138

Cap product

Def: Let $\sigma : \Delta^n \to X$, $\varphi \in C^q(X;\mathbb{Z})$, $q \leq n$
The cap product $\sigma \cap \varphi \in C_{n-q}(X)$ is given by:

$$\sigma \cap \varphi = \varphi(\sigma \lambda_p) \cdot \sigma \rho_{n-p}$$

This defines a bilinear map

$$\cap : C_n(X) \times C^q(X;\mathbb{Z}) \longrightarrow C_{n-p}(X)$$

Note: The cup product induces a bilinear map:

$$\cap : H_n(X) \times H^q(X;\mathbb{Z}) \longrightarrow H_{n-q}(X)$$
$$([x], [\varphi]) \longmapsto [x \cap \varphi] =: [x] \cap [\varphi]$$

Proposition:
Let $H_*(X) = \bigoplus_n H_n(X)$. The multiplication
$$\cap : H_*(X) \times H^*(X,\mathbb{Z}) \to H_*(X)$$
defines a right $H^*(X;\mathbb{Z})$-module structure on $H_*(X)$
In particular for $[x] \in H_*(X)$, $[\varphi], [\psi] \in H^*(X;\mathbb{Z})$ we have:

$$[x] \cap ([\varphi] \cup [\psi]) = ([x] \cap [\varphi]) \cap [\psi]$$

Proposition:
If $f : X \to Y$ - map of spaces $[x] \in H_*(X)$, $[\varphi] \in H^*(Y,\mathbb{Z})$
then:

$$f_*([x]) \cap [\varphi] = f_*([x] \cap f^*([\varphi]))$$

Orientation on manifolds and Poincaré duality

Recall: If M - n-dimensional manifold, $n \geq 1$ then for any $x \in M$ we have:

$$H_q(M, M \setminus x) \cong \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases}$$

Note: For any $x \in M$ we have a homomorphism:

$$i_x : H_n(M) \longrightarrow H_n(M, M \setminus x)$$

Def: Let M - n-dim. compact, conn. manifold. A fundamental class of M is a homology class $[z] \in H^n(M)$ s.t. $i_x([z])$ is a generator of $H_n(M, M \setminus x)$ for each $x \in M$.

Note: If $[z] \in H^n(M)$ is a fundamental class of M then $-[z]$ is also a fundamental class.

Def: If M - compact, conn. mfld such a fundamental class of M exists then we say that M is orientable.

Proposition: Let M - n-dim. compact, connected mfld.

1) If M is orientable then $H_n(M) \cong \mathbb{Z}$ and fundamental classes of M are generators of $H_n(M)$
2) If M is not orientable then $H_n(M) = 0$

Exercise: Show that if M - n-dim compact mfld then $H_q(M) = 0$ for $q > n$

Example:

1) S^n is orientable $\forall n \geq 1$
2) If M, N - orientable then $M \times N$ - orientable.
   In particular $T = S^1 \times S^1$ - orientable.
3) Recall:

$$H_n(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{if } n\text{-odd} \\ 0 & \text{if } n\text{-even} \end{cases}$$

Thus $\mathbb{R}P^n$ is orientable iff n - odd.

Poincare Duality Theorem

Let M - n-dimensional compact, conn., orientable manifold, and let $[z] \in H_n(M)$ be a fundamental class of M. Then for any $q \geq 0$ the homomorphism

$$[z]\cap : H^q(M;\mathbb{Z}) \longrightarrow H_{n-q}(M)$$
$$[\varphi] \longmapsto [z]\cap[\varphi]$$

is an isomorphism.

Application:

Recall: If X - space, $H_q(X)$ - finitely generated $\forall q$, $H_q(X) = 0$ for all $q > n$ for some n then

$$\chi(X) = \Sigma (-1)^q \operatorname{rank}(H_q(X))$$

Fact: If M - compact manifold then $H_q(M)$ is finitely generated $\forall q$.

Proposition

If M - n-dim. compact, connected, orientable manifold and n - odd then $\chi(M) = 0$.

Proof: For $0 \leq q \leq n$ we have

$$\begin{aligned} \operatorname{rank}(H^q(M;\mathbb{Z})) &= \operatorname{rank}(\operatorname{Hom}(H_q(M), \mathbb{Z})) \\ &= \operatorname{rank}(H_q(M)) \end{aligned}$$

By Poincare' duality: $H^q(M;\mathbb{Z}) \cong H_{n-q}(M)$, so:

$$\operatorname{rank}(H_q(M)) = \operatorname{rank}(H_{n-q}(M))$$

If $n = 2k+1$ this gives:

$$\begin{aligned} \operatorname{rank}(H_0(M)) &= \operatorname{rank}(H_{2k+1}(M)) \\ \operatorname{rank}(H_1(M)) &= \operatorname{rank}(H_{2k}(M)) \\ &\vdots \\ \operatorname{rank}(H_k(M)) &= \operatorname{rank}(H_{k+1}(M)) \end{aligned}$$

and so

$$\begin{aligned} \chi(M) &= \sum_{q=0}^{2k+1} (-1)^q \operatorname{rank}(H_q(M)) \\ &= \sum_{q=0}^{k} (-1)^q (\operatorname{rank}(H_q(M) - \operatorname{rank}(H_{n-q}(M)) \\ &= 0 \end{aligned}$$

$\blacksquare$