MTH 628 ALGEBRAIC TOPOLOGY

Homework 4 (due Thu. 2017.04.20)

1. Let X be a CW-complex and let $C_*^{CW}(X)$ denote the cellular chain complex of X. Show that $H_n(C_*^{CW}(X) \otimes G) \cong H_n(X; G)$ for any abelian group G and any $n \geq 0$.

2. If F is a field then homology groups $H_n(X; F)$ have structure of vector spaces over F, so it makes sense to talk about dimension of $H_n(X; F)$. Show that for any finite CW-complex X and any field F we have:

$$\chi(X) = \sum_n (-1)^n \dim H_n(X; F)$$

where $\chi(X)$ is the Euler characteristic of X.

3. a) Let X be a space such that for all $n > 0$ we have $H_n(X; G) = 0$ for $G = \mathbb{Q}$ and for $G = \mathbb{Z}/p\mathbb{Z}$ for any prime number p. Show that $H_n(X; \mathbb{Z}) = 0$ for all $n > 0$.

b) Let $f\colon X \to Y$ be a map of path connected spaces such that for all $n > 0$ the homomorphism $f_*\colon H_n(X; G) \to H_n(Y; G)$ is an isomorphism for $G = \mathbb{Q}$ and for $G = \mathbb{Z}/p\mathbb{Z}$ for any prime number p. Show that the homomorphism $f_*\colon H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$ is an isomorphism for all $n > 0$.

4. For an abelian group G let $T(G)$ denote the torsion subgroup of G, i.e. the subgroup consisting of all elements of finite order. For a space X let $T_n(X) = T(\widetilde{H}_n(X))$. This defines a function from the category of topological spaces to the category of abelian groups. Do the functors T_n define a generalized reduced homology theory? Justify your answer.