TANGLE EQUATIONS, THE JONES CONJECTURE, AND QUANTUM CONTINUED FRACTIONS*

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Abstract. We study systems of 2-tangle equations

$$\begin{cases} N(X+T_1)=L_1 \\ N(X+T_2)=L_2. \end{cases}$$

which play an important role in the analysis of enzyme actions on DNA strands.

We show the benefits of considering such systems in the context of framed tangles and, in particular, we conjecture that in this setting each such system has at most one solution X. We prove a version of this statement for rational tangles.

More importantly, we show that the Jones conjecture implies that if a system of tangle equations has a rational solution then that solution is unique among all tangles. This result opens a door to a purely topological line of attack on the Jones conjecture.

Additionally, we relate systems of tangle equations to the Cosmetic Surgery Conjecture.

Furthermore, we establish a number of properties of the Kauffman bracket $([T]_0, [T]_\infty)$ of 2-tangles T for the purpose of the proofs of the above results, but which are of their own independent interest. In particular, we show that the Kauffman bracket ratio $Q(T) = [T]_\infty/[T]_0$ quantizes continued fraction expansions of rationals. Furthermore, we prove that for algebraic tangles $Q(T)$ determines the slope of incompressible surfaces in $D^3 \smallsetminus T$.

Contents

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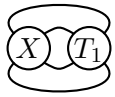
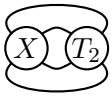
*In memory of John Conway, 1937–2020.

1 Introduction. This paper considers a number of interrelated topics, all motivated by the question of uniqueness of solutions of systems of tangle equations:

(1) 

$= L_1$ and 

$= L_2$,

where $T_1 \neq T_2$ are rational 2-tangles, L_1, L_2 are links, and X is an unknown tangle. Such systems play an important role in the analysis of enzyme actions on DNA strands, cf. Sec. 3.

We show the benefits of considering equations (1) in the context of framed tangles and, in particular, we conjecture that (1) has always at most one solution among all framed 2-tangles up to certain symmetries. In Theorem 4 we prove a version of that conjecture for framed rational tangles.

We study (1) through surgery theory for 3-manifolds and by developing a theory of the Kauffman bracket

$$[T] = ([T]_0, [T]_\infty) \in \mathbb{Z}[A^{\pm 1}]^2$$

for framed 2-tangles T. In particular, we establish a connection between tangle equations and the Jones conjecture, asserting that the Jones polynomial distinguishes all non-trivial knots from the trivial one, [Jo]. Specifically, we show in Theorem 7 that if the Jones conjecture holds and (1) has a rational solution then that solution is unique among all tangles. This may be considered the main result of the paper.

The connections between the Jones polynomials and the topology of the link complements are still obscure, despite much research effort devoted to them in the last decades. Hence, it is pleasing to see a purely topological consequence of the Jones conjecture. In particular, it opens a door to a line of attack on the Jones conjecture through the methods of geometric topology, without utilizing any properties of the Jones polynomial itself.

For the purpose of proving the above results, we establish several statements about the Kauffman bracket of tangles, of their own independent interest. In particular, we study the Kauffman bracket ratio, $Q(T) =$

$[T]_\infty/[T]_0$ which does not depend on the framing of T. We show in Section 9 that for algebraic T, $Q(T)$ determines the slope of incompressible surfaces in $D^3 - T$.

In Section 8 we prove a number of properties of the Kauffman bracket of 2-tangles which are equivalent to the Jones conjecture. In Section 11, we show that for rational tangles these ratios are quantized continued expansions of rationals.

Taking double branched covers of tangles and links translates equations (1) into the language of surgery theory of 3-manifolds. Apart from the Kauffman bracket considerations, this is the second main source of methods utilized throughout the paper. In particular, we show that (1) has no (unframed) algebraic, non-rational solutions when $T_1 \neq T_2$ and $L_1 = L_2$ is a Montesinos link. Furthermore, assuming the Cosmetic Surgery conjecture, (1) has no non-rational solutions among all 2-tangles for any $L_1 = L_2$.

Finally, in Section 12 and 13 we construct a tangle model for $\mathrm{PSL}(2,\mathbb{Z})$ and its action on $\mathbb{Q} \cup \{\infty\}$ and a framed tangle model for an infinite cyclic extension of $\mathrm{PSL}(2,\mathbb{Z})$.

2 Rational and Algebraic Tangles and Links. In order to state the results we will need some preliminaries. Throughout the paper, 2-tangles will be called tangles for brevity. They will be represented by diagrams in round disks D^2 with ends at NE, SE, SW, NW points of ∂D^2. The $\langle -1\rangle, \langle 0\rangle, \langle 1\rangle$ and $\langle \infty\rangle$ tangles and the tangle addition are defined in Fig. 1.

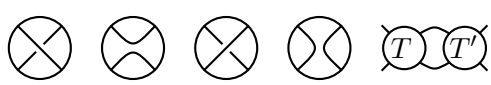


Figure 1. The $-1, 0, 1$ and ∞ tangles and the tangle addition.

The result of adding n tangles $\langle 1\rangle$ (respectively: $\langle -1\rangle$) together is denoted by $\langle n\rangle$ (respectively $\langle -n\rangle$), for $n = 1, 2, 3...$ These tangles, together with $\langle 0\rangle$, are called integral.

The mirror image $-T$ of T is obtained by switching all crossings of T. The tangle rotation $R(T)$ means the 90^0 clockwise rotation and the tangle inversion means the tangle rotation followed by the mirror image.

All tangles obtained from integral ones by operations of addition and rotation are called algebraic. (This class is closed under the mirror image and inversion.) Among them are rational tangles defined as follows:

By $\langle a_n, .., a_1\rangle$ we denote the tangle obtained from $\langle 0\rangle$ by adding $\langle a_1\rangle$ followed by the inversion, then by adding $\langle a_2\rangle$ followed by the inversion, and so on, until this construction is finished by adding a_n at the end, as in Figure 2(left). Tangles of this form, for $a_1, ..., a_n \in \mathbb{Z}$, are called rational because of the following observation by John Conway:

Theorem 1 ([Co]). *The continued fraction*

$$\langle a_n, .., a_1\rangle \to a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \frac{1}{a_1}}} \tag{2}$$

defines a bijection between rational tangles (up to tangle isotopy) and $\mathbb{Q} \cup \{\infty\} = \widehat{\mathbb{Q}}$. *Under this bijection, tangle inversion corresponds to fraction inversion,* $x \to 1/x$, *and the mirror image operation corresponds to the negation,* $x \to -x$.

[Co] does not contain a proof of this result. However, it can be found for example in [Mo2, BZ, KL]. It is important to remember that this map does not preserve addition.

By applying a numerator or denumerator closure (defined in Fig. 2) to a rational tangle we obtain a rational link, also referred to as a 2-bridge link or 4-plat.

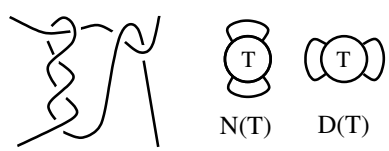


FIGURE 2. Rational Tangle $\langle 2, 3, -2\rangle$, the numerator and denumerator closures

3 Tangle Equations. Prime knots K in S^3 are characterized by their complements, [GL]. Clearly, that is not the case for tangles. For example the complements of all rational tangles are homeomorphic to the genus 2 handlebody. However, one might try to characterize a tangle X by the following equation

$$N(X + T) = L \tag{3}$$

with a given tangle T and a link L. This is a tangle equation. We will always assume that T is rational. Often, one assumes that L is rational as well. In that case, the rational solutions X are classified in [ES, Thm. 2.2]: If $T = \langle a_1, ..., a_{2n}\rangle$ and $L = D(\langle c_1, ..., c_{2k+1}\rangle)$ is not the unlink on 2 components, then

$$X = \langle c_1, ..., c_{2k+1}, r, -a_1, ..., -a_{2n}\rangle \quad \text{or} \quad X = \langle c_{2k+1}, ..., c_1, r, -a_1, ..., -a_{2n}\rangle,$$

for some $r \in \mathbb{Z}$. (Hence, there are infinitely many rational solutions). For the unlink on 2-components L, $X = \langle -a_1, ..., -a_{2n}\rangle$ is the unique rational solution.

Sums of rational tangles $R_1 + ... + R_k$ are called Montesinos tangles. Solutions to (3) among such tangles are discussed in [Er1]. For example, $X = \langle \frac{1}{n}\rangle + \langle \frac{-1}{n+1}\rangle$, for $n \in \mathbb{Z}$, is an infinite family of non-rational solutions to $N(X + \langle 0\rangle)$=unknot, cf. Fig. 3.

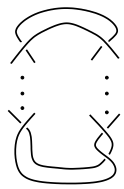


Figure 3. The numerator closures of tangles $\langle \frac{1}{n} \rangle + \langle \frac{-1}{n+1} \rangle$ are the unknot.

More generally, given a tangle X_0 such that $N(X_0+T)$ is the unknot, the connected sum, $X = X_0 \# L$ is a solution of (3) for any L. A tangle X not of this form is called locally unknotted. Finding locally unknotted solutions of (3) is a much harder and generally unsolved problem.

Tangle equations play a crucial role in the analysis of recombination of DNA molecules. The reason for that is that certain enzymes (called recombinase) separate circular DNA substrate molecules into two tangles: T_1, consisting of the part of the DNA molecule bound to the enzyme, and the other part, X, not bound to the enzyme. Then the enzyme replaces T_1 by a tangle T_2. That leads to equations:

$$\begin{cases} N(X+T_1) = L_1 \\ N(X+T_2) = L_2. \end{cases} \tag{4}$$

The substrate knot L_1 is controlled by the experiment. L_2 is called the product knot and it is observable in the experiment. The tangles T_1 and T_2 are known. We will always assume that $T_1 \neq T_2$, since otherwise this system has either no solutions or reduces to a single equation. See e.g. [ADV, BM, Da2, DS, ES, Su, SECS, VLNSLDL, Ya] for further discussion of such systems.

The necessary and sufficient conditions for the existence and uniqueness of solutions for (4) are not known. However, here are some partial results in that direction:

- [ES, Thm 3.7] states necessary conditions for existence of a solution of (4) for $T_1 = \langle 0 \rangle$ or $\langle \infty \rangle$ and L_1 the unknot.
- Any system (4) has at most two rational solutions X, [ES, Cor. 2.3].
- Solutions of (4) among Montesinos tangles are discussed in [Er2, Da1].
- Given rational T_1, L_1, L_2, computer program TopoICE-R looks for solutions (X, T_2) to (4) for rational T_2 and for X obtained from a Montesinos tangle by integral additions and inversions, [DS]. Another, similar computer program is TangleSolve by [ZGHV].

By taking double branched covers of tangles, in Section 6 we relate the question of solutions to (4) to surgery problems on 3-manifolds. In particular, we will relate the existence of solutions to (4) to the Cosmetic Surgery Conjecture.

In general, both the existence and the uniqueness of solutions of (4) are difficult problems.

Let consider the following simple example, which will be needed later:

$$N(X + \langle 1 \rangle) = U = N(X + \langle 0 \rangle), \tag{5}$$

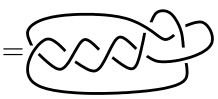
where U denotes the unknot, is satisfied by $X_1 = \langle \infty \rangle$ and by $X_2 = \langle -1/2 \rangle$. By [ES, Cor. 2.3] these are the only rational solutions. However, more generally, we have

Lemma 2. *(Proof in Sec. 14) $X = \langle \infty \rangle$ and $\langle -1/2 \rangle$ are the only unframed solutions to*

$$N(X + \langle 0 \rangle) = U = N(X + \langle 1 \rangle),$$

where U denotes the unknot as before.

Here is a more complicated example:

Example 3. $N(X + \langle -1 \rangle) = U, \quad N(X + \langle 0 \rangle) =$ 

has at least four solutions

$$\langle 3/4 \rangle + \langle 1/3 \rangle, \langle 1/3 \rangle + \langle 3/4 \rangle, \langle 3/7 \rangle + \langle 1/2 \rangle, \langle 1/2 \rangle + \langle 3/7 \rangle.$$

[Er2] *shows that these are the only solutions among Montesinos tangles.*

We are going to see soon that the uniqueness of solutions problem becomes more manageable when one considers framed tangles instead.

4 Framed Tangles. Framed links are tame embeddings of annuli $S^1 \times I \cup ... \cup S^1 \times I$ into $\mathbb{R}^3$. Similarly, framed tangles are embeddings

$$J_1 \times I \cup J_2 \times I \cup S^1 \times I \cup ... \cup S^1 \times I \hookrightarrow D^2 \times I,$$

where J_1, J_2 are intervals and the end arcs $\partial J_1 \times I$, $\partial J_2 \times I$ are horizontal in $D^2 \times I$ at hight $1/2$, each containing a different point from among $(NE, 1/2), (SE, 1/2), (SW, 1/2), (NW, 1/2)$ in $\partial D^2 \times I$. Clearly, every link diagram and tangle diagram defines a framed link or tangle with its framing parallel to the page. We require that every framed link and tangle can be represented in that way. (Hence, components with a half-twist framing are not allowed).

For our purposes it will be convenient to consider framed links and tangles up to balanced isotopy given by the balanced Reidemeister moves, as in Fig. 4. Note that this is a somewhat more flexible isotopy than the regular one,

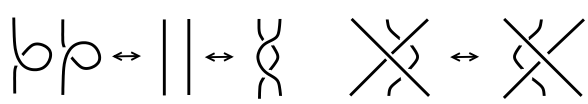


FIGURE 4. Balanced Reidemeister moves

which allows second and third Reidemester moves only.

A rational tangle with an arbitrary framing is a framed rational tangle. Given a diagram D of a tangle or a link, we will denote by D^n the framed tangle or diagram obtained from the page framing by adding $|n|$ positive or negative twists, depending on the sign of n. Note that location of these twists does not matter up to balanced Reidemeister moves.

Definitions of tangle addition and of numerator and denumerator closures generalize immediately to framed tangles. Consequently, systems (4) can be considered in the context of framed tangles and links as well.

5 Uniqueness of Tangle Solutions. Framed links and tangles appear more appropriate for modeling DNA strands which are double stranded, as in Fig. 5. Furthermore, the setting of framed links turns out to make

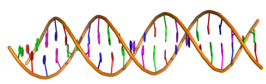


Figure 5. Double stranded DNA with colored bases. Credit: `https://commons.wikimedia.org/wiki/File:Double_stranded_DNA_with_coloured_bases.png`

it possible to prove certain uniqueness results for the solutions of tangle equations. As before, we will always assume that T_1, T_2 are rational and unequal as unframed tangles. (Note that if T_1 and T_2 in (4) differ by framing twists only, then by adjusting the framing of L_2, one can reduce (4) to a form in which $T_1 = T_2$ as framed tangles. In that form, (4) is either inconsistent or reduces to a single equation.)

We have seen in Example 3, that solutions to (4) are non-unique in general. However, for framed tangles we have:

Theorem 4 (Uniqueness of Rational Tangle Solutions, Proof in Sec. 14)**.** *Equations (4) have at most one framed rational solution X for any T_1, T_2, L_1, L_2.*

Let us consider for example the a framed version of equations of Example 3(a):

$$N(X) = U^n \text{ and } N(X + \langle 1 \rangle) = U^m, \tag{6}$$

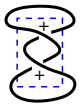
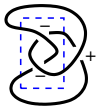
for some $n, m \in \mathbb{Z}$, where as we defined above, U^n denotes the unknot with $|n|$ twists, positive or negative, depending on the sign of n.

Lemma 5. *(1) If $m = n + 1$ then the infinity tangle with n kinks, $\langle \infty \rangle^n$, is the only solution to (6).*
(2) If $m = n - 3$ then $\langle -1/2 \rangle^{n+2}$ is the only solution to (6)
(3) If $m - n$ is neither 1 or -3 then (6) has no framed solution.

Proof. Suppose that X satisfies the above equations. Then stripped of its framing, it is either $X = \langle \infty \rangle$ or $\langle -1/2 \rangle$, by Lemma 2. Since

 $= U^f$ and  $= U^{f+1}$

where the dashed square marks the tangle $\langle\infty\rangle^f$, $X = \langle\infty\rangle$ yields a solution only iff $m - n = 1$ and that solution is $\langle\infty\rangle^f$, where $f = n$. Similarly,

 $= U^{f+2}$ and  $= U^{f-1}$

shows that $X = \langle -1/2\rangle$ yields a solution only iff $m - n = -3$ and that solution is $\langle\infty\rangle^f$, $f = n - 2$.

For $m - n \neq 1, -3$, these equations are contradictory. □

One may ask if framed solutions to (4) are always unique, even among non-rational tangles. There are not known any general bounds on the total number of non-rational solutions of (4) in the unframed or framed setting. However, we know only two types of non-uniqueness (in the framed setting):

- If X is a solution to (4) then so is $H(X)$ obtained by a 180^0 rotation along a horizontal axis. (That rotation does not change the crossing signs of X.) The reason for it is that H preserves rational tangles and, consequently, if $N(X + T) = L$ then

$$N(H(X) + T) = N(H(X) + H(T)) = N(H(X + T)) = N(X + T) = L.$$

- If X is a solution of

$$\begin{cases} N(X + \langle n_1\rangle) = L_1 \\ N(X + \langle n_2\rangle) = L_2, \end{cases}$$

  where $n_1, n_2 \in \mathbb{Z}$ are of the same parity, then the image $R^2(X)$ of X under 180^0 rotation (in page) is a solution as well. Proof: if $L = N(X + \langle n\rangle)$ then by applying R^2, we have

$$L = N(R^2(X + \langle n\rangle)) = N(R^2(\langle n\rangle) + R^2(X)) = N(\langle n\rangle + R^2(X)).$$

  If n is even then $\langle n\rangle + T = T + \langle n\rangle$ for any T and the proof is completed. If n is odd then $\langle n\rangle + T = H(T) + \langle n\rangle$ and, hence $H(R^2(T))$ is a solution. However, the previous discussion of H implies that then $R^2(T)$ is a solution as well.

We propose the following Uniqueness of Tangle Equations Solutions Conjecture:

Conjecture 6. *(1) If $T_1 = \langle n_1\rangle$ and $T_2 = \langle n_2\rangle$ where n_1, n_2 are of the same parity ($n_1 \neq n_2$) then (4) has at most one framed solution up to the horizontal rotation H and up to R^2.*
(2) Otherwise, (4) has at most one framed solution up to the horizontal rotation H.

Note that $H(X) = X = R^2(X)$ for rational X. Consequently, this conjecture implies that any rational solution to (4) is unique (among all tangles). Although it may sound unexpected, we claim that this purely topological claim is implied by the Jones conjecture.

Recall that the Jones conjecture asserts that the Jones polynomial distinguishes all non-trivial knots from the trivial one, [Jo]. Up to date, it has been verified for all knots up to 24 crossings, [TS2].

Theorem 7. *(Proof in Sec. 14.) If the Jones conjecture holds for knots up to n crossings then (4) cannot have two different framed solutions X, X', at least one of them rational, with the sum of their crossing numbers less than n. (These crossing numbers do not include framing twisting kinks.)*

If no topological proof of Theorem 7 is found, independent of the Jones conjecture, then this result may be utilized to disprove the Jones conjecture by purely topological means.

6 Surgery Methods, Relation to Cosmetic Surgery. The methods employed in literature to study tangle equations relay on the following idea: The complement of any rational tangle T in D^3 is a genus 2-handlebody. The double cover of D^3 branched along T, $\Sigma(T)$, is a solid torus. Similarly, the double cover of D^3 branched along any tangle X, $\Sigma(X)$, is a 3-manifold whose boundary is the double cover of ∂D^3 branched along 4 points, i.e. a torus.

By the same symbol Σ we will denote double branched covers of links in S^3. For L rational, $\Sigma(L)$ is a lens space, since for $L = N(\langle x \rangle)$, $x \in \widehat{\mathbb{Q}}$, the space $\Sigma(L)$ is a Dehn filling of the solid torus, $\Sigma(\langle x \rangle)$.

Consequently, (3) and (4) can be studied through the sophisticated methods of surgery theory on 3-manifolds. (For example, [ES] applies the Cyclic Surgery Theorem, [CGLS].)

However, we were unable to prove Conjecture 6 by the methods of surgery theory. Part of the difficulty is that those methods do not take into account the framing and Conjecture 6 does not hold for unframed tangles. Also, most results about tangle equations take advantage of the specific form of $\Sigma(X)$ for rational, or Montesions tangles X. Our conjecture lays no restrictions on X.

Let us now discuss a relation between systems of tangle equations and the Cosmetic Surgery Conjecture. We say that two different surgeries on a knot in an orientable 3-manifold are purely cosmetic if they yield manifolds homeomorphic through a preserving orientation homeomorphism.

The following appears as a conjecture in [Go, Conjecture 6.1] and in [BHW, NZ] and, as Problem 1.81(A) in Kirby's problem list [Ki]:

Conjecture 8 (Cosmetic Surgery Conjecture)**.** *Suppose K is a knot in a closed oriented 3-manifold M such that $M \setminus K$ is irreducible and not homeomorphic to the solid torus. If two different Dehn surgeries on K are purely cosmetic, then there is a homeomorphism of $M \setminus K$ which takes one slope to the other.*

Proposition 9. *Cosmetic Surgery Conjecture implies that systems (4) do not have any (unframed) locally unknotted non-rational solutions X for $L_1 = L_2$.*

Proof. If system (4) has an unframed solution X then $\Sigma(N(X+T_1))$ and $\Sigma(N(X+T_2))$ are two different Dehn fillings of $\Sigma(X)$ yielding $\Sigma(L_1) = \Sigma(L_2)$. By [Li, Thm. 5], $\Sigma(X)$ is irreducible. Consequently, $\Sigma(X)$ must be a solid torus, implying that X is rational. □

Therefore, Cosmetic Surgery Conjecture implies that if

$$N(X+T_1) = N(X+T_2)$$

then X must be either rational or not locally unknotted.

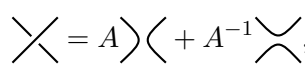
It is worth noting that the case $T_1 = \langle +1 \rangle, T_2 = \langle -1 \rangle$ of this statement is the subject of the Nugatory Crossing Conjecture, which states that in that case the crossing in T_1, T_2 is nugatory, [BFKP, BK, Ka, LM, To]. Hence, in particular, X is locally knotted then.

Taking advantage of partial results towards the Cosmetic Surgery Conjecture we have a part of the conclusion of Proposition 9:

Proposition 10. *(1) System (4) does not have any (unframed) non-rational algebraic solutions X for any Montesinos links $L_1 = L_2$.*
(2) It has a rational solution X only if $L_1 = L_2$ is a rational knot $\mathfrak{b}(p,q)$ with $q^2 = \pm 1$ mod p.

Proof. Suppose that an algebraic solution X exists. Then by [Mo1] the double branched cover, $\Sigma(X)$, is a graph manifold, and, hence, it is non-hyperbolic. Furthermore, $\Sigma(L_1)$ is Seifert, [BZ]. By [Ma, Thm. 1.3], $\Sigma(X)$ may have two different Dehn fillings yielding (in an orientation preserving way) the same lens space only if $\Sigma(X)$ is a solid torus and $\Sigma(L_1)$ is a lens space $L(p,q)$ with $q^2 = \pm 1$ mod p. The double cover $\Sigma(X)$ can be a solid torus only if X is rational. $\Sigma(L_1) = L(p,q)$ implies that $L_1 = \mathfrak{b}(p,q)$. □

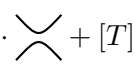
7 Kauffman bracket for tangles and the KB-ratio. The Kauffman bracket $[L] \in \mathbb{Z}[A^{\pm 1}]$ is an invariant of framed links L up to balanced isotopy, satisfying the skein relations



$$= A \quad + A^{-1} \quad , \qquad \bigcirc = \delta,$$

where $\delta = -A^2 - A^{-2}$, normalized so that the bracket of the trivially framed unknot U^0 is $[U^0] = 1$.

The Kauffman bracket can be extended to framed tangles, since each of them can be expressed as



$$T = [T]_0 \cdot \quad + [T]_\infty \cdot \quad ,$$

where $[T]_0, [T]_\infty \in \mathbb{Z}[A^{\pm 1}]$ are uniquely defined. We call the vector $[T] = \begin{pmatrix} [T]_0 \\ [T]_\infty \end{pmatrix}$ the <u>Kauffman bracket</u> of T. Note that it is preserved by balanced isotopies of T.

We call

$$Q(T) = [T]_\infty / [T]_0 \in \mathbb{Q}(A) \cup \{\infty\}$$

the Kauffman bracket ratio, or the KB-ratio of T, for short. Note that it is preserved by the first Reidemeister move and, hence, it is an invariant of unframed tangles. We will discuss algebraic properties of KB-ratios further in Sec. 11. In particular, we will observe that KB-ratios distinguish all (unframed) rational tangles.

Theorem 11. *For any tangle T,*
(1) $Q(T) \in \mathbb{Q}(t) \cup \{\infty\}$, where $t = A^2$.
(2) $t^{-1}Q(T) \in \mathbb{Q}(t^2) \cup \{\infty\}$.
(3) The KB-ratio of the mirror image of T is $Q(T)$ with t^{-1} substituted for t.
(4) The KB-ratio of the rotation R of T is $1/Q(T)$.
(5) For any tangle T', $Q(T+T') = Q(T) + Q(T') + Q(T)Q(T')\delta$.

Proof. (1) It is known that the Kauffman bracket $[D]$ of any link diagram D with c crossings has exponents congruent to c mod 2. It is easy to see that this property generalizes to $[T]_0$ and $[T]_\infty$ for any tangle T. Hence, $Q(T) = [T]_\infty/[T]_0$ is a rational function in $t = A^2$. (For T with odd number of crossings, multiply the numerator and denominator of $Q(T)$ by A.)
(2) The Kauffman bracket $[D]$ of any link diagram D has exponents congruent mod 4. That generalizes to $[T]_0$ and $[T]_\infty$. More specifically, we claim that for any tangle T:

- the exponents of A in $[T]_0$ are congruent mod 4 and the exponents of A in $[T]_\infty$ are congruent mod 4.
- the exponents of A in $[T]_0$ differ from the exponents of A in $[T]_\infty$ by a number congruent to 2 mod 4.

To see that note that $[T]_0$ and $[T]_\infty$ are given by sums of states,

$$\sum_S A^{a(S)-b(S)}\delta^{l(S)},$$

where $l(S)$ is the number of loops in a state S. Each state is of the 0- or ∞-type. A change of smoothing in a state, changes the exponent of A by 2 and it either changes $l(S)$ by one or it changes the state type (alternating between the 0- and ∞-type). That implies the first bullet point above.

To make the proof complete, note that T either (a) contains a crossing whose two different smoothings yield to two different state types or (b) T contains two non-intersecting arcs and $Q(T) = 0$ or ∞. Both cases imply the second bullet point.
(3) and (4) are obvious.
(5) For tangle sums we have,

$= [T]_0[T']_0$

$+ ([T]_0[T']_\infty + [T]_\infty[T']_0 + [T]_\infty[T']_\infty\delta)$

.

Hence,

$$Q(T+T') = \frac{[T]_0[T']_\infty + [T]_\infty[T']_0 + [T]_\infty[T']_\infty\delta}{[T]_0[T']_0}$$

which leads to the desired equality. □

In the next sections we will see the utility of the Kauffman bracket for analyzing systems of tangle equations. A general criterion for the existence of solutions of such systems in terms of the Kauffman bracket is formulated in Corollary 26.

8 Jones Conjecture for Tangles.

Lemma 12 (Kauffman bracket version of Jones Conjecture)**.**
The Jones conjecture (JC) is equivalent to its Kauffman bracket version (KB-JC), stating that if $[K] = r \cdot A^k$, for some $r, k \in \mathbb{Z}$ then $K = U^n$ (the unknot with framing n) for some $n \in \mathbb{Z}$. In particular, $[K] = (-A^3)^n$.

Proof. KB-JC $\Rightarrow$ JC: Suppose that the Jones polynomial of K is $J(K) = 1$ for some knot K. Then K with some framing has its Kauffman bracket equal to $(-A^3)^n$ for some $n \in \mathbb{Z}$. By KB-JC, $K = U^n$. Hence, K is trivial as an unframed knot.

JC $\Rightarrow$ KB-JC: Suppose that $[K] = r \cdot A^k$, for some $r, k \in \mathbb{Z}$. Then $J(K) = r(-A)^{-3w(K)}A^k$. By [Gan, Cor. 3], $J(K) = 1$. Hence, K is (unframed) trivial, by the Jones conjecture. □

Now we can formulate three versions of the Jones conjecture for tangles:

Theorem 13. *(Proof in Sec 14.) The Jones conjecture is equivalent to each of the following statements:*
(a) For any framed tangle T, if $[T] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $T = \langle 0 \rangle$.
(b) If $[T] = \begin{pmatrix} r \cdot A^n \\ 0 \end{pmatrix}$ for some $r, n \in \mathbb{Z}$ then $T = \langle 0 \rangle$ as an unframed tangle.
(c) If $[T'] = [T]$ and T is rational then $T' = T$ (as framed tangles).

Furthermore, if the Jones conjecture holds for knots up to n crossings then (a) and (b) hold for tangles with fewer than n crossings (not counting framing twisting kinks.)

The version (c) of the conjecture does not extend to rational knots, since there are examples of distinct rational knots with coinciding Jones polynomials, [Kan].

9 Slopes of Tangles. We will denote the torus in Fig. 6 by a calligraphic $\mathcal{T}$ to distinguish it from the letter T for tangles. Let r be the 180^0 rotation of $\mathcal{T}$ around the horizontal axis intersecting $\mathcal{T}$ at 4 points. The quotient of $\mathcal{T}$ by the $\mathbb{Z}/2 = \langle e, r \rangle$ group action is a sphere and the quotient map $\mathcal{T} \to S^2$ has 4 branching points. Let us call them SW, NW, NE, SE as in Fig. 6.

Consider a surface S properly embedded in $D^3 \setminus T$ with $\partial S \subset \partial D^3$. We can assume that ∂S has no contractible components, by capping them off and pushing them inside D^3 if necessary. Then ∂S lifts to a collection of non-trivial parallel loops in $\mathcal{T}$. Their slope is called the slope of S. Given the longitude and meridian of $\mathcal{T}$, as in Fig. 6, that slope is represented by a number in $\widehat{\mathbb{Q}}$.

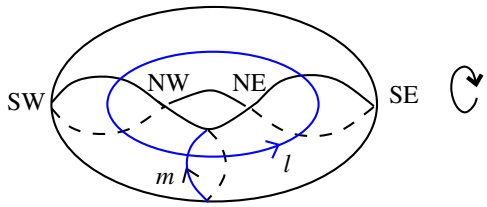


Figure 6. Torus double branched cover of a sphere

A surface S properly embedded in $D^3 \setminus T$ is <u>m-essential</u> (or meridionally essential) if it is incompressible, meridionally incompressible and not boundary-parallel in $D^3 \setminus T$, cf. [Oz1, Oz2] for details.

Ozawa proves that every algebraic tangle T contains an m-essential surface and that all such surfaces have the same slope, [Oz2]. This is the <u>slope of T</u>, $s(T) \in \widehat{\mathbb{Q}}$.

For example, the complement of $\langle 0 \rangle$ has a horizontal disk $S = D^2$ in its complement which is m-essential. Since ∂D^2 lifts to $l \subset \partial \mathcal{T}$, the slope of $\langle 0 \rangle$ is 0. One can construct any rational tangle from $\langle 0 \rangle$ by the operations of rotation and addition of ± 1 which modify S accordingly. Consequently, it is easy to see that the slope of any rational tangle $\langle x \rangle$ is x.

Ozawa proves that

$$s(T_1 + T_2) = s(T_1) + s(T_2), \text{ and } s(\langle n \rangle) = n,$$

for $n \in \mathbb{Z}$. Furthermore, $s(-T) = -s(T)$ and the slope of the rotated tangle T is $s(R(T)) = -1/s(T)$.

By Theorem 11(2), $t^{-1}Q(T)$ is a rational function in $q = -t^2$. We will denote it by $\{T\}_q$. We will further abbreviate $\{\langle x \rangle\}_q$ to $\{x\}_q$ for $x \in \widehat{\mathbb{Q}}$.

Theorem 14. *For every algebraic tangle T, $\{T\}_1 = s(T)$.*

Proof. By Theorem 11(5),

$$\{T + T'\}_q = \{T\}_q + \{T'\}_q + t^{-1}(-t - t^{-1})Q(T)Q(T') =$$
$$\{T\}_q + \{T'\}_q - (1 - q^{-1})Q(T)Q(T') = \{T\}_1 + \{T'\}_1$$

for $q = 1$. Furthermore, by Theorem 11(4),

$$\{R(T)\}_q = t^{-1}/Q(T) = t^{-2}/\{T\} = -q^{-1}/\{T\},$$

we have $\{R(T)\}_1 = -1/\{T\}_1$, for $q = 1$. Finally,

$$\{\langle 0 \rangle\}_1 = 0, \ \{\langle 1 \rangle\}_1 = 1, \ \{\langle -1 \rangle\}_1 = -1.$$

Since these properties coincide with those of the slope and since all algebraic tangles can be constructed from $\langle 0 \rangle$ and $\langle \pm 1 \rangle$ by the operations of addition and reflection, the statement follows. □

Note that $\{T\}_1 \in \widehat{\mathbb{Q}}$ for all tangles. We call it the <u>algebraic slope</u> of T. The following provides an interpretation of the algebraic slope in terms of algebraic topology:

Proposition 15. $|\{T\}_1| = \det(N(T))/\det(D(T))$ *where* det *denotes link determinant and* $N(T)$ *and* $D(T)$ *have arbitrary orientations.*

Proof. The determinant of $N(T)$ is

$$|J(N(T),-1)| = |(-A^3)^{-w(N(T))}[N(T)]| = |[N(T)]|,$$

where $A = e^{\pi i/4}$, $w(N(T))$ is the writhe of $N(T)$, and

$[N(T)] =$  $[T]_\infty +$  $[T]_0 = [T]_\infty.$

Taking an analogous formula for $[D(T)]$ we have

$$\det(N(T))/\det(D(T)) = |[T]_\infty/[T]_0| =$$
$$|Q(T)| = |t\{T\}_1| = |\{T\}_1|,$$

since $t = i$. (Note that our t is not the usual t of the Jones polynomial.) □

Question 16. *Does Theorem 14 extend to non-algebraic tangles? Specifically, are there m-essential surfaces in* $D^3 \setminus T$ *with slope other than* $\{T\}_1$ *for non-algebraic tangles* T*?*

Note that

and, therefore, a tangle T embeds into a link L only if $gcd([T]_0, [T]_\infty)$ divides $[L]$ in $\mathbb{Z}[A^{\pm 1}]$, cf. [TS1]. (Formally, one needs framed tangles and links for that, but since framing affects the bracket by a multiplicative factor of $(-A^3)^n$ which is a unit in $\mathbb{Z}[A^{\pm 1}]$, that statement makes sense in the unframed setting as well.) A version of this observation for $A = e^{\pi i/4}$ was discussed in [Kr].

Note that $|[T]_0|, |[T]_\infty|$ are non-negative integers for that A. In this context, it is interesting to ask:

Question 17. *What is the topological meaning of* $gcd(|[T]_0|, |[T]_\infty|)$ *for* $A = e^{\pi i/4}$*?*

This quantity seems very related to the order of the torsion of the double branched cover of T, however it is not always equal it, as observed in [Ru].

10 Conjecture on the Kauffman bracket of algebraic tangles. Since rational tangles are classified by their slopes, we see that $\{\cdot\}_t$ distinguishes all of them. That is not the case for algebraic tangles because $Q(T_1 + T_2) = Q(T_2 + T_1)$, by Theorem 11(5), while the tangle addition is non-commutative. However, we believe that is the only relation between the KB-ratios of algebraic tangles.

Specifically, let $\sim$ be the smallest equivalence relation on the set of all algebraic tangles $\mathcal{AT}$ such that

- $T_1 + T_2 \sim T_2 + T_1$ for any T_1, T_2 and

- if $T_1 \sim T_1'$ then $T_1 + T_2 \sim T_1' + T_2$ and
- if $T \sim T'$ then $R(T) \sim R(T')$ for any T'.

(Note that any equivalent tangles $T \sim T'$ are related by a sequence of mutations.) Then the KB-ratio is a function on $\mathcal{AT}/\sim$.

Conjecture 18. $Q : \mathcal{AT}/\sim \to \widehat{\mathbb{Q}}$ *is* 1-1.

Note that if $T_1 + T_2$ is a subtangle of a tangle T then the operation $T_1 + T_2 \to T_2 + T_1$ is a composition of the mutation $T_1 + T_2 \to {}_{\varsigma}\Upsilon + {}_{\imath}\Upsilon$ and with mutations $T_1 \to {}_{\imath}\Upsilon$ and $T_2 \to {}_{\varsigma}\Upsilon$.

11 Quantum Rational Numbers and Continued Fractions.

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} = 1 + q + ... + q^{n-1} & \text{for } n > 0 \\ 0 & \text{for } n = 0 \\ \frac{1-q^{-n}}{1-q} = -(q^{-1} + ... + q^{-n}) & \text{for } n < 0 \end{cases} \tag{7}$$

for $n \in \mathbb{Z}$ are called quantum integers. They are ubiquitous in the study of quantum groups and in quantum topology, [Kas], as well as in quantum calculus, [KC]. However, they appear already in the 1808 work of Gauss on binomial coefficients, [Gau].

Proposition 19. $\{n\}_q = [n]_q$ *for every* $n \in \mathbb{Z}$.

Proof. Let us prove it for $n \geq 0$ by induction: For $n = 0$ we have $\{0\}_q = 0 = [0]_q$. Assume that the statement holds for n. Then by Theorem 11(5),

$$\{n+1\}_q = t^{-1}Q(n+1) = t^{-1}Q(n) + t^{-1} \cdot t + t^{-1}Q(n)t\delta,$$

which by the inductive assumption equals

$$t^{-1}Q(n)(1+t\delta) + 1 = \frac{1-q^n}{1-q}(1-t^2-1) + 1 = \frac{1-q^n}{1-q}q + 1 =$$

$$\frac{q - q^{n+1} + (1-q)}{1-q} = [n+1]_q.$$

The proof for $n < 0$ is analogous. (It also follows from the discussion below.) □

For the above reason, one can consider $\{x\}_q$, for $x \in \widehat{\mathbb{Q}}$, as "quantum rational numbers." However, that name was utilized already to denote the values of $[n]_q$ for $q \in \mathbb{Q}$, for example in [LQ, Na]. Note that their quantum n/m belongs to $\mathbb{Q}[q^{1/m}] \cup \{\infty\}$, while our $\{x\}_q$ is in $\mathbb{Q}(q) \cup \{\infty\}$. (These notions should not to be confused with symmetric quantum integers, $\frac{q^n - q^{-n}}{q - q^{-1}}$.)

Let us denote $Q(\langle x \rangle)$ by $\langle x \rangle_t$, for $x \in \widehat{\mathbb{Q}}$, for simplicity. (As indicated by the subscript, we consider $\langle x \rangle_t$ as a function of t.) Hence,

$$\langle \cdot \rangle_t : \widehat{\mathbb{Q}} \to \mathbb{Q}(t) \cup \{\infty\}, \quad \langle x \rangle_t = t\{x\}_{-t^2}.$$

Note that

$$[-n]_q = -q^{-1}[n]_{q^{-1}}$$

and that $\langle n\rangle_t = t\cdot [n]_q$, where $q=-t^2$, is precisely such normalization of $[n]_q$ that

$$\langle -n\rangle_t = \langle n\rangle_{t^{-1}}.$$

By Theorem 11, we immediately have

$$\langle -x\rangle_t = \langle x\rangle_{t^{-1}}, \quad \langle 1/x\rangle_t = 1/\langle -x\rangle_t = 1/\langle x\rangle_{t^{-1}} \tag{8}$$

for any $x\in\widehat{\mathbb{Q}}$.

Furthermore, we have

Proposition 20. *$\langle n+x\rangle_t = \langle n\rangle_t + q^n\langle x\rangle_t$ for any $n\in\mathbb{Z}$.*

Proof. By Theorem 11(5) and by Proposition 19,

$$\langle n+x\rangle_t = \langle n\rangle_t + \langle x\rangle_t + \langle n\rangle_t\langle x\rangle_t\delta = \langle n\rangle_t + \langle x\rangle_t - t\frac{1-q^n}{1-q}\langle x\rangle_t(t+t^{-1}) =$$

$$\langle n\rangle_t + \langle x\rangle_t - (1-q)\frac{1-q^n}{1-q}\langle x\rangle_t = \langle n\rangle_t + q^n\langle x\rangle_t. \qquad \square$$

Since each rational has a continued fraction expansion, equation (8) and Proposition 20 determine $\langle x\rangle_t$ uniquely. For example,

$$\left\langle n+\frac{1}{m}\right\rangle_t = \langle n\rangle_t + \frac{q^n}{\langle m\rangle_{t^{-1}}},$$

$$\left\langle n+\frac{1}{m+\frac{1}{l}}\right\rangle_t = \langle n\rangle_t + \frac{q^n}{\langle m\rangle_{t^{-1}} + \frac{q^{-m}}{\langle l\rangle_t}},$$

and so on.

For that reason, we will call $\langle x\rangle_t$ the <u>quantum continued fraction</u> of x.

As a consequence of Theorem 14 we have:

Corollary 21. *$\langle\cdot\rangle_t : \widehat{\mathbb{Q}} \to \mathbb{Q}(t)\cup\{\infty\}$ is 1-1.*

By Proposition 20 and equation (8),

$$\{n+x\}_q = \{n\}_q + q^n\{x\}_q, \quad \{-x\}_q = -q^{-1}\cdot\{x\}_{q^{-1}}, \tag{9}$$
$$\{1/x\}_q = \frac{-1}{q\{-x\}_q} = \frac{1}{\{x\}_{q^{-1}}}.$$

and

$$\{x\}_{q=1} = x$$

for any $x\in\widehat{\mathbb{Q}}$ by Theorem 14.

As in the case of $\langle x\rangle_t$, these rules determine $\{\cdot\}_q$ uniquely.

Remark 22. *One can compute power series expansions of $\{x\}_q$ for rationals $x=n+1/m$, and for $q=e^h$ using the identities (9):*

$$\{x\}_q = x + (x^2 - x + 1/2 - 1/2m^2)h + O(h^2),$$

for $n, m \in \mathbb{Z}_{>0}$, *while*

$$[x]_q = \frac{1-q^x}{1-q} = x + (x^2 - x)h + O(h^2).$$

Therefore, our quantum rational number $\{x\}_q$ *do not coincide with* $[x]_q$.

Finally, let us show a further connection between our quantum rationals and continued fractions.

Theorem 23 ([Fr])**.** *For any* $a_1, ..., a_n \in \mathbb{Z}$,

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}, \quad \textit{where}$$

$$p_k/q_k = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_k}}}, \quad p_0 = 1, q_0 = 0.$$

(Note that the indices of a in this statement are increasing rather than decreasing, as in the rest of the paper.) Analogously for quantum rational numbers we have:

Theorem 24. *For any* $a_1, ..., a_n \in \mathbb{Z}$,

$$\begin{pmatrix} [a_1]_q & q^{a_1} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} [a_n]_q & q^{a_n} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} P_n & q^{a_n} P_{n-1} \\ Q_n & q^{a_n} Q_{n-1} \end{pmatrix}, \quad \textit{where}$$

$$P_k/Q_k = \left\{ a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_k}}} \right\}_q \quad P_0 = 1, Q_0 = 0.$$

Proof. Let us define P_n and Q_n as the coefficients of the first column of the product of the matrices above. There are two parts of the statement:
(1) the second column of the matrix on the right is $\begin{pmatrix} q^{a_n} P_{n-1} \\ q^{a_n} Q_{n-1} \end{pmatrix}$ and
(2) the value of P_n/Q_n is the quantum rational number as above.

We prove both by induction on n: The statement is trivial for $n = 1$. Assume that it holds for $n-1$. Then

$$\begin{pmatrix} [a_1]_q & q^{a_1} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} [a_n]_q & q^{a_n} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} P_{n-1} & \cdot \\ Q_{n-1} & \cdot \end{pmatrix} \begin{pmatrix} [a_n]_q & q^{a_n} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cdot & q^{a_n} P_{n-1} \\ \cdot & q^{a_n} Q_{n-1} \end{pmatrix},$$

showing the inductive step for first part of the statement. To show the inductive step for the second part of the statement, let

$$\begin{pmatrix} [a_2]_q & q^{a_2} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} [a_n]_q & q^{a_n} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & v \\ w & z \end{pmatrix}.$$

Then by the inductive assumption, $u/w = \{x\}_q$, where $x = a_2 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}$.

Since

$$\begin{pmatrix} [a_1]_q & q^{a_1} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} [a_n]_q & q^{a_n} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} [a_1]_q u + q^{a_1} w & [a_1]_q v + q^{a_1} z \\ u & v \end{pmatrix},$$

$$\frac{P_n}{Q_n} = \frac{[a_1]_q u + q^{a_1} w}{u} = [a_1]_q + \frac{q^{a_1}}{\{x\}_q} = \{a_1 + 1/x\}_q,$$

by (9). Hence, we established the inductive step and the proof follows. □

12 Tangle model of $\mathrm{PSL}(2,\mathbb{Z})$**-action on** $\widehat{\mathbb{Q}}$**.** The group of Möbius transformations

$$z \to \frac{az+b}{cz+d} \tag{10}$$

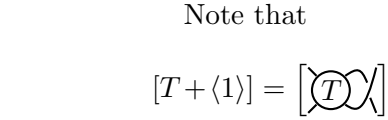
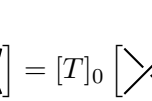
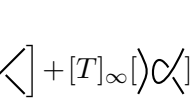
for $a, b, c, d \in \mathbb{Z}$, $ad - bc \neq 0$, acts on $\widehat{\mathbb{Q}}$. Assigning $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to (10) identifies that group with $\mathrm{PSL}(2,\mathbb{Z})$.

The clockwise 90^0 rotation, R, and $P(\langle x \rangle) = \langle x \rangle + \langle 1 \rangle$ act on the set of rational tangles and they correspond to operations

$$z \to -1/z \quad \text{and} \quad z \to z+1$$

on $\widehat{\mathbb{Q}}$ through the Conway's bijection (2). These two transformations generate $\mathrm{PSL}(2,\mathbb{Z})$, cf. eg. [Al] and, hence, provide a topological model of the $\mathrm{PSL}(2,\mathbb{Z})$ action on $\widehat{\mathbb{Q}}$.

Note that

$[T + \langle 1 \rangle] =$  $= [T]_0$  $+ [T]_\infty$  $= [T]_0 \begin{pmatrix} A^{-1} \\ A \end{pmatrix} + [T]_\infty \begin{pmatrix} 0 \\ -A^3 \end{pmatrix}.$

Hence, the operations R and P on tangles induce matrix transformations of the Kauffman brackets:

$$\begin{aligned} [R(T)] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} [T], \\ [P(T)] &= [T + \langle 1 \rangle] = A \cdot M \cdot [T], \end{aligned} \tag{11}$$

where $M = \begin{pmatrix} A^{-2} & 0 \\ 1 & -A^2 \end{pmatrix} = \begin{pmatrix} t^{-1} & 0 \\ 1 & -t \end{pmatrix}$. Since for unframed tangles the Kauffman bracket is defined up to scalar multiples only, the above yields a representation

$$\psi : \mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Q}(t)). \tag{12}$$

13 Framed Möbius Group. Let us consider framed tangles now and let Γ be a subgroup of the symmetric group on all framed tangles generated by R and P. Forgetting framing operation defines an epimorphism $\phi : \Gamma \to \mathrm{PSL}(2, \mathbb{Z})$. This is not an isomorphism, because RP^{-1} corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ of order 3 in $\mathrm{PSL}(2, \mathbb{Z})$, but $(RP^{-1})^3$ is not the identity in Γ. As demonstrated in Fig. 7, $F(T) = (RP^{-1})^3(T)$ adds $+1$ framing twist to T.

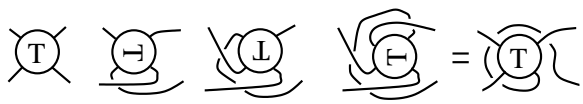


Figure 7. T, $RP^{-1}(T)$, $(RP^{-1})^2(T)$, and $(RP^{-1})^3(T)$

Since the kernel of ϕ contains only framing changing operations,

$$Ker\,\phi = \langle F \rangle = \mathbb{Z}.$$

and this group is central in Γ. Consequently, Γ is an infinite cyclic central extension

$$\{e\} \to \langle F \rangle \to \Gamma \to \mathrm{PSL}(2, \mathbb{Z}) \to \{e\},$$

with a presentation

$$\Gamma = \langle R, P \mid R^2, R(RP^{-1})^3 = (RP^{-1})^3 R \rangle.$$

(The second relation implies that $P(RP^{-1})^3 = (RP^{-1})^3 P$ and, hence, $(RP^{-1})^3$ is indeed central in Γ.)

Such central extensions are classified by $H^2(\mathrm{PSL}(2, \mathbb{Z}), \mathbb{Z}) = \mathbb{Z}/6$. One of them is trivial, $\mathbb{Z} \times \mathrm{PSL}(2, \mathbb{Z})$. Another one, called the universal central extension of $\mathrm{PSL}(2, \mathbb{Z})$ is isomorphic with the braid group on three strands, B_3. Our group Γ however is neither of these two. It cannot be the trivial extension because $(RP^{-1})^3 \neq e$ in Γ, nor can it be B_3 because it contains a finite order element R.

Finally, observe that the representation (12) lifts to a representation

$$\Gamma \to \mathrm{SL}(2, \mathbb{Q}(t))$$

defined through (11).

14 Proofs.

Proof of Lemma 2: Let $\Sigma(X)$ be the double cover of D^3 branched along X, as in Sec. 6. The numerator closure $N(\cdot)$ and the operation $N(\cdot + \langle 1 \rangle)$, on the level of the double cover correspond to two different Dehn fillings of $\Sigma(X)$. Each of them yields the double cover of S^3 branched along U, i.e. S^3. By a theorem of Gordon and Luecke, [GL, Thm. 2], $\Sigma(X)$ must be a solid torus. That implies that X must be a rational tangle, $X = \langle p/q \rangle$, [Li].

By a theorem of Schubert,

$$N(\langle p/q\rangle) = U = N(\langle 1\rangle)$$

only if $p = 1$, [KL, Sch]. Because $N(\langle \frac{1}{q}\rangle + \langle 1\rangle)$ is the unknot, it is easy to see that q is either 0 or $-1/2$. □

Proof of Theorem 13: JC implies (a): Assume the Jones conjecture holds and that $[T] = (1, 0)$. (For convenience, in this proof we will write all vertical vectors horizontally.) Then $[R(T)] = (0, 1)$ and it is easy to check that

$$[N(R(T))] = 1 \text{ and } [N(R(T) + \langle 1\rangle)] = -A^3.$$

By Lemma 12,

$$N(R(T)) = U^0 \text{ and } N(R(T) + \langle 1\rangle) = U^1.$$

Now, by Lemma 5, $R(T) = \langle\infty\rangle$ and, hence, $T = \langle 0\rangle$.

(a) implies JC: Let K be a knot with trivial Jones polynomial. Let us frame it so that $[K] = 1$. Let $K\#\langle 0\rangle$ be the connected sum of K with the lower strand of $\langle 0\rangle$. Then $[K\#\langle 0\rangle] = (1, 0)$ and $K\#\langle 0\rangle = \langle 0\rangle$, by (b), implying that K is trivial.

(b) implies (a): Suppose $[T] = (1, 0)$. Then by (b), $T = \langle 0\rangle$ as an unframed tangle. Since $[T] = (1, 0)$, the framing of T must be trivial.

(a) implies (c): Suppose that T is rational and $[T'] = [T]$. Then T can be transformed

$$T = T_1 \to ... \to T_k = \langle 0\rangle$$

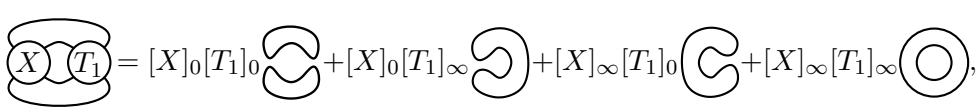
by the operations of rotation, $R(\cdot)$, and of addition of $\langle\pm 1\rangle$, $P^{\pm 1}(\cdot)$, of Sec. 12. Let us apply the same operations to T' :

$$T' = T'_1 \to ... \to T'_k.$$

By (11), $[T_i] = [T'_i]$ for every i and, hence, $[T'_k] = (0, 1)$. By (a), $T'_k = \langle 0\rangle$, implying that $T' = T$.

(c) implies (b): Assume that $[T] = (r \cdot A^n, 0)$, for some $r, n \in \mathbb{Z}$. Then $[D(T)] = r \cdot A^n$ and by [Gan, Cor 3] (as in the proof of Lemma 12), $J(D(T)) = 1$. That implies that $[T] = ((-A^3)^k, 0)$ has the bracket of $\langle 0\rangle$ with some framing $k \in \mathbb{Z}$. Hence, by (c), $T = \langle 0\rangle$, as unframed tangle. That completes the proof of (b). □

Consider now any solution X of (4). Then



and analogously for T_2. Hence,

$$\begin{pmatrix} [N(X + T_1)] \\ [N(X + T_2)] \end{pmatrix} = B \cdot \begin{pmatrix} [X]_0 \\ [X]_\infty \end{pmatrix}$$

where

$$B = \begin{pmatrix} [T_1]_0\delta + [T_1]_\infty & [T_1]_0 + [T_1]_\infty\delta \\ [T_2]_0\delta + [T_2]_\infty & [T_2]_0 + [T_2]_\infty\delta \end{pmatrix}.$$

Note that

$$\det B = (\delta^2 - 1) \cdot \det \begin{pmatrix} [T_1]_0 & [T_1]_\infty \\ [T_2]_0 & [T_2]_\infty \end{pmatrix}.$$

Given a system of equations (4), let

$$q_\mu = \frac{\det \begin{pmatrix} [L_1] & [T_1]_0 + \mu[T_1]_\infty \\ [L_2] & [T_2]_0 + \mu[T_2]_\infty \end{pmatrix}}{(\delta - \mu) \cdot \det \begin{pmatrix} [T_1]_0 & [T_1]_\infty \\ [T_2]_0 & [T_2]_\infty \end{pmatrix}} \text{ for } \mu = \pm 1.$$

Then, by Cramer's Rule, one can verify that

$$[X]_0 = \det \begin{pmatrix} [L_1] & [T_1]_0 + [T_1]_\infty\delta \\ [L_2] & [T_2]_0 + [T_2]_\infty\delta \end{pmatrix} / \det B = \frac{1}{2}(q_1 - q_{-1})$$

and

$$[X]_\infty = \det \begin{pmatrix} [T_1]_0\delta + [T_1]_\infty & [L_1] \\ [T_2]_0\delta + [T_2]_\infty & [L_2] \end{pmatrix} / \det B = \frac{1}{2}(q_1 + q_{-1}).$$

Corollary 25. *For any rational T_1 and T_2 (as always, unequal as unframed tangles) and any L_1 and L_2, all framed solutions X to (4) have the same Kauffman bracket $[X]$ (given by the formulas above).*

Proof. For any rational T_1 and T_2, unequal as unframed tangles, the slopes of T_1 and T_2 differ and, hence, their KB ratios are different, by Corollary 21. That implies that $\det B \neq 0$. □

The above equations provide necessary algebraic conditions for the existence of a framed solution of (4). In particular, we have

Corollary 26. *A necessary condition for the existence of a framed solution to (4) is that*

$$p_0 = \frac{1}{2}(q_1 - q_{-1}), p_\infty = \frac{1}{2}(q_1 + q_{-1}) \in \mathbb{Z}[A^{\pm 1}]$$

and that $A^2 p_\infty / p_0 \in \mathbb{Q}(A^4)$.

Proof of Theorem 4 follows immediately from Corollary 25 and from Corollary 21. □

Proof of Theorem 7: Suppose X, X' are solutions to (4) for some T_1 and T_2, unequal as unframed tangles, and for some L_1 and L_2. Then $[X'] = [X]$ by Corollary 25. Suppose that X is rational. Consider a sequence of additions of $\langle \pm 1 \rangle$ and of rotations which reduces X to $\langle 0 \rangle$. It will transform X' to X''. By (11) $[X''] = [\langle 0 \rangle] = (1, 0)$. By the assumption of JC, $X'' = \langle 0 \rangle$, by Theorem 13. That implies that $X' = X$.

Note that if X and X' have c and c' crossings, respectively, then X'' has at most $c + c'$ crossings. The above argument uses $JC \Rightarrow (a)$ implication of

Theorem 13. The proof of that implication assumes that JC holds for knots up to $c + c' + 1$ crossings. □

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