ESTIMATING THE SIZE OF SKEIN HOMOLOGIES

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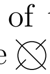
Abstract. We give a new method of distinguishing elements of skein homologies. Using this method we show that each skein homology module is at least as big as the Kauffman bracket skein module of a given 3-manifold.

1. Introduction

Skein modules are constructed by considering a free module spanned by ambient isotopy classes of links (possibly with some sort of decoration - like framing) in a 3-manifold, and then dividing it by a submodule generated by some skein relations [P-1]. More generally, using the same skein relations, one can define a chain complex whose 0-th homology is the original skein module, and whose higher degree homologies are some more complicated invariants of a given 3-manifold. This concept was introduced and described for the Kauffman bracket skein module in [BFK-2]. In this note we generalize some techniques developed in [BFK-2] to construct cycles and to identify non-boundary elements[1]. We use these techniques to show that each skein homology module of an arbitrary 3-manifold is at least as big as the Kauffman bracket skein module of the manifold.

For completeness, we recall main definitions from [BFK-2]. Given an oriented 3-manifold M, a framed link is an embedding of a disjoint union of annuli in M. Let $\mathcal{L}(M)$ denote the set of ambient isotopy classes of framed, unoriented links, including the empty link. A crossing ball for a framed link is an embedding of the pair (B^3, D^2) so that inside the crossing ball the link looks like  with D^2 lying in the page. A framed link with n ordered crossing balls is one where the balls are ordered and disjoint. We number the crossing balls from 1 to n, corresponding to their ordering. Two such objects are equivalent

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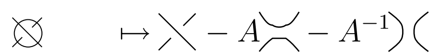
[1]This note originated from the discussion which took place during the first author's lecture on skein homologies at GWU, April 23, 1998. We would like to thank all the students who participated in this discussion.

if there is an ambient isotopy of the links that carries crossing balls to crossing balls in an order preserving fashion. The set of ambient isotopy classes excluding links with trivial components is denoted by $\mathcal{L}^n(M)$.

Let R be a ring with identity and A an invertible element of R. The i-th ball operator

$$\partial_i : R\mathcal{L}^n(M) \to R\mathcal{L}^{n-1}(M)$$

is defined locally at the i-th crossing ball by



along with any necessary applications of $L \cup \bigcirc = -(A^2 + A^{-2})L$. The remaining crossing balls inherit the original ordering.

Define $d_0 : \mathcal{L}^0(M) \to R$ to be the zero map, and for $n \geq 1$ let

$$d_n : R\mathcal{L}^n(M) \to R\mathcal{L}^{n-1}(M)$$

be given by

$$d_n = \sum_{i=1}^{n} (-1)^i \partial_i.$$

It is easy to see that $(R\mathcal{L}^n(M), d_n)$ is a chain complex. Its homology is called the Kauffman bracket homology of a manifold M, and it is denoted by $K_*(M)$. The 0-th homology, $K_0(M)$, is the Kauffman bracket skein module of M. This module has been studied in several papers, e.g. [B], [BFK-1], [HP], [P-2], [PS].

2. The main result

In order to understand homologies one must be able to identify non-boundary cycles in the chain complex. We expand the methods introduced in [BFK-2]. In particular, we prove the following result.

Theorem 1. *Let M be an oriented 3-manifold. If $\frac{1}{2} \in R$ then for any $n = 0, 1, 2...$ there is an epimorphism of R-modules $\phi : K_n(M) \to K_0(M)$.*

Recently, there has been developed a connection between character varieties and the Kauffman bracket skein modules, [B, BFK-1, P-2, PS]. This connection implies an interesting corollary to the above theorem.

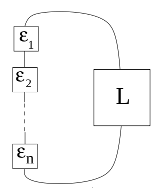
Corollary 2. *If one of the following conditions holds then $K_n(M)$ is infinitely generated for every n :*

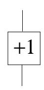
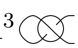
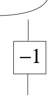
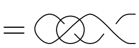
- $\frac{1}{2} \in R$ *and* $H_1(M, \mathbb{Z})$ *is infinite;*

- *R is an algebraically closed field of characteristic 0 and the SL_2-character variety of $\pi_1(M)$ has at least one component of positive dimension.*

Proof. Each of these conditions guarantees that K_0 is infinitely generated. □

Proof of Theorem 1: Consider a map $\phi : \mathcal{L}^n(M) \to \mathcal{L}(M)$ defined by forgetting the crossing balls. This map induces a homomorphism of R-modules $\phi : R\mathcal{L}^n(M) \to K_0(M)$. Since each element of $im\ d_n$ is a linear combination of skein relations, the above homomorphism yields $\phi : R\mathcal{L}^n(M)/im(d_n) \to K_0(M)$. For brevity of notation, we denote the restriction of this homomorphism to homology by the same symbol, $\phi : K_n(M) \to K_0(M)$. We will prove that this map is an epimorphism. Let L be an arbitrary unoriented framed link in M without trivial components. We construct an element $\hat{L} \in \mathcal{L}^n(M)$ in the manner described in the proof of Proposition 2 [BFK-2]: Let $L_{\varepsilon_1,\ldots,\varepsilon_n} \in \mathcal{L}^n(M)$ be obtained from L by inserting n crossing balls in the pattern shown below.

$L_{\varepsilon_1,\ldots,\varepsilon_n} =$  $, \varepsilon_1, \ldots, \varepsilon_n = \pm 1,$

where  $= -A^3$  ,  $=$  .

It does not matter which component of L has been selected for this operation. Let $\hat{L} = \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} L_{\varepsilon_1,\ldots,\varepsilon_n}$. An easy argument (identical to that given in [BFK-2]) shows that $\partial_i \hat{L} = 0$, for $i = 1, 2, \ldots$ Therefore $d_n(\hat{L}) = 0$ and $\hat{L}$ is an n-cycle. Observe that $\phi(L_{\varepsilon_1,\ldots,\varepsilon_n}) = L$ for any choice of $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$. Thus $\phi(\frac{1}{2^n}\hat{L}) = L \in K_0(M)$. This implies that ϕ is an epimorphism. □

There is yet another method of constructing n-cycles in $K_n(M)$. Observe that the symmetric group S_n acts on the space $\mathcal{L}^n(M)$ by permuting the crossing balls in each link. Let $L^{sym} = \Sigma_{\pi \in S_n} \pi L$ be the symmetrization of a link L, with n crossing balls. Similarly, we can define L^{cyc} to be $\Sigma_{\pi \in Z_n} \pi L$, where $Z_n \subset S_n$ is generated by the cycle $(1, 2, \ldots, n)$.

Proposition 3. (i) *L^{sym} is an n-cycle for $n > 1$.*
(ii) *L^{cyc} is an n-cycle for any even n.*
(iii) $\phi(L^{sym}) = n!L \in K_0(M)$

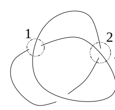
(iv) $\phi(L^{cyc}) = nL \in K_0(M)$. □

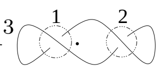
Example 4. *Let L be a positive trefoil with three balls, where the balls are placed around the three crossings in the alternating projection with the blackboard framing, and ordered in an arbitrary way. An easy calculation shows that L is a 3-cycle* [2]. *Since $\phi(L) \neq 0$, L represents a nontrivial element of $K_3(S^3)$.*

We have an exact sequence

$$0 \to Ker\ \phi \to K_n(S^3) \xrightarrow{\phi} K_0(S^3) \to 0.$$

Since $K_0(S^3) = R$ is a free R-module, the sequence splits and R is a direct summand of $K_n(S^3)$, $K_n(S^3) = R \oplus ker\ \phi$. The next example shows that $ker\ \phi$ is non-trivial for $n = 2$.

Example 5. *Let $L =$* 

$+ A^3$ 

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An elementary calculation shows that L is a 2-cycle and that $\phi(L) = 0 \in K_0(M)$. However, L is a non-zero element of the 2-nd skein homology of S^3. This can be shown by considering a modification of ϕ defined as follows. Let $\psi : R\mathcal{L}^n(S^3) \to \mathcal{L}(S^3)$ be the map replacing each crossing ball with the opposite crossing (without the ball). Exactly as before, ψ induces a homomorphism from $K_n(S^3)$ to $K_0(S^3)$. Moreover, it is easy to see that $\psi(L) \neq 0$.

In this paper we have studied an ordered skein homology. Alternatively, one can consider an oriented skein homology, constructed out of links with crossing balls ordered up to an even permutation. We do not know if these homology theories coincide.

The methods of distinguishing nontrivial cycles presented in this paper can be sharpened by considering refinements of the Kauffman bracket skein module based on singular links.

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