Investment Science – Math 458/558

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Investment is a commitment of resources for future benefit.

Examples • Savings account
• US Treasury bond
• A stock
• An option (a right to purchase a stock)
• "Forever" stamps
• lottery tickets

We are only interested in investments with quantitative benefits (with cash value). Hence, we do not discuss:
• benefits having children

Goal Invest in a way which "maximizes" future benefit.

Main problem: How to do this.

What does "maximize" mean?

Eg. What is better? • 5% interest guaranteed by US government, or
• a 10% bond issued by a company which has 3% default chance (probability).

Eg. Lottery tickets are considered a poor investment, since usually only 50% of ticket proceeds is spent on lottery prizes. However, if your life depends on collecting 1 million USD in a very short time period, then playing a lottery may be the most rational investment.

Chapter 2

Def. "Cash flow" or "cash flow stream" Net receipts over a certain time period. Cash flow is positive if one has a net gain. (Buying something creates a negative flow).

An investment of N dollars at interest r will result in $N \cdot (1 + r)$.
Eg. If $r = 10\% = 0.1$ then
future return $= N \cdot 1.1$.

Exercise Analyze cash flows of:
• Investment of \$1000 in a 1 year CD with interest 5%.
• Investment of \$1000 at 10%, paid quarterly, no compounding, for 18 months.

Compounding

N dollars invested at interest r for T years with annual compounding yields

$$N \cdot (1 + r)^T.$$

Exercise Analyze cash flows of and investment of \$1000 in a 3 year CD with interest 5% compounded annually.

• Bank loan of \$ 1000 at 5% is to be paid back in 3 equal payments, payed yearly. How much is the annual payment?

N dollars invested at interest r for T years, compounded quarterly yields

$$N \cdot (1 + r/4)^{4T}.$$

N dollars invested at interest r for T years, compounded daily yields

$$N \cdot (1 + r/365)^{365 \cdot T}.$$

N dollars invested at interest r for T years, with continuous compounding yields

$$N \cdot lim_{k \to \infty} (1 + r/k)^{k \cdot T} =$$
$$= N \cdot e^{r \cdot T}.$$

Eg. N dollars invested at 100% (r=1) for T year yields
$N \cdot 2^T$ with annual compounding
$N \cdot e^T$ with continuous compounding.

Recall that $e^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \ldots \frac{x^n}{n!}$

Ideal Bank
A bank which applies the same interest rate to deposits and loans and has no transaction fees. Its interest rate is the same for any size of principal. You have an unlimited access to banks loans.

This def does not imply that interest rates for 1 year and 2 year loans are the same.

Constant Ideal Bank
is an ideal bank whose interest rates are independent of the length of the time for which they apply.

Present Value refers to present value of future obligations. If the CIB interest rate is r, then the present value of \$ 100 to be received in T years is $\$100/(1+r)^T$.

For simplicity, we assume that all loans and deposits of the ideal bank are compounded annually.

Eg. Consider a cash flow stream $-2, 1, 1, 1$, when the periods are years and the CIB rate is 10%. The future value of this investment is

$$FV = -2 \cdot (1.1)^3 + 1 \cdot (1.1)^2 + 1 \cdot 1.1 + 1$$

The current value is

$$CV = -2 + 1/1.1 + 1/(1.1)^2 + 1/(1.1)^3.$$

Note that $CV = FV/(1.1)^3$.

Main theorem on present value

Two cash flow streams

$$x_0, x_1, ..., x_n \quad \text{and} \quad y_0, y_1, ..., y_n$$

are equivalent if their present values, evaluated at CIB interest rate, are equal.

You can replace "present value" by "future value".

Quiz

A person invests today in a 2-year CD with 10% interest compounded quarterly.
What is the present value of the money (s)he will receive?
The Constant Ideal Bank rate is 5%.

Def. Internal rate of return
The internal rate of return of an annual cash flow stream $x_0, x_1, ..., x_n$ is a number r satisfying the equation

$$x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + ... \frac{x_n}{(1+r)^n} = 0.$$

Eg. Consider a CD with interest r, compounded yearly.
Its internal rate of return is r.

Thm. Among two cash flows, the more profitable is the one with higher internal rate of return.

Eg. The internal rate of return for the cash flow $-2, 1, 1, 1$ is $r = 0.23$.

Problem: In general r defined above is not unique.

Eg. The cash flow

$$1, -2.6, 1.65$$

has two rates of return: r=% 10 and % 50!

Main theorem of internal rate of return

- Suppose a cash flow $x_0, x_1, ..., x_n$ has
  - x_0 negative, and
  - $x_1, ..., x_n \geq 0$, with at least one positive.

  Then the internal rate of return, r, is defined uniquely.

- If, furthermore, $x_0 + x_1 + ... + x_n > 0$ then $r > 0$.

General Principles of Investing

• Comparison
If no risk involved, choose an investment with the best return.

• Risk aversion
Given two investments with the same expected return, choose the one with smaller risk.

Def. Arbitrage is a way to make money with no risk and using none of your own money.

Eg. One bank loans at 4%, another gives 5% on deposits.

Eg. The same stock being traded on two different exchanges (perhaps in different currencies) may create arbitrage opportunities.

Def. A closed-end fund is a mutual fund which does not create new shares. Shares of closed-end funds are often traded on stock exchanges.

Short selling

Short selling or **shorting** is a process of selling a borrowed asset, with the intent of repurchasing it later at lower price.
It is like purchasing a negative quantity of that asset.

Short selling allows an investor to profit from a bear market.

Owning a negative quantity of a stock, is called a "short position."
A positive quantity is called a "long position."

The risk of long position is limited, since you cannot loose more than what you invested.
The risk of short position is unlimited.
If you shorted a share of GE on May 6, 2009 you received $6.66. Now it costs almost $14.

In a simple ”buy – sell” investment total return is

$$R = \frac{\text{amount received}}{\text{amount invested}}.$$

Rate of return is

$$r = \frac{\text{amount received} - \text{amount invested}}{\text{amount invested}}.$$

Eg. For a 5% CD (with no compounding), $R = 1.05$, $r = 0.05$.

Clearly $R = 1 + r$.

We will often shorten “total return” and “rate of return” to “return”.

Eg. A person 1 stock of company X for $100 and sells it a week later for $90.
$R = 0.9$, $r = -10\%$.
Another person shorts that stock at $100 and covers it at $90.
$R = \frac{-90}{-100} = 0.9$, $r = \frac{(-90)-(-100)}{-100} = -10\%$.

Portfolio is a collection of assets.

Suppose a portfolio P is composed of n assets of values $A_1, ..., A_n$.
Then the weight of i-th asset in the portfolio is $w_i = A_i/P$.

Eg. Our \$10,000 portfolio is composed of
- 1-year 5% CD for \$3,000, and
- 500 shares of GE, each of current value \$14.

The weight of CD is 30% in the portfolio.
The weight of GE shares is 70%.

Thm. Suppose that after 1 year, the total return of i-th asset is R_i. Then the total return of the portfolio is

$$R = \sum_{i=1}^{n} w_i \cdot R_i$$

and the rate of return is

$$r = \sum_{i=1}^{n} w_i \cdot r_i.$$

Random Variables – Intuitive approach

Wikipedia: A random variable can be thought of as an unknown value that may change every time it is inspected.
A <u>discrete random variable</u> takes values from a specific discrete set (eg. a finite set or the set of integers).

Eg. Coin toss. Let $X = \begin{cases} 1 & \text{if heads} \\ 0 & \text{if tails.} \end{cases}$

X takes each value with probability $1/2$.
This is called a probability distribution. It is a function from the set of values of X to $[0, 1]$.

We write $P(X = 0) = \frac{1}{2}$.

If ρ is the probability distribution of X then
$P(a < X < b) = \sum \rho(c)$,
where the sum is over all values c of X such that $a \leq c \leq b$

An example of a continuous random variable is a spinner which can choose any value (angle) $\alpha \in [0, 2\pi)$.
$P(a < \alpha < b) = \frac{b-a}{2\pi}$.

Def. Probability distribution of a continuous random variable X is a function $\rho : \mathbb{R} \to [0, 1]$ such that $P(a < X < b) = \int_a^b \rho dx$.

The distribution of α is $\rho(\alpha) = \begin{cases} \frac{1}{2\pi} & \text{for } 0 \leq \alpha \leq 2\pi \\ 0 & \text{otherwise.} \end{cases}$

Let X be the number of heads in a sequence of N coin tosses. If N is very large, then X behaves like a continuous random variable with a normal distribution.

You should distinguish between a random variable and its probability distribution.
For example, let X be the number of heads and Y be the number of tails in a series of 3 coin tosses. Then X and Y have the same distributions, but $X \neq Y$. In fact, $Y = 3 - X$.

If X takes values $x_1, ..., x_n$ with probabilities $p_1, ..., p_n$ then $\sum_{i=1}^{n} 1$ and the expected value of X is

$$E(X) = \sum_{i=1}^{n} x_i p_i.$$

Another notation: $\overline{X}$.

Eg. Let $X \in \{1, ..., 6\}$ be the result of a roll of the die. Then $E(X) = 3.5$