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<u>Next goal</u>: Develop tools for calculating $H_*(X)$:

1) For $X = A \cup B$ relate the groups $H_*(A)$, $H_*(B)$, $H_*(A \cap B)$ and $H_*(X)$.
2) For $A \subseteq X$ relate the groups $H_*(X)$, $H_*(A)$, $H_*(X/A)$

<u>More homological algebra</u>

<u>Def</u>: A sequence of abelian groups and homomorphisms

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

is <u>exact</u> if $\operatorname{Im}(\alpha_{n+1}) = \operatorname{Ker}(\alpha_n)$ $\forall n$

<u>Note</u>: If (C_*, ∂) is a chain complex then $\operatorname{Im}(\partial_{n+1}) \subseteq \operatorname{Ker}(\partial_n)$ and $H_n(C_*) = \operatorname{Ker}(\partial_n) / \operatorname{Im}(\partial_{n+1})$

Thus the homology groups $H_n(C_*)$ measure how far (C_*, ∂) is from being exact.

<u>Example</u>:

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{0} \cdots$$

is an exact sequence.

<u>Note</u>:

$0 \to A \xrightarrow{f} B$ is exact iff f is 1-1

$A \xrightarrow{f} B \to 0$ is exact iff f is onto

$0 \to A \xrightarrow{f} B \to 0$ is exact iff f is an isomorphism

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Def: A short exact sequence is an exact sequence of the form:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

(i.e: f is 1-1
g is onto
$\mathrm{Im}(f) = \mathrm{Ker}(g)$)

Example: $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ is exact

Future applications:

I Long exact sequence of a cofibration

Cofibration = "nice" inclusion $i: A \hookrightarrow X$ of a subspace

Example: X - CW-complex, $A \subseteq X$ - subcomplex then $i: A \hookrightarrow X$ is a cofibration

Thm: If $i: A \to X$ is a cofibration then there is a long exact sequence of homology groups:

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\delta} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \cdots$$
$$\cdots \to \tilde{H}_0(A) \xrightarrow{i_*} \tilde{H}_0(X) \xrightarrow{q_*} \tilde{H}_0(X/A) \to 0$$

Here:

$i_* : \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X)$ - homom. induced by $i : A \hookrightarrow X$

$q_* : \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X/A)$ - homom. induced by the quotient map $q : X \to X/A$

$$\tilde{H}_n(X) = \operatorname{Ker}\left(H_n(X) \to H_n(*)\right)$$

= the reduced n^{th} homology group of X

Note:

$$H_n(*) \cong \begin{cases} 0 & n > 0 \\ \mathbb{Z} & n = 0 \end{cases}$$

so:

$$\tilde{H}_n(X) \cong \begin{cases} H_n(X) & n > 0 \\ \operatorname{Ker}\left(H_0(X) \to H_0(*)\right) \cong \bigoplus_{i=1}^{N-1} \mathbb{Z} \end{cases}$$

$$\begin{array}{ccc} H_0(X) & \longrightarrow & H_0(*) \\ \cong & & \cong \\ \bigoplus_{i=1}^{N} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ (n_i) & \longmapsto & \Sigma n_i \end{array}$$

In particular if X - path conn. then

$$\tilde{H}_n(X) \cong \begin{cases} H_n(X) & n > 0 \\ 0 & n = 0 \end{cases}$$

Theorem: For $n \geqslant 0$ we have:

$$\tilde{H}_q(S^n) = \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases}$$

Proof: By induction with respect to n.

For $n = 0$ we have:

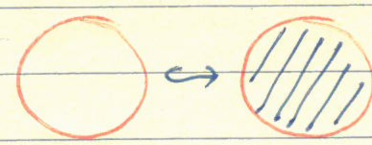
$$\tilde{H}_q(S^0) = \begin{cases} 0 & q > 0 \\ \mathbb{Z} & q = 0 \end{cases}$$

Inductive step: assume that

$$\tilde{H}_q(S^{n-1}) \cong \begin{cases} \mathbb{Z} & q = n-1 \\ 0 & q \neq n-1 \end{cases}$$

Note: 1) we have a cofibration: $i: S^{n-1} \hookrightarrow D^n$

$\uparrow$ n-dim. disc



2) $D^n / S^{n-1} \cong S^n$

3) $D^n \simeq *$ so $\tilde{H}_q(D^n) = 0 \quad \forall q$

Take the long exact sequence of the cofibr. $i: S^{n-1} \hookrightarrow D^n$:

$$\cdots \to \tilde{H}_q(S^{n-1}) \to \underset{\underset{0}{\parallel}}{\tilde{H}_q(D^n)} \to \underset{\underset{\tilde{H}_q(S^n)}{\cong}}{\tilde{H}_q(D^n/S^{n-1})} \xrightarrow{\delta} \tilde{H}_{q-1}(S^{n-1}) \to \underset{\underset{0}{\parallel}}{H_{q-1}(D^n)} \to \cdots$$

$$\cdots \to \underset{\underset{0}{\parallel}}{\tilde{H}_1(D^n)} \to \underset{\underset{\tilde{H}_1(S^n)}{\cong}}{\tilde{H}_1(D^n/S^{n-1})} \xrightarrow{\delta} \tilde{H}_0(S^{n-1}) \to \underset{\underset{0}{\parallel}}{\tilde{H}_0(D^n)} \to \underset{\underset{\tilde{H}_0(S^n)}{\cong}}{\tilde{H}_0(D^n/S^{n-1})} \to 0$$

This gives: $\tilde{H}_0(S^n) = 0$

$\tilde{H}_q(S^n) \cong \tilde{H}_{q-1}(S^{n-1})$ for $q > 0$

so:

$$\tilde{H}_q(S^n) \cong \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases}$$

▨

II The Mayer-Vietoris sequence

Thm: Let $A, B \subseteq X$ be subspaces s.t.

$$X = \text{int}(A) \cup \text{int}(B)$$

interior of A

and let

$$\begin{tikzcd} & A \arrow[dr, "j_A"] & \\ A \cap B \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & X \\ & B \arrow[ur, "j_B"'] & \end{tikzcd}$$

be inclusion maps.

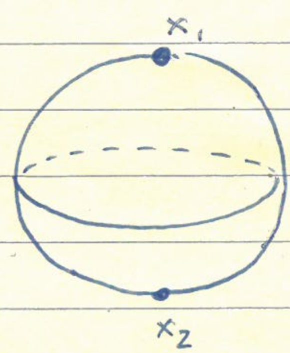
The following sequence is exact:

$$\cdots \to H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0$$

$$x \longmapsto (i_{A*}(x), i_{B*}(x))$$

$$(y, z) \longmapsto j_{A*}(y) - j_{B*}(z)$$

Exercise: Compute $H_q(S^n)$ using the M-V sequence:



$A = S^n \setminus \{x_1\}$

$B = S^n \setminus \{x_2\}$

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From short to long exact sequences

Def: Let A_*, B_*, C_* be chain complexes and let $f_*: A_* \to B_*$, $g_*: B_* \to C_*$ be chain maps. The sequence

$$0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$$

is a short exact sequence of chain complexes if

$$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$$

is a short exact sequence of abelian groups $\forall n$.

Proposition:

If $0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$ is a short ex. seq. of chain complexes then $\forall n$ there is a homomorphism

$$\delta: H_n(C_*) \to H_{n-1}(A_*)$$

s.t. the following sequence is exact:

$$\dots \to H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{f_*} H_{n-1}(B_*) \to \dots$$

Proof: (diagram chasing)

We have a comm. diagram:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_n & \xrightarrow{f_*} & B_n & \xrightarrow{g_*} & C_n & \to & 0 \\
 & & \downarrow \partial^A & & \downarrow \partial^B & & \downarrow \partial^C & & \\
0 & \to & A_{n-1} & \xrightarrow{f_*} & B_{n-1} & \xrightarrow{g_*} & C_{n-1} & \to & 0 \\
 & & \downarrow \partial^A & & \downarrow \partial^B & & \downarrow \partial^C & & \\
0 & \to & A_{n-2} & \xrightarrow{f_*} & B_{n-2} & \xrightarrow{g_*} & C_{n-2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

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where: - all rows are exact
- columns are chain complexes
- all squares commute

Construction of $\delta : H_n(C_*) \to H_{n-1}(A_*)$

Take $[c] \in H_n(C_*)$

We want $\delta[c] \in H_{n-1}(A_*)$

$c \in Z_n(C_*) = \mathrm{Ker}(\partial^C : C_n \to C_{n-1})$

g_* - onto $\Rightarrow \exists\, b \in B_n$ s.t. $g_*(b) = c$

Note: $g_* \circ \partial^B(b) = \partial^C g_*(b) = \partial^C(c) = 0$

So: $\partial^B(b) \in \mathrm{Ker}(g_* : B_{n-1} \to C_{n-1})$

$\qquad\qquad\qquad\qquad \| \leftarrow$ exactness

$\qquad\qquad \mathrm{Im}(f_* : A_{n-1} \to B_{n-1})$

Note:

1) f_* - mono $\Rightarrow \dot{\exists}\, a \in A_{n-1}$ s.t. $f_*(a) = \partial^B(b)$

2) $f_* \circ \partial^A(a) = \partial^B \circ f_*(a) = \partial^B \partial^B(b) = 0$

since f_* - mono this gives: $\partial^A(a) = 0$

This means that $a \in Z_{n-1}(A_*)$ and so it represents an element $[a] \in H_{n-1}(A_*)$.

Define: $\delta[c] := [a]$

Check (exercise):

1) $\delta : H_n(C_*) \to H_{n-1}(A_*)$ is a well defined homomorphism

2) the resulting sequence is exact:

$$\ldots \to H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{f_*} \ldots$$

<u>Back to topological spaces</u>

X - space

$A \subseteq X$ - subspace

The inclusion $i: A \hookrightarrow X$ induces a map $i_*: C_*(A) \to C_*(X)$

We have a chain complex:

$$C_*(X,A) := \left(\dots \to C_n(X)/C_n(A) \xrightarrow{\partial} C_{n-1}(X)/C_{n-1}(A) \xrightarrow{\partial} \dots \right)$$

<u>Def</u>: The chain complex $C_*(X,A)$ is called the <u>relative singular chain complex</u> of the pair (X,A).

The homology groups

$$H_n(X,A) := H_n(C_*(X,A))$$

are the <u>relative singular homology groups</u> of (X,A).

<u>Note</u>:

For a pair of spaces $A \subseteq X$ we have a short exact sequence of singular chain complexes:

$$0 \to C_*(A) \xrightarrow{i_*} C_*(X) \to C_*(X,A) \to 0$$

This gives a long exact sequence of homology groups

$$\dots \to H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} \dots \to H_0(X,A) \to 0$$

This is the <u>long exact sequence of the pair</u> (X,A).

<u>Note</u>:

1) If A is a contractible space then $H_n(X,A) \cong \tilde{H}_n(X)$ $\forall n$ (exercise). In particular if $x_0 \in X$ then $H_n(X,x_0) \cong \tilde{H}_n(X)$

2) If X is contractible then $H_n(X,A) \cong \tilde{H}_{n-1}(A)$ for $n>0$ and $H_0(X,A) = 0$ (exercise).

3) We will see that if $A \hookrightarrow X$ is a cofibration then $H_n(X,A) \cong \tilde{H}_n(X/A)$. This and the long exact sequence of the pair (X,A) will give the long exact sequence of a cofibration.

4) The long exact sequence of a pair is natural: if $A \subseteq X$, $B \subseteq Y$, and $f: X \to Y$ is a map s.t. $f(A) \subseteq B$ the we have a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\delta"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\delta"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

(exercise).

<u>Sidenote:</u> Natural transformations

<u>Def:</u> Let $F, G: \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\eta: F \Rightarrow G$ is a collection of morphisms $\eta = \{\eta_c : F(c) \to G(c)\}_{c \in \mathcal{C}}$ such that for every morphism $f: c \to c'$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd}
F(c) \arrow[r, "\eta_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\
F(c') \arrow[r, "\eta_{c'}"'] & G(c')
\end{tikzcd}$$

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Example:

Recall: If G - group then $G/[G,G]$ - the abelianization of G. This defines a functor

$$F: Gr \longrightarrow Gr$$
$$G \longmapsto G/[G,G]$$

the category of groups

Let $Id: Gr \to Gr$ - the identity functor.

For any $G \in Gr$ we have the quotient homomorphism:

$$\eta_G : \underset{\underset{Id(G)}{\shortparallel}}{G} \longrightarrow \underset{\underset{F(G)}{\shortparallel}}{G/[G,G]}$$

This defines a natural transformation $\eta : Id \Rightarrow F$

Example:

Let Top_2 = the category of pairs of top. spaces.

Objects: (X,A) s.t. $A \subseteq X$

Morphisms: $f: (X,A) \to (Y,B)$ - cont. map $f: X \to Y$ s.t. $f(A) \subseteq B$

For $n \geq 0$ we have functors

$$H_n : Top_2 \longrightarrow Ab$$
$$(X,A) \longmapsto H_n(X,A)$$

$$H_n^{sub} : Top_2 \longrightarrow Ab$$
$$(X,A) \longrightarrow H_n(A)$$

For $n > 0$ the boundary maps $\delta: H_n(X,A) \to H_{n-1}(A)$ define a natural transformation $\delta: H_n \to H_{n-1}^{sub}$

Excision Theorem

closure of Z in X interior of A in X

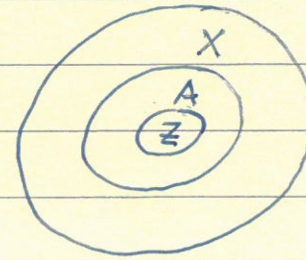
Let $Z \subseteq A \subseteq X$ be subspaces s.t. $\overline{Z} \subseteq \mathrm{int}(A)$.

Then the inclusion

$$(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$$

induces isomorphisms:

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A) \quad \forall n$$



Corollary (Excision II)

If $A, B \subseteq X$, $\mathrm{int}(A) \cup \mathrm{int}(B) = X$ then the inclusion

$$(B, A \cap B) \hookrightarrow (X, A)$$

induces isomorphisms

$$H_n(B, B \cap A) \xrightarrow{\cong} H_n(X, A) \quad \forall n$$

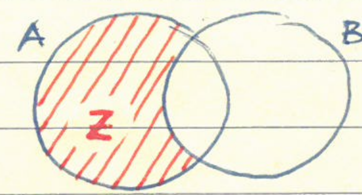
Proof of Corollary:

Take $Z = X \setminus B = A \setminus (A \cap B)$

Then $Z \subseteq A \subseteq X$

Also: $\overline{Z} = \overline{X \setminus B} \subseteq X \setminus \mathrm{int}(B) \subseteq \mathrm{int}(A)$

actually $=$ since $\mathrm{int}(A) \cup \mathrm{int}(B) = X$



Thus we can apply the excision theorem.

$\blacksquare$

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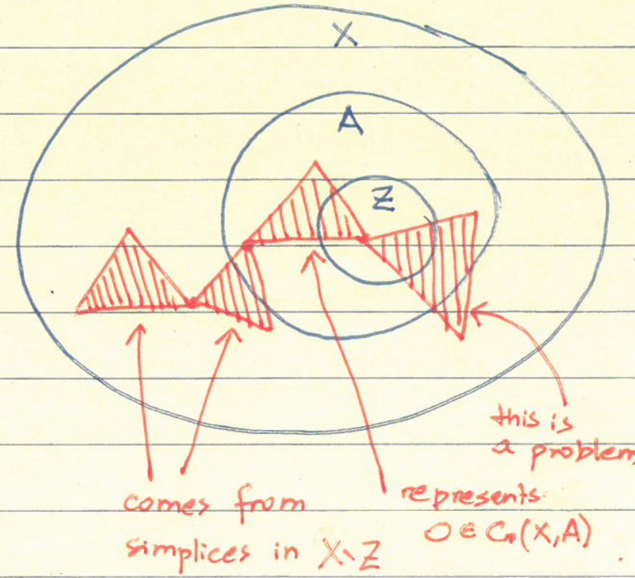
Idea of the proof of Excision Thm:

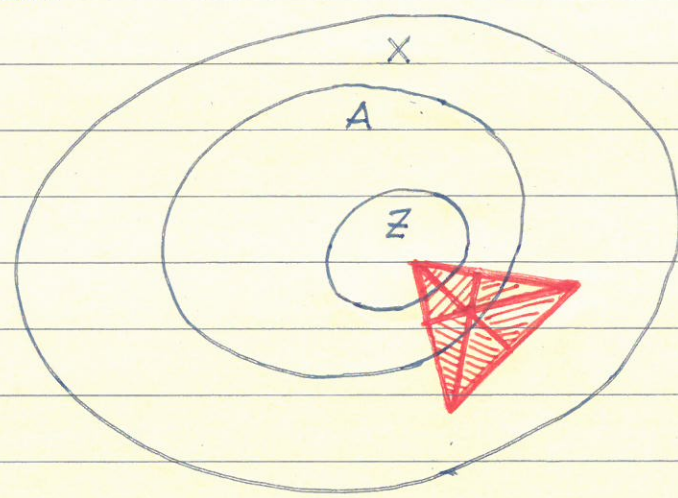
- Each chain in $C_*(X,A)$ is represented by some combination of singular simplices in X:



- simplices that do not touch Z come from $C_*(X \smallsetminus Z, A \smallsetminus Z)$ and simplices contained in A represent $0 \in C_*(X,A)$

- the only problem comes from simplices that intersect Z and are not contained in A.

- <u>idea</u>: subdivide simplices so that this last situation does not happen:



- this lets us replace a chain in $C_*(X,A)$ by a chain in $C_*(X \smallsetminus Z, A \smallsetminus Z)$

(38)

Details

Let X - top. space, $\mathcal{U} = \{\mathcal{U}_i\}$ - open cover of X:
$\mathcal{U}_i \subseteq X$ - open $\forall_i$, $\bigcup_i \mathcal{U}_i = X$.

Define: $C_n^{\mathcal{U}}(X)$ = the subgroup of $C_n(X)$ generated by all singular simplices $\sigma: \Delta^n \to X$ s.t. $\sigma(\Delta^n) \subseteq \mathcal{U}_i$ for some $i \in I$.

This defines a sub-chain complex $C_*^{\mathcal{U}}(X) \subseteq C_*(X)$:

$$\ldots \to C_n^{\mathcal{U}}(X) \xrightarrow{\partial} C_{n-1}^{\mathcal{U}}(X) \to C_{n-2}^{\mathcal{U}}(X) \to \ldots$$

(*) Proposition:
If X is a space and $\mathcal{U} = \{\mathcal{U}_i\}$ is an open cover of X then the inclusion map

$$i_*: C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$$

induces isomorphisms of homology groups:

$$i_*: \underbrace{H_*(C^{\mathcal{U}}(X))}_{\substack{\shortparallel \\ H_*^{\mathcal{U}}(X)}} \to \underbrace{H_*(C_*(X))}_{\substack{\shortparallel \\ H_*(X)}}$$

Proof: See e.g. Hatcher Prop. 2.21 p.119.

(**) Corollary: Let X - space, $\mathcal{U} = \{\mathcal{U}_i\}$ - open cover of X, $A \subseteq X$. Define: $C_n^{\mathcal{U}}(X, A) = C_n^{\mathcal{U}}(X) / C_n^{\mathcal{U}}(A)$.
The inclusion

$$i_*: C_*^{\mathcal{U}}(X, A) \to C_*(X, A)$$

induces isomorphisms of homology groups:

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$$i_*: \underbrace{H_n(C_*^{\mathcal{U}}(X,A))}_{\substack{\| \\ H_n^{\mathcal{U}}(X,A)}} \longrightarrow \underbrace{H_n(C_*(X,A))}_{\substack{\| \\ H_n(X,A)}}$$

Proof of the excision theorem:

Take $Z \subseteq A \subseteq X$. Since $\bar{Z} \subseteq \mathrm{int}(A)$ we have an open cover $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$ of X where $\mathcal{U}_1 = X \setminus \bar{Z}$, $\mathcal{U}_2 = \mathrm{int}(A)$.

We have a comm. diagram of homology groups:

$$\begin{tikzcd}
H_n(X\setminus Z, A\setminus Z) \arrow[r, "j_*"] & H_n(X,A) \\
H_n^{\mathcal{U}}(X\setminus Z, A\setminus Z) \arrow[u] \arrow[r, "j_*"'] & H_n^{\mathcal{U}}(X,A) \arrow[u]
\end{tikzcd}$$

By Corollary (**) the vertical maps are isomorphisms. It suffices then to show that $j_*: H^{\mathcal{U}}(X\setminus Z, A\setminus Z) \to H^{\mathcal{U}}(X,A)$ is an isomorphism.

1. j_* is onto:

Indeed: if $x \in H_n^{\mathcal{U}}(X,A)$

then $x = [\sum_i n_i \sigma_i + \sum_j m_j \tau_j]$

where

$$\sigma_i : \Delta^n \longrightarrow \mathcal{U}_1 = X \setminus \bar{Z} \subseteq X \setminus Z \quad \forall_i$$
$$\tau_j : \Delta^n \longrightarrow \mathcal{U}_2 = \mathrm{int}(A) \subseteq A \quad \forall_j$$

Note that

$$[\sum n_i \sigma_i + \sum m_j \tau_j] = [\sum n_i \sigma_i] \text{ in } H_n^{\mathcal{U}}(X,A)$$

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Also: $[\Sigma n_i \sigma_i] \in H^{\mathcal{U}}_n(X \setminus Z, A \setminus Z)$

Thus $x = j_*([\Sigma n_i \sigma_i])$.

② <u>j_* is 1-1</u>:

Indeed: assume that $[\sum_i n_i \sigma_i] \in H^{\mathcal{U}}_n(X \setminus Z, A \setminus Z)$ and $j_*([\Sigma n_i \sigma_i]) = [\Sigma n_i \sigma_i] = 0 \in H^{\mathcal{U}}_n(X, A)$.

<u>Then</u>: $\sum_i n_i \sigma_i = \partial w_1 + w_2$

where: $w_1 \in C^{\mathcal{U}}_{n+1}(X)$, $w_2 \in C^{\mathcal{U}}_n(A)$

↑ we can assume: $w_1 \in C^{\mathcal{U}}_{n+1}(X \setminus Z)$

This gives:

$$\underbrace{(\Sigma n_i \sigma_i) - \partial w_1}_{\text{singular chain in } X \setminus Z} = \underbrace{w_2}_{\text{singular chain in } A}$$

It follows that w_2 is a singular chain in $(X \setminus Z) \cap A = A \setminus Z$

<u>We obtain</u>:

$$\Sigma n_i \sigma_i = \partial w_1 + w_2$$

where:

$w_1 \in C^{\mathcal{U}}_{n+1}(X \setminus Z)$

$w_2 \in C^{\mathcal{U}}_n(A \setminus Z)$

This gives:

$$[\Sigma n_i \sigma_i] = 0 \text{ in } H^{\mathcal{U}}_n(X \setminus Z, A \setminus Z)$$

(41) Application: the Mayer-Vietoris sequence.

Thm: Let $A, B \subseteq X$ - subspaces s.t.

$$X = \mathrm{int}(A) \cup \mathrm{int}(B)$$

There exists an exact sequence:

$$\ldots \to H_n(A\cap B) \xrightarrow{(i_{A*}, i_{B*})} H_n(A) \oplus H_n(B) \xrightarrow{j_{A*} - j_{B*}} H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B) \to \ldots$$

$$x \longmapsto (i_{A*}(x), i_{B*}(x))$$

$$(y, z) \longmapsto j_{A*}(y) - j_{B*}(z)$$

where:

$$\begin{tikzcd} & A \arrow[dr, "j_A"] & \\ A\cap B \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & X \\ & B \arrow[ur, "j_B"'] & \end{tikzcd}$$

- inclusion maps, and $\delta : H_n(X) \xrightarrow{\delta} H_{n-1}(A\cap B)$
is given by:

$$H_n(X) \longrightarrow H_n(X, B) \xrightarrow{\cong} H_n(A, A\cap B) \longrightarrow H_{n-1}(A \cap B)$$

↑ Excision II ↑ the boundary homomorphism

Barratt - Whitehead Lemma:

Consider a comm. diagram of abelian groups:

$$\begin{tikzcd} \ldots \arrow[r] & A_n \arrow[r, "f"] \arrow[d, "\alpha"] & B_n \arrow[r, "g"] \arrow[d, "\beta"] & C_n \arrow[r, "h"] \arrow[d, "\gamma", "\cong"'] & A_{n-1} \arrow[r, "f"] \arrow[d, "\alpha"] & B_{n-1} \arrow[r] \arrow[d, "\beta"] & \ldots \\ \ldots \arrow[r] & A'_n \arrow[r, "f'"'] & B'_n \arrow[r, "g'"'] & C'_n \arrow[r, "h'"'] & A'_{n-1} \arrow[r, "f'"'] & B'_{n-1} \arrow[r] & \ldots \end{tikzcd}$$

If rows are exact and γ is an isomorphism then the following sequence is exact:

(42) $\cdots \to A_n \xrightarrow{(\alpha, f)} A'_n \oplus B_n \xrightarrow{f'-\beta} B'_n \xrightarrow{h\circ\gamma\circ g'^{-1}} A'_{n-1} \to \cdots$

$x \longmapsto (\alpha(x), f(x))$

$(y, z) \longmapsto f'(y) - \beta(z)$

Proof: Exercise. ▨

Proof of Theorem (A):

Apply the Barratt-Whitehead lemma to the following diagram:

long exact seq. of $(A, A\cap B)$

long exact seq. of (X, B)

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A\cap B) \arrow[r] \arrow[d] & H_n(A) \arrow[r] \arrow[d] & H_n(A, A\cap B) \arrow[r] \arrow[d, "\cong"'] & H_{n-1}(A\cap B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(X) \arrow[r] & H_n(X, B) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

Excision II

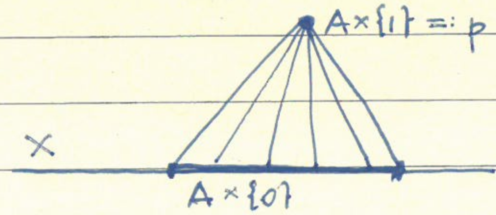
▨

Application: Long exact sequence of a cofibration

Recall:

If $A \subseteq X$ then

$$X \cup CA = X \sqcup A \times [0,1] \Big/ \begin{array}{l} a \sim (a,0) \ \forall a \in A \\ (a,1) \sim (a',1) \ \forall a, a' \in A \end{array}$$



Note: $X \cup CA$ is the mapping cone of the inclusion $i: A \hookrightarrow X$.

Proposition: If $A \subseteq X$ then we have an isomorphism:

$$H_n(X, A) \cong \tilde{H}(X \cup CA) \quad \forall_{n \geq 0}$$

Proof:

① $\tilde{H}_n(X \cup CA) \cong H_n(X \cup CA, CA)$

- holds since $CA \simeq *$

② $H_n(X \cup CA, CA) \cong H_n(X \cup CA \setminus \{p\}, CA \setminus \{p\})$

where $p = A \times \{1\} \subseteq CA$

- holds by excision.

③ $H_n(X \cup CA \setminus \{p\}, CA \setminus \{p\}) \cong H_n(X, A)$

Indeed: the inclusion

$$i: X \hookrightarrow \underbrace{X \cup CA \setminus \{p\}}_{X'} \cong X \cup A \times [0,1)$$

is a homotopy equivalence.

Also: $i|_A : A \hookrightarrow CA \setminus \{p\} \cong A \times [0,1)$

is a homotopy equivalence.

(44) This gives a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "i|_{A*}", "\cong"'] & H_n(X) \arrow[r] \arrow[d, "i_*", "\cong"'] & H_n(X,A) \arrow[r, "\delta"] \arrow[d, "i_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "i|_{A*}", "\cong"'] & H_{n-1}(X) \arrow[r] \arrow[d, "i_*", "\cong"'] & \cdots \\
\cdots \arrow[r] & H_n(A') \arrow[r] & H_n(X') \arrow[r] & H_n(X',A') \arrow[r, "\delta"] & H_{n-1}(A') \arrow[r] & H_{n-1}(X') \arrow[r] & \cdots
\end{tikzcd}$$

The fact that $i_*: H_n(X,A) \to H_n(X',A')$ is an isomorphism follows directly from:

The Five Lemma:
Consider a commutative diagram of abelian groups:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"', "\cong"] & A_2 \arrow[r] \arrow[d, "f_2"', "\cong"] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"', "\cong"] & A_5 \arrow[d, "f_5"', "\cong"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

Assume that:
- both rows are exact
- f_1, f_2, f_4, f_5 are isomorphisms.

Then f_3 is also an isomorphism

Pf: exercise
(Note: it suffices to assume that f_1 is onto and f_5 is 1-1)

$\blacksquare$

(45) **Corollary:** For $A \subseteq X$ we have a long exact sequence:

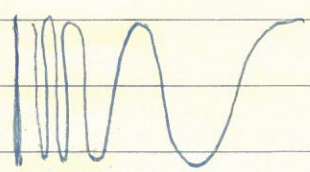
$$\ldots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X \cup CA) \to \tilde{H}_{n-1}(A) \to \ldots$$

Proof:

Follows from the long exact sequence of the pair (X, A) and the isomorphisms $H_n(X,A) \cong \tilde{H}_n(X \cup CA)$.

Note: In general $X \cup CA \not\simeq X/A$

Example:

$X =$  $\subseteq \mathbb{R}^2$

$$\underbrace{\{0\} \times [-1,1]}_{A} \cup \{(x, \sin \tfrac{1}{x}),\ x > 0\}$$

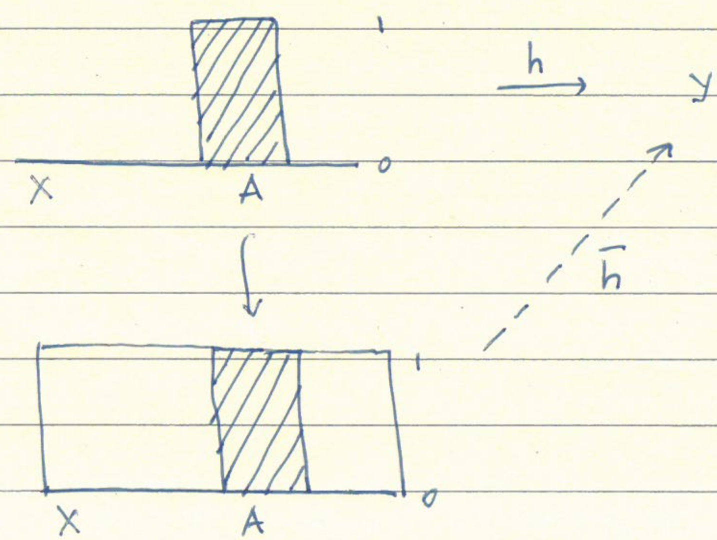
Check: $X/A \cong [0, \infty) \subseteq \mathbb{R}$ - path connected

$X \cup CA$ - not path connected

so: $X/A \simeq X \cup CA$.

Cofibrations

Def: Let X - top. space, $A \subseteq X$.
The pair (X,A) has the homotopy extension property (HEP) if any map
$$h : X\times\{0\} \cup A\times[0,1] \longrightarrow Y$$
can be extended to a map $\bar{h} : X\times[0,1] \longrightarrow Y$



Proposition:
(X,A) has HEP iff $X\times\{0\}\cup A\times[0,1]$ is a retract of $X\times[0,1]$

Proof: Exercise.

Proposition: If (X,A) has HEP and X is Hausdorff then A is closed in X.

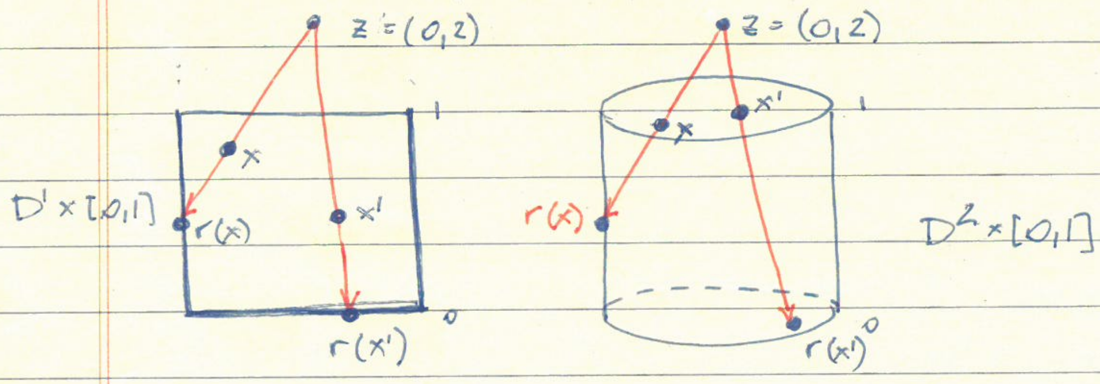
Def: If (X,A) has HEP then we say that the inclusion map $i : A \hookrightarrow X$ is a cofibration

(47) Proposition 1:

$S^{n-1} \hookrightarrow D^n$ is a cofibration $\forall_n$

Pf. (idea): it suffices to show that there exists a retraction $r: D^n \times [0,1] \longrightarrow D^n \times \{0\} \cup S^{n-1} \times [0,1]$

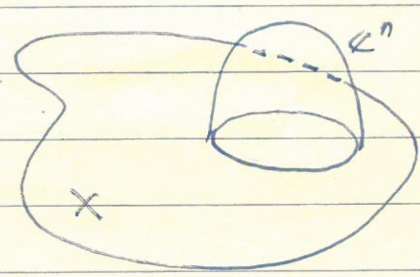
r can be defined as a radial projection:



Recall: cell attachment

$$\varphi: S^{n-1} \longrightarrow X \quad \text{- attaching map}$$

$$X \cup_\varphi e^n = X \sqcup D^n \big/ x \sim \varphi(x) \ \text{ for } x \in S^{n-1} \subseteq D^n$$



Proposition 2:

If $Y = X \cup_\varphi e^n$ then $i: X \hookrightarrow Y$ is a cofibration.

Proof: It is enough to construct a retraction

$$R: Y \times [0,1] \longrightarrow X \times [0,1] \cup Y \times \{0\}$$

$$Y \times [0,1] = X \times [0,1] \sqcup D^n \times [0,1] \big/ (x,t) \sim (\varphi(x), t) \text{ for } (x,t) \in S^{n-1} \times [0,1]$$

$$X \times [0,1] \cup Y \times \{0\} \cong X \times [0,1] \sqcup D^n \times \{0\} \cup S^{n-1} \times [0,1] \big/ (x,t) \sim (\varphi(x), t)$$

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Set: $R|_{X\times[0,1]} = id|_{X\times[0,1]}$

$$R|_{D^n\times[0,1]} = r : D^n\times[0,1] \longrightarrow D^n\times\{0\}\times S^{n-1}\times[0,1]$$

From Prop. 1.

$\blacksquare$

Note: The same argument shows that if $Y = X\cup$(collection of n-cells) then $i : X \longrightarrow Y$ is a cofibration.

Recall: (Y, X) is a relative CW-complex if $Y = \bigcup_{n=-1}^{\infty} Y^{(n)}$ where:

- $Y^{(-1)} = X$
- for $n\geq 0$ $Y^{(n)}$ is obtained from $Y^{(n-1)}$ by attaching some number of n-cells

weak topology:

$f: Y \longrightarrow Z$ is continuous iff $f|_{Y^{(n)}} : Y^{(n)} \longrightarrow Z$ is continuous $\forall n$

Theorem: If (Y, X) - relative CW-complex then $i: X \hookrightarrow Y$ is a cofibration

Proof:

By Proposition 2 for every $n \geq 0$ we have a retraction

$$R_n : Y^{(n)}\times[0,1] \longrightarrow Y^{(n)}\times\{0\}\cup Y^{(n-1)}\times[0,1]$$

This extends to a retraction

$$\overline{R}_n : Y\times\{0\}\cup Y^{(n)}\times[0,1] \longrightarrow Y\times\{0\}\cup Y^{(n-1)}\times[0,1]$$

$$(y,0) \longmapsto (y,0)$$

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Notice that $Y \times [0,1] = \bigcup_{n=-1}^{\infty} \underbrace{(Y \times \{0\} \cup Y^{(n)} \times [0,1])}^{Z_n}$

Define $R: Y \times [0,1] \to Y \times \{0\} \cup X \times [0,1]$

by: $R|_{Z_n} = R_0 \circ R_1 \circ \dots \circ R_n$

Note: 1) R is well defined

2) R is a retraction

3) R is continuous (since $R|_{Z_n}$ is continuous $\forall_n$ and $Y \times [0,1]$ has weak topology).

$\blacksquare$

Example:

If X - CW-complex, $X^{(n)}$ - n^{th} skeleton of X then $X^{(n)} \hookrightarrow X$ is a cofibration.

Proposition:

If $A \hookrightarrow X$ is a cofibration and $A \simeq *$ then the quotient map

$$q: X \to X/A$$

is a homotopy equivalence.

Proof:

$A \simeq * \Rightarrow \exists\, \tilde{h}: A \times [0,1] \to A$

$\tilde{h}|_{A \times \{0\}} = id_A$, $h|_{A \times \{1\}} =$ constant map

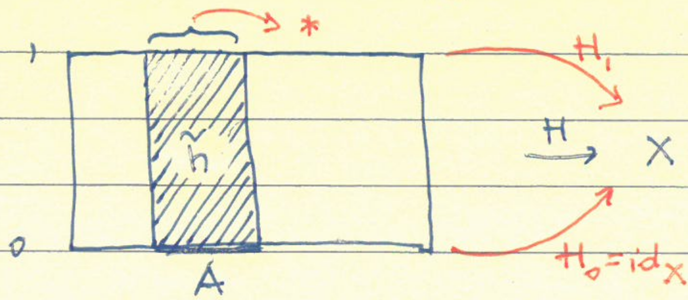
Take $h: X \times \{0\} \cup A \times [0,1] \to X$

$(x,0) \longmapsto x$

$(a,t) \longmapsto \tilde{h}(a,t)$

(50) Since $A \hookrightarrow X$ is a cofibration this extends to

$$H : X \times [0,1] \longrightarrow X$$



Note: 1) $H_1(A) = * \in A$

so H_1 admits a factorization

$$\begin{tikzcd} X \arrow[rr, "H_1"] \arrow[dr, "q"'] & & X \\ & X/A \arrow[ur, "\overline{H}_1"'] & \end{tikzcd}$$

2) H gives a homotopy $\overline{H}_1 \circ q \simeq id_X$

3) Since $H_t(A) = \tilde{h}_t(A) \subseteq A \quad \forall t$

thus H admits a factorization

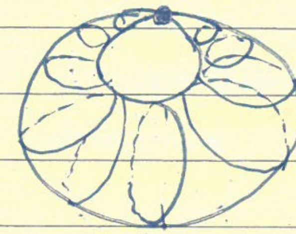
$$\overline{\overline{H}} : X/A \times [0,1] \longrightarrow X/A$$

this map gives a homotopy $q \circ \overline{H}_1 \simeq id_{X/A}$.

$\blacksquare$

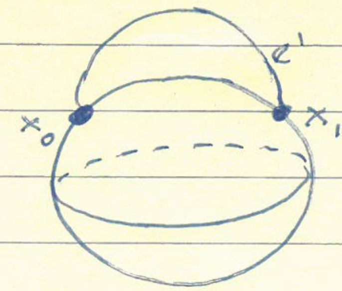
Example:

Let $x_0, x_1 \in S^n$ $\quad Y = S^n/\{x_0, x_1\}$
$x_0 \neq x_1$



Q: $Y \simeq\ ?$

Note: Let $X = S^n \cup e^1$

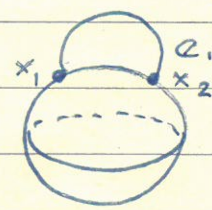


1) $Y \cong X/\overline{e}_1$
closure of e_1

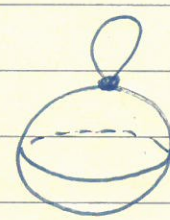
2) $\overline{e}_1 \hookrightarrow X$ is a cofibration

3) $\overline{e}_1 \cong [0,1] \simeq *$

Thus: $Y \cong X/\overline{e}_1 \simeq X$

4) 

$\simeq$



This gives:

$$S^n/\{x_1, x_2\} \simeq S^n \vee S^1$$

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Back to homology

<u>Recall</u>: For $A \subseteq X$ we have a long exact sequence:

$$\ldots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X \cup CA) \to \tilde{H}_{n-1}(A) \to \ldots$$

where $i : A \hookrightarrow X$

$j : X \hookrightarrow X \cup CA$

<u>Proposition</u>: If $i : A \hookrightarrow X$ cofibration then we have a long exact sequence:

$$\ldots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \ldots$$

where $q : X \to X/A$ - the quotient map.

<u>Proof</u>:

① since $i : A \hookrightarrow X$ cofibration thus

$j : CA \hookrightarrow X \cup CA$ also is a cofibration.

② $CA \simeq * \Rightarrow X \cup CA \xrightarrow{\bar{q}} X \cup CA/_{CA} \cong X/A$

- homotopy equiv.

③ We have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "q"'] & X \cup CA \arrow[d, "\bar{q}"] \\ & X/A \end{tikzcd}$$

This gives an exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_n(A) \arrow[r, "i_*"] & \tilde{H}_n(X) \arrow[r, "q_*"] & \tilde{H}_n(X/A) \arrow[r] & \tilde{H}_{n-1}(A) \arrow[r] & \cdots \\
\cdots \arrow[r] & \tilde{H}_n(A) \arrow[r, "i_*"] \arrow[u, "="] & \tilde{H}_n(X) \arrow[r] \arrow[u, "="] & \tilde{H}_n(X \cup CA) \arrow[r] \arrow[u, "\cong"', "\bar{q}_*"] & \tilde{H}_{n-1}(A) \arrow[r] \arrow[u, "="] & \cdots
\end{tikzcd}$$

<u>Exact sequence of a map</u>

<u>Goal:</u>

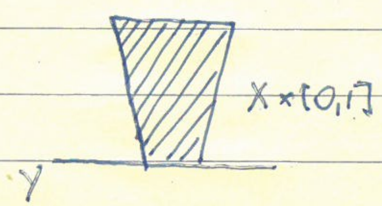
Let $f: X \to Y$ - arbitrary map.

Find a homology exact sequence such that $H_n(X) \xrightarrow{f_*} H_n(Y)$ is a part of this sequence.

<u>Recall:</u>

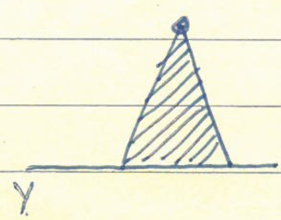
If $f: X \to Y$ - a map then

- the <u>mapping cylinder</u> of f is the space



$$M_f = Y \cup X \times [0,1] \Big/ (x,0) \sim f(x)$$

- the <u>mapping cone</u> of f is the space



$$C_f = M_f \big/ X \times \{1\}$$

(54)

Note:

1) The map

$$\begin{aligned} j : X &\hookrightarrow M_f \\ x &\longmapsto (x,1) \end{aligned}$$

is a cofibration and $C_f = M_f / X$

2) The map

$$\begin{aligned} g : M_f &\longrightarrow Y \\ y &\longmapsto y \\ (x,t) &\longmapsto f(x) \end{aligned}$$

is a homotopy equivalence with a homotopy inverse given by the inclusion $i : Y \hookrightarrow M_f$

Upshot 1: Any map $f : X \to Y$ admits a factorization:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"', "\text{cofibration}"] & & Y \\ & M_f \arrow[ur, "g", "\simeq"'] & \end{tikzcd}$$

Upshot 2: Any map $f : X \to Y$ fits into a long exact sequence:

$$\begin{tikzcd} & & \tilde{H}_n(M_f) \arrow[dr, "q_*"] & & \\ \cdots \arrow[r] & \tilde{H}_n(X) \arrow[r, "f_*"] & \tilde{H}_n(Y) \arrow[u, "i_*"] \arrow[r, "k_*"] & \tilde{H}_n(C_f) \arrow[r] & \tilde{H}_{n-1}(X) \arrow[r] & \cdots \\ \cdots \arrow[r] & \tilde{H}_n(X) \arrow[u, "="] \arrow[r] & \tilde{H}_n(M_f) \arrow[u, "g_*", "\cong"'] \arrow[r, "q_*"'] & \tilde{H}_n(C_f) \arrow[u, "="] \arrow[r] & \tilde{H}_{n-1}(X) \arrow[u, "="] \arrow[r] & \cdots \end{tikzcd}$$

where $k : Y \hookrightarrow C_f$ - inclusion

$q : M_f \to C_f$ - the quotient map.