Cup Product

From now on: R = commutative ring with 1.

Goal: Define multiplication of cohomology classes:

$$\cup : H^p(X;R) \times H^q(X;R) \longrightarrow H^{p+q}(X;R)$$

$$([\varphi], [\psi]) \longmapsto [\underbrace{\varphi \cup \psi}_{\uparrow}]$$

cup product of φ and ψ

Define: $H^*(X;R) = \bigoplus_{n=0}^{\infty} H^n(X;R)$.

Cup product gives a multiplication

$$H^*(X;R) \times H^*(X;R) \to H^*(X;R)$$

that makes $H^*(X;R)$ into a ring.

Any map $f: X \to Y$ induces a ring homomorphism:

$$f^*: H^*(Y;R) \to H^*(X;R)$$

This gives a better way of distinguishing spaces:

even if $H^n(X;R) \cong H^n(Y;R)$ $\forall_n$

iso of groups

it may happen that $H^*(X;R) \not\cong H^*(Y;R)$

non-isomorphic rings.

which implies that $X \not\simeq Y$.

Construction of cup product

Recall:

1) X, Y – spaces

The chain complex $C_*(X)\otimes C_*(Y)$ is given by:

$$(C_*(X)\otimes C_*(Y))_n = \bigoplus_{p+q=n} C_p(X)\otimes C_q(Y)$$

$$\partial : (C_*(X)\otimes C_*(Y))_n \longrightarrow (C_*(X)\otimes C_*(Y))_{n-1}$$

$$c\otimes d \longmapsto \partial c\otimes d + (-1)^{|c|} c\otimes d$$

This defines a cochain complex $\mathrm{Hom}(C_*(X)\otimes C_*(Y); R)$

2) There is a natural chain homotopy equivalence

$$\Phi_* : C_*(X\times Y) \xrightarrow{\simeq} C_*(X)\otimes C_*(Y)$$

$$\sigma \longmapsto \sum_{p+q=n} \underbrace{\sigma_X \lambda_p}_{\text{front face}} \otimes \underbrace{\sigma_Y \rho_q}_{\text{back face}}$$

where:

$$\sigma : \Delta^n \longrightarrow X\times Y$$

$$\sigma_X : \Delta^n \xrightarrow{\sigma} X\times Y \xrightarrow{pr_1} X$$

$$\sigma_Y : \Delta^n \longrightarrow X\times Y \xrightarrow{pr_2} Y$$

This gives a cochain homotopy equivalence:

$$\Phi^* : \mathrm{Hom}(C_*(X)\otimes C_*(Y); R) \to \mathrm{Hom}(C_*(X\times Y); R)$$

$$\mathrm{Hom}(C_*(X\times Y); R) = C^*(X\times Y; R)$$

Define:

$$\begin{array}{ccc} \gamma: C^p(X;R) \times C^q(Y;R) & \longrightarrow & \mathrm{Hom}(C_*(X)\otimes C_*(Y);R)_{p+q} \\ \| & & \| \\ \mathrm{Hom}(C_p(X);R) \times \mathrm{Hom}(C_q(X);R) & & \mathrm{Hom}(\bigoplus_{k+\ell=p+q} C_k(X)\otimes C_\ell(Y);R) \\ (\varphi,\psi) & \longmapsto & \varphi\otimes\psi \end{array}$$

where:

$$\varphi\otimes\psi\,(x\otimes y) = \begin{cases} \varphi(x)\cdot\psi(y) & \text{if } x\otimes y \in C_p(X)\otimes C_q(Y) \\ 0 & \text{otherwise} \end{cases}$$

Proposition:

γ defines a bilinear map:

$$\begin{array}{c} \gamma: H^p(X;R) \times H^q(Y;R) \longrightarrow H^{p+q}(C_*(X)\otimes C_*(Y);R) \\ \cong \big\downarrow \Phi^* \\ H^{p+q}(X\times Y;R) \end{array}$$

Proof: Exercise.

■

Theorem (Künneth Formula for cohomology)

Let X, Y - spaces of finite type and let R - PID.

There exists a split short exact sequence:

$$0 \to \bigoplus_{p+q=n} H^p(X;R)\otimes_R H^q(Y;R) \xrightarrow{\Phi^*\gamma} H^n(X\times Y;R) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H^p(X;R), H^q(Y;R)) \to 0$$

Note: For any space X we get a bilinear map

$$H^p(X;R)\times H^q(X;R) \xrightarrow{\Phi^*\gamma} H^{p+q}(X\times X;R)$$

Let $\Delta: X \longrightarrow X\times X$ - the diagonal map. It induces
$x \longmapsto (x,x)$

$$\Delta^*: H^n(X\times X;R) \longrightarrow H^n(X;R)$$

Def.: If $[\varphi]\in H^p(X;R)$, $[\psi]\in H^q(X;R)$ then

$$\underbrace{[\varphi]\cup[\psi]}_{\uparrow\text{cup product of } [\varphi],[\psi]} = \Delta^*\Phi^*\gamma([\varphi],[\psi]) \in H^{p+q}(X;R)$$

Note: Cup product can be described explicitly as follows.
Take $\varphi\in C^p(X;R)$, $\psi\in C^q(X;R)$
Then $\varphi\cup\psi = \Delta^*\Phi^*\gamma(\varphi,\psi) \in C^{p+q}(X;R) = \mathrm{Hom}(C_{p+q}(X);R)$
is the cochain given by

$$\varphi\cup\psi(\sigma) = \varphi(\sigma\lambda_p)\cdot\psi(\sigma\rho_q)$$

for $\sigma: \Delta^{p+q}\longrightarrow X$.
If φ,ψ - cocycles then

$$[\varphi]\cup[\psi] = [\varphi\cup\psi]$$

Properties of cup product

Proposition 1

Cup product is associative: if $\varphi \in C^p(X;R)$
$\psi \in C^q(X;R)$
$\zeta \in C^r(X;R)$

then: $(\varphi \cup \psi) \cup \zeta = \varphi \cup (\psi \cup \zeta) \in C^{p+q+r}(X;R)$

Proof: For $\sigma : \Delta^{p+q+r} \to X$ we have:

$$\begin{aligned}((\varphi \cup \psi) \cup \zeta)(\sigma) &= (\varphi \cup \psi)(\sigma \lambda_{p+q}) \cdot \zeta(\sigma \rho_r) \\ &= \varphi(\sigma \underbrace{\lambda_{p+q} \lambda_p}_{= \lambda_p}) \cdot \psi(\sigma \lambda_{p+q} \rho_q) \cdot \zeta(\sigma \rho_r)\end{aligned}$$

$$\begin{aligned}(\varphi \cup (\psi \cup \zeta))(\sigma) &= \varphi(\sigma \lambda_p) \cdot (\psi \cup \zeta)(\sigma \rho_{q+r}) \\ &= \varphi(\sigma \lambda_p) \cdot \psi(\sigma \underbrace{\rho_{q+r} \lambda_q}_{= \lambda_{p+q} \rho_q}) \cdot \zeta(\sigma \underbrace{\rho_{q+r} \rho_r}_{= \rho_r})\end{aligned}$$

∎

Proposition 2

Let $\varepsilon \in C^0(X;R) = \mathrm{Hom}(C_0(X);R)$
be the cochain given by $\varepsilon(\sigma) = 1 \ \forall \sigma \in C_0(X)$.
Then for any $\varphi \in C^p(X;R)$ we have:

$$\varphi \cup \varepsilon = \varepsilon \cup \varphi = \varphi$$

<u>Proof:</u> For $\sigma : \Delta^p \to X$ we have:

$$\varphi \cup \varepsilon(\sigma) = \varphi(\underbrace{\sigma\lambda_p}_{=\sigma}) \cdot \underbrace{\varepsilon(\sigma\rho_0)}_{=1} = \varphi(\sigma)$$

Similarly $\varepsilon \cup \varphi(\sigma) = \varphi(\sigma)$

$\blacksquare$

<u>Note:</u>

1) ε is a cocycle, so it represents a cohomology class $[\varepsilon] \in H^0(X;R)$. By Proposition 2 we have:
$$[\varphi] \cup [\varepsilon] = [\varepsilon] \cup [\varphi] = [\varphi]$$
for any $[\varphi] \in H^p(X;R)$

2) Recall: if X - path connected then we have an isomorphism $H^0(X;R) \cong R$. Under this isomorphism $[\varepsilon]$ corresponds to $1 \in R$.
If X is not path connected then
$$H^0(X;R) \cong \prod_{i \in I} H^0(X_i;R) \cong \prod_{i \in I} R$$
where $\{X_i\}_{i \in I}$ - the set of path conn. components of X. Under this isomorphism $[\varepsilon]$ corresponds the element $(1)_{i \in I} \in \prod_{i \in I} R$

1 on each coordinate

<u>Proposition 3:</u>
If $f : X \to Y$ - map of spaces and $f^* : C^*(Y;R) \to C^*(X;R)$ - the induced map of cochain complexes then for $\varphi \in C^p(Y;R)$, $\psi \in C^q(Y;R)$ we have:
$$f^*(\varphi \cup \psi) = (f^*\varphi) \cup (f^*\psi)$$

Proof: Exercise.

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Note: Let $H^*(X;R) = \bigoplus_n H^n(X;R)$. The cup product defines a multiplication:

$$\cup : H^*(X;R) \times H^*(X;R) \to H^*(X;R)$$

By Proposition 1 and 2 $H^*(X;R)$ taken with this multiplication is a ring with the multiplicative identity given by $[\varepsilon] \in H^0(X;R)$.

By Proposition 3 a map of spaces $f: X \to Y$ induces a homomorphism of rings

$$f^*: H^*(Y;R) \to H^*(X;R)$$

Example: $H^*(S^n;\mathbb{Z}) \cong \underset{\langle[\varepsilon]\rangle}{\underset{\shortparallel}{H^0(S^n;\mathbb{Z})}} \oplus \underset{\langle[\alpha]\rangle}{\underset{\shortparallel}{H^n(S^n;\mathbb{Z})}} = \{n[\varepsilon] + m[\alpha] \mid m, n \in \mathbb{Z}\}$

generator of $H^n(S^n,\mathbb{Z})$

We have: $[\varepsilon] \cup [\varepsilon] = [\varepsilon]$

$[\varepsilon] \cup [\alpha] = [\alpha] \cup [\varepsilon] = [\alpha]$

$[\alpha] \cup [\alpha] \in H^{2n}(S^n;\mathbb{Z}) = 0$

so: $[\alpha] \cup [\alpha] = 0$

Thus:

$$(n_1[\varepsilon] + m_1[\alpha]) \cup (n_2[\varepsilon] + m_2[\alpha])$$
$$= n_1 \cdot n_2[\varepsilon] + (n_1 m_2 + m_1 n_2)[\alpha]$$

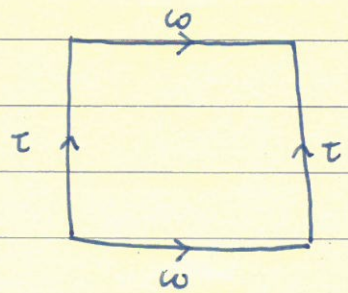
We get: $H^*(S^n;\mathbb{Z}) \cong \mathbb{Z}[t]/(t^2)$

where $\deg(t) = n$

Example: Cohomology ring of $T = S^1 \times S^1$

Recall:

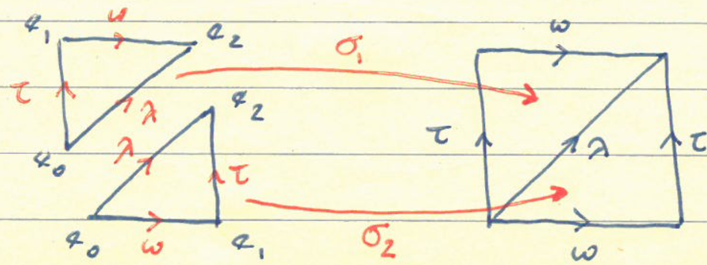
$$H_q(T) = \begin{cases} 0 & q > 2 \\ \mathbb{Z} & q = 2 \\ \mathbb{Z}\oplus\mathbb{Z} & q = 1 \\ \mathbb{Z} & q = 0 \end{cases}$$



Note: We can consider ω and τ as singular 1-simplices in T that represent homology classes $[\omega], [\tau] \in H_1(T)$. By Hurewicz Theorem the group $H_1(T)$ is freely generated by $[\omega]$ and $[\tau]$.

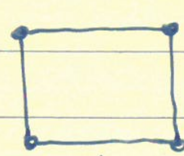
(*) **Lemma**

Let $\sigma_1, \sigma_2 \colon \Delta^2 \to T$ be singular simplices defined as follows:



Then $\gamma = \sigma_1 - \sigma_2$ is a cycle in $C_2(T)$ such that $[\gamma]$ generates $H_2(T)$.

Proof: Let $X = [0,1] \times [0,1]$ with the CW-complex structure given by vertices, edges and the interior:



Let $q \colon X \to T$ be the quotient map.

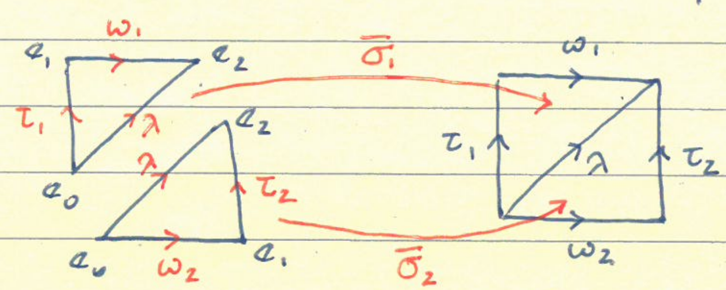
Consider the diagram:

$$\begin{array}{ccccc} & & H_2(X, X^{(1)}) & \xrightarrow[\cong]{\delta} & H_1(X^{(1)}) \\ & & \cong \big\downarrow q_* & & \\ H_2(T) & \xrightarrow[j_*]{\cong} & H_2(T, T^{(1)}) & & \\ [\gamma] & \longmapsto & j_*[\gamma] & & \end{array}$$

All maps in this diagram are isomorphisms: For horizontal maps this follows from long exact sequences of pairs. For the vertical map it holds by commutativity of the square

$$\begin{array}{ccc} H_2(X, X^{(1)}) & \xrightarrow{\cong} & \tilde{H}_2(X/X^{(1)}) \\ \big\downarrow q_* & & \big\downarrow \cong \\ H_2(T, T^{(1)}) & \xrightarrow[\cong]{} & H_2(T/T^{(1)}) \end{array}$$

Let $\bar{\sigma}_1, \bar{\sigma}_2 : \Delta^2 \to X$ be simplices given by:



Let $\bar{\gamma} = \bar{\sigma}_1 - \bar{\sigma}_2$. Since $q_*[\bar{\gamma}] = j_*[\gamma]$ it suffices to show that $[\bar{\gamma}]$ is a generator of $H_2(X, X^{(1)})$. This in turn is equivalent to showing that $\delta[\bar{\gamma}]$ is a generator of $H_1(X^{(1)})$.

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We have:

$$\delta[\bar{\gamma}] = [\partial\bar{\gamma}] = [\partial\bar{\sigma}_1 - \partial\sigma_2]$$
$$= [(\omega_1 - \lambda + \tau_1) - (\tau_2 - \lambda + \omega_2)]$$
$$= [\omega_1 - \tau_2 - \omega_2 + \tau_1]$$
$$= [\underbrace{\omega_1 * \bar{\tau}_2 * \bar{\omega}_2 * \tau_1}_{\text{concatenation of paths}}]$$

and by Hurewicz Theorem $[\omega_1 * \bar{\tau}_2 * \bar{\omega}_2 * \tau_1]$ generates $H_1(X^{(1)})$.

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Recall:

1) For any space X we have a homomorphism:

$$\beta : H^n(X;\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_n(X);\mathbb{Z})$$
$$[\varphi] \longmapsto \bar{\varphi} : H_n(X) \to \mathbb{Z}$$
$$[x] \mapsto \varphi(x)$$

2) Since $H_n(T)$ - free abelian $\forall_n$ by the Universal Coefficient Theorem for cohomology the homomorphism

$$\beta : H^n(T;\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_n(T);\mathbb{Z})$$

is an isomorphism $\forall_n$.

Note: 1) Since $H_1(T)$ is free abelian on generators $[\omega], [\tau]$ it follows that $\mathrm{Hom}(H_1(T);\mathbb{Z})$ is free abelian on generators $\omega^*, \tau^* : H_1(T) \to \mathbb{Z}$ where:

$\omega^*([\omega]) = 1$, $\omega^*([\tau]) = 0$

$\tau^*([\omega]) = 0$, $\tau^*([\omega]) = 1$

2) Since $\beta: H^1(T, \mathbb{Z}) \to \mathrm{Hom}(H_1(T); \mathbb{Z})$ is an isomorphism we get that $H^1(T, \mathbb{Z})$ is freely generated by elements $[\varphi_\omega], [\varphi_\tau]$ s.t. $\beta([\varphi_\omega]) = \omega^*$, $\beta([\varphi_\tau]) = \tau^*$

Notice that by definition of β we get that $\varphi_\omega, \varphi_\tau \in C^2(T; \mathbb{Z})$ are cocycles s.t.

$$\varphi_\omega(\omega) = 1 \quad \varphi_\omega(\tau) = 0$$
$$\varphi_\tau(\omega) = 0 \quad \varphi_\tau(\tau) = 1$$

We will show that $[\varphi_\tau] \cup [\varphi_\omega]$ generates $H^2(T; \mathbb{Z})$.

We have an isomorphism

$$\beta: H^2(T; \mathbb{Z}) \longrightarrow \mathrm{Hom}(H_2(T); \mathbb{Z})$$

Let $\gamma = \sigma_1 - \sigma_2$ - defined as in Lemma (*).

Since $[\gamma]$ generates $H_2(T)$, thus $\mathrm{Hom}(H_2(T); \mathbb{Z})$ is generated by the homomorphism $\gamma^*: H_2(T) \to \mathbb{Z}$ s.t. $\gamma^*(\gamma) = 1$. It will be enough to show that $\beta([\varphi_\tau] \cup [\varphi_\omega]) = \gamma^*$. By the definition of β this amounts to showing that $(\varphi_\tau \cup \varphi_\tau)(\gamma) = 1$.

<u>We have:</u>

$$\begin{aligned}(\varphi_\tau \cup \varphi_\omega)(\gamma) &= \varphi_\tau(\sigma_1\lambda_1) \cdot \varphi_\omega(\sigma_1\rho_1) - \varphi_\tau(\sigma_2\lambda_1) \cdot \varphi_\omega(\sigma_2\rho_1) \\ &= \varphi_\tau(\tau) \cdot \varphi_\omega(\omega) - \varphi_\tau(\omega) \cdot \varphi_\omega(\tau) \\ &= 1 - 0 = 1\end{aligned}$$

<u>Note:</u> 1) $(\varphi_\omega \cup \varphi_\omega)(\gamma) = \varphi_\omega(\tau) \cdot \varphi_\omega(\omega) - \varphi_\omega(\omega) \cdot \varphi_\omega(\tau) = 0$

<u>So:</u> $[\varphi_\omega] \cup [\varphi_\omega] = 0$. Similarly: $[\varphi_\tau] \cup [\varphi_\tau] = 0$.

2) $(\varphi_\omega \cup \varphi_\tau)(\gamma) = \varphi_\omega(\tau) \cdot \varphi_\tau(\omega) - \varphi_\omega(\omega) \cdot \varphi_\tau(\tau) = -1$

This gives: $[\varphi_\omega] \cup [\varphi_\tau] = -[\varphi_\tau] \cup [\varphi_\omega]$.

<u>Upshot</u>: Let $[\varphi_\gamma]$ - the generator of $H^2(T;\mathbb{Z})$ s.t. $\beta([\varphi_\gamma]) = \gamma^*$. Then:

$$H^*(T,\mathbb{Z}) = \langle \underbrace{[\varepsilon]}_{\deg 0}, \underbrace{[\varphi_\omega], [\varphi_\tau]}_{\deg 1}, \underbrace{[\varphi_\gamma]}_{\deg 2} \rangle$$

$[\varepsilon]$ - multiplicative unit

$[\varphi_\gamma] = [\varphi_\tau] \cup [\varphi_\omega] = -[\varphi_\omega] \cup [\varphi_\tau]$

$[\varphi_\omega]^2 = [\varphi_\tau]^2 = 0$

<u>Application</u>

Take $X = S^1 \vee S^1 \vee S^2$

Notice that

$$H_n(X) = \begin{cases} 0 & n > 2 \\ \mathbb{Z} & n = 2 \\ \mathbb{Z}\oplus\mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0 \end{cases}$$

<u>so</u>: $H_n(X) \cong H_n(T)\ \forall n$

This also gives: $H^n(X;\mathbb{Z}) \cong H^n(T,\mathbb{Z})\ \forall n$.

We will show, however that the cohomology ring $H^*(X;\mathbb{Z})$ is not isomorphic to $H^*(T;\mathbb{Z})$. Since homotopy equivalence induces an isomorphism of cohomology rings it implies that $X \not\simeq T$.

It will suffice to show that for any $[x],[y] \in H^1(X;\mathbb{Z})$ we have $[x]\cup[y] = 0$.

Let $q\colon X \to X/S^2 \cong S^1\vee S^1$ be the quotient map.

Notice that the map $q^*: H^1(S^1\vee S^1;\mathbb{Z}) \to H^1(X;\mathbb{Z})$ is an isomorphism so $\exists\, [x'],[y'] \in H^1(S^1\vee S^1,\mathbb{Z})$ s.t. $q^*[x'] = [x]$, $q^*[y'] = [y]$. This gives:

$$[x]\cup[y] = q^*[x'] \cup q^*[y'] = q^*([x']\cup[y'])$$

Since $[x']\cup[y'] \in H^2(S^1\vee S^1;\mathbb{Z}) = 0$ we get that $[x]\cup[y] = 0$.

Note: For $[\varphi_\omega], [\varphi_\tau] \in H^1(T, \mathbb{Z})$ we had:

$$[\varphi_\omega] \cup [\varphi_\tau] = -[\varphi_\tau] \cup [\varphi_\omega]$$

In general we have:

Theorem: If $[\varphi] \in H^p(X;R)$, $[\psi] \in H^q(X;R)$ then:

$$[\varphi] \cup [\psi] = (-1)^{pq} [\psi] \cup [\varphi]$$

Application of cup product:

The Hopf invariant and homotopy groups of spheres.

Recall:

$$\pi_m(X) = \left(\begin{array}{l} \text{homotopy classes} \\ \text{of basepoint pres.} \\ \text{maps } S^n \longrightarrow X \end{array} \right)$$

Goal:

(*) Theorem: If $n > 0$ then $\pi_{4n-1}(S^{2n}) = 0$.

Let $f: S^{4n-1} \longrightarrow S^{2n}$ and let X_f be the CW-complex obtained by attaching a cell to S^{2n} along the map f:

$$X_f = S^{2n} \cup_f e^{4n}$$

We have:

$$H^q(X_f; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & q = 4n \\ \mathbb{Z} & q = 2n \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Let u - generator of $H^{2n}(X_f, \mathbb{Z})$
v - generator of $H^{4n}(X_f, \mathbb{Z})$.

We have: $u \cup u \in H^{4n}(X_f, \mathbb{Z})$, so $u \cup u = h(f) v$ for some $h(f) \in \mathbb{Z}$.

Def: The integer $h(f)$ is called the Hopf invariant of f.

Note:

1) $h(f)$ depends only on the homotopy class of f. Indeed, if $f \simeq g$ then there is a homotopy equivalence $X_f \simeq X_g$ which induces an isomorphism of cohomology rings.

2) If $f \simeq *$ then $h(f) = 0$. Indeed in such case $X_f \simeq S^{2n} \vee S^{4n}$ which implies that $u \cup u = 0$ (check!)

Theorem (*) will follow immediately from:

Proposition:

For any $n > 1$ there exists $f: S^{4n-1} \longrightarrow S^{2n}$ such that $h(f) \neq 0$.

Recall: We had: there are generators $u_1, u_2 \in H^1(S^1 \times S^1; \mathbb{Z})$ such that $u_1 \cup u_2$ generates $H^2(S^1 \times S^1; \mathbb{Z})$ and $u_1^2 = u_2^2 = 0$

We will use:

Lemma: If $n \geq 1$ then

$$H^q(S^n \times S^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & q = 2n \\ \mathbb{Z}\oplus\mathbb{Z} & q = n \\ \mathbb{Z} & q = 0 \\ 0 & \text{otherwise} \end{cases}$$

and there are generators u_1, u_2 of $H^n(S^n \times S^n; \mathbb{Z})$ such that $u_1^2 = u_2^2 = 0$, and $u_1 \cup u_2$ generates $H^{2n}(S^n \times S^n; \mathbb{Z})$

Proof of Proposition

Let $x_0 \in S^{2n}$. Take

$$X = S^{2n} \times S^{2n} \Big/ (x, x_0) \sim (x, x_0) \quad \forall x \in S^n$$

Note: X has a structure of a CW-complex with single cells in dimensions 0, $2n$ and $4n$.

We get: $X^{(2n)} \cong S^{2n}$. Let $f: S^{4n-1} \longrightarrow S^{2n}$ be the attaching map of the $4n$-cell. Then $X = X_f$.

We will show that $h(f) \neq 0$.

Let $q: S^{2n} \times S^{2n} \longrightarrow X$ be the quotient map

Notice that

$$H_k(S^{2n} \times S^{2n}) \cong \begin{cases} \mathbb{Z} & k = 0, 4n \\ \mathbb{Z}\oplus\mathbb{Z} & k = 2n \\ 0 & \text{otherwise} \end{cases} \qquad H_k(X) \cong \begin{cases} \mathbb{Z} & k = 0, 4n \\ \mathbb{Z} & k = 2n \\ \mathbb{Z} & k = 0 \end{cases}$$

Moreover, we can choose generators

$$u_1, u_2 \in H_{2n}(S^{2n}\times S^{2n}), \quad v \in H_{4n}(S^{2n}\times S^{2n})$$
$$\bar{u} \in H_{2n}(X), \quad \bar{v} \in H_{4n}(X)$$

such that

$$q_*(u_1) = q_*(u_2) = \bar{u}$$
$$q_*(v) = \bar{v}$$

Since $H_k(S^{2n}\times S^{2n})$, $H_k(X)$ - free abelian we have isomorphisms:

$$\beta: H^k(S^{2n}\times S^{2n}; \mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(H_k(S^{2n}\times S^{2n}); \mathbb{Z})$$
$$\beta: H^k(X;\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(H_k(X);\mathbb{Z})$$

Let $u_1^*, u_2^* \in H^{2n}(S^{2n}\times S^{2n};\mathbb{Z})$ be elements such that $\beta u_i^*(u_j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$. Then u_1^*, u_2^* generate $H^{2n}(S^{2n}\times S^{2n};\mathbb{Z})$

Similarly we define $v^*, \bar{u}^*, \bar{v}^*$.
Using commutativity of the diagram

$$\begin{tikzcd}
H^k(X;\mathbb{Z}) \arrow[r, "q^*"] \arrow[d, "\beta"', "\cong"] & H^k(S^{2n}\times S^{2n};\mathbb{Z}) \arrow[d, "\beta", "\cong"'] \\
\mathrm{Hom}(H_k(X);\mathbb{Z}) \arrow[r] & \mathrm{Hom}(H_k(S^{2n}\times S^{2n}); \mathbb{Z}) \\
\varphi \arrow[r, mapsto] & \varphi q_*
\end{tikzcd}$$

<u>we obtain:</u>

$$q^*(\bar{v}^*) = v^*$$
$$q^*(\bar{u}^*) = u_1^* + u_2^*$$

This gives:

$$\begin{aligned} q^*(\bar{u}^* \cup \bar{u}^*) &= q^*(\bar{u}^*) \cup q^*(\bar{u}^*) \\ &= (u_1^* + u_2^*) \cup (u_1^* + u_2^*) \\ &= (u_1^*)^2 + (u_2^*)^2 + 2\, u_1^* \cup u_2^* \\ &= 2 u_1^* \cup u_2^* \\ &= (\pm 2) \cdot v^* \end{aligned}$$

This gives $\bar{u}^* \cup \bar{u}^* = (\pm 2) \cdot \bar{v}^*$

Therefore $h(f) = \pm 2$

$\blacksquare$

Note:

1) One can show that the assignment

$$h : \pi_{4n-1}(S^{2n}) \longrightarrow \mathbb{Z}$$
$$[f] \longmapsto h(f)$$

is a homomorphism of groups. By the last Proposition this homomorphism is non-trivial. This implies that $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus \mathrm{Ker}(h)$. It is also known that $\mathrm{Ker}(h)$ is always a finite group. We have:

$n=1$	$\pi_3(S^2) \cong \mathbb{Z}$
$n=2$	$\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12$
$n=3$	$\pi_{11}(S^6) \cong \mathbb{Z}$
$n=4$	$\pi_{15}(S^8) \cong \mathbb{Z} \oplus \mathbb{Z}/120$

2) In general we have

$$\pi_m(S^n) = \begin{cases} 0 & m < n \\ \mathbb{Z} & n = m \\ \mathbb{Z} \oplus (\text{finite}) & n = 2k,\ m = 4k-1 \\ \text{finite} & \text{otherwise.} \end{cases}$$