Cohomology

Def: A cochain complex of is a sequence of abelian group and group homomorphisms

$$C^* = (\dots \to C^{q-1} \xrightarrow{\delta^{q-1}} C^q \xrightarrow{\delta^q} C^{q+1} \xrightarrow{\delta^{q+1}} \dots)$$

such that $\delta^{q+1}\delta^q = 0 \ \forall q$.

We define:

$Z^q(C^*) = \mathrm{Ker}(\delta^q : C^q \to C^{q+1})$ - q-cocyles of C^*

$B^q(C^*) = \mathrm{Im}(\delta^{q-1} : C^{q-1} \to C^q)$ - q-coboundaries of C^*

$H^q(C^*) = Z^q(C^*)/B^q(C^*)$ - q-cohomology of C^*

Note: For a cochain complex C^* define: $C_q := C^{-q}$, $\partial_q = \delta^{-q}$.
This gives a chain complex

$$C_* := (\dots \to C_{-q+1} \xrightarrow{\partial_{-q+1}} C_{-q} \xrightarrow{\partial_{-q}} C_{-q-1} \to \dots)$$

s.t. $H_{-q}(C_*) = H^q(C_*)$. Using this correspondence we can translate facts about chain complexes into facts about cochain complexes.

Note: If A, B - abelian groups,

$\mathrm{Hom}(A,B)$ = the set of all homomorphisms $A \to B$

then $\mathrm{Hom}(A,B)$ has a structure of an abelian group:

if $\varphi, \psi : A \to B$ then

$$\varphi + \psi : A \to B$$
$$a \mapsto \varphi(a) + \psi(b)$$

A group homomorphism $f: A \to A'$ induces a homomorphism

$$f^* : \mathrm{Hom}(A', B) \to \mathrm{Hom}(A, B)$$
$$\varphi \longmapsto \varphi \circ f$$

Upshot:

1) For C_* - chain complex, G - abelian group we can define a cochain complex:

$$\mathrm{Hom}(C_*;G) = (\dots \to \mathrm{Hom}(C_q;G) \xrightarrow{\delta^q} \mathrm{Hom}(C_{q+1};G) \to \dots)$$

$$\varphi \longmapsto \varphi \circ \partial_{q+1}$$

2) A chain map $f_*: C_* \to D_*$ defines a map of cochain complexes:

$$f^*: \mathrm{Hom}(D_*;G) \to \mathrm{Hom}(C_*;G)$$

$$\varphi \longmapsto \varphi \circ f$$

Def: The singular cochain complex of a space X with coefficients in a group G is the cochain complex

$$C^*(X;G) := \mathrm{Hom}(C_*(X);G)$$

The q-th singular cohomology group of X with coefficients in G is the group

$$H^q(X;G) := H^q(C^*(X;G))$$

Note: 1) The assignment $X \mapsto C^*(X;G)$ defines a contravariant functor

$$C^*(-,G): \mathrm{Top} \to \underbrace{\text{co-}\mathrm{Ch}(\mathbb{Z})}$$

the category of cochain complexes of abelian groups

2) The assignment $X \mapsto H^q(X;G)$ defines a contravariant functor:

$$H^q(-;G): \mathrm{Top} \to \mathrm{Ab}$$

Note: We will denote $H^q(X) := H^q(X;\mathbb{Z})$

Example:

Recall: chain complex of a point:

$$C_\bullet(*) \cong (\dots \xrightarrow{id} \underset{3}{\mathbb{Z}} \xrightarrow{0} \underset{2}{\mathbb{Z}} \xrightarrow{id} \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \to 0)$$

This gives:

$$C^\bullet(*;G) \cong (\mathrm{Hom}(0,G) \to \mathrm{Hom}(\mathbb{Z},G) \to \mathrm{Hom}(\mathbb{Z},G) \to \mathrm{Hom}(\mathbb{Z},G) \to \dots)$$

$$\begin{array}{ccccccccc} \cong & & \cong & & \cong & & \cong & & \\ 0 & \xrightarrow{0} & G & \xrightarrow{0} & G & \xrightarrow{id} & G & \xrightarrow{0} & \dots \end{array}$$

So: $H^0(*;G) \cong G$

$H^q(*;G) \cong 0$ for $q > 0$

Proposition

If X- path connected then $H^0(X,G) \cong G$

Proof:

We have:

$$0 \to C^0(X,G) \xrightarrow{\delta} C^1(X,G)$$

$$H^0(X;G) = Z^0(X;G) = \mathrm{Ker}(\delta)$$

Note:

1) We have a bijection:

$$\left(\begin{array}{c} \text{elements of} \\ C^0(X;G) = \mathrm{Hom}(C_0(X);G) \end{array}\right) \cong \left(\begin{array}{c} \text{maps of sets} \\ \varphi: X \longrightarrow G \\ x \longmapsto gx \end{array}\right)$$

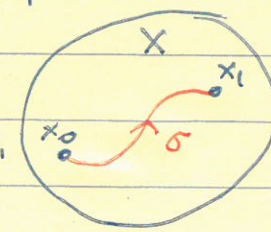
2) $\varphi \in Z^0(X,G)$ if $\varphi(\partial\sigma) = 0$ for any 1-simplex σ in X

$\varphi(\partial\sigma) = \varphi(\sigma(e_1)) - \varphi(\sigma(e_0))$ so this gives:

$\varphi(\sigma(e_1)) = \varphi(\sigma(e_0)) \quad \forall\sigma$

Note: Since X- path connected for any $x_0, x_1 \in X$ there is $\sigma : \Delta^1 \to X$ jointing x_0 and x_1.

This gives:



$$\varphi \in Z^0(X;G) \text{ iff } \varphi(x_0) = \varphi(x_1) \ \forall_{x_0, x_1}$$

We obtain:

$$\begin{aligned} H^0(X;G) &= Z^0(X;G) \\ &= (\text{constant functions } X \to G) \\ &\cong G \end{aligned}$$

▨

Proposition: If $\{X_i\}_{i\in I}$ is a set of path conn. components of X then

$$H^n(X;G) \cong \prod_{i\in I} H^n(X_i;G)$$

In particular $H^0(X;G) \cong \prod_{i\in I} G$.

Proof: 1) $C^*(X;G) \cong \prod_{i\in I} C^*(X_i, G)$ ← product of cochain complexes of X_i

2) Cohomology commutes with products:

$$H^n\left(\prod_{i\in I} C^*_{(i)}\right) \cong \prod_{i\in I} H^n(C^*_{(i)})$$

Further properties of singular cohomology

Proposition

If $f, g : X \to Y$, $f \simeq g$ then $f^* = g^* : H^q(Y;G) \to H^q(X;G)$ $\forall q$.

Proof: Take chain maps $f_*, g_* : C_*(X) \to C_*(Y)$. Since $f \simeq g$ there is a chain homotopy Φ_* between f_* and g_*. This defines a cochain homotopy

$$\begin{array}{ccc} \Phi^* : C^{*+1}(Y;G) & \longrightarrow & C^*(X;G) \\ \| & & \| \\ \mathrm{Hom}(C_{*+1}(Y), G) & & \mathrm{Hom}(C_*(X);G) \\ \varphi & \longmapsto & \varphi \circ \Phi \end{array}$$

between f^* and g^*.

$\blacksquare$

For $A \subseteq X$ define:

$$C^*(X,A;G) = \mathrm{Hom}(C_*(X,A);G)$$
$$H^q(X,A;G) = H^q(C^*(X,A);G)$$

(*) <u>Proposition</u>:

Let $A \subseteq X$ and let $i: A \hookrightarrow X$ be the inclusion. There is a long exact sequence of cohomology groups:

$$\ldots \to H_q(X,A;G) \to H_q(X) \xrightarrow{i^*} H_q(A) \xrightarrow{\delta} H_{q+1}(X,A;G) \to \ldots$$

<u>Lemma:</u>

Let

(**) $$0 \to C \xrightarrow{f} C' \xrightarrow{g} C'' \to 0$$

be a short exact sequence of abelian groups.

1) For any group G the sequence
$$0 \to \mathrm{Hom}(C'',G) \xrightarrow{g^*} \mathrm{Hom}(C',G) \xrightarrow{f^*} \mathrm{Hom}(C;G)$$
is exact

2) If the sequence (**) splits (e.g. if C'' is a free abelian group) then the sequence
$$0 \to \mathrm{Hom}(C'',G) \xrightarrow{g^*} \mathrm{Hom}(C',G) \to \mathrm{Hom}(C;G) \to 0$$
is exact.

<u>Proof</u>: Exercise. $\blacksquare$

<u>Note:</u> 2) is not true in general for non-split sequences

E.g.: for $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ and $G = \mathbb{Z}$ we get:

$$\varphi \longmapsto \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}$$

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) & \to & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) & \to & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \\ & \cong & & \cong & & \cong \\ & 0 & & \mathbb{Z} & \xrightarrow{\quad \cdot p \quad} & \mathbb{Z} \end{array}$$

↑ not onto if $p > 1$

<u>Proof of Proposition (*)</u>

For each q we have a split short exact sequence:

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \to C_n(X,A) \to 0$$

By Lemma this gives a short exact sequence of cochain complexes:

$$0 \to C^*(X,A;G) \to C^*(X;G) \xrightarrow{i^*} C^*(A,G) \to 0$$

which gives the long exact sequence of cohomology groups.

∎

<u>Excision Theorem for Cohomology</u>

If X - space, $Z \subseteq A \subseteq X$ - subspaces s.t. $\overline{Z} \subseteq \mathrm{int}(A)$ then the inclusion map $i: (X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism

$$i^*: H^n(X,A;G) \xrightarrow{\cong} H^n(X \setminus Z, A \setminus Z; G)$$

for any $n \geq 0$.

<u>Proof:</u> Similar as for excision for homology. ∎

<u>Thm</u> (Mayer-Vietoris sequence for cohomology)
Let $A, B \subseteq X$ s.t. $X = \mathrm{int}(A) \cup \mathrm{int}(B)$ and let

$$\begin{tikzcd} & A \arrow[dr, hook, "j_A"] & \\ A \cap B \arrow[ur, hook, "i_A"] \arrow[dr, hook, "i_B"'] & & X \\ & B \arrow[ur, hook, "j_B"'] & \end{tikzcd}$$

be the inclusion maps. The following sequence is exact:

$$\ldots \to H^n(X,G) \xrightarrow{(j_A^*, j_B^*)} H^n(A,G) \oplus H^n(B,G) \xrightarrow{i_A^* - i_B^*} H^n(A\cap B, G) \xrightarrow{\delta} H^{n+1}(X,G) \to \ldots$$

$$x \longmapsto (j_A^*(x), j_B^*(x))$$

$$(y, z) \longmapsto i_A^*(y) - i_B^*(z)$$

<u>Proof:</u> Similar as for homology. $\blacksquare$

<u>Def:</u> Let $c : X \to *$ be the constant map.
The <u>n^{th} reduced cohomology group</u> of X with coefficients in G is the group

$$\tilde{H}^n(X;G) := \mathrm{Coker}(c^* : H^n(*;G) \to H^n(X;G))$$

<u>Note:</u> Recall that $H^n(*,G) \cong \begin{cases} 0 & n>0 \\ G & n=0 \end{cases}$ this gives

$$\tilde{H}^n(X;G) \cong H^n(X;G) \quad \text{for } n>0$$

We also have:

$$H^0(X;G) \cong \bigoplus^N G$$

where N = the number of path conn. components of X (exercise). This gives:

$$\tilde{H}^0(X;G) \cong \bigoplus^{N-1} G.$$

(114)

<u>Thm:</u> (Cofibration sequence for cohomology)
Let $i: A \hookrightarrow X$ - cofibration, $q: X \to X/A$ - the quotient map. The following sequence is exact:

$$\cdots \to \tilde{H}^n(X/A; G) \xrightarrow{q^*} \tilde{H}^n(X; G) \xrightarrow{i^*} \tilde{H}^n(A; G) \xrightarrow{\delta} \tilde{H}^{n+1}(X/A; G) \to \cdots$$

<u>Proof:</u> Similar as for homology.

▨

115

Homology vs Cohomology groups

Def: Let $0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} G \to 0$ be a presentation of an abelian group G. For an abelian group H we define:

$$\mathrm{Ext}(G,H) = \mathrm{Coker}\left(\mathrm{Hom}(F_0,H) \xrightarrow{d_1^*} \mathrm{Hom}(F_1,H)\right)$$

Proposition:
$\mathrm{Ext}(G,H)$ depends only on the groups G and H, and not on the choice of presentation of G.

Proof: Exercise. ▨

Example: $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ - presentation of $\mathbb{Z}/p\mathbb{Z}$. We have:

$$\varphi \longmapsto \varphi \circ p$$

$$\mathrm{Ext}(\mathbb{Z}/p\mathbb{Z}, H) = \mathrm{Coker}\big(\underset{\underset{H}{\cong}}{\mathrm{Hom}(\mathbb{Z},H)} \xrightarrow{\ \cdot p\ } \underset{\underset{H}{\cong}}{\mathrm{Hom}(\mathbb{Z},H)}\big)$$

$$\cong H/pH$$

Proposition: If G is free abelian then $\mathrm{Ext}(G,H) = 0 \ \forall H$.

Proof: G has a presentation: $0 \to \underset{F_1}{0} \to \underset{F_0}{G} \xrightarrow{id} G \to 0$ ▨

Note: In general $\mathrm{Ext}(G,H) \not\cong \mathrm{Ext}(H,G)$.
E.g. $\mathrm{Ext}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$

<u>Def</u>: A group H is <u>divisible</u> if for any $h \in H$ and $n = 1, 2, \dots$ there is $h' \in H$ s.t. $nh' = h$.

<u>Example</u>: $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Q}/\mathbb{Z}$ are divisible group.

<u>Proposition</u>:
If H is divisible then $\mathrm{Ext}(G, H) = 0$ $\forall H$.

<u>Proposition</u>:

$$\mathrm{Ext}(\bigoplus_i G_i, H) \cong \prod_i \mathrm{Ext}(G_i, H)$$

$$\mathrm{Ext}(G, \prod_i H) \cong \prod_i \mathrm{Ext}(G, H_i)$$

<u>Note</u>: For any space X and group G there is a homomorphism

$$\tilde{\beta}: Z^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X), G)$$

defined as follows.
We have:

$$\begin{aligned} Z^n(X;G) &= \{\varphi \in C^n(X;G) \mid \delta^n(\varphi) = 0\} \\ &= \{\varphi \in \mathrm{Hom}(C_n(X), G) \mid 0 = \varphi\partial_{n+1}: C_{n+1}(X) \to G\} \\ &= \{\varphi \in \mathrm{Hom}(C_n(X), G) \mid \varphi|_{B_n(X)} = 0\} \end{aligned}$$

Thus if $\varphi \in Z_n(X, G)$ then $\varphi|_{Z_n(X)}: Z_n(X) \to G$ and $B_n(X) \in \mathrm{Ker}(\varphi|_{Z_n(X)})$. This defines a homomorphism

$$\begin{aligned} \bar{\varphi}: H_n(X) = Z_n(X)/B_n(X) &\longrightarrow G \\ [c] &\longmapsto \varphi(c) \end{aligned}$$

We set: $\tilde{\beta}(\varphi) = \bar{\varphi}$.

Note: If $\varphi \in B^n(X,G)$ then $\varphi = \varphi' \circ \partial_n : C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\varphi'} G$

$Z_n(X) \subset C_n(X)$, $Z_n(X) \to 0$

so $\varphi|_{Z_n(X)} = 0$.

This shows that $\tilde{\beta}(\varphi) = 0$ for $\varphi \in B^n(X;G)$, and so $\tilde{\beta}$ induces a homomorphism:

$$\beta : H^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X), G)$$

$$[\varphi] \longmapsto \bar{\varphi} : H_n(X) \to G$$

$$[x] \mapsto \varphi(x)$$

Theorem: (The Universal Coefficient Theorem for Cohomology) For any space X, any abelian group and $n \geq 0$ there is a split short exact sequence:

$$(*) \quad 0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X;G) \xrightarrow{\beta} \mathrm{Hom}(H_n(X), G) \to 0$$

As a consequence we get:

$$H_n(X;G) \cong \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}(X), G)$$

Proof: Similar to the proof of the Universal Coefficient Theorem for homology. ▨

Note: The sequence $(*)$ is natural, but its splitting is not.

Corollary: If G is a divisible group (e.g. $G = \mathbb{Q}, \mathbb{R}, \mathbb{C}$) then $\beta : H^n(X;G) \to \mathrm{Hom}(H_n(X), G)$ is an isomorphism $\forall_n$.

Example: $H^q(S^n, \mathbb{Q}) \cong \mathrm{Hom}(H_q(S^n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & q = 0, n \\ 0 & \text{otherwise} \end{cases}$

<u>Def:</u> A space X is of <u>finite type</u> if $H_n(X)$ is finitely generated for each $n \geq 0$.

<u>E.g.</u> If X - CW-complex with finitely many cells in each dimension then X is of finite type.

If X is of finite type then

$$H_n(X) \cong \underbrace{T_n(X)}_{\substack{\text{torsion}\\ \text{subgroup}}} \oplus \underbrace{F_n(X)}_{\cong\, \mathbb{Z}^k}$$

<u>Corollary:</u>
If X is of finite type then

$$H^n(X;\mathbb{Z}) \cong T_{n-1}(X) \oplus F_n(X)$$

In particular, if X is of finite type and $H_n(X;\mathbb{Z})$ is free abelian $\forall_n$ then $H^n(X;\mathbb{Z}) \cong H_n(X;\mathbb{Z})$ $\forall_n$.

<u>Proof:</u> Exercise.

<u>Example:</u>

$$H^q(S^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & \text{otherwise} \end{cases}$$

<u>Recall:</u>

$$H_q(\mathbb{RP}^\infty) \cong \begin{cases} 0 & q>0,\ q\text{-even} \\ \mathbb{Z}/2 & q>0,\ q\text{-odd} \\ \mathbb{Z} & q=0 \end{cases}$$

This gives:

$$H^q(\mathbb{RP}^\infty) \cong \begin{cases} \mathbb{Z}/2 & q>0\ q\text{-even} \\ 0 & q>0\ q\text{-odd} \\ \mathbb{Z} & q=0 \end{cases}$$

119

<u>Note</u>: If X, Y - spaces s.t. $H_n(X) \cong H_n(Y)$ $\forall n \geq 0$ $\Rightarrow$ $H^n(X) \cong H^n(Y)$ $\forall n \geq 0$. The converse is not true.

Take e.g. X - CW-complex s.t.

$$H_n(X) \cong \begin{cases} 0 & n > 1 \\ \mathbb{Q} & n = 1 \\ \mathbb{Z} & n = 0 \end{cases}$$

Then:

$$H^n(X) \cong \begin{cases} 0 & n > 2 \\ \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \overset{\text{tricky}}{\cong} \bigoplus_{t \in \mathbb{R}} \mathbb{Q} & n = 2 \\ \mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) = 0 & n = 1 \\ \mathbb{Z} & n = 0 \end{cases}$$

Take $Y = X \vee X$. We have:

$$H_n(Y) \cong \begin{cases} 0 & n > 1 \\ \mathbb{Q} \oplus \mathbb{Q} & n = 1 \\ \mathbb{Z} & n = 0 \end{cases}$$

<u>Also</u>:

$$H^n(Y) \cong \begin{cases} 0 & n > 2 \\ \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \left(\bigoplus_{t \in \mathbb{R}} \mathbb{Q}\right) \oplus \left(\bigoplus_{t \in \mathbb{R}} \mathbb{Q}\right) \cong \bigoplus_{t \in \mathbb{R}} \mathbb{Q} & n = 2 \\ \mathrm{Hom}(\mathbb{Q} \oplus \mathbb{Q}, \mathbb{Z}) = 0 & n = 1 \\ \mathbb{Z} & n = 0 \end{cases}$$

<u>Thus</u>: $H^n(X) \cong H^n(Y)$ $\forall n$ but $H_1(X) \not\cong H_1(Y)$.

On the other hand we have:

<u>Proposition</u>: Let $f: X \to Y$ - a map of spaces. The following conditions are equivalent:

1) $f_*: H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$
2) $f^*: H^n(X) \to H^n(Y)$ is an isomorphism $\forall n$

(120)

Proof:

Exercise. Use the fact that if G is an abelian group s.t. $\mathrm{Hom}(G,\mathbb{Z})=0$ and $\mathrm{Ext}(G,\mathbb{Z})=0$ then $G=0$.

▨