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Axioms for homology

Let ne-Top be the category of non-empty topological spaces and continuous maps.

Def: A generalized reduced homology theory is a sequence of functors

$$\tilde{h}_n : \text{ne-Top} \to Ab$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

1) If $f, g : X \to Y$ and $f \simeq g$ then $f_* = g_* : \tilde{h}_n(X) \to \tilde{h}_n(Y)$ for all n.
2) For any cofibration $A \xrightarrow{i} X \xrightarrow{q} X/A$ and $n \in \mathbb{Z}$ the following sequence is exact:
$$\tilde{h}_n(A) \xrightarrow[i_*]{} \tilde{h}_n(X) \xrightarrow[q_*]{} \tilde{h}_n(X/A)$$

3) For any space X and $n \in \mathbb{Z}$ there is a natural isomorphism:
$$s_X : \tilde{h}_n(X) \longrightarrow \tilde{h}_{n+1}(\Sigma X)$$
4) If (X_α, x_α) are well-pointed spaces then the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism
$$\bigoplus_\alpha i_{\alpha *} : \bigoplus_\alpha \tilde{h}_n(X_\alpha) \to \tilde{h}_n(\bigvee_\alpha X_\alpha)$$
for $n \in \mathbb{Z}$.

<u>Note</u>: Using 3) and 4) for a cofibration $A \xrightarrow{i} X$ we obtain a long exact sequence:

$$\begin{tikzcd}
\ldots \arrow[r] & \tilde{h}_n(A) \arrow[r] & \tilde{h}_n(X) \arrow[r] & \tilde{h}_n(X/A) \arrow[r] & h_{n-1}(A) \arrow[r] & \ldots \\
& & \tilde{h}_n(X) \arrow[u, "\|"] \arrow[r] & \tilde{h}_n(X \cup CA) \arrow[u, "\cong"] \arrow[r] & h_n(\Sigma A) \arrow[u, "s_A^{-1}"] &
\end{tikzcd}$$

$X \cup CA \simeq X/A$

$X \hookrightarrow X \cup CA$ is a cofibration and $X \cup CA / X \cong \Sigma A$

<u>Theorem</u>: Let $\{\tilde{h}_n\}$ be a generalized homology theory, such that

$$(*) \qquad \tilde{h}_n(S^0) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Then for CW-complex $X \neq \emptyset$ there is a natural isomorphism:

$$\varphi_X : \tilde{h}_n(X) \xrightarrow{\cong} \underbrace{\tilde{H}_n(X)}_{\text{reduced singular homology of } X}$$

(<u>Note</u>: For $n < 0$ we set: $\tilde{H}_n(X) = 0$)

Homology with coefficients in a group G

Let X - space, G - abelian group

Define:

$C_n(X;G) = \{\sum_\sigma g_\sigma \sigma \mid \sigma: \Delta^n \to X,\ g_\sigma \in G,\ g_\sigma \neq 0$ for fin. many σ only$\}$

$C_n(X;G)$ is an abelian group with addition given by

$$\sum_\sigma g_\sigma \sigma + \sum_\sigma g'_\sigma \sigma = \sum_\sigma (g_\sigma + g'_\sigma)\sigma$$

Note: $C_n(X;G) \cong \bigoplus_\sigma G$

Define: $\partial_n : C_n(X;G) \longrightarrow C_{n-1}(X;G)$

$$g\sigma \longmapsto \sum_{i=0}^{n} (-1)^i g\sigma^{(i)}$$

This gives a chain complex:

$$C_*(X;G) = (\dots \xrightarrow{\partial_{n+1}} C_n(X;G) \xrightarrow{\partial_n} C_{n-1}(X;G) \to \dots)$$

Define: $H_n(X;G) := H_n(C_*(X;G))$

↑ n-th singular homology group of X with coefficients in G

$\tilde{H}_n(X;G) := \mathrm{Ker}(H_n(X;G) \to H_n(*;G))$

Note: $H_n(X) = H_n(X;\mathbb{Z})$, $\tilde{H}_n(X) := \tilde{H}_n(X;\mathbb{Z})$

Theorem: The functors $\{X \longmapsto \tilde{H}_n(X;G)\}$ define a generalized reduced homology theory such that $\tilde{H}_0(S^0;G) \cong G$ and $\tilde{H}_n(S^0;G) = 0$ for $n > 0$.

Next goal: How to relate $H_*(X;G)$ to $H_*(X) = H_*(X;\mathbb{Z})$?

A different description of $C_*(X;G)$, $H_*(X;G)$

Let $C_*(X)$ - the usual singular chain complex of X
G - abelian group

Let $C_*(X)\otimes G$ be the chain complex given by:

$$C_*(X)\otimes G = (\dots \to C_n(X)\otimes G \xrightarrow{\partial_n \otimes id} C_{n-1}(X)\otimes G \xrightarrow{\partial_{n-1}\otimes id} \dots)$$

$$x\otimes g \longmapsto (\partial x)\otimes g$$

Note: For each n we have an isomorphism:

$$C_n(X)\otimes G \longrightarrow C_n(X;G)$$

$$\Sigma(n_i\sigma_i)\otimes g_i \longmapsto \Sigma(n_i g_i)\sigma_i$$

This defines an isomorphism of chain complexes:

$$C_*(X)\otimes G \xrightarrow{\cong} C_*(X;G)$$

and so an isomorphism of homology groups

$$H_n(C_*(X)\otimes G) \xrightarrow[\cong]{} H_n(X;G)$$

Algebraic question:

$C_* = (\dots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \dots)$ - chain complex of free abelian groups

G - abelian group

How to relate $H_*(C_*)$ to $H_*(C_*\otimes G)$

Notation:

$C_* = (\ldots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots)$ - chain complex of free abelian groups

G - abelian group

$Z_n = \mathrm{Ker}(\partial_n)$

$B_n = \mathrm{Im}(\partial_{n+1})$

$B_n \overset{i_n}{\hookrightarrow} Z_n \overset{j_n}{\hookrightarrow} C_n$ - inclusions

$i_n \otimes id : B_n \otimes G \to Z_n \otimes G$

$j_n \otimes id : Z_n \otimes G \to C_n \otimes G$

Lemma 1

In the above notation for every n there is a short exact sequence

$$0 \to \mathrm{Coker}(i_n \otimes id) \to H_n(C_* \otimes G) \to \mathrm{Ker}(i_{n-1} \otimes id) \to 0$$

Moreover, this sequence splits which gives an isomorphism:

$$H_n(C_* \otimes G) \xrightarrow{\cong} \mathrm{Coker}(i_n \otimes id) \oplus \mathrm{Ker}(i_{n-1} \otimes id)$$

Proof: We have a short exact sequence:

$$0 \to Z_n \overset{j_n}{\hookrightarrow} C_n \xrightarrow{\partial_n} B_{n-1} \to 0$$

Moreover, since this sequence splits the sequence

$$0 \to Z_n \otimes G \xrightarrow{j_n \otimes id} C_n \otimes G \xrightarrow{\partial_n \otimes id} B_{n-1} \otimes G \to 0$$

is also exact.

Consider $Z_* \otimes G = (\ldots \to Z_n \otimes G \to Z_{n-1} \otimes G \to \ldots)$

and $B_* \otimes G = (\ldots \to B_n \otimes G \to B_{n-1} \otimes G \to \ldots)$

as chain complexes with trivial differentials.

This gives a short ex. seq. of chain complexes:

$$0 \to Z_* \otimes G \to C_* \otimes G \to B_{*-1} \otimes G \to 0$$

and so a long ex. seq. of homology groups:

$$\begin{array}{ccccccccc}
\cdots \to H_{n+1}(B_{*-1}\otimes G) & \to & H_n(Z_*\otimes G) & \to & H_n(C_*\otimes G) & \to & H_n(B_{*-1}\otimes G) & \to & H_{n-1}(Z_*\otimes G) \to \cdots \\
\| & & \| & & \| & & \| & & \| \\
B_n\otimes G & \xrightarrow{i_n\otimes id} & Z_n\otimes G & \longrightarrow & H_n(C_*\otimes G) & \to & B_{n-1}\otimes G & \xrightarrow{i_{n-1}\otimes id} & Z_{n-1}\otimes G
\end{array}$$

This gives a short exact seq.:

(*) $\quad 0 \to \mathrm{Coker}(i_n\otimes id) \to H_n(C_*\otimes G) \to \mathrm{Ker}(i_{n-1}\otimes id) \to 0.$

In order to show that this sequence splits it is enough to construct a homomorphism

$$s : \mathrm{Ker}(i_{n-1}\otimes id) \to H_n(C_*\otimes G)$$

s.t. the composition

$$\mathrm{Ker}(i_{n-1}\otimes id) \xrightarrow{s} H_n(C_*\otimes G) \to \mathrm{Ker}(i_{n-1}\otimes id)$$

is the identity homomorphism.

Since B_{n-1} is free there is $\bar{s} : B_{n-1} \to C_n$ s.t. $(B_{n-1} \xrightarrow{\bar{s}} C_n \xrightarrow{\partial_n} B_{n-1}) = id$. This gives $(B_{n-1}\otimes G \xrightarrow{\bar{s}\otimes id} C_n\otimes G \xrightarrow{\partial_n\otimes id} B_{n-1}\otimes G) = id$

In particular $\partial_n\otimes id(\bar{s}\otimes id(\mathrm{Ker}(i_{n-1}\otimes id)) \subseteq \mathrm{Ker}(i_{n-1}\otimes id)$

which gives $\bar{s}\otimes id(\mathrm{Ker}(i_{n-1}\otimes id)) \subseteq Z_n(C_*\otimes G)$. As a consequence we obtain a homomorphism

$$s : \mathrm{Ker}(i_{n-1}\otimes id) \xrightarrow{\bar{s}\otimes id} Z_n(C_*\otimes G) \to H_n(C_*\otimes G)$$

which gives the desired splitting of (*). ▨

<u>Lemma 2</u>: In the above notation we have

$$\mathrm{Coker}(i_n\otimes id) \cong H_n(C_*)\otimes G$$

Proof:

Recall: If $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ - exact sequence then the sequence

$$A\otimes G \xrightarrow{f\otimes id} B\otimes G \xrightarrow{g\otimes id} C\otimes G \to 0$$

is also exact.

Applying this to the sequence

$$B_n \xrightarrow{i_n} Z_n \longrightarrow H_n(C_*) \to 0$$

we obtain an exact sequence:

$$B_n\otimes G \xrightarrow{i_n\otimes id} Z_n\otimes G \longrightarrow H_n(C_*)\otimes G \to 0$$

which gives

$$\mathrm{Coker}(i_n\otimes id) \cong H_n(C_*)\otimes G$$

▨

Next: $\mathrm{Ker}(i_{n-1}\otimes id) = ?$

Def: Let H - abelian group. A presentation of H is a short exact sequence of the form:

$$0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} H \to 0$$

where F_1, F_2 - free abelian groups.

Proposition: For any abelian group H there exists a presentation of H.

Proof: Choose an epimorphism $F_0 \xrightarrow{d_0} H$ where F_0 - free abelian. Since subgroups of free abelian groups are free taking $F_1 = \mathrm{Ker}(d_0)$ we get a presentation:

$$0 \to F_1 \xhookrightarrow{d_1} F_0 \xrightarrow{d_0} H \to 0$$

Def: If $0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} H \to 0$ is a presentation of H then

$$\mathrm{Tor}(H,G) := \mathrm{Ker}(F_1 \otimes G \xrightarrow{d_1 \otimes id} F_0 \otimes G)$$

Note: $\mathrm{Tor}(H,G)$ is a group s.t. the following sequence is exact:

$$0 \to \mathrm{Tor}(H,G) \to F_1 \otimes G \to F_0 \otimes G \to H \otimes G \to 0$$

Proposition: $\mathrm{Tor}(H,G)$ is well defined: if

$$0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} H \to 0$$
$$0 \to F_1' \xrightarrow{d_1'} F_0' \xrightarrow{d_0'} H \to 0$$

- two presentations of H then

$$\mathrm{Ker}(F_1 \otimes G \xrightarrow{d_1 \otimes id} F_0 \otimes G) \cong \mathrm{Ker}(F_1' \otimes G \xrightarrow{d_1' \otimes id} F_0' \otimes G)$$

Proof: Exercise. ▨

Example:

$H = \mathbb{Z}/n\mathbb{Z}$

Presentation of H:

$$0 \to \mathbb{Z} \xrightarrow[\cdot n]{} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

This gives:

$$k \otimes g \mapsto nk \otimes g$$

$$\begin{array}{ccccc} \mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G) = \mathrm{Ker}(& \mathbb{Z} \otimes G & \longrightarrow & \mathbb{Z} \otimes G &) \\ k \otimes g & \downarrow \cong & & \downarrow \cong & \\ \downarrow & G & \xrightarrow{\cdot n} & G & \\ kg & g & \longmapsto & ng & \end{array}$$

So: $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G) \cong \{g \in G \mid ng = 0\}$

e.g.:

$$\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$$
$$\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$$

Proposition:

1) $\mathrm{Tor}(H,G) \cong \mathrm{Tor}(G,H)$
2) $\mathrm{Tor}(\bigoplus_i H_i, G) \cong \bigoplus_i \mathrm{Tor}(H_i, G)$
3) $\mathrm{Tor}(H, G) \cong \mathrm{Tor}(T(H), G)$

where $T(H)$ = the torsion subgroup of H
$= \{h \in H \mid nh = 0 \ \exists n > 0.\}$

Theorem: (The Universal Coefficients Theorem)
If C_* is a chain complex of free abelian groups then for any abelian group G and $n \in \mathbb{Z}$ there is a short exact sequence

$$0 \to H_n(C_*) \otimes G \to H_n(C_* \otimes G) \longrightarrow \mathrm{Tor}(H_{n-1}(C_*), G) \to 0$$

Moreover, this sequence splits giving an isomorphism:

$$H_n(C_* \otimes G) \cong H_n(C_*) \otimes G \oplus \mathrm{Tor}(H_{n-1}(C_*), G)$$

Proof: By Lemma 1 and 2 we have a split short exact seq.:

$$0 \to H_n(C_*) \otimes G \to H_n(C_* \otimes G) \to \mathrm{Ker}(i_{n-1} \otimes id) \to 0$$

where $i_{n-1} : B_{n-1} \hookrightarrow Z_{n-1}$.

Notice that $0 \to B_{n-1} \xhookrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C_*) \to 0$ is a presentation of $H_{n-1}(C_*)$. Therefore

$$\mathrm{Ker}(i_{n-1} \otimes id) = \mathrm{Tor}(H_{n-1}(C_*), G)$$

∎

<u>Corollary</u>: For a space X and an abelian group G there is a split short exact seq.:

(*) $\quad 0 \to H_n(X) \otimes G \to H_n(X;G) \to \mathrm{Tor}(H_{n-1}(X), G) \to 0$

This gives an isomorphism:

$$H_n(X;G) \cong H_n(X) \otimes G \oplus \mathrm{Tor}(H_{n-1}(X), G)$$

<u>Note</u>: 1) The sequence (*) is natural: a map $f: X \to Y$ gives a comm. diagram:

$$\begin{array}{ccccccccc} 0 & \to & H_n(X)\otimes G & \to & H_n(X;G) & \to & \mathrm{Tor}(H_{n-1}(X);G) & \to & 0 \\ & & \downarrow{\scriptstyle f_*\otimes id} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ 0 & \to & H_n(Y)\otimes G & \to & H_n(Y,G) & \to & \mathrm{Tor}(H_{n-1}(Y);G) & \to & 0 \end{array}$$

2) The splitting of (*) is not natural. As a consequence for a map $f: X \to Y$ the diagram

$$\begin{array}{ccc} H_n(X;G) & \xrightarrow{\cong} & H_n(X)\otimes G \oplus \mathrm{Tor}(H_{n-1}(X), G) \\ \downarrow{\scriptstyle f_*} & & \downarrow \\ H_n(Y;G) & \xrightarrow[\cong]{} & H_n(Y)\otimes G \oplus \mathrm{Tor}(H_{n-1}(Y), G) \end{array}$$

in general will not commute.

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Example:

Recall:

$$H_q(\mathbb{RP}^n) = \begin{cases} 0 & q > n \\ 0 & q = n\text{-even} \quad \text{or} \quad \mathbb{Z} \quad q = n\text{-odd} \\ \mathbb{Z}/2 & 0 < q < n \quad q\text{-odd} \\ 0 & 0 < q < n \quad q\text{-even} \\ \mathbb{Z} & q = 0 \end{cases}$$

We have:

$$H_q(\mathbb{RP}^n; \mathbb{Z}/2) \cong H_q(\mathbb{RP}^n) \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}(H_{q-1}(\mathbb{RP}^n), \mathbb{Z}/2)$$

This gives:

$$H_0(\mathbb{RP}^n; \mathbb{Z}/2) \cong H_0(\mathbb{RP}^n) \otimes \mathbb{Z}/2 \cong \mathbb{Z} \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$$

For $0 < q < n$, q-even:

$$\begin{aligned} H_q(\mathbb{RP}^n; \mathbb{Z}/2) &\cong H_q(\mathbb{RP}^n) \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}(H_{q-1}(\mathbb{RP}^n), \mathbb{Z}/2) \\ &\cong 0 \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}(\mathbb{Z}/2, \mathbb{Z}/2) \\ &\cong \mathbb{Z}/2 \end{aligned}$$

For $0 < q < n$, q-odd:

$$\begin{aligned} H_q(\mathbb{RP}^n; \mathbb{Z}/2) &\cong H_q(\mathbb{RP}^n) \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}(H_{q-1}(\mathbb{RP}^n), \mathbb{Z}/2) \\ &\cong \mathbb{Z}/2 \otimes \mathbb{Z}/2 \oplus \begin{cases} \mathrm{Tor}(0, \mathbb{Z}/2) & q-1 \neq 0 \\ \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2) & q-1 = 0 \end{cases} \\ &\cong \mathbb{Z}/2 \end{aligned}$$

For $q = n$:

$$\begin{aligned} H_n(\mathbb{RP}^n, \mathbb{Z}/2) &\cong H_n(\mathbb{RP}^n) \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}(H_{n-1}(\mathbb{RP}^n); \mathbb{Z}/2) \\ &\cong \begin{cases} 0 \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}(\mathbb{Z}/2, \mathbb{Z}/2) & n\text{-even} \\ \mathbb{Z} \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}(0, \mathbb{Z}/2) & n\text{-odd} \end{cases} \\ &\cong \mathbb{Z}/2 \end{aligned}$$

For $q = n+1$:

$$H_{n+1}(\mathbb{RP}^n, \mathbb{Z}/2) \cong H_{n+1}(\mathbb{RP}^n) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}(H_n(\mathbb{RP}^n), \mathbb{Z}/2)$$

$$\cong 0 \otimes \mathbb{Z}/2 \oplus \begin{cases} \operatorname{Tor}(\mathbb{Z}, \mathbb{Z}/2) & n\text{- odd} \\ \operatorname{Tor}(0, \mathbb{Z}/2) & n\text{- even} \end{cases}$$

$$\cong 0$$

For $q > n+1$:

$$H_q(\mathbb{RP}^n) \cong H_{q-1}(\mathbb{RP}^n) = 0 \quad \text{so} \quad H_q(\mathbb{RP}^n; \mathbb{Z}/2) = 0$$

<u>Upshot:</u>

$$H_q(\mathbb{RP}^n; \mathbb{Z}/2) \cong \begin{cases} 0 & q > n \\ \mathbb{Z}/2 & 0 \leq q \leq n \end{cases}$$

<u>Note:</u> We can also compute this using the cellular chain complex with $\mathbb{Z}/2$ coefficients:

<u>Recall:</u> The usual cellular chain complex for $\mathbb{RP}^n$:
(say n- even)

$$\cdots \to 0 \to \underset{n}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{n-1}{\mathbb{Z}} \xrightarrow{\cdot 0} \underset{n-2}{\mathbb{Z}} \xrightarrow{\cdot 2} \cdots \to \underset{2}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{1}{\mathbb{Z}} \xrightarrow{\cdot 0} \underset{0}{\mathbb{Z}} \to 0$$

This gives the cellular chain complex with coeff. in $\mathbb{Z}/2$:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \otimes \mathbb{Z}/2 \arrow[r] \arrow[d, "\cong"] & \mathbb{Z} \otimes \mathbb{Z}/2 \arrow[r, "2 \otimes 1"] \arrow[d, "\cong"] & \mathbb{Z} \otimes \mathbb{Z}/2 \arrow[r, "0 \otimes 1"] \arrow[d, "\cong"] & \mathbb{Z} \otimes \mathbb{Z}/2 \arrow[r, "2 \otimes 1"] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathbb{Z} \otimes \mathbb{Z}/2 \arrow[r, "2 \otimes 1"] \arrow[d, "\cong"] & \mathbb{Z} \otimes \mathbb{Z}/2 \arrow[r, "0 \otimes 1"] \arrow[d, "\cong"] & \mathbb{Z} \otimes \mathbb{Z}/2 \arrow[r] \arrow[d, "\cong"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "0"] & \cdots \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

<u>So:</u> $H_q(\mathbb{RP}^n) \cong \begin{cases} 0 & q > n \\ \mathbb{Z}/2 & 0 \leq q \leq n \end{cases}$