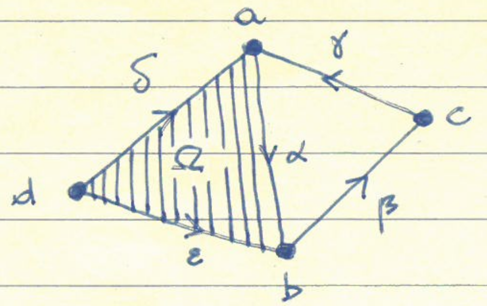
1

Homology

Motivation:

A space X with triangulation:



a, b, c, d = 0-simplices

$\alpha, \beta, \gamma, \delta$ = 1-simplices

Ω = 2-simplex

$C_i(X)$ = the free abelian group generated by i-simplices of X:

$$C_0(X) \cong \overset{a}{\mathbb{Z}} \oplus \overset{b}{\mathbb{Z}} \oplus \overset{c}{\mathbb{Z}} \oplus \overset{d}{\mathbb{Z}}$$

$$C_1(X) \cong \overset{\alpha}{\mathbb{Z}} \oplus \overset{\beta}{\mathbb{Z}} \oplus \overset{\gamma}{\mathbb{Z}} \oplus \overset{\delta}{\mathbb{Z}} \oplus \overset{\varepsilon}{\mathbb{Z}}$$

$$C_2(X) \cong \overset{\Omega}{\mathbb{Z}}$$

Consider a homomorphism:

$$\text{(edge)} \longmapsto \text{(end} - \text{start)}$$

$$C_1(X) \xrightarrow{\partial_1} C_0(X)$$

$$\begin{array}{ccccccccc}
\alpha & \beta & \gamma & \delta & \varepsilon & & a & b & c & d \\
\mathbb{Z} \oplus & \mathbb{Z} \oplus & \mathbb{Z} \oplus & \mathbb{Z} \oplus & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus & \mathbb{Z} \oplus & \mathbb{Z} \oplus & \mathbb{Z} \\
1 & & & & & \longmapsto & (-1, & 1, & 0, & 0) \\
& 1 & & & & \longmapsto & (0, & -1, & 1, & 0) \\
& & 1 & & & \longmapsto & (1, & 0, & -1, & 0) \\
& & & 1 & & \longmapsto & (1, & 0, & 0, & -1) \\
& & & & 1 & \longmapsto & (0, & 1, & 0, & -1)
\end{array}$$

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Note: $\alpha\beta\gamma$ is a loop $\rightsquigarrow$ $\partial_1(\overset{\alpha}{1},\overset{\beta}{1},\overset{\gamma}{1},0,0) = 0$

$\delta\alpha\varepsilon$ is a loop $\rightsquigarrow$ $\partial_1(1,0,0,1,1) = 0$

Upshot: elements of $\mathrm{Ker}(\partial_1) \longleftrightarrow$ loops in X

Another homomorphism:

$$(2\text{-simplex}) \longrightarrow (\text{sum of edges around the simplex})$$

$$C_2(X) \xrightarrow{\partial_2} C_1(X)$$

$$\overset{\Omega}{\mathbb{Z}} \longrightarrow \overset{\alpha}{\mathbb{Z}} \oplus \overset{\beta}{\mathbb{Z}} \oplus \overset{\gamma}{\mathbb{Z}} \oplus \overset{\delta}{\mathbb{Z}} \oplus \overset{\varepsilon}{\mathbb{Z}}$$

$$1 \longmapsto (1, 0, 0, 1, -1)$$

Note: Elements of $\mathrm{Im}(\partial_1) \longleftrightarrow$ conctractible loops in X.

We have:

1) $\mathrm{Im}(\partial_1) \subseteq \mathrm{Ker}(\partial_1)$ $\longleftrightarrow$ the group of loops which are homotopically non-trivial

2) $\underbrace{\mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2)}_{\parallel}$

$H_1(X)$

1st homology group of X

1st generalization: higher homology groups

X - space with triangulation

$C_n(X)$ = the free abelian group generated by n-simplices of X

(3)

We have homomorphisms:

$$\dots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \dots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

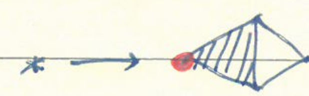
s.t. $\underbrace{\mathrm{Im}(\partial_{n+1})}_{n\text{- boundaries}} \subseteq \underbrace{\mathrm{Ker}(\partial_n)}_{n\text{- cycles}}$

$$\underbrace{H_n(X)}_{\text{the } n^{th} \text{ homology group of } X.} = \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1})$$

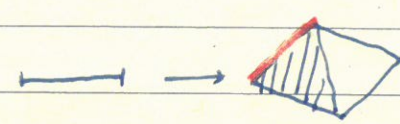
<u>2^{nd} generalization: homology for non-triangulated spaces</u>

<u>Idea</u>: If X - triangulated space then:

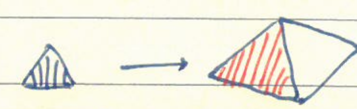
0- simplex of X = certain map $* \to X$



1- simplex of X = certain map $\vdash\!\dashv \;\to X$



2- simplex of X = certain map $\Delta \to X$



If X - any (non-triangulated) space define:

singular 0- simplex of X = <u>any</u> map $* \to X$
(= point in X)

singular 1- simplex of X = <u>any</u> map $\vdash\!\dashv \;\to X$
(= path in X)

singular 2- simplex of X = <u>any</u> map $\Delta \to X$

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Then define $C_i(X)$, $H_i(X)$ as before

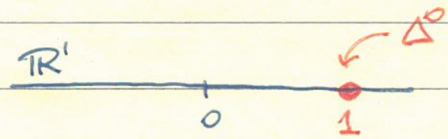
Details:

Def: Let $e_0 = (1,0,\dots,0)$, $e_1 = (0,1,0,\dots,0)$, $e_n = (0,\dots,0,1)$ be the standard basis of $\mathbb{R}^{n+1}$. The standard n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ is given by:

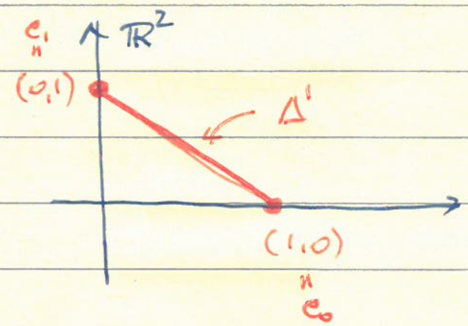
$$\Delta^n = \{ \sum_{i=0}^{n} t_i e_i \mid \sum_{i=1}^{n} t_i = 1,\ t_i \geq 0\ \forall i \}$$

Example:

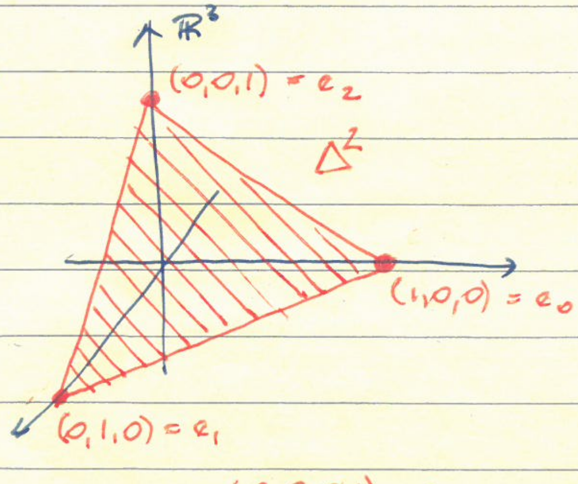
n = 0



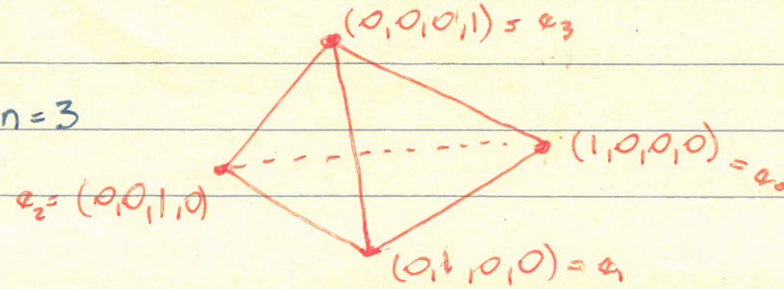
n = 1



n = 2



n = 3



$\Delta^3 \subseteq \mathbb{R}^4$

Note: If $A \subseteq \mathbb{R}^m$ is a convex set and $a_0, \ldots, a_n \in A$ then there exists a unique affine map $f : \Delta^n \to A$ s.t. $f(e_i) = a_i$. Explicitly:

$$f(\Sigma t_i e_i) = \Sigma t_i a_i$$

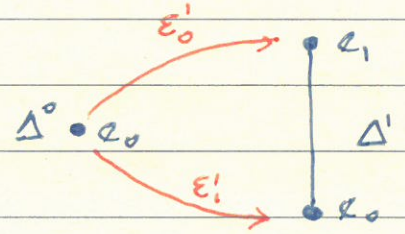
Def: For $k = 0, 1, \ldots, n$ the k^{th}-face of Δ^n is the affine map $\varepsilon^n_k : \Delta^{n-1} \hookrightarrow \Delta^n$ s.t.

$$\begin{aligned} e_0 &\to e_0 \\ &\vdots \\ e_{k-1} &\to e_{k-1} \\ e_k &\to e_{k+1} \\ &\vdots \\ e_{n-1} &\to e_n \end{aligned}$$

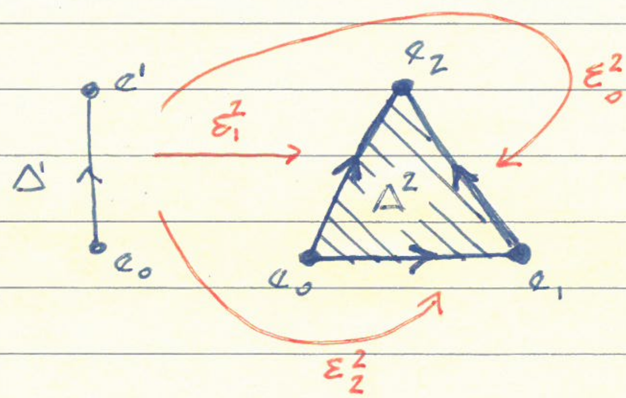
← skip the k^{th} vertex

Example:

n = 1



n = 2



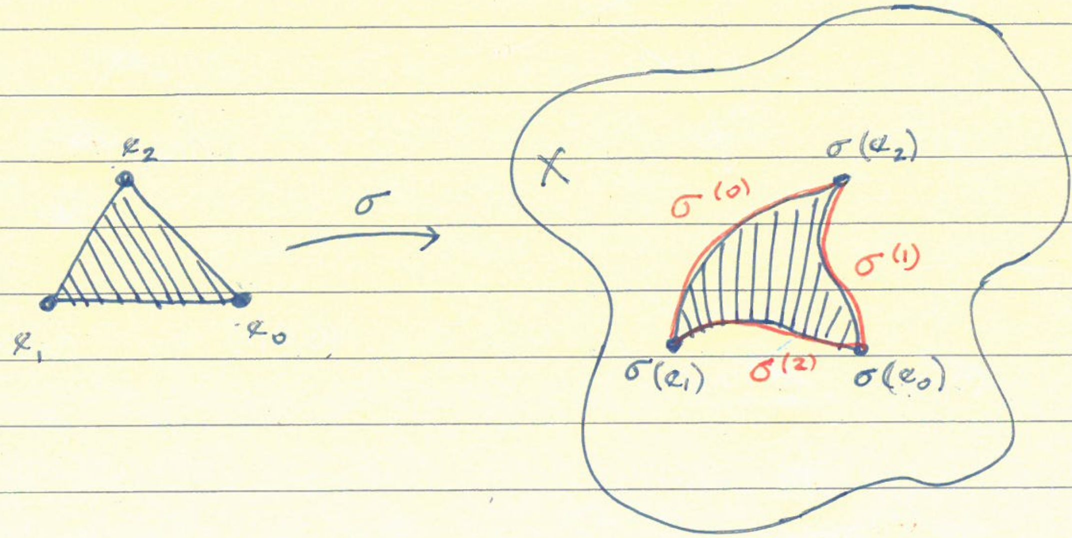
Def: 1) A singular n-simplex in a space X is a map

$$\sigma : \Delta^n \to X$$

2) The k^{th} face of a singular n-simplex σ is the singular $(n-1)$-simplex

$$\sigma(k) : \Delta^{n-1} \xrightarrow{\varepsilon^n_k} \Delta^n \xrightarrow{\sigma} X$$

⑥



Singular chain complex

Def: $C_n(X)$ = the free abelian group generated by all singular n-simplices of X

$$= \bigoplus_{(\sigma: \Delta^n \to X)} \mathbb{Z}$$

$C_n(X)$ is called the group of singular n-chains of X.

Elements of $C_n(X)$:

$$\sum_\sigma n_\sigma \sigma \quad , \quad n_\sigma \in \mathbb{Z} \ , \ n_\sigma \neq 0 \text{ for a finite number of } \sigma \text{ only}$$

addition in $C_n(X)$:

$$\sum n_\sigma \sigma + \sum m_\sigma \sigma = \sum (n_\sigma + m_\sigma)\sigma$$

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<u>Def:</u> For $n > 0$ the <u>boundary map</u>

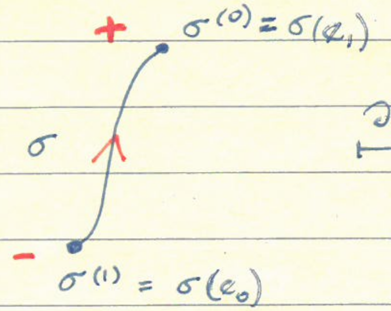
$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

is the homomorphism defined by

$$\partial_n(\sigma) = \sum_{k=0}^{n} (-1)^k \sigma^{(k)}$$

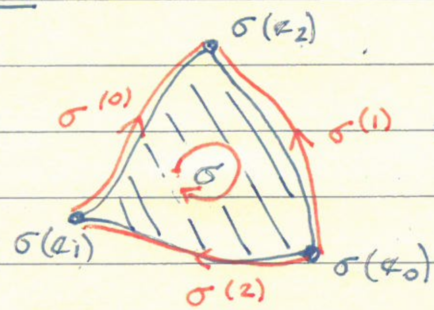
<u>Example:</u>

<u>n=1</u>



$\overset{\partial_1}{\longmapsto}$ $\sigma^{(0)} - \sigma^{(1)}$

<u>n=2</u>



$\overset{\partial_2}{\longmapsto}$ $\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}$

<u>Note:</u> We define : $\partial_0 : C_0(X) \to 0$ - the trivial map

<u>Proposition:</u>

For any $n \geq 0$ $\partial_n \circ \partial_{n+1} : C_{n+1}(X) \to C_{n-1}(X)$ is the trivial homomorphism.

(Equivalently: $\operatorname{Im}(\partial_{n+1}) \subseteq \operatorname{Ker}(\partial_n)$)

Proof: Enough to check:

$$\partial_n \circ \partial_{n+1}(\sigma) = 0 \quad \forall \sigma \in C_{n+1}(X)$$

We have:

$$\partial_n \circ \partial_{n+1}(\sigma) = \sum_{k=0}^{n+1} (-1)^k \partial_n(\sigma^{(k)})$$

$$= \sum_{k=0}^{n+1} (-1)^k \sum_{\ell=0}^{n} (-1)^\ell \left(\sigma^{(k)}\right)^{(\ell)}$$

Exercise: if $k > \ell$ then $\left(\sigma^{(k)}\right)^{(\ell)} = \left(\sigma^{(\ell)}\right)^{(k-1)}$

This gives:

$$\partial_{n+1} \circ \partial_n(\sigma) = \sum_{k=0}^{n+1} \sum_{\ell \geq k} (-1)^{k+\ell} \left(\sigma^{(k)}\right)^{(\ell)} + \sum_{k=0}^{n+1} \sum_{\ell < k} (-1)^{k+\ell} \left(\sigma^{(\ell)}\right)^{(k-1)}$$

$$= \sum_{k=0}^{n} \sum_{\ell \geq k} (-1)^{k+\ell} \left(\sigma^{(k)}\right)^{(\ell)} + \sum_{k'=0}^{n} \sum_{\ell' \geq k'} (-1)^{k'+\ell'+1} \left(\sigma^{(k')}\right)^{(\ell')}$$

$k' = \ell$

$\ell' = k-1$

$$= 0.$$

Upshot:

For a space X we obtain a sequence of abelian groups and group homomorphisms:

$$\to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots \to C_0(X) \xrightarrow{\partial_0} 0$$

such that $\partial_n \circ \partial_{n+1} = 0 \quad \forall n$.

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Algebraic setup: chain complexes

Def: A chain complex C_* is a sequence

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

where:

C_n - abelian group

$\partial_n : C_n \to C_{n-1}$ - group homomorphism ("differential") such that $\partial_n \circ \partial_{n+1} = 0 \ \forall n$.

If C_* is a chain complex then

- the group of <u>n-cycles</u> of C_* is the group
  $Z_n(C_*) = \mathrm{Ker}(C_n \xrightarrow{\partial_n} C_{n-1}) \leq C_n$
- the group of <u>n-boundaries</u> of C_* is the group
  $B_n(C_*) = \mathrm{Im}(C_{n+1} \xrightarrow{\partial_{n+1}} C_n) \leq C_n$

Note: Since $\partial_n \circ \partial_{n+1} = 0$ we have:

$B_n(C_*) \leq Z_n(C_*)$

Def: The group $H_n(C_*) := Z_n(C_*)/B_n(C_*)$ is called the <u>n-th homology group</u> of C_*

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Example

$$C_* = (\dots \to \overset{C_3}{0} \xrightarrow{\partial_3} \overset{C_2}{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\partial_2} \overset{C_1}{\mathbb{Z}} \xrightarrow{\partial_1} \overset{C_0}{\mathbb{Z}/2 \oplus \mathbb{Z}/3} \xrightarrow{\partial_0} \overset{C_{-1}}{0} \to \dots)$$

$$(m,n) \longmapsto 2m$$

$$1 \longmapsto (1,0)$$

$Z_0(C_*) = \mathbb{Z}/2 \oplus \mathbb{Z}/3$

$B_0(C_*) = \mathbb{Z}/2 \oplus 0$

$H_0(C_*) \cong \mathbb{Z}/3$

$Z_1(C_*) = 2\mathbb{Z}$

$B_1(C_*) = 2\mathbb{Z}$

$H_1(C_*) = 0$

$Z_2(C_*) = 0 \oplus \mathbb{Z}$

$B_2(C_*) = 0 \oplus 0$

$H_2(C_*) = 0 \oplus \mathbb{Z} \cong \mathbb{Z}$

Back to spaces:

Def: If X is a space then the chain complex

$$C_*(X) = (\dots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \to \dots \to C_0(X) \xrightarrow{\partial_0} 0)$$

is called the <u>singular chain complex</u> of X.

Notation: $\underbrace{Z_n(X)} := Z_n(C_*(X))$ ...

the group of singular n-cycles of X.

$\underbrace{B_n(X)} := B_n(C_*(X))$

the group of singular n-boundaries of X

$\underbrace{H_n(X)} := H_n(C_*(X))$

the n^{th} singular homology group of X

Example:

$X = \{*\}$

Note: 1) For $n \geq 0$ X has only one singular n-simplex:

$$\sigma_n : \Delta^n \longrightarrow * = X$$

Thus $C_n(*) \cong \mathbb{Z} \quad \forall n \geq 0$

2) $\sigma_n^{(k)} = \sigma_{n-1} \quad \forall_{k=0,1,\dots,n}$

Therefore:

$$\partial \sigma_n = \sum_{k=0}^{n} (-1)^k \sigma_n^{(k)} = \sum_{k=0}^{n} (-1)^k \sigma_{n-1} = \begin{cases} 0 & \text{if } n\text{-odd} \\ \sigma_{n-1} & \text{if } n\text{-even} \end{cases}$$

We obtain:

$$C_*(*) \cong (\dots \to \overset{4}{\mathbb{Z}} \to \overset{3}{\mathbb{Z}} \to \overset{2}{\mathbb{Z}} \to \overset{1}{\mathbb{Z}} \to \overset{0}{\mathbb{Z}} \to 0)$$

$$1 \longmapsto 0$$

$$1 \longmapsto 1$$

$$1 \longmapsto 0$$

$$1 \longmapsto 1$$

$H_0(*) \cong \mathbb{Z}$

$H_1(*) = 0$

$H_2(*) = 0$

$H_3(*) = 0$

We get:

Proposition: (the dimension axiom)

$$H_n(*) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

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Low dimensional homology

$H_0(X)$

Proposition: If X is a path conn. space and $X \neq \emptyset$ then $H_0(X) \cong \mathbb{Z}$

Proof: We have:

$$\ldots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

$$Z_0(X) = C_0(X) = \left\{ \sum_{x \in X} n_x x \;\middle|\; \begin{matrix} x \in X \\ n_x \neq 0 \text{ for fin. many } x \end{matrix} \right\}$$

0-simplices = points in X

Take:

$$\begin{aligned} \theta : Z_0(X) &\longrightarrow \mathbb{Z} \qquad \text{- homomorphism onto} \\ \sum n_x x &\longmapsto \sum n_x \end{aligned}$$

We will show that $\operatorname{Ker}(\theta) = B_0(X)$.

Then: $H_0(X) = Z_0(X) / B_0(X) = Z_0(X) / \operatorname{Ker}(\theta) \cong \mathbb{Z}$

1) $B_0(X) \subseteq \operatorname{Ker}(\theta)$:

$$\begin{aligned} B_0(X) &= \operatorname{Im}(\partial_1) \\ &= \{ \partial_1 (\sum m_\sigma \sigma) \mid \sum m_\sigma \sigma \in C_1(X) \} \\ &= \{ \sum m_\sigma (\sigma^{(0)} - \sigma^{(1)}) \mid \sum m_\sigma \sigma \in C_1(X) \} \\ &= \{ \sum m_\sigma \sigma^{(0)} - \sum m_\sigma \sigma^{(1)} \} \end{aligned}$$

$$\theta \left(\sum m_\sigma \sigma^{(0)} - \sum m_\sigma \sigma^{(1)} \right) = \sum m_\sigma - \sum m_\sigma = 0$$

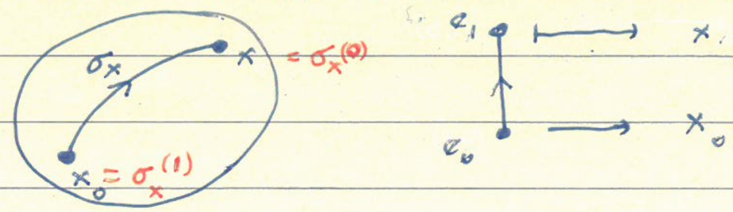
so: $B_0(X) \subseteq \operatorname{Ker}(\theta)$

2) $\operatorname{Ker}(\theta) \subseteq B_0(X)$

Let $\sum n_x x \in \operatorname{Ker}(\theta)$ i.e. $\sum n_x = 0$

Choose $x_0 \in X$

X - path conn. $\Rightarrow \exists_{x \in X}\ \exists\, \sigma_x \colon \Delta^1 \longrightarrow X$



We have:

$$\partial_1\left(\sum n_x \sigma_x\right) = \sum n_x \left(\underset{\substack{\parallel \\ x}}{\sigma_x(0)} - \underset{\substack{\parallel \\ x_0}}{\sigma_x(1)}\right)$$

$$= \sum n_x x - \sum n_x x_0$$

$$= \sum n_x x - \underbrace{\left(\sum n_x\right) x_0}_{\substack{\parallel \\ 0}} = \sum n_x x$$

Thus: $\sum n_x x \in \partial_1(C_1(X)) = B_1(X)$

$\blacksquare$

Q: What if X is not path connected?

Proposition: If $\{X_i\}_{i \in I}$ is the set of path conn. components of X then

$$H_n(X) \cong \bigoplus_{i \in I} H_n(X_i)$$

In particular $H_0(X) \cong \bigoplus_{i \in I} \mathbb{Z}$

Proof: Enough to notice:

1) $C_*(X) \cong \underbrace{\bigoplus_{i\in I} C_*(X_i)}$

direct sum of singular chain complexes of the components X_i

2) Homology of chain complexes commutes with direct sums:

$$H_n\Big(\bigoplus_{i\in I} C_*^{(i)}\Big) \cong \bigoplus_{i\in I} H_n\big(C_*^{(i)}\big)$$

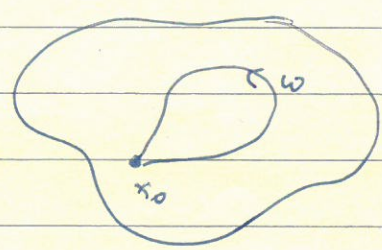
$\blacksquare$

$H_1(X)$

Let X - top. space, $x_0 \in X$ - basepoint

Recall:

$\pi_1(X, x_0)$ - fundamental gp of X

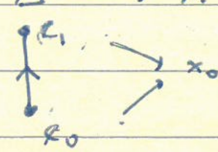


elements: homotopy classes of loop

$$\omega : [0,1] \longrightarrow X$$
$$0,1 \longmapsto x_0$$

multiplication: concatenation of loops

Note: 1) Since $\Delta^1 \cong [0,1]$ we can consider a loop ω in X as a singular 1-simplex $\omega : \Delta^1 \longrightarrow X$



2) We have: $\partial_1 \omega = x_0 - x_0 = 0$ so $\omega \in Z_1(X)$, and so it represents an element $[\omega] \in H_1(X)$

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Proposition

The function

$$\varphi : \pi_1(X, x_0) \longrightarrow H_1(X)$$
$$[\omega] \longmapsto [\omega]$$

homotopy class homology class

is a well defined

Proof:

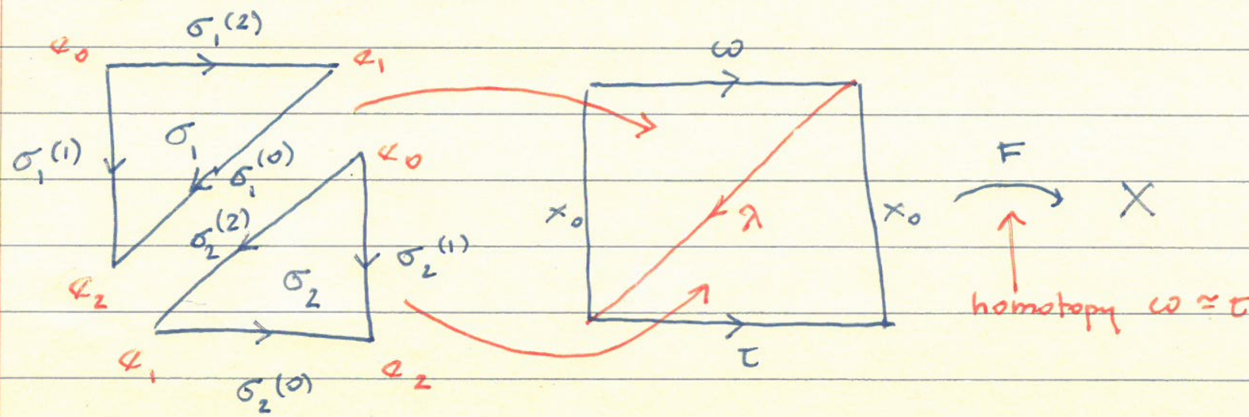
1) φ is well defined

Assume that ω, τ - loops in X, $\omega \simeq \tau$.

We need to show that $[\omega] = [\tau]$ in $H_1(X) \in Z_1(X)/B_1(X)$.

This is equivalent to showing that $\omega = \tau + \partial b$

for some $b \in C_2(X)$.



Take $b = \sigma_1 - \sigma_2$

We have:

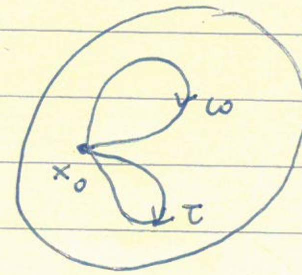
$$\begin{aligned}
\partial b &= \partial\sigma_1 - \partial\sigma_2 \\
&= (\cancel{\lambda} - \cancel{x_0} + \omega) - (\tau - \cancel{x_0} + \cancel{\lambda}) \\
&= \omega - \tau
\end{aligned}$$

So: $\omega = \tau + \partial b$.

2) <u>φ is a homomorphism</u>

Let ω, τ - loops in X, $\omega * \tau$ - concatenation of ω, τ. We want:

$$\varphi[\omega * \tau] = \varphi[\omega] + \varphi[\tau]$$

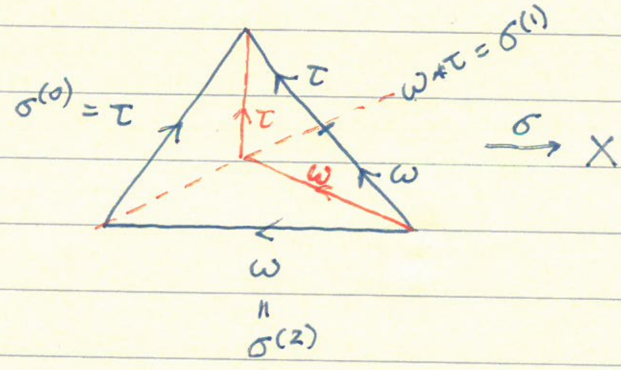


This is equivalent to showing that

$$\omega * \tau = \omega + \tau + \partial b$$

for some $b \in C_2(X)$.

Define $\sigma : \Delta^2 \to X$ as follows:



<u>We have:</u> $\partial\sigma = \tau - \omega * \tau + \omega$

so: $\tau + \omega = \omega * \tau + \partial\sigma$

$\blacksquare$

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Hurewicz Theorem

<u>Recall</u>: 1) If G - group, $g, h \in G$ then

$$[g,h] = ghg^{-1}h^{-1}$$

- the commutator of g, h.

2) $[G,G]$ = the subgroup of G generated by all commutators $[g,h]$, $g,h \in G$.

<u>We have</u>:

1) $[G,G]$ is a normal subgroup of G
2) $G/[G,G]$ is an abelian group. Denote: $G_{ab} = G/[G,G]$
3) If $f: G \longrightarrow H$ - homomorphism and H is abelian then $[G,G] \subseteq \mathrm{Ker}(f)$. This induces a homomorphism:

$$f_*: G_{ab} \longrightarrow H$$

<u>Thm</u>: (Hurewicz):

If X is path connected then the homomorphism

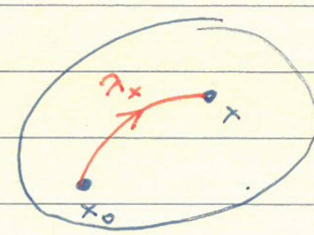
$$\varphi_*: \pi_1(X, x_0)_{ab} \longrightarrow H_1(X)$$

is an isomorphism.

<u>Proof</u>:

For $x \in X$ fix a path $\lambda_x: [0,1] \longrightarrow X$

$$0 \longmapsto x_0$$
$$1 \longmapsto x$$

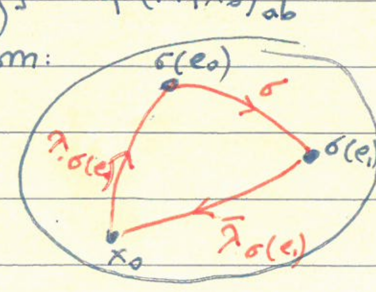


For a singular 1-simplex σ define:

$$\psi(\sigma) = [\lambda_{\sigma(e_0)} * \sigma * \bar{\lambda}_{\sigma(e_1)}] \in \pi_1(X, x_0)_{ab}$$

This uniquely extends to a homomorphism:

$$\psi: C_1(X) \longrightarrow \pi_1(X, x_0)_{ab}$$
$$\Sigma n_i \sigma_i \longmapsto \Sigma n_i \psi(\sigma_i)$$



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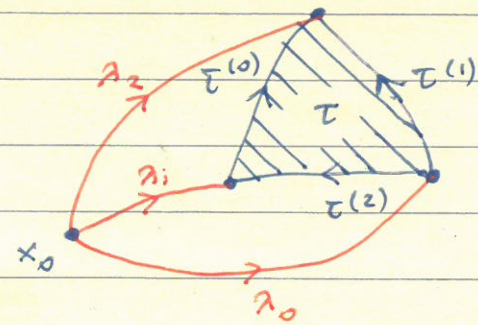
Claim 1:

$B_1(X) \subseteq \mathrm{Ker}(\psi)$

Indeed, it is enough to show that for any singular 2-simplex $\tau: \Delta^2 \longrightarrow X$ we have $\psi(\partial\tau) = 0 \in \pi_1(X, x_0)_{ab}$



We have:

$$\begin{aligned}
\psi(\partial\tau) &= \psi(\tau^{(0)} - \tau^{(1)} + \tau^{(2)}) \\
&= \psi(\tau^{(0)}) - \psi(\tau^{(1)}) + \psi(\tau^{(2)}) \\
&= [\lambda_1 * \tau^{(0)} * \bar{\lambda}_2] - [\lambda_0 * \tau^{(1)} * \bar{\lambda}_2] + [\lambda_0 * \tau^{(2)} * \bar{\lambda}_1]
\end{aligned}$$

Note: in $\pi_1(X, x_0)_{ab}$ we have:

$$\begin{aligned}
[\lambda_0 * \tau^{(1)} * \bar{\lambda}_2] &= [\lambda_0 * \tau^{(2)} * \tau^{(0)} * \bar{\lambda}_2] \\
&= [\lambda_0 * \tau^{(2)} * \bar{\lambda}_1 * \lambda_1 * \tau^{(0)} * \bar{\lambda}_2] \\
&= [\lambda_0 * \tau^{(2)} * \bar{\lambda}_1] + [\lambda_1 * \tau^{(0)} * \bar{\lambda}_2]
\end{aligned}$$

This gives:

$$\psi(\partial\tau) = 0$$

Upshot: ψ induces a homomorphism:

$$\psi_*: H_1(X) \longrightarrow \pi_1(X, x_0)$$

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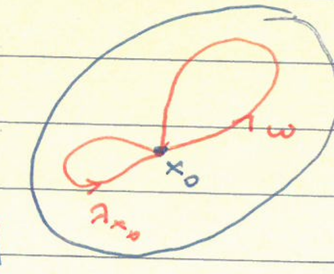
Claim 2:

$$\psi_* \varphi_* : \pi_1(X, x_0)_{ab} \longrightarrow \pi_1(X, x_0)_{ab} \text{ is the identity map}$$

Indeed: if $[\omega] \in \pi_1(X, x_0)$ then

$$\begin{aligned}\psi_*\varphi_*[\omega] &= [\lambda_{x_0} * \omega * \bar{\lambda}_{x_0}] \\ &= [\lambda_{x_0}] + [\omega] + [\bar{\lambda}_{x_0}] \\ &= [\cancel{\lambda_{x_0}}] + [\omega] - [\cancel{\lambda_{x_0}}] \\ &= [\omega]\end{aligned}$$

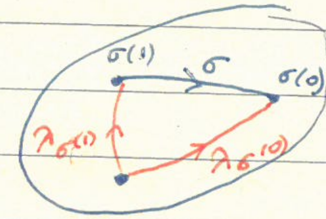


Claim 3:

$$\varphi_* \psi_* : H_1(X) \longrightarrow H_1(X) \quad \text{is the identity map}$$

Indeed: notice first that if σ is a 1-simplex in X then

$$\begin{aligned}\varphi_*\psi(\sigma) &= [\lambda_{\sigma(1)} * \sigma * \bar{\lambda}_{\sigma(0)}] \\ &\underset{\text{check}}{=} [\lambda_{\sigma(1)} + \sigma - \lambda_{\sigma(0)}] \\ &= [\sigma - \lambda_{\partial\sigma}]\end{aligned}$$



Therefore if $c \in C_1(X)$ then

$$\varphi_*\psi(c) = [c - \lambda_{\partial c}]$$

In particular if c is a cycle (e.i. $\partial c = 0$) then $\varphi_*\psi(c) = [c]$.

∎

Upshot:

$H_1(S^1) \cong \mathbb{Z}$

$H_1(S^n) \cong 0$ for $n > 1$

$H_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$

$H_1(S^1 \vee S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$

$H_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$

Functoriality of homology:

Algebra:

Def: A chain map of chain complexes $f_*: C_* \to D_*$ is a sequence of of homomorphisms $f_n: C_n \to D_n$ s.t. the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots \to & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \to \cdots \\
& \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & \\
\cdots \to & D_{n+1} & \xrightarrow[\partial_{n+1}]{} & D_n & \xrightarrow[\partial_n]{} & D_{n-1} & \to \cdots
\end{array}$$

Proposition: If $f_*: C_* \to D_*$ is a chain map then $f_n(Z_n(C_*)) \subseteq Z_n(D_*)$ and $f_n(B_n(C_*)) \subseteq B_n(D_*)$ $\forall n$. As a result f_* induces a homomorphism of homology groups:

$$\begin{array}{rcl}
f_*: H_n(C_*) & \longrightarrow & H_n(D_*) \\
\| & & \| \\
Z_n(C_*)/B_n(C_*) & \longrightarrow & Z_n(D_*)/B_n(D_*) \\
[c] & \longmapsto & [f_*(c)]
\end{array}$$

Note: 1) Chain complexes of abelian groups and chain maps define a category $Ch(\mathbb{Z})$.

2) For any n the assignment

$$C_* \longmapsto H_n(C_*)$$

defines a functor

$$H_n: Ch(\mathbb{Z}) \longrightarrow \underline{Ab}$$

↑ the category of abelian groups

Back to topology:

If $f: X \to Y$ - map of spaces,

$\sigma: \Delta^n \to X$ - singular simplex in X

then: $f \circ \sigma: \Delta^n \to Y$ - singular simplex in Y

This gives a homomorphism

$$f_n: C_n(X) \longrightarrow C_n(Y)$$
$$\sigma \longmapsto f \circ \sigma$$

Check: these homomorphisms define a map of singular chain complexes:

$$f_*: C_*(X) \longrightarrow C_*(Y)$$

In effect we obtain a functor

$$C_*: \mathrm{Top} \longrightarrow Ch(\mathbb{Z})$$
$$X \longmapsto C_*(X)$$

Since composition of functors is a functor for any $n \geq 0$ we get a functor

$$X \longmapsto H_n(X)$$

$$\begin{tikzcd} \mathrm{Top} \arrow[rr, "H_n"] \arrow[dr, "C_*"'] & & Ab \\ & Ch(\mathbb{Z}) \arrow[ur, "H_n"'] & \end{tikzcd}$$

$$X \longmapsto C_*(X) \qquad C_* \longmapsto H_n(C_*)$$

Corollary:

If $f: X \to Y$ is a homeomorphism then $f_*: H_n(X) \to H_n(Y)$ is an isomorphism $\forall_{n \geq 0}$.

(22)

Homotopy invariance

Goal: If $f, g : X \to Y$, $f \simeq g$ then $f_* = g_* : H_n(X) \to H_n(Y)$.

Algebraic setup:

Def: Let $f_*, g_* : C_* \to D_*$ be chain maps of chain complexes. A chain homotopy between f_* and g_* is a sequence of homomorphisms

$$\Phi_n : C_n \longrightarrow D_{n+1}$$

s.t. $g_n - f_n = \partial_{n+1} \circ \Phi_{n+1} + \Phi_{n-1} \circ \partial_n \quad \forall n$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d, shift right, "f_{n+2}"'] \arrow[d, shift left, "g_{n+1}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] \arrow[dl, "\Phi_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "\Phi_n"'] & C_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "\Phi_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+2} \arrow[r, "\partial_{n+2}"'] & D_{n+1} \arrow[r, "\partial_{n+1}"'] & D_n \arrow[r, "\partial_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Proposition:

If $f_*, g_* : C_* \to D_*$ are chain homotopic maps then they induce the same homomorphisms of homology groups:

$$f_* = g_* : H_n(C_*) \longrightarrow H_n(D_*) \qquad \forall n$$

Proof: We need to show:

if $z \in Z_n(C_*)$ then $f_*(z) = g_*(z) + b$ for some $b \in B_n(D_*)$.

Let Φ - chain homotopy between f_* and g_*.
We have:

$$f_n(z) = g_n(z) + \underbrace{\partial\Phi(z)}_{\in B_n(D_*)} + \underbrace{\Phi\partial(z)}_{= 0} \leftarrow z \in Z_n(C_*)$$

$\blacksquare$

(*) Theorem:

If $f, g: X \to Y$ - maps of spaces, $f \simeq g$ then the induced maps $f_*, g_*: C_*(X) \to C_*(Y)$ are chain homotopic.

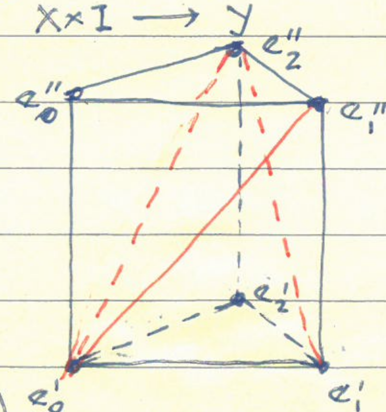
In particular f, g induce the same homomorphisms of singular homology groups:

$$f_* = g_*: H_n(X) \to H_n(Y) \qquad \forall n$$

Proof (sketch):

Let $F: X \times I \to Y$ be a homotopy between f and g.
For any singular simplex $\sigma: \Delta^n \to X$ we have a map

$$\sigma^F: \Delta^n \times I \xrightarrow{\sigma \times id} X \times I \longrightarrow Y$$



Denote: $e_i' = (e_i, 0) \in \Delta^n \times I$

$e_i'' = (e_i, 1) \in \Delta^n \times I$

For $i = 0, \dots, n$ let
$\tau_i: \Delta^{n+1} \to \Delta^n \times I$ be the affine map given by
$(e_0, \dots, e_{n+1}) \mapsto (e_0', \dots, e_i', e_i'', \dots, e_n'')$
and let τ_i^σ be the singular $(n+1)$-simplex in Y given by:

$$\tau_i^\sigma: \Delta^{n+1} \xrightarrow[\tau_i]{} \Delta^n \times I \xrightarrow[\sigma^F]{} Y$$

(24) Let $\Phi_n : C_n(X) \to C_{n+1}(X)$ be the homomorphism given by

$$\Phi_n(\sigma) = \sum_{i=0}^{n} (-1)^i \tau_i^\sigma$$

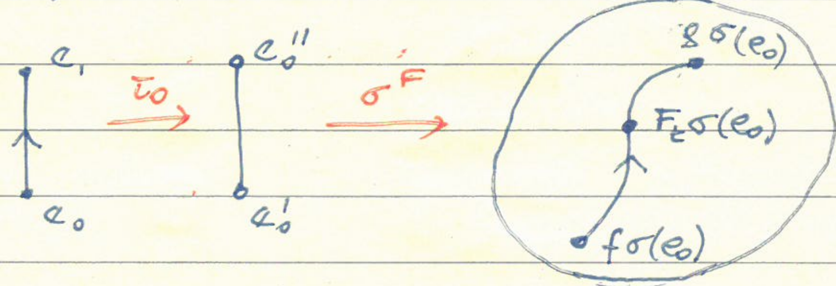
<u>Check</u>: these homomorphisms give a chain homotopy between the chain maps $f_*, g_* : C_*(X) \to C_*(Y)$.

<u>E.g.</u>:

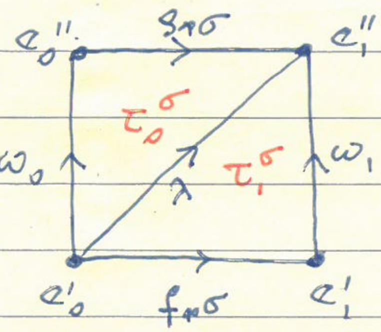
$$\begin{tikzcd}
C_2(X) \arrow[r, "\partial"] & C_1(X) \arrow[r, "\partial"] \arrow[dl, "\Phi_1"'] \arrow[d, shift right, "f_*"'] \arrow[d, shift left, "g_*"] & C_0(X) \arrow[dl, "\Phi_0"] \\
C_2(Y) \arrow[r] & C_1(Y) \arrow[r] & C_0(Y)
\end{tikzcd}$$

<u>Φ_0:</u> $\sigma : \Delta^0 \to X$

$\Phi_0(\sigma)$:



Φ_1: $\sigma : \Delta^1 \to X$



$$\Phi(\sigma) = \tau_0^\sigma - \tau_1^\sigma$$

$$\begin{aligned}
\partial\Phi_1(\sigma) &= \partial(\tau_0^\partial) - \partial(\tau_1^\partial) \\
&= (g_*\sigma - \cancel{\lambda} + \omega_0) - (\omega_1 - \cancel{\lambda} + f_*\sigma) \\
&= (g_*\sigma - f_*\sigma) + \underbrace{(\omega_0 - \omega_1)}_{\substack{\shortparallel \\ -\Phi_0(\partial\sigma)}}
\end{aligned}$$

$\blacksquare$

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Corollary: If $f: X \to Y$ is a homotopy equivalence then $f_*: H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$.

Example:

$$X \simeq * \quad \Rightarrow \quad H_n(X) \cong \begin{cases} \mathbb{Z} & n=0 \\ 0 & n>0 \end{cases}$$