SKEIN MODULES AND TQFT

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ABSTRACT

We prove that the Kauffman bracket skein module of a 3-manifold M at a $4r$-th root of unity divided by the Jones-Wenzl idempotent depends only on ∂M. [1]

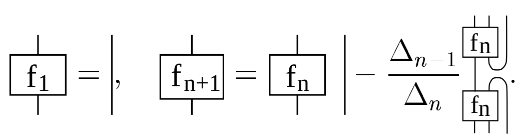
Keywords: Kauffman bracket, skein module, TQFT.

Let r be a fixed natural number, $r = 2, 3, ...$, and let A be a primitive $4r$-th root of $1 \in \mathbb{C}$. The Kauffman bracket skein module of an oriented 3-manifold M, denoted by $\mathcal{S}(M)$, is $\mathbb{C}$-linear space of formal linear combinations of framed, unoriented links in M, considered up to an ambient isotopy in M and the following skein relations:

Let Δ_n be the value of the n-th Chebyshev polynomial:

$$\Delta_n = d\Delta_{n-1} - \Delta_{n-2},\ \Delta_0 = 1,\ \Delta_1 = d = -(A^2 + A^{-2}),$$

and let f_n be the n-th Jones-Wenzl idempotent:

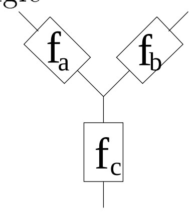


We denote by $K_r(M)$ the quotient of $S(M)$ by the submodule generated by all linear skeins involving f_{r-1}. Obviously, $K_r(S^3) = \mathcal{S}(S^3) \simeq \mathbb{C}$.

[1] This fact was proved by J. Roberts first, and then by myself. Since the result has created much interest and Robert's proof has not been published yet, I decided to publish my proof.

Theorem 1 (Roberts)
If M is a compact 3-manifold then $K_r(M)$ depends on ∂M only, i.e. if M_1, M_2 are compact 3-manifolds such that $\partial M_1 \simeq \partial M_2$ then $K_r(M_1)$ and $K_r(M_2)$ are isomorphic $\mathbb{C}$-linear spaces.

A 3-vertex $\{a, b, c\}$ is a tangle



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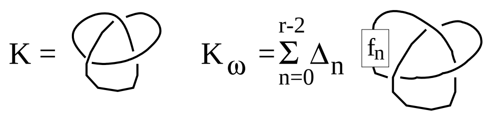
Any 3-vertex satisfies the relations: $a + b + c = 0$ mod 2, $a + b \geq c, a + c \geq b, b + c \geq a$. A 3-vertex is admissible if $a + b + c \leq 2r - 4$. J. Roberts proved that all non-admissible 3-vertices vanish in $K_r(M)$, for any 3-manifold M. It is also possible to prove that $K_r(M)$ is isomorphic as a $\mathbb{C}$-linear space to the vector space assigned to ∂M by the topological quantum field theory associated with the Reshetikhin-Turaev quantum SU_2 invariant.

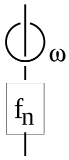
We precede the proof of Theorem 1 with a few standard facts and definitions. For any framed knot K in M, K_ω denotes a linear skein defined as in the example:



If L is a framed link then L_ω denotes a linear skein formed by decorating all the components of L with ω.

Lemma 2
Let L and K be a framed link and a framed knot in M respectively. Let L' be a framed link obtained from L by sliding one of its components along K. Then $L \cup K_\omega = L' \cup K_\omega$ in $K_r(M)$.

Lemma 3



$= \begin{cases} 0 & \text{if } 1 \leq n \leq r-2 \\ \Omega & \text{if } n = 0 \end{cases}$ *in $K_r(M)$, where $\Omega = \sum_{i=0}^{r-2} \Delta_i^2$.*

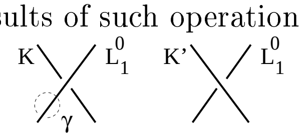
Moreover, $\Omega \neq 0$.

The Lemmas 2 and 3 are well-known for $M = S^3$. However, the proofs of these lemmas given for example in [L] and [P-S] for $M = S^3$ hold for any 3-manifold M.

Proof of Theorem 1:
For any framed link $L \subset M$ we will denote a regular neighborhood of L in M by $V(L)$ and the surgery along L in M by M_L. $V(L)$ is always a disjoint union of solid tori in M. Let M_1, M_2 be two 3-manifolds such that $\partial M_1 \simeq \partial M_2$. Then there exist framed links $L_1 \subset M_1$, $L_2 \subset M_2$ which satisfy the following conditions:

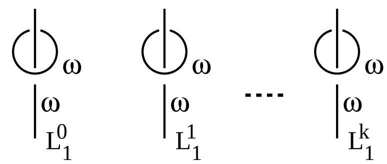
- There is a homeomorphism $\phi : M_1 \setminus V(L_1) \to M_2 \setminus V(L_2)$
- $(M_1)_{L_1} = M_2, \ (M_2)_{L_2} = M_1$.

We define a homomorphism $f_1 : K_r(M_1) \to K_r(M_2)$ in the following way. Let K be any framed link in M_1. We can isotope K outside of $V(L_1)$ and therefore consider it as an element of $K_r(M_1 \setminus V(L_1))$. Then $\phi(K)$ is a framed link in $M_2 \setminus V(L_2)$ and we define $f_1(K)$ to be $\phi(K) \cup (L_2)_\omega \in K_r(M_2)$. At a first sight it seems that f_1 is ill-defined since the operation of pushing $K \subset M_1$ outside $V(L_1) \subset M_1$ is not unique. The picture presents two (generally inequivalent) results of such operation.



L_1^0 is a component of L_1. Notice that we can get K' by sliding K along the meridian γ going around $V(L_1^0)$. This closed curve will correspond under the homeomorphism ϕ to a closed curve $\phi(\gamma)$ parallel to the boundary of $V(L_2^0)$, for some connected component L_2^0 of L_2. Moreover, the requirement $M_1 = (M_2)_{L_2}$ implies that $\phi(\gamma)$ is parallel to one of the boundary components of L_2^0. (L_2^0 is considered here as an annulus embedded into M_2.) Therefore $\phi(K')$ is $\phi(K)$ slid along L_2^0. Since $f_1(K)$ is a union of L_2 decorated by ω and of $\phi(K)$, Lemma 2 implies that sliding K along L_2^0 will not change the value of $f_1(K) \in K_r(M_2)$. Hence $f_1(K) = f_1(K')$ and f is a well defined homomorphism $K_r(M_1) \to K_r(M_2)$.

Let $f_2 : K_r(M_2) \to K_r(M_1)$ be defined analogously. We want to prove that $f_2 \circ f_1$ and $f_1 \circ f_2$ are of the from $c \cdot Id_{K_r(M_1)}$, $c \cdot Id_{K_r(M_2)}$, respectively, for some non-zero $c \in \mathbb{C}$. If K is a framed link in M_1 then $f_1(K) = K \cup (L_2)_\omega \in K_r(M_2)$. Let us push $(L_2)_\omega$ outside $V(L_2)$ in the direction parallel to the framing of L_2. Now $K \cup (L_2)_\omega$ lies in $M_2 \setminus V(L_2)$. What is $\phi^{-1}(L_2) \in M_1 \setminus V(L_1)$? The condition $M_1 = (M_2)_{L_2}$ implies that each of the components of $\phi^{-1}(L_2)$ is a framed knot surrounding the meridian of one of the tori $\partial V(L_1) \subset \partial(M_1 \setminus V(L_1))$. Therefore $f_2 \circ f_1(K) = K \cup (L_1)_\omega \cup (\phi^{-1}(L_2))_\omega$ where $(L_1)_\omega \cup (\phi^{-1}(L_2))_\omega$ is of the form



where $L_1^0, L_1^1, ..., L_1^k$ are the connected components of L_1. By Lemma 3 $(L_1)_\omega \cup (L_2)_\omega = \Omega^{k+1} \in \mathbb{C}, \Omega^{k+1} \neq 0$. Therefore $f_2 \circ f_1$ is an automorphism of $K_r(M_1)$. This implies that f_1 is a monomorphism and f_2 is an epimorphism. The same argument for $f_1 \circ f_2$ shows that f_1 is epi and f_2 is mono. Therefore both f_1 and f_2 are isomorphisms. □

References

[L] W. B. R. Lickorish, An Introduction to Knot Theory, Graduate Texts in Mathematics, Springer 1997.

[P-S] V. V. Prasolov, A.B. Sossinsky, Knots, Links, Braids and 3-manifolds, Translations of Mathematical Monographs, AMS, Vol 154.