9 | Separation Axioms

Separation axioms are a family of topological invariants that give us new ways of distinguishing between various spaces. The idea is to look how open sets in a space can be used to create “buffer zones” separating pairs of points and closed sets. Separations axioms are denoted by $T_1$, $T_2$, etc., where $T$ comes from the German word *Trennungsaxiom*, which just means “separation axiom”. Separation axioms can be also seen as a tool for identifying how close a topological space is to being metrizable: spaces that satisfy an axiom $T_i$ can be considered as being closer to metrizable spaces than spaces that do not satisfy $T_i$.

9.1 Definition. A topological space $X$ satisfies the axiom $T_1$ if for every points $x, y \in X$ such that $x \neq y$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

9.2 Example. If $X$ is a space with the antidiscrete topology and $X$ consists of more than one point then $X$ does not satisfy $T_1$.

9.3 Proposition. Let $X$ be a topological space. The following conditions are equivalent:

1) $X$ satisfies $T_1$.

2) For every point $x \in X$ the set $\{x\} \subseteq X$ is closed.

Proof. Exercise.

9.4 Definition. A topological space $X$ satisfies the axiom $T_2$ if for any points $x, y \in X$ such that $x \neq y$
there exist open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

A space that satisfies the axiom $T_2$ is called a **Hausdorff space**.

**9.5 Note.** Any metric space satisfies $T_2$. Indeed, for $x, y \in X$, $x \neq y$ take $U = B(x, \varepsilon)$, $V = B(y, \varepsilon)$ where $\varepsilon < \frac{1}{2}d(x, y)$. Then $U, V$ are open sets, $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**9.6 Note.** If $X$ satisfies $T_2$ then it satisfies $T_1$.

**9.7 Example.** The real line $\mathbb{R}$ with the Zariski topology satisfies $T_1$ but not $T_2$.

The following is a generalization of Proposition 5.13

**9.8 Proposition.** Let $X$ be a Hausdorff space and let $\{x_n\}$ be a sequence in $X$. If $x_n \to y$ and $x_n \to z$ for some then $y = z$.

**Proof.** Exercise. \qed

**9.9 Definition.** A topological space $X$ satisfies the axiom $T_3$ if $X$ satisfies $T_1$ and if for each point $x \in X$ and each closed set $A \subseteq X$ such that $x \notin A$ there exist open sets $U, V \subseteq X$ such that $x \in U$, $A \subseteq V$, and $U \cap V = \emptyset$.

A space that satisfies the axiom $T_3$ is called a **regular space**.

**9.10 Note.** Since in spaces satisfying $T_1$ sets consisting of a single point are closed (9.3) it follows that if a space satisfies $T_3$ then it satisfies $T_2$.

**9.11 Example.** Here is an example of a space $X$ that satisfies $T_2$ but not $T_3$. Take the set $K = \{\frac{1}{n} \mid n = 1, 2, \ldots\} \subseteq \mathbb{R}$
Define a topological space $X$ as follows. As a set $X = \mathbb{R}$. A basis $\mathcal{B}$ of the topology on $X$ is given by
\[ \mathcal{B} = \{ U \subseteq \mathbb{R} \mid U = (a, b) \text{ or } U = (a, b) \setminus K \text{ for some } a < b \} \]

Notice that the set $X \setminus K$ is open in $X$, so $K$ is a closed set.

The space $X$ satisfies $T_2$ since any two points can be separated by some open intervals. On the other hand we will see that $X$ does not satisfy $T_3$. Take $x = 0 \in X$ and let $U, V \subseteq X$ be open sets such that $x \in U$ and $K \subseteq V$. We will show that $U \cap V \neq \emptyset$. Since $x \in U$ and $U$ is open there exists a basis element $U_1 \in \mathcal{B}$ such that $x \in U_1$ and $U_1 \subseteq U$. By assumption $U_1 \cap K = \emptyset$, so $U_1 = (a, b) \setminus K$ for some $a < 0 < b$. Take $n$ such that $\frac{1}{n} < b$. Since $\frac{1}{n} \in V$ and $V$ is open there is a basis element $V_1 \in \mathcal{B}$ such that $\frac{1}{n} \in V_1$ and $V_1 \subseteq V$. Since $V_1 \cap K \neq \emptyset$ we have $V_1 = (c, d)$ for some $c < \frac{1}{n} < d$. For any $z \in \mathbb{R}$ such that $c < z < \frac{1}{n}$ and $z \notin K$ we have $z \in U_1 \cap V_1$, and so $z \in U \cap V$.

\[ \mathbb{R} \]
\[ 0 \quad \frac{1}{n+1} \quad z \quad \frac{1}{n} \quad \frac{1}{n-1} \]

\[ U_1 = (a, b) \setminus K \quad V_1 = (c, d) \]

9.12 Definition. A topological space $X$ satisfies the axiom $T_4$ if $X$ satisfies $T_1$ and if for any closed sets $A, B \subseteq X$ such that $A \cap B = \emptyset$ there exist open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

A space that satisfies the axiom $T_4$ is called a normal space.

9.13 Note. If $X$ satisfies $T_4$ then it satisfies $T_3$.

9.14 Theorem. Every metric space is normal.

The proof of this theorem will rely on the following fact:

9.15 Proposition. Let $X$ be a topological space satisfying $T_1$. If for any pair of closed sets $A, B \subseteq X$ satisfying $A \cap B = \emptyset$ there exists a continuous function $f: X \to [0, 1]$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$ then $X$ is a normal space.
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9.16 Definition. Let \((X, \rho)\) be a metric space. The distance between a point \(x \in X\) and a set \(A \subseteq X\) is the number
\[
\rho(x, A) := \inf \{ \rho(x, a) \mid a \in A \}
\]

9.17 Lemma. If \((X, \rho)\) is a metric space and \(A \subseteq X\) is a closed set then \(\rho(x, A) = 0\) if and only if \(x \in A\).

Proof. Exercise.

9.18 Lemma. Let \((X, \rho)\) be a metric space and \(A \subseteq X\). The function \(\phi: X \to \mathbb{R}\) given by
\[
\phi(x) = \rho(x, A)
\]
is continuous.

Proof. Let \(x \in X\). We need to check that for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(\rho(x, x') < \delta\) then \(|\phi(x) - \phi(x')| < \varepsilon\). It will be enough to show that
\[
|\phi(x) - \phi(x')| \leq \rho(x, x')
\]
for all \(x, x' \in X\) since then we can take \(\delta = \varepsilon\).

For \(a \in A\) we have
\[
\rho(x, A) \leq \rho(x, a) \leq \rho(x, x') + \rho(x', a)
\]
This gives
\[
\rho(x, A) \leq \rho(x, x') + \rho(x', A)
\]
and so
\[
\phi(x) - \phi(x') = \rho(x, A) - \rho(x', A) \leq \rho(x, x')
\]
In the same way we obtain \(\phi(x') - \phi(x) \leq \rho(x', x)\), and so \(|\phi(x) - \phi(x')| \leq \rho(x, x')\).

\(\square\)

Proof of Theorem 9.14. Let \((X, \rho)\) be a metric space and let \(A, B \subseteq X\) be closed sets such that \(A \cap B = \emptyset\). By Proposition 9.15 it will suffice to show that there exists a continuous function \(f: X \to [0, 1]\) such that \(A \subseteq f^{-1}\{0\}\) and \(B \subseteq f^{-1}\{1\}\). Take \(f\) to be the function given by
\[
f(x) = \frac{\rho(x, A)}{\rho(x, A) + \rho(x, B)}
\]
By Lemma 9.17 \(\rho(x, A) = 0\) only if \(x \in A\), and \(\rho(x, B) = 0\) only if \(x \in B\). Since \(A \cap B = \emptyset\) we have \(\rho(x, A) + \rho(x, B) \neq 0\) for all \(x \in X\), so \(f\) is well defined. From Lemma 9.18 it follows that \(f\) is a
continuous function. Finally, for any \( x \in A \) we have
\[
f(x) = \frac{q(x, A)}{q(x, A) + q(x, B)} = \frac{0}{0 + q(x, B)} = 0
\]
and for any \( x \in B \) we have
\[
f(x) = \frac{q(x, A)}{q(x, A) + q(x, B)} = \frac{q(x, A)}{q(x, A) + 0} = 1
\]

Notice that the function \( f \) constructed in the proof of Theorem 9.14 satisfies a condition that is stronger than the assumption of Proposition 9.15: we have \( f(x) = 0 \) if and only if \( x \in A \) and \( f(x) = 1 \) if and only if \( x \in B \). Thus we obtain:

**9.19 Corollary.** If \((X, q)\) is a metric space and \( A, B \subseteq X \) are closed sets such that \( A \cap B = \emptyset \) then there exists a continuous function \( f: X \to [0, 1] \) such that \( A = f^{-1}(\{0\}) \) and \( B = f^{-1}(\{1\}) \).

**9.20 Note.** The results described above can be summarized by the following picture:

![Diagram of separation axioms]

Each rectangle represents the class of topological spaces satisfying the corresponding separation axiom. No area of this diagram is empty. Even though we have not seen here an example of a space that satisfies \( T_3 \) but not \( T_4 \) such spaces do exist.

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**Exercises to Chapter 9**

**E9.1 Exercise.** Prove Proposition 9.3.


E9.5 Exercise. Let $X$ be a topological space and let $Y$ be a subspace of $X$.

a) Show that if $X$ satisfies $T_1$ then $Y$ satisfies $T_1$.

b) Show that if $X$ satisfies $T_2$ then $Y$ satisfies $T_2$.

c) Show that if $X$ satisfies $T_3$ then $Y$ satisfies $T_3$.

Note: It may happen that $X$ satisfies $T_4$ but $Y$ does not.

E9.6 Exercise. Show that if $X$ is a normal space and $Y$ is a closed subspace of $X$ then $Y$ is a normal space.

E9.7 Exercise. Let $X$ be a Hausdorff space. Show that the following conditions are equivalent:

(i) every subspace of $X$ is a normal space.

(ii) for any two sets $A, B \subseteq X$ such that $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$ there exists open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

E9.8 Exercise. This is a generalization of Exercise 6.9. Recall that a retract of a topological space $X$ is a subspace $Y \subseteq X$ for which there exists a continuous function $r : X \to Y$ such that $r(x) = x$ for all $x \in Y$. Show that if $X$ is a Hausdorff space and $Y \subseteq X$ is a retract of $X$ then $Y$ is a closed in $X$.

E9.9 Exercise. Let $X$ be a space satisfying $T_{1}$. Show that the following conditions are equivalent:

(i) $X$ is a normal space.

(ii) For any two disjoint closed sets $A, B \subseteq X$ there exist closed sets $A', B' \subseteq X$ such that $A \cap A' = \emptyset$, $B \cap B' = \emptyset$ and $A' \cup B' = X$.

E9.10 Exercise. Let $X$ be a topological space and $Y$ be a Hausdorff space. Let $f, g : X \to Y$ be continuous functions and let $A \subseteq X$ be given by

$$A = \{ x \in X \mid f(x) = g(x) \}$$

Show that $A$ is closed in $X$.

E9.11 Exercise. Let $X$ be a topological space, $Y$ be a Hausdorff space, and let $A$ be a set dense in $X$. Let $f, g : X \to Y$ be continuous functions. Show that if $f(x) = g(x)$ for all $x \in A$ then $f(x) = g(x)$ for all $x \in X$.

E9.12 Exercise. Let $f : X \to Y$ be a continuous function. Assume that $f$ is onto and that for any closed set $A \subseteq X$ the set $f(A) \subseteq Y$ is closed. Show that if $X$ is a normal space then $Y$ is also normal.