that such a generalization can be achieved by employing rational approximations, along similar lines to what is done in the case of zero boundary conditions [35]. The issue is nontrivial, however, because the definition of ω in (5.44) for the solution in the modulated elliptic wave region requires the evaluation of $r(k)$ along the branch cut $\widetilde{B} = [\overline{\alpha}, \alpha]$. A similar requirement is present for the phase g_∞ in the same region. The condition (3.1) with $\epsilon = q_o$ ensures that the reflection coefficient is analytic in the strip $|\Im(k)| < q_o$, which contains $\widetilde{B}$ for all times and therefore allows ω to be well-defined. (It is straightforward to show that the solution of the modulation equations (1.8) satisfies $|\alpha_{\mathrm{im}}| < q_o$.) In this case, the leading-order asymptotics in Theorems 1.1 and 1.2 is uniform for all $x \in \mathbb{R}$. On the other hand, if the initial conditions only decay to q_o slowly as $x \to \pm\infty$ (and as a result the reflection coefficient is not analytic) the process of rational approximation could lead to $O(1)$ changes in ω.

(11) A related question is therefore whether the dynamics of modulationally unstable systems with initial conditions that are only slowly decaying to the background as $x \to \pm\infty$ is also described by an asymptotic state similar to the one described in this work. While the observations in the preceding paragraph cast some doubts that this will be the case, one should also note that changes in ω and g_∞ only affect the local position shift X_o and the overall phase of the solution, and not the structure of the modulated elliptic wave, which, as discussed earlier, is independent of the initial condition. Note also that, for any fixed x, $\xi \to 0$ as $t \to \infty$ and, correspondingly, $\alpha \to iq_o$. As a result, the analyticity requirements needed in order to perform the deformations in Section 5 become progressively weaker as $t \to \infty$. (That is, essentially any $\epsilon > 0$ is enough for sufficiently large t.) While in this case the resulting leading-order asymptotics is not uniform in x, it is conceivable that some form of the modulated elliptic wave described in this work could

also describe those kinds of situations. But of course a definitive answer to this question will require further work.

(12) Another interesting problem is the generalization of the results to the long-time asymptotics to initial conditions which generate a nontrivial discrete spectrum. Apart from the intrinsic interest of this problem from a mathematical point of view, the analysis of such kinds of problems is also important from a physical point of view, since it will allow one for the first time to study the interactions between solitons and radiation in a modulationally unstable medium. The same framework will also allow one to address another related important open question, concerning the stability of solitons on nonzero background in modulationally unstable media.

(13) Finally, returning to the more general issue of the behavior of modulationally unstable systems, there are of course many other important open questions, and it should be clear that much more work is needed until a comprehensive picture can be pieced together. For example, it remains to be determined what is the precise role, if any, played in the asymptotic dynamics by the Kuznetsov-Ma solitons, by the Akhmediev breathers and the Peregrine solitons, and by the NLS-Whitham equations, in particular regarding how the ellipticity of the NLS-Whitham equations contributes to the overall picture. Whether and how the results on the infinite line reduce to those with periodic boundary conditions as the characteristic width of the perturbations increases without bounds, and the understanding of the precise connections between the phenomena discussed in this work and the theory of integrable turbulence [5, 77], also remain as important open questions. Nonetheless, it should also be clear that the results of this work represent a significant step forward on a problem that has remained essentially open for fifty years. In particular, we find it very remarkable that a broad class of perturbations of the constant background under the effect of modulational instability gives rise to a universal asymptotic state corresponding to the slow modulations of the elliptic solutions of the focusing NLS equation.

Appendix: Estimation of the Error

We now establish estimate (4.36) for the term M^{err} that appears in formula (4.33) for the leading-order asymptotics in the plane wave region. Note that the proof also carries over to the estimate (5.83) for M^{err} in the modulated elliptic wave region once the appropriate parametrices are employed. (See [18] for the construction of such parametrices.)

Since M^{err} solves the Riemann-Hilbert problem (4.31), using Plemelj's formulae we may express the $O(1/k)$ coefficient M_1^{err} in the large-k expansion of M^{err}

as

$$M_1^{\mathrm{err}}(x,t) = -\frac{1}{2i\pi}\int_{\bigcup_j \check{L}_j \cup \partial D^{\varepsilon}_{k_1}} M^{\mathrm{err}-}(\nu)[V^{\mathrm{err}}(\nu) - I]d\nu.$$

By the Cauchy-Schwarz inequality we then find

$$\text{(A.1)}\quad \begin{aligned} |M_1^{\mathrm{err}}(x,t)| \leqslant \frac{1}{2\pi}\Big(&\|M^{\mathrm{err}-} - I\|_{L^2_k(\bigcup_j \check{L}_j \cup \partial D^{\varepsilon}_{k_1})}\|V^{\mathrm{err}} - I\|_{L^2_k(\bigcup_j \check{L}_j \cup \partial D^{\varepsilon}_{k_1})} \\ &+ \|V^{\mathrm{err}} - I\|_{L^1_k(\bigcup_j \check{L}_j \cup \partial D^{\varepsilon}_{k_1})}\Big). \end{aligned}$$

We will estimate each of the terms involved in the right-hand side of the above inequality separately. We begin with $\|V^{\mathrm{err}} - I\|_{L^p_k(\check{L}_j)}$. Along the contours $\check{L}_j$, we have $V^{\mathrm{err}} - I = M^B\big(V_j^{(4)} - I\big)(M^B)^{-1}$. Thus, we find

$$\text{(A.2)}\quad \|V^{\mathrm{err}} - I\|_{L^p_k(\check{L}_j)} \leqslant \|M^B\|_{L^\infty_k(\check{L}_j)}\big\|V_j^{(4)} - I\big\|_{L^p_k(\check{L}_j)}\|(M^B)^{-1}\|_{L^\infty_k(\check{L}_j)}.$$

Since M^B is analytic away from the branch cut B and $M^B = O(1)$ as $k \to \infty$, we have

$$\|M^B\|_{L^\infty_k(\check{L}_j)}\|(M^B)^{-1}\|_{L^\infty_k(\check{L}_j)} < c < \infty, \quad c > 0.$$

Hence, inequality (A.2) becomes

$$\text{(A.3)}\quad \|V^{\mathrm{err}} - I\|_{L^p_k(\check{L}_j)} \leqslant c\,\big\|V_j^{(4)} - I\big\|_{L^p_k(\check{L}_j)}, \quad c > 0.$$

The right-hand side of inequality (A.3) can be estimated by exploiting the exponential decay along $\check{L}_j$. For example, for $j = 2$ inequality (A.3) reads

$$\text{(A.4)}\quad \|V^{\mathrm{err}} - I\|_{L^p_k(\check{L}_2)} \leqslant c\begin{pmatrix} 0 & 0 \\ \Big\|\frac{re^{-2i(\theta t - g)}}{1 + r\overline{r}}\,\delta^{-2}\Big\|_{L^p_k(\check{L}_2)} & 0 \end{pmatrix}.$$

Recall that the function δ defined by equation (4.13) is nonzero, analytic away from $(-\infty, k_1]$ and $O(1)$ as $k \to \infty$. Similarly, the function g defined by equation (4.23) is analytic away from B and $O(1)$ as $k \to \infty$. Thus,

$$\text{(A.5)}\quad \left\|\frac{re^{-2i(\theta t - g)}}{1 + r\overline{r}}\,\delta^{-2}\right\|_{L^p_k(\check{L}_2)} \leqslant c\left\|\frac{r}{1 + r\overline{r}}\,e^{-2i\theta t}\right\|_{L^p_k(\check{L}_2)}, \quad c > 0.$$

Furthermore, since $\check{L}_2$ lies outside the disk $D^{\varepsilon}_{k_1}$, we have $\|\Re(i\theta)\|_{L^\infty(\check{L}_2)} > C > 0$. Hence, exploiting also the analyticity of the reflection coefficient r provided by Lemma 3.1, we find

$$\text{(A.6)}\quad \left\|\frac{r}{1 + r\overline{r}}\,e^{-2i\theta t}\right\|_{L^p_k(\check{L}_2)} < \widetilde{C}e^{-Ct} \quad \forall t > T > 0,\ C, \widetilde{C} > 0.$$

Combining inequalities (A.4)–(A.6) we then obtain the estimate

$$\|V^{\mathrm{err}} - I\|_{L^p_k(\check{L}_2)} \leqslant \widetilde{C}e^{-Ct} \quad \forall t > T > 0,\ C, \widetilde{C} > 0.$$

Similar estimates hold along the remaining contours. Overall, we have

$$\|V^{\mathrm{err}} - I\|_{L^p_k(\check{L}_j)} \leqslant \widetilde{C} e^{-Ct} \quad \forall t > T > 0,\ C, \widetilde{C} > 0,\ j = 1,2,3,4. \tag{A.7}$$

Next, we wish to estimate $V^{\mathrm{err}} - I$ along the circle $\partial D^{\varepsilon}_{k_1}$. Recall, however, that now

$$V^{\mathrm{err}} = M^{\mathrm{asymp}-} (V_D^{\mathrm{asymp}})^{-1} (M^{\mathrm{asymp}-})^{-1}, \tag{A.8}$$

where V_D^{asymp} is the *unknown* jump of M^{asymp} across $\partial D^{\varepsilon}_{k_1}$. Thus, we first need to estimate this unknown jump.

Lemma. The jump V_D^{asymp} of M^{asymp} across the circle $\partial D^{\varepsilon}_{k_1}$ admits the estimate

$$V_D^{\mathrm{asymp}} = I + O\big(t^{-\frac{1}{2}}\big), \quad t \to \infty. \tag{A.9}$$

PROOF. Similarly to [34], for $|k - k_1| \leqslant \varepsilon$ with ε sufficiently small we take the Taylor series of the function θ about the point k_1 to write

$$\theta(k)t = \theta(k_1)t + z^2, \tag{A.10}$$

where

$$z = \sqrt{t}(k - k_1)\bigg(\sum_{n=0}^{\infty} \theta_{n+2}(k - k_1)^n\bigg)^{\frac{1}{2}}, \quad \theta_n = \frac{\theta^{(n)}(k_1)}{n!}. \tag{A.11}$$

Furthermore, as $k \in \overline{D^{\varepsilon}_{k_1}}$ we can write $\big(\sum_{n=0}^{\infty} \theta_{n+2}(k - k_1)^n\big)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \alpha_n (k - k_1)$, $\alpha_n \in \mathbb{C}$; hence, equation (A.11) implies

$$\frac{z}{\sqrt{t}} = \sum_{n=1}^{\infty} \alpha_{n-1}(k - k_1)^n, \quad k \in \overline{D^{\varepsilon}_{k_1}}. \tag{A.12}$$

Moreover, for ε sufficiently small, equation (A.12) can be inverted via recursive approximations to yield

$$k = k_1 + \sum_{n=1}^{\infty} \beta_n \bigg(\frac{z}{\sqrt{t}}\bigg)^n, \quad k \in \overline{D^{\varepsilon}_{k_1}}.$$

Next, we let

$$M^D(k) = M^B(k)\, m^D\bigg(\sqrt{t} \sum_{n=1}^{\infty} \alpha_{n-1}(k - k_1)^n\bigg), \quad k \in D^{\varepsilon}_{k_1}, \tag{A.13}$$

which equivalently reads

$$\begin{aligned} m^D(z) = {} & (M^B)^{-1}\bigg(k_1 + \sum_{n=1}^{\infty} \beta_n \bigg(\frac{z}{\sqrt{t}}\bigg)^n\bigg) \\ & \cdot M^D\bigg(k_1 + \sum_{n=1}^{\infty} \beta_n \bigg(\frac{z}{\sqrt{t}}\bigg)^n\bigg), \quad z \in D^{\varepsilon}_0. \end{aligned} \tag{A.14}$$

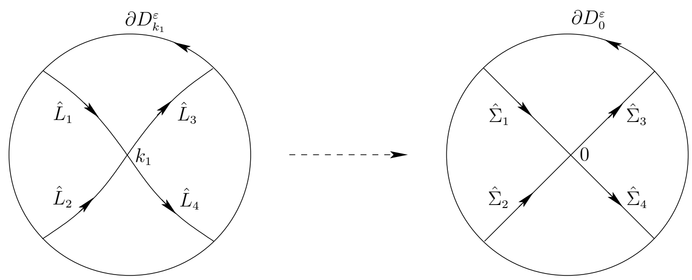

FIGURE A.1. The transformation from the contours $\widehat{L}_j$ to the contours $\widehat{\Sigma}_j$, $j = 1, 2, 3, 4$.

Note that the presence of M^B does not affect the jump conditions of M^D inside the disk $D^{\varepsilon}_{k_1}$ since M^B is analytic there. Furthermore, the mapping

$$\overline{D^{\varepsilon}_{k_1}} \ni k \mapsto z = \sqrt{t} \sum_{n=1}^{\infty} \alpha_{n-1} (k - k_1)^n \in \overline{D^{\varepsilon}_0} \tag{A.15}$$

is conformal and, as such, it preserves angles locally. Hence, there exists an appropriate choice of the jump contours $\widehat{L}_j$ that lie inside the disk $D^{\varepsilon}_{k_1}$, which is mapped under (A.15) to certain contours $\widehat{\Sigma}_j$ (see Figure A.1) that lie inside the disk D^{ε}_0 and form a cross centered at $z = 0$ such that its extension does not intersect with the branch cut B. Thus, for $k \in D^{\varepsilon}_{k_1}$ we infer from equations (4.30) and (A.14) that the function $m^D(z) = m^D(x, t, z)$ is analytic in $D^{\varepsilon}_0 \setminus \bigcup_{j=1}^4 \widehat{\Sigma}_j$ and satisfies the jump conditions

$$\begin{gathered} m^{D+}(z) = m^{D-}(z) V_j^{(4)}\left(k_1 + \sum_{n=1}^{\infty} \beta_n \left(\frac{z}{\sqrt{t}}\right)^n\right), \\ z \in \widehat{\Sigma}_j,\ j = 1, 2, 3, 4. \end{gathered} \tag{A.16}$$

Extending the jump conditions (A.16) in the whole complex k-plane implies that m^D is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^4 \Sigma_j$ and satisfies the jump conditions

$$\begin{gathered} m^{D+}(z) = m^{D-}(z) V_j^{(4)}\left(k_1 + \sum_{n=1}^{\infty} \beta_n \left(\frac{z}{\sqrt{t}}\right)^n\right), \\ z \in \Sigma_j,\ j = 1, 2, 3, 4, \end{gathered} \tag{A.17}$$

and the normalization condition $m^D(z) = I + O(1/z)$, $z \to \infty$, where the contours Σ_j are the extensions of the contours $\widehat{\Sigma}_j$ outside the disk D^{ε}_0.

Since $r(k) = O(1/k)$ as $k \to \infty$ (see [12]), we may now proceed similarly to [34]. Eventually, we find

$$m^D(x,t,z) = \delta_0^{\hat{\sigma}_3}\, \widetilde{m}_\infty^D(x,z) + O\big(t^{-\frac{1}{2}} \ln t\big), \quad t \to \infty,$$

where for

$$\nu = -\frac{1}{2\pi} \ln(1 + |r(k_1)|^2), \quad \chi(k) = -\frac{1}{2i\pi} \int_{-\infty}^{k_1} \ln(k - \nu) d[\ln(1 + |r(\nu)|^2)]$$

we define

$$\delta_0 = \beta_1^{i\nu} t^{-\frac{i\nu}{2}} e^{\chi(k_1)} e^{i\theta(k_1)t - ig(k_1)}.$$

Moreover, the t-independent function $\widetilde{m}_\infty^D(z) = \widetilde{m}_\infty^D(x,z)$ is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^4 \Sigma_j$ and satisfies the jump conditions

$$\widetilde{m}_\infty^{D+}(z) = \widetilde{m}_\infty^{D-}(z)\, V_j^\infty(z), \quad z \in \Sigma_j,\ j = 1,2,3,4,$$

and the normalization condition $\widetilde{m}_\infty^D(z) = I + O(1/z)$, $z \to \infty$, with

$$V_1^\infty = \begin{pmatrix} 1 & \frac{\overline{r}(k_1)}{1+|r(k_1)|^2}\, z^{2i\nu} e^{2iz^2} \\ 0 & 1 \end{pmatrix}, \quad V_2^\infty = \begin{pmatrix} 1 & 0 \\ \frac{r(k_1)}{1+|r(k_1)|^2}\, z^{-2i\nu} e^{-2iz^2} & 1 \end{pmatrix},$$

$$V_3^\infty = \begin{pmatrix} 1 & 0 \\ r(k_1) z^{-2i\nu} e^{-2iz^2} & 1 \end{pmatrix}, \quad V_4^\infty = \begin{pmatrix} 1 & \overline{r}(k_1)\, z^{2i\nu} e^{2iz^2} \\ 0 & 1 \end{pmatrix}.$$

Thus, since $M^{\text{asymp}+} = M^D$ and $M^{\text{asymp}-} = M^B$ for $k \in \partial D_{k_1}^\varepsilon$, using equation (A.14) we find

$$V_D^{\text{asymp}} = m^D\left(\sqrt{t} \sum_{n=1}^\infty \alpha_{n-1}(k - k_1)^n\right), \quad k \in \partial D_{k_1}^\varepsilon.$$

Then, since $z \to \infty$ in the limit $t \to \infty$ for $k \in \partial D_{k_1}^\varepsilon$, we use the large-$z$ expansion of $m^D(z)$ evaluated at $\sqrt{t} \sum_{n=1}^\infty \alpha_{n-1}(k - k_1)^n$ to obtain

$$\begin{aligned} V_D^{\text{asymp}} = I &+ \frac{m_1^D}{\sqrt{t} \sum_{n=1}^\infty \alpha_{n-1}(k - k_1)^n} \\ &+ O\left(\left[\sqrt{t} \sum_{n=1}^\infty \alpha_{n-1}(k - k_1)^n\right]^{-2}\right), \quad t \to \infty, \end{aligned}$$

from which we infer estimate (A.9). $\square$

Returning to equation (A.8) we have

$$\begin{aligned} \|V^{\text{err}} - I\|_{L_k^p(\partial D_{k_1}^\varepsilon)} &\leqslant \|V^{\text{err}} - I\|_{L_k^\infty(\partial D_{k_1}^\varepsilon)} \left(\int_{k \in D_{k_1}^\varepsilon} |dk|\right)^{\frac{1}{p}} \\ &\leqslant (2\pi\varepsilon)^{\frac{1}{p}}\, O\big(t^{-\frac{1}{2}}\big). \end{aligned} \tag{A.18}$$

Combining inequalities (A.7) and (A.18) we thereby obtain the estimate

$$\|V^{\mathrm{err}} - I\|_{L_k^p(\bigcup_j \check{L}_j \cup \partial D_{k_1}^{\varepsilon})} \leqslant O\big(t^{-\frac{1}{2}}\big), \quad t \to \infty. \tag{A.19}$$

Returning to inequality (A.1), it remains to estimate $M^{\mathrm{err}-} - I$. This is done by using standard estimates that involve the Cauchy transform (see [18]). Eventually, we find

$$\|M^{\mathrm{err}-} - I\|_{L_k^2(\bigcup_j \check{L}_j \cup \partial D_{k_1}^{\varepsilon})} \leqslant C\|V^{\mathrm{err}} - I\|_{L_k^2(\bigcup_j \check{L}_j \cup \partial D_{k_1}^{\varepsilon})}, \quad C > 0. \tag{A.20}$$

Inequalities (A.1), (A.19), and (A.20) complete the proof of estimate (4.36).

Acknowledgment. The authors thank M. J. Ablowitz, P. Deift, G. El′, R. Jenkins, B. Prinari, and M. Ramachandran for useful discussions and comments. We also thank the anonymous referees for their constructive criticism on the manuscript. This work was partially supported by the National Science Foundation under grants DMS-1311847 and DMS-1614623. The second author also acknowledges an AMS-Simons Travel Grant.

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Received March 2016.