14 | Compact Spaces

14.1 Definition. Let $X$ be a topological space. A cover of $X$ is a collection $\mathcal{Y} = \{Y_i\}_{i \in I}$ of subsets of $X$ such that $\bigcup_{i \in I} Y_i = X$.

If the sets $Y_i$ are open in $X$ for all $i \in I$ then $\mathcal{Y}$ is an open cover of $X$. If $\mathcal{Y}$ consists of finitely many sets then $\mathcal{Y}$ is a finite cover of $X$.

14.2 Definition. Let $\mathcal{Y} = \{Y_i\}_{i \in I}$ be a cover of $X$. A subcover of $\mathcal{Y}$ is a cover $\mathcal{Y}'$ of $X$ such that every element of $\mathcal{Y}'$ is in $\mathcal{Y}$.

14.3 Example. Let $X = \mathbb{R}$. The collection

$$\mathcal{Y} = \{(m, n) \subseteq \mathbb{R} | m, n \in \mathbb{Z}, m < n\}$$

is an open cover of $\mathbb{R}$, and the collection

$$\mathcal{Y}' = \{(-n, n) \subseteq \mathbb{R} | n = 1, 2, \ldots\}$$

is a subcover of $\mathcal{Y}$.

14.4 Definition. A space $X$ is compact if every open cover of $X$ contains a finite subcover.

14.5 Example. A discrete topological space $X$ is compact if and only if $X$ consists of finitely many points.
14.6 Example. Let $X$ be a subspace of $\mathbb{R}$ given by
$$X = \{0\} \cup \{\frac{1}{n} \mid n = 1, 2, \ldots\}$$

The space $X$ is compact. Indeed, let $\mathcal{U} = \{U_i\}_{i \in I}$ be any open cover of $X$ and let $0 \in U_0$. Then there exists $N > 0$ such that $\frac{1}{n} \in U_i$ for all $n > N$. For $n = 1, \ldots, N$ let $U_{i_n} \in \mathcal{U}$ be a set such that $\frac{1}{n} \in U_{i_n}$. We have:
$$X = U_{i_0} \cup U_{i_1} \cup \cdots \cup U_{i_N}$$
so $\{U_{i_0}, U_{i_1}, \ldots, U_{i_N}\}$ is a finite subcover of $\mathcal{U}$.

14.7 Example. The real line $\mathbb{R}$ is not compact since the open cover
$$\mathcal{Y} = \{(n - 1, n + 1) \subseteq \mathbb{R} \mid n \in \mathbb{Z}\}$$
does not have any finite subcover.

14.8 Proposition. Let $f: X \to Y$ be a continuous function. If $X$ is compact and $f$ is onto then $Y$ is compact.

Proof. Exercise. \hfill $\square$

14.9 Corollary. Let $f: X \to Y$ be a continuous function. If $A \subseteq X$ is compact then $f(A) \subseteq Y$ is compact.

Proof. The function $f|_A: A \to f(A)$ is onto, so this follows from Proposition 14.8. \hfill $\square$

14.10 Corollary. Let $X, Y$ be topological spaces. If $X$ is compact and $Y \cong X$ then $Y$ is compact.

Proof. Follows from Proposition 14.8. \hfill $\square$

14.11 Example. For any $a < b$ the open interval $(a, b) \subseteq \mathbb{R}$ is not compact since $(a, b) \not\cong \mathbb{R}$.

14.12 Proposition. For any $a < b$ the closed interval $[a, b] \subseteq \mathbb{R}$ is compact.

Proof. Let $\mathcal{U}$ be an open cover of $[a, b]$ and let
$$A = \{x \in [a, b] \mid \text{the interval } [a, x] \text{ can be covered by a finite number of elements of } \mathcal{U}\}$$
Let $x_0 := \sup A$.

Step 1. We will show that $x_0 > a$. Indeed, let $U \in \mathcal{U}$ be a set such that $a \in U$. Since $U$ is open we have $[a, a + \varepsilon) \subseteq U$ for some $\varepsilon > 0$. It follows that $x \in A$ for all $x \in [a, a + \varepsilon)$. Therefore $x_0 \geq a + \varepsilon$. 
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Step 2. Next, we will show that \( x_0 \in A \). Let \( U_0 \in \mathcal{U} \) be a set such that \( x_0 \in U_0 \). Since \( U_0 \) is open and \( x_0 > a \) there exists \( \varepsilon_1 > 0 \) such that \( (x_0 - \varepsilon_1, x_0) \subseteq U_0 \). Also, since \( x_0 = \sup A \) there is \( x \in A \) such that \( x \in (x_0 - \varepsilon_1, x_0] \). Notice that

\[
[a, x_0] = [a, x] \cup (x_0 - \varepsilon_1, x_0]
\]

By assumption the interval \([a, x_0]\) can be covered by a finite number of sets from \( \mathcal{U} \) and \((x_0 - \varepsilon_1, x_0]\) is covered by \( U_0 \in \mathcal{U} \). As a consequence \([a, x_0]\) can be covered by a finite number of elements of \( \mathcal{U} \), and so \( x_0 \in A \).

Step 3. In view of Step 2 it suffices to show that \( x_0 = b \). To see this take again \( U_0 \in \mathcal{U} \) to be a set such that \( x_0 \in U_0 \). If \( x_0 < b \) then there exists \( \varepsilon_2 > 0 \) such that \([x_0, x_0 + \varepsilon_2] \subseteq U_0 \). Notice that for any \( x \in (x_0, x_0 + \varepsilon_2) \) the interval \([a, x]\) can be covered by a finite number of elements of \( \mathcal{U} \), and thus \( x \in A \). Since \( x > x_0 \) this contradicts the assumption that \( x_0 = \sup A \).

\[ \square \]

14.13 Proposition. Let \( X \) be a compact space. If \( Y \) is a closed subspace of \( X \) then \( Y \) is compact.

Proof. Exercise. \[ \square \]

14.14 Proposition. Let \( X \) be a Hausdorff space and let \( Y \subseteq X \). If \( Y \) is compact then it is closed in \( X \).

Proposition 14.14 is a direct consequence of the following fact:

14.15 Lemma. Let \( X \) be a Hausdorff space, let \( Y \subseteq X \) be a compact subspace, and let \( x \in X \setminus Y \). There exists open sets \( U, V \subseteq X \) such that \( x \in U, Y \subseteq V \) and \( U \cap V = \emptyset \).

Proof. Since \( X \) is a Hausdorff space for any point \( y \in Y \) there exist open sets \( U_y \) and \( V_y \) such that \( x \in U_y, y \in V_y \) and \( U_y \cap V_y = \emptyset \). Notice that \( Y \subseteq \bigcup_{y \in Y} V_y \). Since \( Y \) is compact we can find a finite number of points \( y_1, \ldots, y_n \in Y \) such that

\[
Y \subseteq V_{y_1} \cup \cdots \cup V_{y_n}
\]

Take \( V = V_{y_1} \cup \cdots \cup V_{y_n} \) and \( U := U_{y_1} \cap \cdots \cap U_{y_n} \).
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Proof of Proposition 14.14. By Lemma 14.15 for each point \( x \in X \setminus Y \) we can find an open set \( U_x \subseteq X \) such that \( x \in U_x \) and \( U_x \subseteq X \setminus Y \). Therefore \( X \setminus Y \) is open and so \( Y \) is closed.

14.16 Corollary. Let \( X \) be a compact Hausdorff space. A subspace \( Y \subseteq X \) is compact if and only if \( Y \) is closed in \( X \).


14.17 Proposition. Let \( f : X \to Y \) be a continuous function, where \( X \) is a compact space and \( Y \) is a Hausdorff space. For any closed set \( A \subseteq X \) the set \( f(A) \) is closed in \( Y \).

Proof. Let \( A \subseteq X \) be a closed set. By Proposition 14.13 \( A \) is a compact space and thus by Corollary 14.9 \( f(A) \) is a compact subspace of \( Y \). Since \( Y \) is a Hausdorff space, using Proposition 14.14 we obtain that \( f(A) \) is closed in \( Y \).

14.18 Proposition. Let \( f : X \to Y \) be a continuous bijection. If \( X \) is a compact space and \( Y \) is a Hausdorff space then \( f \) is a homeomorphism.

Proof. This follows from Proposition 6.13 and Proposition 14.18.

14.19 Theorem. If \( X \) is a compact Hausdorff space then \( X \) is normal.

Proof. Step 1. We will show first that \( X \) is a regular space (9.9). Let \( A \subseteq X \) be a closed set and let \( x \in X \setminus A \). We need to show that there exists open sets \( U, V \subseteq X \) such that \( x \in U \), \( A \subseteq V \) and \( U \cap V = \emptyset \). Notice that by Proposition 14.13 the set \( A \) is compact. Since \( X \) is Hausdorff existence of the sets \( U \) and \( V \) follows from Lemma 14.15.

Step 2. Next, we show that \( X \) is normal. Let \( A, B \subseteq X \) be closed sets such that \( A \cap B = \emptyset \). By Step 1 for every \( x \in A \) we can find open sets \( U_x \) and \( V_x \) such that \( x \in U_x \), \( B \subseteq V_x \) and \( U_x \cap V_x = \emptyset \). The collection \( \mathcal{U} = \{ U_x \}_{x \in A} \) is an open cover of \( A \). Since \( A \) is compact there is a finite number of points...
$x_1, \ldots, x_m \in A$ such that \{\(U_{x_1}, \ldots, U_{x_m}\)\} is a cover of \(A\). Take \(U := \bigcup_{i=1}^{m} U_{x_i}\) and \(V := \bigcap_{i=1}^{m} V_{x_i}\). Then \(U\) and \(V\) are open sets, \(A \subseteq U, B \subseteq V\) and \(U \cap V = \emptyset\). \(\square\)

### Exercises to Chapter 14

**E14.1 Exercise.** Prove Proposition 14.8.


**E14.3 Exercise.** Let \(X\) be a Hausdorff space and let \(A \subseteq X\). Show that the following conditions are equivalent:

(i) \(A\) is compact

(ii) \(A\) is closed in \(X\) and in any open cover \{\(U_i\)\}_{i \in I} of \(X\) there exists a finite number of sets \(U_{i_1}, \ldots, U_{i_n}\) such that \(A \subseteq \bigcup_{k=1}^{n} U_{i_k}\).

**E14.4 Exercise.** a) Let \(X\) be a compact space and for \(i = 1, 2, \ldots\) let \(A_i \subseteq X\) be a non-empty closed set. Show that if \(A_{i+1} \subseteq A_i\) for all \(i\) then \(\bigcap_{i=1}^{\infty} A_i = \emptyset\).

b) Give an example of a (non-compact) space \(X\) and closed non-empty sets \(A_i \subseteq X\) satisfying \(A_{i+1} \subseteq A_i\) for \(i = 1, 2, \ldots\) such that \(\bigcap_{i=1}^{\infty} A_i = \emptyset\).

**E14.5 Exercise.** a) Let \(X\) be a compact Hausdorff space and for \(i = 1, 2, \ldots\) let \(A_i \subseteq X\) be a closed, connected set. Show that if \(A_{i+1} \subseteq A_i\) for all \(i\) then \(\bigcap_{i=1}^{\infty} A_i\) is connected.

b) Give an example of a space \(X\) and subspaces \(A_1 \subseteq A_2 \subseteq \ldots \subseteq X\) such that \(A_i\) is connected for each \(i\) but \(\bigcap_{i=1}^{\infty} A_i\) is not connected.

**E14.6 Exercise.** The goal of this exercise is to show that if \(f : X \to \mathbb{R}\) is a continuous function and \(X\) is a compact space then there exist points \(x_1, x_2 \in X\) such that \(f(x_1)\) is the minimum value of \(f\) and \(f(x_2)\) is the maximum value.

Let \(X\) be a compact space and let \(f : X \to \mathbb{R}\) be a continuous function.

a) Show that there exists \(C > 0\) such that \(|f(x)| < C\) for all \(x \in X\).

b) By part a) there exists \(C > 0\) such that \(f(X) \subseteq [-C, C]\). This implies that \(\inf f(X) \neq -\infty\) and \(\sup f(X) \neq +\infty\). Show that there are points \(x_1, x_2 \in X\) such that \(f(x_1) = \inf f(X)\) and that \(f(x_2) = \sup f(X)\).

**E14.7 Exercise.** Let \((X, g)\) be a compact metric space, and let \(f : X \to X\) be a function such that \(q(f(x), f(y)) < q(x, y)\) for all \(x, y \in X, x \neq y\).

a) Show that the function \(\varphi : X \to \mathbb{R}\) given by \(\varphi(x) = q(x, f(x))\) is continuous.
b) Show that there exists a unique point \( x_0 \in X \) such that \( f(x_0) = x_0 \).

**E14.8 Exercise.** Let \( f: X \to Y \) be a continuous map such for any closed set \( A \subseteq X \) the set \( f(A) \) is closed in \( Y \).

a) Let \( y \in Y \). Show that if \( U \subseteq X \) is an open set and \( f^{-1}(y) \subseteq U \) then there exists an open set \( V \subseteq Y \) such that \( y \in V \) and \( f^{-1}(V) \subseteq U \).

b) Show that if \( Y \) is compact and \( f^{-1}(y) \) is compact for each \( y \in Y \) then \( X \) is compact.

**E14.9 Exercise.** Let \( X, Y \) be topological spaces, and let \( p_1: X \times Y \to X \) be the projection map: \( p_1(x, y) = x \). Show that if \( Y \) is compact then for any closed set \( A \subseteq X \times Y \) the set \( p_1(A) \subseteq X \) is closed in \( X \).

**E14.10 Exercise.** A continuous function \( f: X \to Y \) is a **local homeomorphism** if for each point \( x \in X \) there exists an open neighborhood \( U_x \subseteq X \) such that \( f(U_x) \) is open in \( Y \) and \( f|_{U_x}: U_x \to f(U_x) \) is a homeomorphism.

a) Assume that \( f: X \to Y \) is a local homeomorphism where \( X \) is a compact space. Show that for each \( y \in Y \) the set \( f^{-1}(y) \) consists of finitely many points.

b) Assume that \( f: X \to Y \) is a local homeomorphism where \( X \) is a compact Hausdorff space and \( Y \) is a Hausdorff space. Let \( y \in Y \) be a point such that \( f^{-1}(y) \) consists of \( n \) points. Show that there exists an open set \( V \subseteq Y \) such that \( y \in V \) and that for each \( y' \in V \) the set \( f^{-1}(y') \) consists of \( n \) points.