64

Degrees of maps of spheres

Recall: If $n \geq 1$ then

$$H_n(S^n) \cong \mathbb{Z}$$

It follows that any map $f: S^n \longrightarrow S^n$ induces a homomorphism

$$\begin{array}{ccc} f_*: H_n(S^n) & \longrightarrow & H_n(S^n) \\ \wr\| & & \wr\| \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

Any such homomorphism is of the form $f_*([z]) = n \cdot [z]$ for some $n \in \mathbb{Z}$

Def: Let $f: S^n \longrightarrow S^n$. The integer n s.t. $f_*([z]) = n \cdot [z]$ for $z \in H_n(S^n)$ is called the degree of the map f and dended by $\deg(f)$.

Properties of $\deg(f)$

1) $\deg(id_{S^n}) = 1$
2) if $f \simeq g$ then $\deg(f) = \deg(g)$
3) if $f \simeq *$ (in particular if f is not onto) then $\deg(f) = 0$.
4) if f is a homotopy equivalence then $\deg(f) = \pm 1$
5) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

Proposition:

Let $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = 1\}$ and let

$$\varphi_k: S^n \longrightarrow S^n$$

be given by $\varphi_k(x_1, \ldots, x_k, \ldots, x_{n+1}) = (x_1, \ldots, -x_k, \ldots, x_{n+1})$.
Then $\deg(\varphi_k) = -1$.

65

<u>Proof:</u>

We can assume: $k=1$

$$\varphi = \varphi_1 : S^n \longrightarrow S^n$$
$$(x_1, x_2, \dots, x_{n+1}) \qquad (-x_1, x_2, \dots, x_{n+1})$$

Take $S^{n-1} = \{(0, x_2, \dots, x_{n+1}) \mid \sum x_i^2 = 1\} \subseteq S^n$

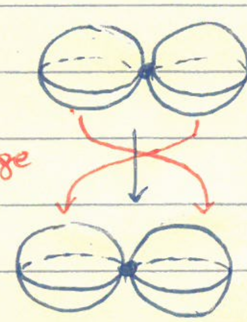
<u>Note:</u> $\varphi|_{S^{n-1}} = id_{S^{n-1}}$

The map φ defines a map of the long exact seq. of the cofibration $i: S^{n-1} \hookrightarrow S^n$:

(*)
$$\begin{tikzcd}
0 \cong \tilde{H}_n(S^{n-1}) \arrow[r, "i_*"] \arrow[d, "id"] & \underset{\mathbb{Z}\,\cong}{\tilde{H}_n(S^n)} \arrow[r, "q_*", "1-1"'] \arrow[d, "\varphi_*"] & \tilde{H}_n(S^n/S^{n-1}) \arrow[r, "\delta", "onto"'] \arrow[d, "\varphi_*"] & \underset{\mathbb{Z}\,\cong}{\tilde{H}_{n-1}(S^{n-1})} \arrow[r] \arrow[d, "id"] & \tilde{H}_{n-1}(S^n) \cong 0 \arrow[d, "\varphi_*"] \\
0 \cong \tilde{H}_n(S^{n-1}) \arrow[r] & \underset{\cong\,\mathbb{Z}}{\tilde{H}_n(S^n)} \arrow[r, "1-1", "q"'] & \tilde{H}_n(S^n/S^{n-1}) \arrow[r, "onto", "\delta"'] & \underset{\cong\,\mathbb{Z}}{\tilde{H}_{n-1}(S^{n-1})} \arrow[r] & \tilde{H}_{n-1}(S^n) \cong 0
\end{tikzcd}$$

<u>Note:</u> We have a commutative diagram:

$$\begin{tikzcd}
S^n/S^{n-1} \arrow[r, "\cong"] \arrow[d, "\varphi"] & S^n \vee S^n \arrow[d, "interchange"] \\
S^n/S^{n-1} \arrow[r, "\cong"'] & S^n \vee S^n
\end{tikzcd}$$



This gives a comm. diagram of homology groups:

$$\begin{tikzcd}
\tilde{H}_n(S^n/S^{n-1}) \arrow[r, "\cong"] \arrow[d, "\varphi_*"] & H_n(S^n \vee S^n) \cong \tilde{H}_n(S^n) \oplus \tilde{H}_n(S^n) \cong \mathbb{Z} \oplus \mathbb{Z} \arrow[d] \\
\tilde{H}_n(S^n/S^{n-1}) \arrow[r] & H_n(S^n \vee S^n) \cong \tilde{H}_n(S^n) \oplus \tilde{H}_n(S^n) \cong \mathbb{Z} \oplus \mathbb{Z}
\end{tikzcd}$$

$(m, n) \mapsto (n, m)$

As a consequence the diagram (*) becomes:

$$(**)\quad \begin{tikzcd} 0 \arrow[r] & \tilde{H}_n(S^n) \arrow[r, "1-1"', "q_*"] \arrow[d, "\varphi_*"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "epi", "\delta"'] \arrow[d, "(m,n) \mapsto (n,m)"] & \mathbb{Z} \arrow[r] \arrow[d, "id_{\mathbb{Z}}"] & 0 \\ 0 \arrow[r] & \tilde{H}_n(S^n) \arrow[r, "q_*", "1-1"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\delta", "epi"'] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

<u>We have:</u> $\delta : \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z}$

$$(m,n) \longmapsto km + ln$$

for some $k, l \in \mathbb{Z}$. By commutativity of (**) we get:

$$\underset{km+ln}{\underset{\shortparallel}{\psi(m,n)}} = \underset{kn+lm}{\underset{\shortparallel}{\psi(n,m)}} \qquad \forall\, m,n \in \mathbb{Z}$$

<u>so</u>: $k = l$, and so $\psi(m,n) = k(m+n)$

Also, since ψ - epi we have $k = \pm 1$.

This gives:

$$\operatorname{Im}(q_*) = \operatorname{Ker}(\psi) = \{(m,-m) \mid m \in \mathbb{Z}\}$$

Therefore for $[z] \in \tilde{H}_n(S^n)$ we obtain:

$$q_*([z]) = (m,-m)$$

for some $m \in \mathbb{Z}$. By commutativity of (**) this gives:

$$q_*\varphi_*([z]) = (-m,m) = -q_*([z]) = q_*(-[z])$$

Since q_* - 1-1 we obtain:

$$\varphi_*([z]) = -[z]$$

and so $\deg(\varphi) = -1$.

∎

67

Corollary

Let $\alpha: S^n \longrightarrow S^n$ be the antipodal map:

$$\alpha(x_1, x_2, \dots, x_{n+1}) = (-x_1, -x_2, \dots, -x_{n+1})$$

Then $\deg(\alpha) = (-1)^{n+1}$

Proof: $\alpha = \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_{n+1}$

so: $\deg(\alpha) = \deg(\varphi_1) \cdot \ldots \cdot \deg(\varphi_{n+1}) = (-1)^{n+1}$

$\blacksquare$

Recall: If $(X, x_0), (Y, y_0)$ - pointed spaces then

$$X \vee Y = X \sqcup Y /_{x_0 \sim y_0}$$

If $f: X \to Z$, $g: Y \to Z$ s.t. $f(x_0) = g(y_0)$ then we have a map

$$f \vee g: X \vee Y \longrightarrow Z$$
$$x \longmapsto f(x)$$
$$y \longmapsto g(y)$$

Note: Let $f, g: \underset{x_0 \longmapsto x_0}{S^n \longrightarrow S^n}$, $\deg(f) = m$, $\deg(g) = n$

We have:

$$H_n(S^n) \oplus H_n(S^n) \xrightarrow{\cong} H_n(S^n \vee S^n) \xrightarrow{(f \vee g)_*} H_n(S^n)$$
$$(k, \ell) \longmapsto (mk + n\ell)$$

Take $S^{n-1} \subseteq S^n$, and let $q: S^n \to S^n/_{S^{n-1}} \cong S^n \vee S^n$ quotient map. By choosing an appropriate homomophism $h: S^n/_{S^{n-1}} \to S^n \vee S^n$ we can assume that the map hq is given on homology by:

$$(hq)_*: H_n(S^n) \longrightarrow H_n(S^n \vee S^n) \xrightarrow{\cong} H_n(S^n) \oplus H_n(S^n)$$
$$k \longmapsto (k, k)$$

In effect we obtain:

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "(hq)_*"] & H_n(S^n \vee S^n) \arrow[r, "(f\vee g)_*"] \arrow[d, shift right, "\cong"'] & H_n(S^n) \\
k \arrow[dr, bend right] & H_n(S^n)\oplus H_n(S^n) \arrow[u, shift right, "\cong"'] & mk + nk = (m+n)k \\
& (k,k) \arrow[ur, bend right] &
\end{tikzcd}$$

Therefore: $\deg((f \vee g) \circ hq) = \deg(f) + \deg(g)$

We obtain:

Proposition: For any $n \in \mathbb{Z}$ there exists a map $f: S^n \to S^n$ s.t. $\deg(f) = n$.

Proof: Assume first that $n \geqslant 0$. We argue by induction w.r.t. n.

$n = 0$: Take $c: S^n \to S^n$ - constant map. Then $\deg(c) = 0$

$n = 1$: Take $id: S^n \to S^n$ - the identity map. Then $\deg(id) = 1$.

inductive step:

Assume that there exists a map $f: S^n \to S^n$ s.t. $\deg(f) = n$. Take $f' = (f \vee id) \circ hq$ where hq - defined as above. Then $\deg(f') = n+1$.

If $n < 0$ we can use a similar argument starting with the fact that $\deg(\varphi_1) = -1$. ▨

69

<u>Upshot:</u> Let $[S^n, S^n]$ = the set of homotopy classes of maps $S^n \longrightarrow S^n$.

1) We have a well-defined function

$$\deg : [S^n, S^n] \longrightarrow \mathbb{Z}$$
$$[f] \longmapsto \deg(f)$$

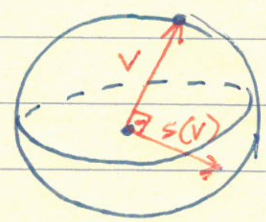
2) This function is onto.

One can also show that deg is 1-1, so it gives a bijection of sets $[S^n, S^n] \cong \mathbb{Z}$. Therefore if $f, g : S^n \longrightarrow S^n$ then $f \simeq g$ iff $\deg(f) = \deg(g)$.

Application: vector fields and division algebras

Recall: $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$.
It follows that we can consider each element $v \in S^n$ as a vector in $\mathbb{R}^{n+1}$.

Def: A vector field on S^n is a map $s: S^n \to \mathbb{R}^{n+1}$ such that for each $v \in S^n$ the vector $s(v)$ is orthogonal to v:



Def: Vector fields $s_1, s_2, \dots, s_k : S^n \to S^n$ are linearly independent if the vectors $s_1(v), s_2(v), \dots, s_k(v)$ are linearly independent for each $v \in S^n$.

Proposition: S^n has a vector field $s: S^n \to \mathbb{R}^{n+1}$ such that $s(v) \neq 0$ for all $v \in S^n$ iff n is odd.

Lemma: Let $f, g : S^n \to S^n$ be two maps such that $f(v) \neq -g(v)$ for all $v \in S^n$. Then $f \sim g$

Proof: exercise. ▨

Proof of Proposition.

($\Rightarrow$) Take $s': S^n \longrightarrow S^n$, $s'(v) = \frac{s(v)}{\|s(v)\|}$.

Let $id: S^n \longrightarrow S^n$ - the identity map

$\alpha: S^n \longrightarrow S^n$ - the antipodal map

We have: $id(v) \perp s'(v)\ \forall_v$, so $id(v) \neq -s'(v)\ \forall_v$

and so by the Lemma $id \sim s'$.

By the same argument $\alpha \sim s'$.

This gives $id \sim \alpha$ and so $1 = \deg(id) = \deg(\alpha) = (-1)^{n+1}$.

Therefore n is odd.

($\Leftarrow$) If $n = 2k-1$ - odd then

$$S^n = \{ (x_1, x_2, \dots, x_{2k-1}, x_{2k}) \in \mathbb{R}^{2k} \mid \Sigma x_i = 1 \}$$

Define $s: S^n \longrightarrow \mathbb{R}^{n+1}$ by

$$s(x_1, x_2, \dots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}).$$

$\blacksquare$

Def: A division algebra structure on $\mathbb{R}^n$ is bilinear map

$$\mu: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x, y \longmapsto x \cdot y$$

such that

1) There exists $e \in \mathbb{R}^n$ satisfying $ex = xe = x$ for all $x \in \mathbb{R}^n$

2) For any $0 \neq x \in \mathbb{R}^n$ there exists $y \in \mathbb{R}^n$ s.t. $xy = yx = e$.

Examples

$\mathbb{R}$ = real numbers – division alg. str. on $\mathbb{R}^1$

$\mathbb{C}$ = complex numbers – ——"—— ——"—— $\mathbb{R}^2$

$\mathbb{H}$ = quaternions – ——"—— ——"—— $\mathbb{R}^4$

non-commutative

$\mathbb{O}$ = octonions – ——"—— ——"—— $\mathbb{R}^8$

non-commutative, non-associative

Q: For which n there exist a division algebra structure on $\mathbb{R}^n$?

Proposition:

If $\mathbb{R}^n$ has a structure of a division algebra then the sphere $S^{n-1} \subseteq \mathbb{R}^n$ admits $n-1$ linearly independent vector fields.

Proof: Assume that $\mathbb{R}^n$ has a division algebra structure with a unit $e \in \mathbb{R}^n$. Choose vectors $e_2, e_3, \ldots, e_n$ s.t. $\{e, e_2, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$.

Notice that for $0 \neq v \in \mathbb{R}^n$ the map

$$\begin{aligned} l_v : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto vx \end{aligned}$$

is a linear isomorphism so the set $\{\underset{v}{\underset{\shortparallel}{ve}}, ve_2, \ldots, ve_n\}$ is linearly independent.

Applying Gram-Schmidt orthogonalization

to this set we get an orthogonal basis of $\mathbb{R}^n$:
$\{\underset{\substack{\| \\ v}}{s_1(v)}, s_2(v), \dots, s_n(v)\}$. The functions

$$\begin{aligned} s_i : S^{n-1} &\longrightarrow \mathbb{R}^n \\ v &\longmapsto s_i(v) \end{aligned}$$

for $i = 2, \dots, n$ give linearly independent vector fields on S^{n-1}. ▨

<u>Corollary:</u>
If $n > 1$ is odd then $\mathbb{R}^n$ does not have a structure of a division algebra.

<u>Proof:</u> It follows from the last proposition since if n - odd then S^{n-1} does not have any non-vanishing vector field. ▨

Homology of CW-complexes

Recall: If X - CW-complex then $X^{(n)}$ = the n-skeleton of X.

$X^{(0)}$ = 0-cells in X

$X^{(n)} = X^{(n-1)} \cup \{n\text{-cells}\}$

Lemma 1

If X is a CW-complex then

$$H_q(X^{(n)}, X^{(n-1)}) \cong \begin{cases} \bigoplus_{c_n} \mathbb{Z} & q = n \quad \text{(where } c_n = \text{the set of } n\text{-cells)} \\ 0 & q \neq n \end{cases}$$

Proof: The inclusion $X^{(n-1)} \hookrightarrow X^{(n)}$ is a cofibration

so:

$$H_q(X^{(n)}, X^{(n-1)}) \cong \tilde{H}_q(X^{(n)}/X^{(n-1)})$$

$$\cong \tilde{H}_q\Big(\bigvee_{\alpha \in c_n} S^n_\alpha\Big)$$

$$\cong \bigoplus_{\alpha \in c_n} \tilde{H}_q(S^n_\alpha)$$

∎

Lemma 2:

If X - CW-complex and $n > 0$ then:

1) $H_q(X^{(n)}) = 0$ for $q > n$
2) the inclusion $i : X^{(n)} \hookrightarrow X$ induces an isomorphism: $i_* : H_q(X^{(n)}) \xrightarrow{\cong} H_q(X)$ for $q < n$

75

<u>Proof:</u> Take the long exact sequence of the pair $(X^{(n)}, X^{(n-1)})$:

$$\ldots \to H_{q+1}(X^{(n)}, X^{(n-1)}) \to H_q(X^{(n-1)}) \to H_q(X^{(n)}) \to H_q(X^{(n)}, X^{(n-1)}) \to \ldots$$

By Lemma 1 $H_q(X^{(n)}, X^{(n-1)}) = 0$ for $q \neq n$ thus $H_q(X^{(n-1)}) \to H_q(X^{(n)})$ is an isomorphism $\forall q \neq n, n-1$

<u>We get:</u>

1) If $q > n$ then

$$H_q(X^{(n)}) \cong H_q(X^{(n-1)}) \cong H_q(X^{(n-2)}) \cong \ldots \cong H_q(\underset{\substack{\uparrow \\ \text{discrete set}}}{X^{(0)}}) = 0$$

2) Assume that $\dim X < \infty$ i.e. $X = X^{(N)}$ for some N

Then for $q < n$ we have:

$$H_q(X^{(n)}) \cong H_q(X^{(n+1)}) \cong \ldots \cong H_q(X^{(N)}) = H_q(X)$$

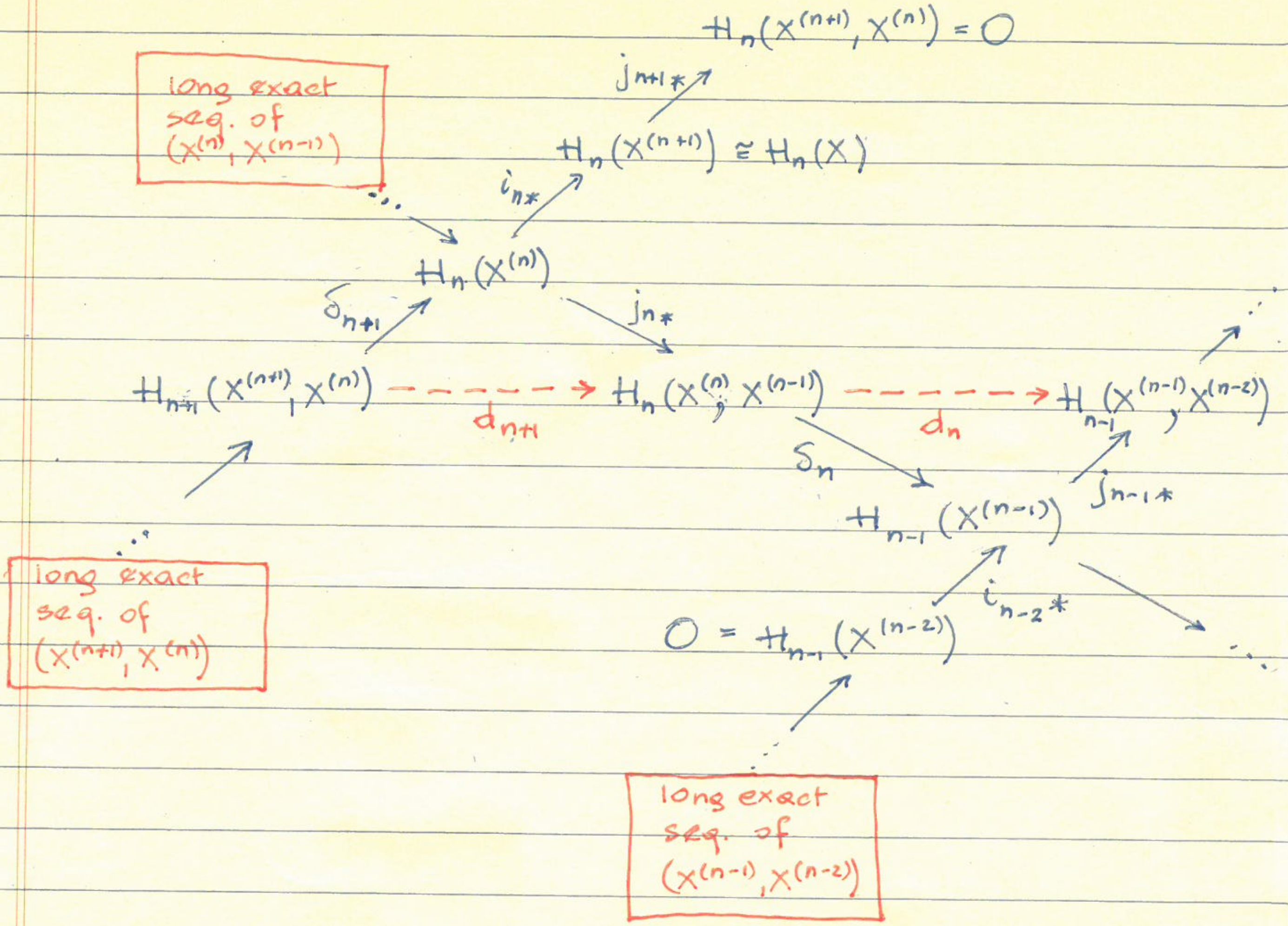
Assume that $\dim X = \infty$. Recall that any compact subspace $A \subseteq X$ is contained in some finite subcomplex of X. It follows that $A \subseteq X^{(m)}$ for some $m \geq 0$.

Since the standard simplex Δ^k is compact $\forall k$, it follows that for any singular simplex $\sigma: \Delta^k \to X$ we have $\sigma(\Delta^k) \subseteq X^{(m)}$ for some n.

If $c = \Sigma n_i \sigma_i \in C_k(X)$ then c is a combination of a finite number of singular simplices, so $c \in C_k(X^{(m)})$ for some m. Using this one can show that the inclusion $i: X^{(n)} \hookrightarrow X$ induces an isomorphism $i_*: H_q(X^{(n)}) \to H_q(X)$ for $q < n$ (exercise) ▨

Cellular chain complex

Let X - CW-complex. We have a diagram:



<u>Define</u>: $d_n := j_{n-1*}\,\delta_n : H_n(X^{(n)}, X^{(n-1)}) \longrightarrow H_{n-1}(X^{(n-1)}, X^{(n-2)})$

<u>Note</u>: $d_n \circ d_{n+1} = j_{n-1*}\underbrace{\delta_n\, j_{n*}}_{= 0}\delta_{n+1} = 0$

<u>Def</u>: The chain complex

$$\ldots \to H_{n+1}(X^{(n+1)}, X^{(n)}) \xrightarrow{d_{n+1}} H_n(X^{(n)}, X^{(n-1)}) \xrightarrow{d_n} H_{n-1}(X^{(n-1)}, X^{(n-2)}) \to \ldots$$

is called the <u>cellular chain complex</u> of X.

77

Define:

$H_q^{CW}(X) :=$ the q^{th} homology group of the cellular chain complex of X.

Theorem: If X is a CW-complex then

$$H_n^{CW}(X) \cong H_n(X) \quad \forall n$$

Proof: We have:

from the long exact seq. of $(X^{(n+1)}, X^{(n)})$

$$H_n(X) \cong H_n(X^{(n+1)}) \cong H_n(X^n)/\mathrm{Im}(\delta_{n+1})$$

since j_{n*} is 1-1

$$\cong \mathrm{Im}(j_{n*})/\mathrm{Im}(j_{n*}\delta_{n+1})$$

$\mathrm{Im}(j_{n*}) = \mathrm{Ker}(\delta_n)$

$$\cong \mathrm{Ker}(\delta_n)/\mathrm{Im}(d_{n+1})$$

since j_{n-1*} is 1-1

$$\cong \mathrm{Ker}(j_{n-1*}\delta_n)/\mathrm{Im}(d_{n+1})$$

$$= \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1})$$

$$= H_n^{CW}(X).$$

∎

Recall:

$$H_q(X^{(n)}, X^{(n-1)}) \cong \bigoplus_{n\text{-cells}} \mathbb{Z}$$

This gives:

Corollary: Let X - CW-complex.

1) If for some n the complex X has no n-cells then $H_n(X) = 0$
2) $H_q(X) = 0$ for $q > \dim X$.
3) If X has finitely many n-cells then $H_n(X)$ is a finitely generated group.

Next goal:

Let X - CW-complex, C_n = the set of n-cells in X we have a comm. diagram:

$$\begin{tikzcd}
\bigoplus_{\alpha \in C_n} \tilde{H}_n(S^n_\alpha) \arrow[r, dashed, "\overline{d}_n"] & \bigoplus_{\beta \in C_{n-1}} \tilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^{(n)}, X^{(n-1)}) \arrow[u, "\cong"] \arrow[r, "d_n"'] & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \arrow[u, "\cong"']
\end{tikzcd}$$

We want to describe the homomorphism $\overline{d}_n$ explicitly

Note: 1) Let u_α - a generator of $\tilde{H}_n(S^n_\alpha)$.
It suffices to compute $\overline{d}_n(u_\alpha)$ $\forall \alpha \in C_n$

2) If v_β - a generator of $\tilde{H}_{n-1}(S^n_\beta)$ then

$$\overline{d}_n(u_\alpha) = \sum_\beta n_{\alpha\beta} v_\beta$$

for some $n_{\alpha\beta} \in \mathbb{Z}$. Thus it suffices to compute $n_{\alpha\beta}$ for all $\alpha \in C_n$, $\beta \in C_{n-1}$:

$$u_\alpha \longmapsto n_{\alpha\beta} v_\beta$$

$$\begin{tikzcd}
\widetilde{H}_n(S^n_\alpha) \arrow[r, dashed] \arrow[d] & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
\bigoplus_{\alpha \in C_n} \widetilde{H}_n(S^n_\alpha) \arrow[r, "\bar{d}_n"'] & \bigoplus_{\beta \in C_{n-1}} \widetilde{H}_{n-1}(S^{n-1}_\beta) \arrow[u] \\
\widetilde{H}_n(X^{(n)}, X^{(n-1)}) \arrow[r, "d_n"'] \arrow[u, "\cong"'] & \widetilde{H}_{n-1}(X^{(n-1)}, X^{(n-2)}) \arrow[u, "\cong"']
\end{tikzcd}$$

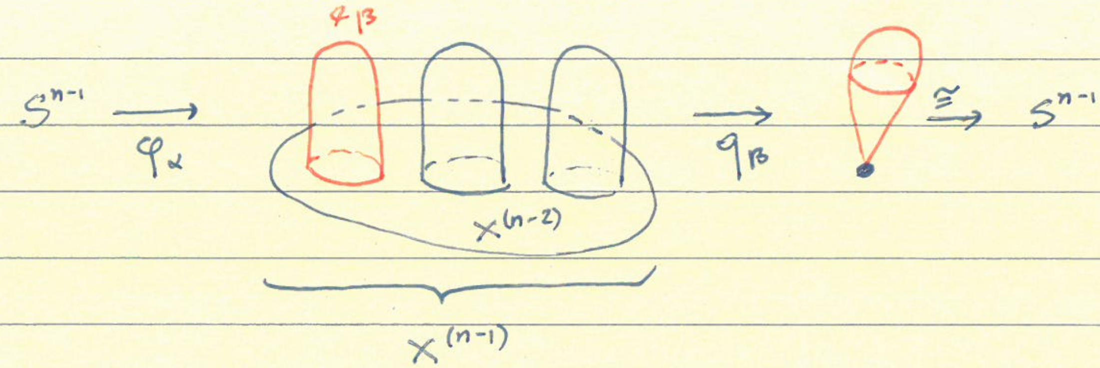
<u>Theorem</u>:

Let e_α be an n-cell in X (for $n>1$) with the attaching map $\varphi_\alpha : S^{n-1} \to X^{(n-1)}$. Let e_β be an $(n-1)$-cell in X and let

$$q_\beta : X^{(n-1)} \longrightarrow X^{(n-1)} / (X^{(n-1)} \smallsetminus e_\beta) \cong S^{n-1}$$

be the quotient map.

Then $n_{\alpha\beta} = \deg(q_\beta \varphi_\alpha : S^{n-1} \to S^{n-1})$



Proof: This follows from the following commutative diagram:

$$\begin{tikzcd}
\tilde{H}_n(S^n_\alpha) \arrow[d] & & & & \tilde{H}_{n-1}(S^{n-1}_\beta) \\
\bigoplus_\alpha \tilde{H}_n(S^n_\alpha) & H_n(D^n_\alpha, S^{n-1}_\alpha) \arrow[l] \arrow[ul, "\cong"'] \arrow[r, "\delta", "\cong"'] \arrow[dl, "\Phi_\alpha"] & H_{n-1}(S^{n-1}_\alpha) \arrow[d, "\varphi_{\alpha *}"] & \tilde{H}_{n-1}\left(X^{(n-1)} / X^{(n-1)} \setminus e_\beta\right) \arrow[ur, "\cong"] & \bigoplus_\beta \tilde{H}_{n-1}(S^{n-1}_\beta) \arrow[u] \\
H_n(X^{(n)}, X^{(n-1)}) \arrow[u, "\cong"] \arrow[rr, "\delta"'] \arrow[rrrr, bend right] & & H_{n-1}(X^{(n-1)}) \arrow[ur, "q_{\beta *}"] \arrow[rr, "j"'] & & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \arrow[u, "\cong"']
\end{tikzcd}$$

$u \longmapsto v \longmapsto \deg(q_\beta \varphi_\alpha)\, v$

where:

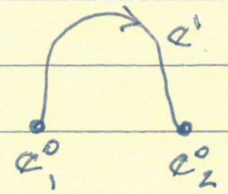
$\Phi_\alpha : D^n \longrightarrow X^{(n)}$ - the characteristic map of e_α

(Note: $\varphi_\alpha = \Phi_\alpha\big|_{S^{n-1}}$)

$\blacksquare$

Note: The differential d_1 can be described as follows:

$$d_1 : \bigoplus_{\alpha \in C_1} H_1(S^1_\alpha) \cong \underbrace{H_1(X^{(1)}, X^{(0)})}_{\substack{\cong \\ \left(\text{free ab. gp. generated by 1-cells}\right)}} \longrightarrow H_0(X^{(0)}, \overset{\emptyset}{\overset{\shortparallel}{X^{(-1)}}}) \cong \underbrace{H_0(X^{(0)})}_{\substack{\cong \\ \left(\text{free ab. gp. generated by 0-cells}\right)}}$$



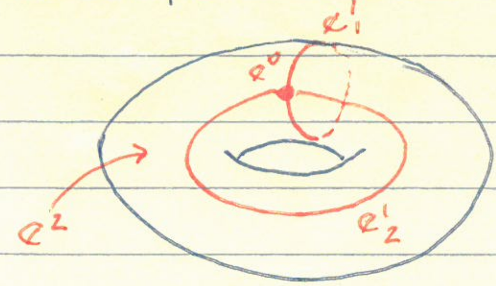
$\longmapsto \quad e^0_2 - e^0_1$

Note: If X has only one 0-cell then $d_1 = 0$.

Example:

$T^2 = S^1 \times S^1$ - torus

CW-complex structure:



1 0-cell

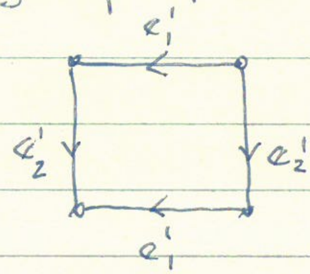
2 1-cells

1 2-cell

Cellular chain complex: $0 \to \overset{\mathbb{Z}}{\underset{\cong}{}}\langle e_2 \rangle \xrightarrow{d_2} \overset{\mathbb{Z}\oplus\mathbb{Z}}{\underset{\cong}{}}\langle e_1^1, e_2^1 \rangle \underset{\underset{0}{\shortparallel}}{\xrightarrow{d_1}} \overset{\mathbb{Z}}{\underset{\cong}{}}\langle e_0 \rangle \to 0$

The attaching map of the 2-cell

$S^1 \cong$



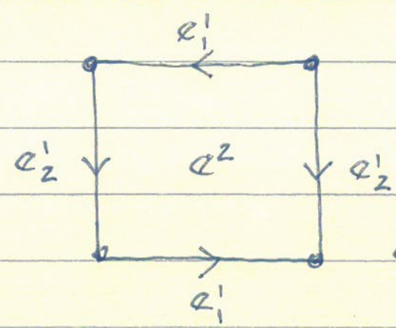
This gives: $d_2(e_2) = 0$

Thus:

$$H_q(T^2) \cong \begin{cases} \mathbb{Z} & q=2 \\ \mathbb{Z}\oplus\mathbb{Z} & q=1 \\ \mathbb{Z} & q=0 \end{cases}$$

Example:

K = the Klein bottle:



Cellular chain complex

$$0 \to \langle e_2 \rangle \xrightarrow{d_2} \langle e_1^1, e_2^1 \rangle \underset{\underset{0}{\shortparallel}}{\xrightarrow{d_1}} \langle e_0 \rangle \to 0$$

$$d_2(e_2) = 2e_1^1 + 0\, e_2^2$$

Thus:

$$H_q(K) \cong \begin{cases} 0 & q=2 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & q=1 \\ \mathbb{Z} & q=0 \end{cases}$$

Projective spaces

Def: The n-dimensional real projective space is the space given by

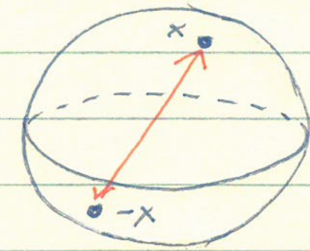
$$\mathbb{R}P^n = \mathbb{R}^{n+1} \smallsetminus \{0\} \big/ \sim$$

where $(x_1, x_2, \dots, x_{n+1}) \sim (\lambda x_1, \lambda x_2, \dots, \lambda x_{n+1})$ for $\lambda \in \mathbb{R} \smallsetminus \{0\}$

(**Note:** $\mathbb{R}P^n$ is the "space of 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$)

Alternative definition:

$$\mathbb{R}P^n = S^n \big/ {x \sim (-x)}$$



Note: $\mathbb{R}P^0 \cong *$

$\mathbb{R}P^1 \cong S^1$

Note: The inclusions $S^1 \subseteq S^2 \subseteq S^3 \subseteq \dots$ give inclusions $\mathbb{R}P^1 \subseteq \mathbb{R}P^2 \subseteq \mathbb{R}P^3 \subseteq \dots$

Define: $\mathbb{R}P^\infty = \bigcup_{n=1}^{\infty} \mathbb{R}P^n$

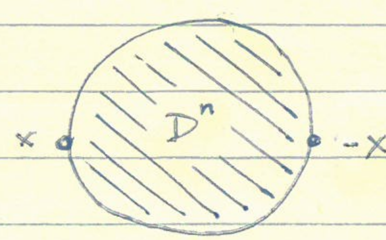
($\mathbb{R}P^\infty$ is taken with the weak topology: $U \subseteq \mathbb{R}P^\infty$ is open iff $U \cap \mathbb{R}P^n$ is open in $\mathbb{R}P^n$ $\forall n$)

CW-complex structure on $\mathbb{R}P^n$

Note: $\mathbb{R}P^n \cong D^n \big/ \sim$

where $x \sim (-x)$

for all $x \in S^{n-1} \subseteq D^n$



Upshot:

$\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching one n-cell:

$$\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup e^n$$

The attaching map of e^n is the quotient map

$$q_n\colon S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$$
$$x \longmapsto [x]$$

We have:

$$\mathbb{RP}^0 = * = e_0$$
$$\mathbb{RP}^n = e_0 \cup e_1 \cup \dots \cup e_n$$
$$\mathbb{RP}^\infty = \underbrace{e_0 \cup e_1 \cup \dots}_{}$$

↑ one cell in each dimension

Cellular chain complex for $\mathbb{RP}^n$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \dots \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & \langle e_n \rangle \arrow[u, phantom, "\cong" rotate=90] \arrow[r, "d_n"] & \langle e_{n-1} \rangle \arrow[u, phantom, "\cong" rotate=90] \arrow[r, "d_{n-1}"] & \dots & \langle e_1 \rangle \arrow[r, "d_1"', "{}"] & \langle e_0 \rangle \arrow[r] & 0
\end{tikzcd}$$

$$d_1 = 0$$

Computation of $d_k\colon \langle e_k \rangle \longrightarrow \langle e_{k-1} \rangle$:

$$f_k\colon \left(S^{k-1} \xrightarrow{q_k} \mathbb{RP}^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1} \right)$$

↑ the quotient map

$$d_k(e_k) = \deg(f_k)\cdot e_{k-1}$$

Note: We have a comm. diagram:

$$\begin{tikzcd}
S^{k-1} \arrow[r, "q_k"] \arrow[d] \arrow[dr, "f_k"] & \mathbb{RP}^{k-1} \arrow[d] \\
S^{k-1}/S^{k-2} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \arrow[d, phantom, "\cong" rotate=90] \\
S^{k-1} \vee S^{k-1} \arrow[r, "\mathrm{id} \vee \alpha"] & S^{k-1}
\end{tikzcd}$$

(α = the antipodal map
$\alpha(x) = -x$)

This gives:

$$\begin{aligned} \deg(f_k) &= \deg(id) + \deg(\alpha) \\ &= 1 + (-1)^k \\ &= \begin{cases} 2 & \text{if } k\text{-even} \\ 0 & \text{if } k\text{-odd} \end{cases} \end{aligned}$$

As a consequence the cellular chain complex of $\mathbb{RP}^n$ looks as follows:

for n-even:

$$\begin{array}{ccccccccccccccccc} & & n & & n-1 & & n-2 & & n-3 & & & & 2 & & 1 & & 0 \\ 0 & \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{\cdot 0} & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{\cdot 0} & \cdots & \xrightarrow{\cdot 0} & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{\cdot 0} & \mathbb{Z} \to 0 \end{array}$$

so: if n-even then

$$H_q(\mathbb{RP}^n) \cong \begin{cases} 0 & q > n \\ 0 & n \geq q > 0,\ q\text{-even} \\ \mathbb{Z}/2 & n > q > 0,\ q\text{-odd} \\ \mathbb{Z} & q = 0 \end{cases}$$

for n-odd

$$\begin{array}{ccccccccccccccccc} & & n & & n-1 & & n-2 & & n-3 & & & & 2 & & 1 & & 0 \\ 0 & \to & \mathbb{Z} & \xrightarrow{\cdot 0} & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{\cdot 0} & \mathbb{Z} & \xrightarrow{\cdot 2} & \cdots & \xrightarrow{\cdot 0} & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{\cdot 0} & \mathbb{Z} \to 0 \end{array}$$

so: if n-odd then:

$$H_q(\mathbb{RP}^n) \cong \begin{cases} 0 & q > n \\ \mathbb{Z} & q = n \\ 0 & n > q > 0,\ q\text{-even} \\ \mathbb{Z}/2 & n > q > 0,\ q\text{-odd} \\ \mathbb{Z} & q = 0 \end{cases}$$

For $\mathbb{RP}^\infty$:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} \overset{\text{even}}{\mathbb{Z}} \xrightarrow{\cdot 2} \overset{\text{odd}}{\mathbb{Z}} \xrightarrow{\cdot 0} \cdots \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \longrightarrow 0$$

So:

$$H_q(\mathbb{RP}^\infty) \cong \begin{cases} 0 & q>0,\ q\text{-even} \\ \mathbb{Z}/2 & q>0,\ q\text{-odd} \\ \mathbb{Z} & q=0 \end{cases}$$

Def: The n-dimensional complex projective space is the space given by:

$$\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\}/\sim$$

where $(z_1, \dots, z_{n+1}) \sim (\lambda z_1, \dots, \lambda z_{n+1})$ for $\lambda \in \mathbb{C} \setminus \{0\}$.

(Note: $\mathbb{CP}^n$ = the space of complex 1-dim linear subspaces of $\mathbb{C}^{n+1}$.)

Alternative definition

$$\mathbb{CP}^n = \overbrace{\{v = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \|v\| = 1\}}^{S^{2n+1}} \Big/ v \sim \lambda v \text{ for } \lambda \in \mathbb{C},\ |\lambda| = 1$$

$$\Big\uparrow \cong$$

$$\underbrace{\{w = (z_2, \dots, z_{n+1}) \in \mathbb{C}^n \mid \|w\| \leq 1\}}_{D^{2n}} \Big/ w \sim \lambda w \text{ for } \lambda \in \mathbb{C},\ |\lambda| = 1,\ \|w\| = 1$$

$$w = (z_2, \dots, z_n) \longmapsto \left(\sqrt{1 - \|w\|^2}, z_2, \dots, z_n\right)$$

Thus: $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a single $2n$-cell e^{2n}. Attaching map is the quotient map $\varphi_n : S^{2n-1} \to \mathbb{CP}^{n-1}$.

<u>We obtain:</u>

$$\mathbb{CP}^n = e^0 \cup e^2 \cup \ldots \cup e^{2n-2} \cup e^{2n}$$

<u>Note:</u> $\mathbb{CP}^0 \subseteq \mathbb{CP}^1 \subseteq \ldots$

<u>Define:</u> $\mathbb{CP}^\infty = \bigcup_{n=1}^{\infty} \mathbb{CP}^n$

We have: $\mathbb{CP}^\infty = e^0 \cup e^2 \cup e^4 \cup \ldots$

<u>Cellular chain complex for $\mathbb{CP}^n$:</u>

$$0 \to \overset{2n}{\mathbb{Z}} \to 0 \to \overset{2n-2}{\mathbb{Z}} \to 0 \to \ldots \quad \to \overset{2}{\mathbb{Z}} \to 0 \to \overset{0}{\mathbb{Z}} \to 0$$

This gives:

$$H_q(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & 0 \leq q \leq 2n \text{, } q\text{-even} \\ 0 & \text{otherwise} \end{cases}$$

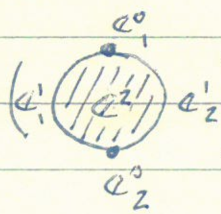
$$H_q(\mathbb{CP}^\infty) \cong \begin{cases} \mathbb{Z} & q \geq 0 \text{, } q\text{-even} \\ 0 & \text{otherwise} \end{cases}$$

Euler characteristic

Def: Let X be a finite CW-complex.
The Euler characteristic of X is the integer

$$\chi(X) = \sum_n (-1)^n \cdot (\#\text{ of } n\text{-cells in } X)$$

Example:

$\chi\Big($  $\Big) = 2 - 2 + 1 = 1$

0-cells 1-cells 2-cells

Recall:
If G - finitely generated abelian group then
$$G \cong \mathbb{Z}^n \oplus (\text{torsion})$$
We define: $\operatorname{rank}(G) = n$.

Note: If X - a finite CW-complex,
C_* - the cellular chain complex of X
Then $\chi(X) = \sum_n (-1)^n \cdot \operatorname{rank}(C_n)$

(*) **Theorem:** If X - a finite CW-complex then
$$\chi(X) = \sum_n (-1)^n \cdot \operatorname{rank}(H_n(X))$$

Corollary: 1) $\chi(X)$ does not depend on the CW-structure of X.
2) if $X \simeq Y$ then $\chi(X) = \chi(Y)$.

Theorem (*) follows directly from the following fact:

Proposition

Let C_* be a chain complex of abelian groups such that

1) $C_n \neq 0$ for finitely many n only
2) C_n is finitely generated for all n

Then

$$\sum_n (-1)^n \operatorname{rank}(C_n) = \sum_n (-1)^n \operatorname{rank}(H_n(C_*))$$

Proof:

Check: If $0 \to A \to B \to C \to 0$ - short exact sequence of abelian groups then

$$\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$$

The chain complex C_* gives for each n short exact sequences:

$$0 \to Z_n(C_*) \to C_n \xrightarrow{\partial} B_{n-1}(C_*) \to 0$$

and:

$$0 \to B_n(C_*) \to Z_n(C_*) \to H_n(C_*) \to 0$$

This gives:

$$\begin{aligned} \operatorname{rank}(C_n) &= \operatorname{rank}(Z_n(C_*)) + \operatorname{rank}(B_{n-1}(C_*)) \\ &= \operatorname{rank}(B_n(C_*)) + \operatorname{rank}(H_n(C_*)) + \operatorname{rank}(B_{n-1}(C_*)) \end{aligned}$$

so:

$$\sum_n (-1)^n \operatorname{rank}(C_n) = \sum_n (-1)^n H_n(C_*)$$

∎