SU_n-QUANTUM INVARIANTS FOR PERIODIC LINKS

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Abstract. Murasugi discovered a formula relating the Jones polynomial of a periodic link to the Jones polynomial of the quotient link. Later a similar formula for SU_3-quantum invariants of links was proved by Chbili. We extend these results to SU_n-quantum invariants of periodic links for all values of n.

1. The SU_n-quantum invariants for periodic links

Let $P_n(L) \in \mathbb{Z}[t^{\pm 1}]$ denote the SU_n-quantum invariant of a link L, defined by the skein relations:

$$t^n P_n(L_+) - t^{-n} P_n(L_-) = (t - t^{-1}) P_n(L_0), \quad P_n(\emptyset) = 1.$$

Note that instead of using the standard variable q, we use here $t = q^{1/2}$. The value of P_n for the trivial link of k components is $[n]^k$, where $[n]$ denotes the nth quantum integer, $[n] = \frac{t^n - t^{-n}}{t - t^{-1}}$.

We will identify $\mathbb{R}^3$ with $\mathbb{R} \times \mathbb{C}$, with coordinates denoted by x and z. A link in $\mathbb{R}^3$ is p-periodic if it is isotopic to a link $L \subset \mathbb{R}^3$ preserved by the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, z) = (x, e^{2\pi i/p})$, which fixes the x-coordinate and rotates the z-plane by the angle $\frac{2\pi}{p}$. Note that in this situation L is disjoint from the x-axis. Let $G = \mathbb{Z}/p\mathbb{Z}$ denote the group of homeomorphisms of $\mathbb{R}^3$ generated by T, and let π denote the covering map $\mathbb{R}^3 \to \mathbb{R}^3/G = \mathbb{R}^3$, $\pi(x, z) = (x, z^p)$, branched along the x-axis. We call $\bar{L} = \pi(L)$ the factor link of L.

Murasugi discovered the following formula relating the SU_2-quantum invariant (ie. the Jones polynomial) of p-periodic links L and their factor links, $\bar{L}$, [M, Thm 1],

$$P_2(L) = P_2(\bar{L})^p \mod (p, [2]^p - [2]).$$

Recently, Chbili proved an analogous formula for SU_3-quantum invariants, [C],

$$P_3(L) = P_3(\bar{L})^p \mod (p, [3]^p - [3]).$$

These results may suggest that $P_n(L) = P_n(\bar{L})^p$ modulo $(p, [n]^p - [n])$, for all n. As we will see below, this intuition is wrong. However, we have the following result generalizing Murasugi-Chbili's formula:

Theorem 1.1. *Let $p \neq 2$ be prime and let L be a p-periodic link.*
(1) $P_n(L) = P_n(\bar{L})^p$ in $\mathbb{Z}[t^{\pm 1}]/(p, [2]^p - [2])$.
(2) If n is odd then $P_n(L) = P_n(\bar{L})^p$ in $\mathbb{Z}[t^{\pm 1}]/(p, [3]^p - [3])$.

Remarks (1) Since the consecutive quantum integers are related by the formula $[n+1] = [2][n] - [n-1]$, we see that the ideal $([3]^p - [3], p)$ is contained in $([2]^p - [2], p)$. Hence, for n odd, the statement (2) in the above theorem is stronger than (1).
(2) Usually $P_n(L) \neq P_n(\bar{L})^p$ modulo $(p, [k]^p - [k])$, for k different than in the theorem above. For example, for the $(3,5)$-torus knot, L, and $p = 5$ $P_4(L) \neq P_4(\bar{L})^5$ modulo $(5, [k]^5 - [k])$, for $k = 3, 4$ and $P_5(L) \neq P_5(\bar{L})^5$ modulo $(5, [k]^5 - [k])$, for $k = 4, 5$. Hence, Theorem 1.1 does not admit any straightforward generalization.
(3) We will see later that the importance of the polynomials $[2]^p - [2]$ and $[3]^p - [3]$ for Theorem 1.1 follows from their particularly simple factorizations:

$$[2]^p - [2] = (t^p - t)(t^p - t^{-1})t^{-p},$$

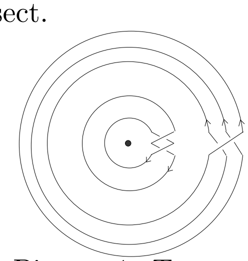
and

$$[3]^p - [3] = (t^p - t)(t^p - t^{-1})(t^p + t)(t^p + t^{-1})t^{-2p}.$$

(4) Finally, let us remark that the above theorem can be used to relate the SU_n-quantum invariants of 3-manifolds M with a $\mathbb{Z}/p\mathbb{Z}$-action to the quantum invariants of the quotient manifolds, M/G, $G = \mathbb{Z}/p\mathbb{Z}$.

2. The proof

Let T_{kl}, for $k \neq 0, l > 0$, denote the closure of a braid on k strands composed of l positive half-twists in $R^3 \setminus \{x\text{-axis}\}$, going clockwise around the x-axis if $k > 0$ and counterclockwise if $k < 0$. Embedded in R^3, T_{kl} is just the (k,l)-torus link. By a generalized torus link we will mean a link $T_{k_1l_1;\ldots;k_sl_s}$ in $R^3 \setminus \{x\text{-axis}\}$ whose components (not necessarily connected), $T_{k_1l_1}, \ldots, T_{k_sl_s}$, are placed in $R^3 \setminus \{x\text{-axis}\}$ in such a way that their diagrams in the z-plane do not intersect.



Picture 1: $T_{2,2;-3,1}$

Lemma 2.1. *For any p-periodic link $L \subset R^3 \setminus \{x\text{-axis}\}$, for p prime, there is a finite sequence of polynomials $v_j \in \mathbb{Z}[t^{\pm 1}]$ and a sequence of generalized torus links, L_j, of the form $T_{k_1p;k_2p;\ldots;k_sp}$, such that*

$$P_n(L) = \sum_j v_j^p P_n(L_j) \quad \textit{mod } p \quad \textit{and} \quad P_n(\bar{L}) = \sum_j v_j P_n(\bar{L}_j).$$

Note that if $L_j = T_{k_1p;k_2p;\ldots;k_sp}$, then $L_j = T_{k_11;k_21;\ldots;k_s1}$.

Proof. If $\bar{L}_+$, $\bar{L}_-$, $\bar{L}_0$, are three skein related links in $R^3 \setminus \{x\text{-axis}\}$ then $\pi^{-1}(\bar{L}_+)$, $\pi^{-1}(\bar{L}_-)$, $\pi^{-1}(\bar{L}_0)$, are three p-periodic links in $R^3 \setminus \{x\text{-axis}\}$ which differ by an entire G-orbit of crossings. By resolving the p positive crossings in $\pi^{-1}(\bar{L}_+)$, lying above the specified positive crossing in $\bar{L}_+$, we get the following skein relation

$$t^{np}P_n(\pi^{-1}(\bar{L}_+)) - t^{-np}P_n(\pi^{-1}(\bar{L}_-)) = (t-t^{-1})^p P_n(\pi^{-1}(\bar{L}_0)) \mod p.$$

Therefore, any skein resolving tree for $\bar{L}$ with leaves $\bar{L}_j$ yields presentations

$$P_n(\bar{L}) = \sum_j v_j P_n(\bar{L}_j),$$

$$P_n(L) = \sum_j v_j^p P_n(L_j) \mod p,$$

for certain v_j's in $\mathbb{Z}[t^{\pm 1}]$ and for $L_j = \pi^{-1}(\bar{L}_j)$. By [P, Thm 0.4], each link $\bar{L}$ in $R^3 \setminus \{x\text{-axis}\}$ has a skein resolving tree with leaves being the generalized torus links $T_{k_1 1; k_2 1; \ldots; k_s 1}$. Since

$$\pi^{-1}(T_{k_1 1; k_2 1; \ldots; k_s 1}) = T_{k_1 p; k_2 p; \ldots; k_s p}$$

the proof is completed. □

Each generalized torus link (in R^3) is a union of unlinked components, $T_{k_1 l_1} \cup \ldots \cup T_{k_s l_s}$. (However, if $(k_i, l_i) \neq 1$ then $T_{k_i l_i}$ is itself a link). Since

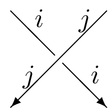
$$P_n(T_{k_1 l_1} \cup \ldots \cup T_{k_s l_s}) = \prod_i P_n(T_{k_i l_i}),$$

Theorem 1.1 follows from Lemma 2.1 and the theorem below.

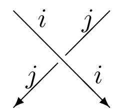
Theorem 2.2. *For any $k \in \mathbb{Z} \setminus \{0\}$ and a prime p,*
(1) $P_n(T_{kp}) = P_n(T_{k1})^p$ mod $(p, [2]^p - [2])$; and
(2) $P_n(T_{kp}) = P_n(T_{k1})^p$ mod $(p, [3]^p - [3])$, if n is odd.

The proof is based on the Jaeger's approach to the SU_n-quantum invariant of links, [J], described below. Consider diagrams of oriented links in R^3 as oriented 4-valent graphs, with vertices marked by the sign of the corresponding crossing. Given a diagram D we will define the rotation number of D, $r(D)$, to be (as usual) the sum of the signs of the Seifert circles of D. The sign of a circle is $+$ if it is oriented counterclockwise, and $-$ otherwise. We denote the writhe (i.e. the sum of the signs of the vertices of D) by $w(D)$.

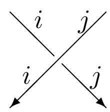
Consider a labeling of all edges of D by numbers $1, 2, \ldots, n$ such that for any i the number of edges labeled by i incident towards a vertex v is the same as the number of edges labeled by i incident from v. Denote the set of all such labelings by $L(D, n)$. Observe that for any labeling $f \in L(D, n)$, the edges labeled by a given number i in D form a new link diagram $D_{f,i}$, after "smoothing out" the two-valent vertices. For each $f \in L(D, n)$ and each vertex v in D we define $< v|D|f >$ to be either $0, 1, z$, or $-z$ depending on which of the following situations occurs:



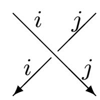
$< v, D, f >= 1$



$< v, D, f >= 1$



if $i < j$ then $< v, D, f >= z$
if $i > j$ then $< v, D, f >= 0$



if $i < j$ then $< v, D, f >= 0$
if $i > j$ then $< v, D, f >= -z$

For a given labeling of a diagram D, we denote the product $\prod_v < v|D|f >$ over all vertices of D by $< D|f >$.

Jaeger defines a polynomial $H'(D, z, a)$ which by his Theorem on page 328 in [J] is related to $P_n(D)$ by the following formula

$$P_n(D) = t^{nw(D)} H'(D, z, t^n), \tag{1}$$

where $z = t - t^{-1}$. Furthermore, by Proposition 2 in [J],

$$H'(D, z, t^n) = t^{-(n+1)r(D)} \sum_{f \in L(D,n)} < D|f > t^{w(D,f)+2s(D,f)}, \tag{2}$$

where $w(D, f) = \sum_{i=1}^{n} w(D_{f,i})$, and $s(D, f) = \sum_{i=1}^{n} ir(D_{f,i})$.

Using the factorizations provided in Remark (3) and the fact that the polynomials $t^p \pm t^{\pm 1}$ are pairwise relatively prime for $p \neq 2$, we obtain the following isomorphisms of rings

$$\mathbb{Z}[t^{\pm 1}]/(p, [2]^p - [2]) \simeq \mathbb{F}_p[t^{\pm 1}]/(t^p - t) \times \mathbb{F}_p[t^{\pm 1}]/(t^p - t^{-1}),$$

$$\mathbb{Z}[t^{\pm 1}]/(p, [3]^p - [3]) \simeq$$
$$\mathbb{F}_p[t^{\pm 1}]/(t^p - t) \times \mathbb{F}_p[t^{\pm 1}]/(t^p - t^{-1}) \times \mathbb{F}_p[t^{\pm 1}]/(t^p + t) \times \mathbb{F}_p[t^{\pm 1}]/(t^p + t^{-1}),$$

where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Therefore, we will prove Theorem 2.2 by showing the following congruences:

$$P_n(T_{kp}) = P_n(T_{k1})^p \mod (p, t^p - t), \tag{3}$$

$$P_n(T_{kp}) = P_n(T_{k1})^p \mod (p, t^p - t^{-1}). \tag{4}$$

and, for n odd:

$$P_n(T_{kp}) = P_n(T_{k1})^p \mod (p, t^p + t), \tag{5}$$

$$P_n(T_{kp}) = P_n(T_{k1})^p \mod (p, t^p + t^{-1}). \tag{6}$$

Proof of (3): Let D_{kl} denote the standard diagram of the torus link T_{kl}, as in Picture 1 for $(k,l) = (2,2)$ and $(k,l) = (-3,1)$. Since $w(D_{kp}) = (k-1)p = pw(D_{k1})$, by (1), it is enough to prove that $H'(D_{kp}, z, t^n) = H'(D_{k1}, z, t^n)^p$. Since $r(D_{kp}) = r(D_{k1})$, the above equation reduces modulo $t^p - t$ to
(7)

$$\sum_{f\in L(D_{kp},n)} < D_{kp}|f > t^{w(D_{kp},f)+2s(D_{kp},f)} = \sum_{f\in L(D_{k1},n)} < D_{k1}|f >^p t^{pw(D_{k1},f)+2ps(D_{k1},f)}.$$

Consider the action of $G = \mathbb{Z}/p\mathbb{Z}$ on the set of labelings of D_{kp} induced by the G-action on the z-plane. An orbit of such action will be either composed of a single G-equivariant labeling or of p different labelings. Note that in the later case, each of these p labelings contributes the same factor in the sum on the left side of (7). Hence, working modulo p, it is enough to consider the G-equivariant labelings of D_{kp} only. Since such labelings are in an obvious bijection with labelings of D_{k1}, we will identify these two sets. Now, the following lemma completes the proof.

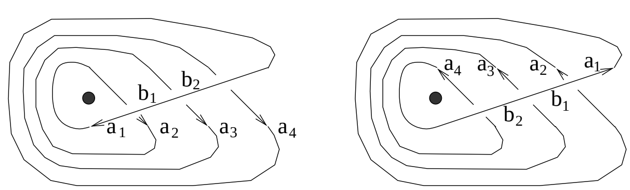
Lemma 2.3. *(1)* $< D_{kp}|f >=< D_{k1}|f >^p$,
(2) $w(D_{kp}, f) = pw(D_{k1}, f)$,
(3) $s(D_{kp}, f) = s(D_{k1}, f)$.

Proof. Since (1) and (2) are obvious, we prove (3) only. It is enough to show that

$$r((D_{kp})_{f,i}) = r((D_{k1})_{f,i}).$$

The left and right sides of the above identity are equal to the numbers of the Seifert circles for $(D_{kp})_{f,i}$ and $(D_{k1})_{f,i}$ respectively. Since the Seifert circles for $(D_{kp})_{f,i}$ and $(D_{k1})_{f,i}$ are identical, the proof is completed. □

The proof of (4) requires comparing the Jaeger state sum summations for the diagrams D_{kp} and D_{-k1}. We will use a natural bijection between labelings of D_{k1} and of D_{-k1}, presented in the example below for $k = 4$:



$T_{4,1}$ $T_{-4,1}$

Using this bijection and the bijection between the $\mathbb{Z}/p\mathbb{Z}$-equivariant labelings of D_{kp} and labelings of D_{k1}, we will identify the equivariant labelings of D_{kp} and the labelings of D_{-k1}. Now the proof of (4) is identical to the proof of (3), and it is based on the following lemma.

Lemma 2.4. *(1)* $r(D_{kp}) = -r(D_{-k1})$, *and hence* $t^{r(D_{kp})} = t^{pr(D_{-k1})}$.
(2) For any labeling $f \in L(D_{kp}, n) \simeq L(D_{-k1}, n)$,

(a) $< D_{kp}|f> = < D_{-k1}|f>^p$
(b) $w(D_{kp,f}) = pw(D_{-k1,f})$
(c) $s(D_{kp,f}) = -s(D_{-k1,f})$ *and hence* $t^{s(D_{kp,f})} = t^{ps(D_{-k1,f})}$.

The proof of the lemma is straightforward and therefore it is left to the reader. For the proof of (5), identify the equivariant labelings of D_{kp} with the labelings of D_{k1}, as before, and use Lemma 2.3. Note, that since $n+1$ is even, $t^{(n+1)r(D_{kp})} = t^{(n+1)r(D_{k1})p}$ modulo t^p+t. We also have $s(D_{kp},f) = s(D_{k1},f)$, and hence

$$t^{2ps(D_{k1},f)} = (-t)^{2s(D_{k1},f)} = t^{2s(D_{k1},f)} = t^{s(D_{kp},f)}.$$

For the proof of (6), identify the equivariant labelings of D_{kp} with the labelings of $D_{k,-1}$, as before, and use Lemma 2.4 again. Notice that $t^{(n+1)r(D_{kp})} = t^{(n+1)r(D_{k,-1})p}$ modulo $t^p + t^{-1}$. We also have $s(D_{kp},f) = -s(D_{k,-1},f)$, and hence

$$t^{2ps(D_{k,-1},f)} = (-t^{-1})^{2s(D_{k,-1},f)} = t^{2s(D_{kp},f)}.$$

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