55

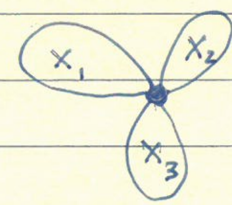
Computations / applications

Homology of wedges of spaces

Recall: 1) A pointed space is a space X with a choice of a basepoint $x_0 \in X$.

2) If $\{(X_i, x_i)\}_{i\in I}$ - a family of pointed spaces then

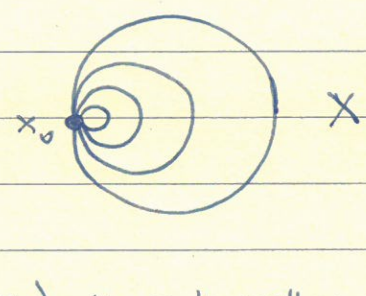
$$\underbrace{\bigvee_{i\in I} X_i}_{\text{wedge of } \{X_i\}} = \coprod X_i \Big/ x_i \sim x_j$$



Def: A well-pointed space is a pointed space (X, x_0) such that the inclusion $\{x_0\} \hookrightarrow X$ is a cofibration.

Example: If X - CW-complex, $x_0 \in X$ then (X, x_0) is well-pointed.

Example: Let X = the Hawaiian earring space:



The space (X, x_0) is not well-pointed. (exercise).

Proposition: If $\{(X_i, x_i)\}_{i\in I}$ is a family of well-pointed spaces then the inclusion maps $j_i : X_i \hookrightarrow \bigvee_{i\in I} X_i$ induce an isomorphism:

$$\bigoplus_{i\in I} j_{i*} : \bigoplus_i \tilde{H}_n(X_i) \to \tilde{H}_n(\bigvee_i X_i)$$

Proof: We have a cofibration:

$$\coprod_i \{x_i\} \hookrightarrow \coprod X_i$$

and $\coprod X_i / \coprod \{x_i\} = \bigvee X_i$. This gives a long exact sequence:

for $n \geq 1$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(\{x_i\}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_n(\coprod X_i) \arrow[r, "\cong"] & \widetilde{H}_n(\bigvee X_i) \arrow[r] & H_{n-1}(\{x_i\}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \cdots \\
& 0 & \bigoplus \widetilde{H}_n(X_i) \arrow[u, "\cong"] \arrow[ur, "\bigoplus j_{i*}"'] & & 0 &
\end{tikzcd}$$

for $n = 1$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_1(\{x_i\}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_1(\coprod X_i) \arrow[r, "\cong"] & \widetilde{H}_1(\bigvee X_i) \arrow[r, "0"] & H_0(\{x_i\}) \arrow[r, "1-1"] & H_0(\coprod X_i) \arrow[r] & \cdots \\
& 0 & \bigoplus \widetilde{H}_n(X_i) \arrow[u, "\cong"] \arrow[ur, "\bigoplus j_{i*}"'] & & & &
\end{tikzcd}$$

for $n = 0$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_0(\{x_i\}) \arrow[r, "1-1"] & H_0(\coprod X_i) \arrow[r] & \widetilde{H}_0(\bigvee X_i) \arrow[r] & 0 \\
& \bigoplus_i H_0(\{x_i\}) \arrow[r] \arrow[u, "\cong"'] & \bigoplus_i H_0(X_i) \arrow[r] \arrow[u, "\cong"'] & \bigoplus_i \left(H_0(X_i) / H_0(\{x_i\}) \right) \arrow[r, "\cong"] \arrow[u, "\cong"'] & \bigoplus_i \widetilde{H}_0(X_i) \arrow[ul, "\bigoplus j_{i*}"']
\end{tikzcd}$$

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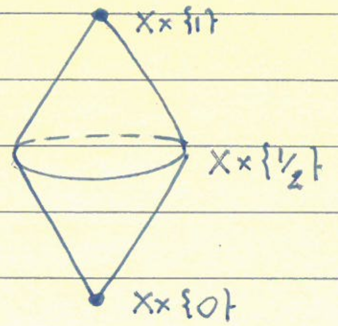
The suspension isomorphism

Def: The suspension of a space X is the space



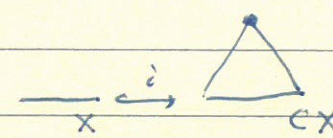
$$\Sigma X = X \times [0,1] \Big/ \begin{smallmatrix} (x,0) \sim (x',0) \\ (x,1) \sim (x',1) \end{smallmatrix} \quad \forall x, x' \in X$$

Example: $\Sigma S^n \cong S^{n+1} \quad \forall n$

Proposition: For any X there is an isomorphism

$$\delta : \tilde{H}_n(\Sigma X) \longrightarrow \tilde{H}_{n-1}(X)$$

Proof: Notice that the map $i : X \longrightarrow CX$ is a cofibration and $\Sigma X = CX/X$.



This gives a long exact sequence:

$$\cdots \to \tilde{H}_n(X) \to \underset{\overset{\cong}{0}}{\tilde{H}_n(CX)} \to \tilde{H}_n(\Sigma X) \xrightarrow{\delta} \tilde{H}_{n-1}(X) \to \underset{\overset{\cong}{0}}{\tilde{H}_{n-1}(CX)} \to \cdots$$

We get: $\delta : \tilde{H}_n(\Sigma X) \longrightarrow \tilde{H}_{n-1}(X)$ is an isomorphism $\forall n$

Note: The suspension isomorphism is natural: any map $f : X \longrightarrow Y$ induces a map $\Sigma f : \Sigma X \to \Sigma Y$, $(x,t) \longmapsto (f(x), t)$ and the following diagram commutes:

$$\begin{tikzcd}
\tilde{H}_n(\Sigma X) \arrow[r, "\delta", "\cong"'] \arrow[d, "\Sigma f_*"'] & \tilde{H}_{n-1}(X) \arrow[d, "f_*"] \\
\tilde{H}_n(\Sigma Y) \arrow[r] & \tilde{H}_{n-1}(Y)
\end{tikzcd}$$

Corollary:

If X, Y are spaces s.t. $\Sigma^k X \simeq \Sigma^k Y$ for some $k \geq 0$ then $\tilde{H}_n(X) \cong \tilde{H}_n(Y)$ $\forall_n$

Proof: We have:

$$\tilde{H}_n(X) \cong \tilde{H}_{n+k}(\Sigma^k X) \cong \tilde{H}_{n+k}(\Sigma^k Y) \cong \tilde{H}_n(Y).$$

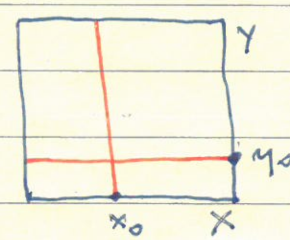
Note:

It may happen that $X \not\simeq Y$ but $\Sigma^k X \simeq \Sigma^k Y$ for some k.

Example: Let $(X, x_0), (Y, y_0)$ - well pointed spaces

We have a cofibration:

$$i: X \vee Y \longrightarrow X \times Y$$
$$x \longmapsto (x, y_0)$$
$$y \longmapsto (x_0, y)$$



The smash product of X and Y is the space

$$X \wedge Y = X \times Y / X \wedge Y$$

E.g. if $X = S^n$, $Y = S^m$ then $X \times Y \cong S^{n+m}$

One can show that for any X, Y there is a homotopy equivalence:

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y) \simeq \Sigma(X \vee Y \vee (X \wedge Y))$$

If $X = Y = S^1$ this gives

$$\Sigma(S^1 \times S^1) \simeq \underbrace{\Sigma S^1 \vee \Sigma S^1 \vee \Sigma(S^1 \wedge S^1)}_{S^2 \vee S^2 \vee S^3} \simeq \Sigma(S^1 \vee S^1 \vee S^2)$$

59

This implies:

$$\tilde{H}_n(S^1 \times S^1) \cong \tilde{H}_n(S^1 \vee S^1 \vee S^2) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & \text{otherwise} \end{cases}$$

On the other hand:

$$\pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$\pi_1(S^1 \vee S^1 \vee S^2) \cong \mathbb{Z} * \mathbb{Z} \quad \text{- free group on 2 generators}$$

so: $S^1 \times S^1 \not\cong S^1 \vee S^1 \vee S^2$.

60

Some applications of homology

Recall: for $n \geq 0$

$$\tilde{H}_q(S^n) \cong \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases}$$

Theorem:

Let D^n - closed n-dimensional disc.

For all $n \geq 1$ S^{n-1} is not a retract of D^n.

Pf: Let $i: S^{n-1} \to D^n$ be the inclusion map.

If $r: D^n \to S^{n-1}$ were a retraction then we would have a commutative diagram:

$$\begin{tikzcd} \mathbb{Z} \cong \tilde{H}_{n-1}(S^{n-1}) \arrow[rr, "id_*"] \arrow[dr, "i_*"'] & & \tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z} \\ & \tilde{H}_{n-1}(D^n) \arrow[ur, "r_*"'] & \\ & 0 \arrow[u, phantom, "\cong" rotate=90] & \end{tikzcd}$$

which is impossible. $\blacksquare$

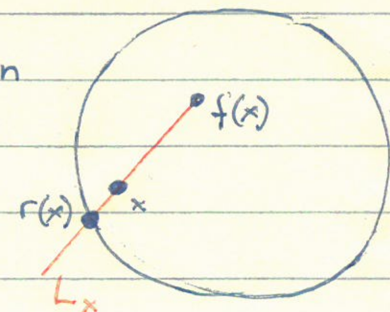
Brouwer Fixed Point Theorem

If $f: D^n \to D^n$ is any continuous map then $\exists x_0 \in D^n$ s.t. $f(x_0) = x_0$.

(61)

Pf: Assume that there exists a map $f: D^n \to D^n$ s.t. $f(x) \neq x$ for all $x \in D^n$. Define $r: D^n \to S^{n-1}$ as follows: For $x \in D^n$ let $L_x \subseteq \mathbb{R}^n$ be the half line that begins at $f(x)$ and passes through x. Take $r(x)$ to be the point of intersection of L_x with S^{n-1}.

The map r is a retraction of D^n onto S^{n-1} which is impossible by the previous theorem.



▨

Theorem:

If $n \neq m$ then $\mathbb{R}^n \not\cong \mathbb{R}^m$.

Proof: Assume that $f: \mathbb{R}^n \to \mathbb{R}^m$ - homeomorphism and $m \neq n$. For $x_0 \in \mathbb{R}^n$ we have a homeomorphism:

$$f|_{\mathbb{R}^n \setminus \{x_0\}} : \mathbb{R}^n \setminus \{x_0\} \longrightarrow \mathbb{R}^m \setminus \{f(x_0)\}$$

This induces an isomorphism:

$$\left(f|_{\mathbb{R}^n \setminus \{x_0\}}\right)_* : \tilde{H}_q(\mathbb{R}^n \setminus \{x_0\}) \xrightarrow{\cong} \tilde{H}_q(\mathbb{R}^m \setminus \{f(x_0)\}) \quad \forall q$$

However $\tilde{H}_{n-1}(\mathbb{R}^n \setminus \{x_0\}) \cong \tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$

$$\tilde{H}_{n-1}(\mathbb{R}^m \setminus \{f(x_0)\}) \cong \tilde{H}_{n-1}(S^{m-1}) \cong 0$$

- contradiction.

▨

62

<u>Recall:</u> A space M is an n-dimensional manifold if M is Hausdorff, has countable basis, and if each point $x \in M$ has an open neighborhood $U_x \subseteq M$ s.t. $U_x \cong \mathbb{R}^n$.

<u>Theorem:</u> Dimension of a topological manifold is well defined: if $M \neq \emptyset$ is a space that has a structure of an m-dimensional manifold and of an n-dimensional manifold then $m = n$.

<u>Proof:</u> Assume that M has a structure of an m-dimensional manifold, let $x_0 \in M$ and let $U_{x_0} \subseteq M$ - open nbhd of x_0 s.t. $U_{x_0} \cong \mathbb{R}^m$.

Notice that $\underbrace{M \setminus U_{x_0}}_{\text{closed in } M} \subseteq \underbrace{M \setminus \{x_0\}}_{\text{open in } M} \subseteq M$

By excision we get:

$$
\begin{aligned}
H_q(M, M \setminus \{x_0\}) &\cong H_q(M \setminus (M \setminus U_{x_0}), (M \setminus \{x_0\}) \setminus (M \setminus U_{x_0})) \\
&\cong H_q(U_{x_0}, U \setminus \{x_0\}) \\
&\cong H_q(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\})
\end{aligned}
$$

We have the long exact sequence of the pair $(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\})$:

<u>for $q \geq 2$:</u>

$$\ldots \to \underset{\underset{0}{\cong}}{H_q(\mathbb{R}^m)} \to H_q(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \to H_{q-1}(\mathbb{R}^m \setminus \{0\}) \to \underset{\underset{0}{\cong}}{H_{q-1}(\mathbb{R}^m)} \to \ldots$$

<u>for $q = 1$:</u>

$$\to \underset{\underset{0}{\cong}}{H_1(\mathbb{R}^m)} \to H_1(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \to \overbrace{H_0(\mathbb{R}^m \setminus \{0\})}^{\cong \begin{cases} 0 & m=0 \\ \mathbb{Z} \oplus \mathbb{Z} & m=1 \\ \mathbb{Z} & m>1 \end{cases}} \to H_0(\mathbb{R}^m) \to \ldots$$

<u>for $q = 0$:</u> $\ldots \to H_0(\mathbb{R}^m \setminus \{0\}) \to H_0(\mathbb{R}^m) \to H_0(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \to 0$

63

<u>This gives:</u>

$$H_q(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \cong \begin{cases} \mathbb{Z} & q = m \\ 0 & q \neq m \end{cases}$$

We obtain:

$$H_q(M, M \setminus \{x_0\}) \cong \begin{cases} \mathbb{Z} & q = m \\ 0 & q \neq m \end{cases}$$

By the same argument, if M has a structure of an n-dimensional manifold then

$$H_q(M, M \setminus \{x_0\}) \cong \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases}$$

Thus we must have: $m = n$.