Homology of product spaces

Problem: X, Y - spaces

$H_q(X \times Y) = ?$

Def: Let C_*, D_* - chain complex. The tensor product $C_* \otimes D_*$ is a chain complex given as follows:

$$(C_* \otimes D_*)_n = \bigoplus_{p+q} C_p \otimes D_q$$

If $c \otimes d \in C_p \otimes D_q \subseteq (C_* \otimes D_*)_n$ then

then $\partial(c \otimes d) = \partial c \otimes d + (-1)^p c \otimes \partial d \in (C_* \otimes D_*)_{n-1}$

Note: If $f_* : C_* \to C'_*$, $g : D_* \to D'_*$ - chain maps the we have a chain map:

$$f_* \otimes g_* : C_* \otimes D_* \to C'_* \otimes D'_*$$
$$c \otimes d \longmapsto f_*(c) \otimes g_*(d)$$

Eilenberg-Zilber Theorem:

For any spaces X, Y there exists a natural chain homotopy equivalence:

$$\Phi_{X,Y} : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$$

This gives natural isomorphisms

$$\Phi_{X,Y} : H_n(X \times Y) \to H_n(C_*(X) \otimes C_*(Y))$$

Proof:

Use the <u>method of acyclic models.</u>

∎

<u>Note:</u> $\Phi_{X,Y}: C_*(X\times Y) \to C_*(X)\otimes C_*(Y)$ can be explicitly defined as follows.

For $p \leq n$ define:

$$\lambda_p: \Delta^p \to \Delta^n \qquad\qquad \rho_p: \Delta^p \to \Delta^n$$

$$\begin{array}{ccc} e_0 & \mapsto & e_0 \\ \vdots & & \vdots \\ e_p & \mapsto & e_p \end{array} \qquad\qquad \begin{array}{ccc} e_0 & \mapsto & e_{n-p} \\ \vdots & & \vdots \\ e_p & \mapsto & e_n \end{array}$$

For $\tau: \Delta^n \to X$ we will say that

$\tau\lambda_p: \Delta^p \to X$ is the <u>p-th front face of τ</u>

$\tau\rho_p: \Delta^p \to X$ is the <u>p-th back face of τ</u>

For $\sigma: \Delta^n \to X\times Y$ let $\sigma_X: \Delta^n \xrightarrow{\sigma} X\times Y \xrightarrow{pr_X} X$

$\sigma_Y: \Delta^n \xrightarrow{\sigma} X\times Y \xrightarrow{pr_Y} Y$

We define:

$$\Phi_{X,Y}: C_n(X\times Y) \longrightarrow (C_*(X)\otimes C_*(Y))_n = \bigoplus_{p+q=n} C_p(X)\otimes C_q(Y)$$

$$\sigma \longmapsto \sum_{p+q=n} \sigma_X\lambda_p \otimes \sigma_Y\rho_q$$

Next: C_*, D_* - chain complexes

$$H_q(C_* \otimes D_*) = ?$$

Proposition

For any chain complexes C_*, D_* and $p, q \in \mathbb{Z}$ there is a well defined homomorphism

$$\alpha : H_p(C_*) \otimes H_q(D_*) \longrightarrow H_{p+q}(C_* \otimes D_*)$$
$$[x] \otimes [y] \longmapsto [x \otimes y]$$

Proof:

It is enough to show that we have a well defined bilinear map

$$\bar{\alpha} : H_p(C_*) \times H_q(D_*) \longrightarrow H_{p+q}(C_* \otimes D_*)$$
$$([x], [y]) \longmapsto [x \otimes y]$$

1) Note that if $x \in Z_p(C_*)$, $y \in Z_q(D_*)$ then $x \otimes y \in Z_{p+q}(C_* \otimes D_*)$. This gives a well defined bilinear map

$$\bar{\bar{\alpha}} : Z_p(C_*) \times Z_q(D_*) \longrightarrow H_{p+q}(C_* \otimes D_*)$$
$$(x, y) \longmapsto [x \otimes y]$$

2) We have:

$$\begin{aligned}
\bar{\bar{\alpha}}(x + \partial b, y + \partial d) &= [(x + \partial b) \otimes (y + \partial d)] \\
&= [x \otimes y + x \otimes \partial d + \partial b \otimes y + \partial b \otimes \partial d] \\
&= [x \otimes y + (-1)^p \partial(x \otimes d) + \partial(b \otimes y) + \partial(b \otimes \partial d)] \\
&= [x \otimes y] \\
&= \bar{\bar{\alpha}}(x, y)
\end{aligned}$$

This shows that $\bar{\bar{\alpha}}(x, y)$ depends only on the homology classes of x and y, and so $\bar{\alpha}$ is well defined. ∎

Theorem: (Künneth Formula)

If C_* - chain complex of free abelian groups

D_* - any chain complex of abelian groups

then there is a split short exact sequence:

$$0 \to \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \xrightarrow{\alpha} H_n(C_* \otimes D_*) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(D_*)) \to 0$$

$$[x] \otimes [y] \longmapsto [x \otimes y]$$

Proof: Similar to the Universal Coefficient Thm. ▨

Theorem:

For any spaces X, Y there is a split short exact sequence:

$$(*) \quad 0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0$$

Proof: Follows from the Eilenberg-Ziber Theorem and the Künneth Formula. ▨

Note: The short exact sequence $(*)$ is natural but its splitting is not.

Example:

Recall: $H_q(S^n) \cong \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & \text{otherwise} \end{cases}$

since $H_\ell(S^n)$ is torsion free for all ℓ.

We have:

$$H_q(X \times S^n) \cong \bigoplus_{k+\ell=q} (H_k(X) \otimes H_\ell(S^n)) \oplus \underbrace{\bigoplus_{k+\ell=q-1} \mathrm{Tor}(H_k(X), H_\ell(S^n))}_{= 0}$$

$$\cong H_q(X) \otimes H_0(S^n) \oplus H_{q-n}(X) \otimes H_n(S^n) \cong H_q(X) \oplus H_{q-n}(X).$$

Note: The Künneth Formula can be generalized:

Def: Let R - commutative ring with 1

M - R-module

A presentation of M is a short exact sequence of the form

$$0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

where F_1, F_0 are free R-modules.

Proposition: If R is a PID then any R-module has a presentation.

Proof: Use the fact that if R - PID then submodules of free R-modules are free. ▨

Def: Let R - PID, M - R-module with presentation $0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$. For an R-module N we set:

$$\mathrm{Tor}_R(M,N) := \mathrm{Ker}(F_1 \otimes R \xrightarrow{d_1 \otimes id} F_0 \otimes R)$$

Proposition: $\mathrm{Tor}_R(M,N)$ depends only on M, N and not on the choice of presentation of M.

(*) Note: If M - free R-module then M has presentation

$$0 \to 0 \to M \xrightarrow{id} M \to 0$$

so $\mathrm{Tor}_R(M,N) = 0$ for all N.

Theorem: (Künneth Formula)

Let R - PID, C_* - chain complex of free R-modules

D_* - chain complex of R-modules.

There exists a split short exact sequence:

$$0 \to \bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*) \longrightarrow H_n(C_* \otimes_R D_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_R(H_p(C_*), H_q(D_*)) \to 0$$

Corollary:

If X, Y - spaces, R - PID then there is a split short exact sequence:

$$0 \to \bigoplus_{p+q=n} H_p(X;R) \otimes_R H_q(X;R) \to H_n(X \times Y; R) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_R(H_p(X;R), H_q(X;R)) \to 0.$$

Note: If R - field then by Note (*) we get:

$\mathrm{Tor}_R(H_p(X;R), H_q(Y;R)) = 0 \quad \forall_{p,q}$ so we have:

$$H_n(X \times Y; R) \cong \bigoplus_{p+q=n} H_p(X;R) \otimes_R H_q(Y;R)$$

Example:

Recall: $H_q(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2 \quad \forall q \geq 0$

This gives: $H_q(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z}/2) \cong \bigoplus_{k+l=q} H_k(\mathbb{R}P^\infty; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H_l(\mathbb{R}P^\infty; \mathbb{Z}/2)$

$$\cong \bigoplus^{q+1} \mathbb{Z}/2$$