Note on the Homfly-pt polynomial and linking numbers*

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Let L be an oriented link of n components $L_1, ..., L_n$. We denote the linking number between the i-th and j-th components (a half of the sum of signs of the crossings between these components) by l_{ij}. It is well known that the Homfly-pt polynomial of a link encodes the global linking number of the link, $\sum_{i \neq j} l_{ij}$, cf. [L-M, Prop 22]. It is an interesting problem whether any further information about the numbers l_{ij} is encoded in the Homfly-pt polynomial. We are going to show that indeed there are $n-1$ algebraically independent polynomials in l'_{ij}s which can be read from the value of the Homfly-pt polynomial of L.

Let Ω be the set of 2-element subsets of $\{1, 2, ..., n\}$, and let

$$c_k(L) = \sum l_{i_1 j_1} \cdot ... \cdot l_{i_k j_k},$$

where the sum is taken over all k element sets

$$I = \{\{i_1, j_1\}, \{i_2, j_2\}, ..., \{i_k, j_k\}\} \subset \Omega$$

such that I does not contain a subset $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_i, a_1\}\}$, where $a_1, a_2, ..., a_i \in \{1, 2, ..., n\}$. We assume that $c_0(L) = 1$. Since I has at most $n-1$ elements, $c_n(L) = c_{n+1}(L) = ... = 0$.

Let the Homfly-pt polynomial $P_L(v, z) \in \mathbb{Z}[v^{\pm 1}, z]$ be defined by the standard conditions:

$$vP_{L_+} - v^{-1}P_{L_-} = zP_{L_0}, \quad P_{T_n} = \left(\frac{v - v^{-1}}{z}\right)^{n-1},$$

where T_n denotes the trivial link of n components.

*This is a translation of a part of the author's Master degree thesis [S]. However, some remarks were added in December 1998.

The result below was discovered by J. Przytycki, and rediscovered by me (before it was published in [P, Thm. 3.1]). Later, it was also observed in [K].

Theorem 1 *1. For any link L of n components the limit*

$$Q_L(q) = lim_{v\to 1}\left(\frac{q}{v-v^{-1}}\right)^{\frac{n-1}{2}} P_L(v, \sqrt{q(v-v^{-1})})$$

exists.

2. Q_L is a polynomial in q and

$$Q_L(q) = \sum c_i(L)q^i.$$

Since the number of components of L is determined by P_L, (cf. [L-M, Prop. 24]) the numbers $c_1(L), ..., c_{n-1}(L)$ are also completely determined by P_L.

Proof of theorem:

1. Observe that

$$\bar{P}_L(q, v) = \left(\frac{q}{v-v^{-1}}\right)^{\frac{n-1}{2}} P_L(v, \sqrt{q(v-v^{-1})})$$

satisfies the conditions

$$v\bar{P}_{L_+} - v^{-1}\bar{P}_{L_-} = \begin{cases} \sqrt{q(v-v^{-1})}\cdot\sqrt{\frac{v-v^{-1}}{q}}\bar{P}_{L_0} & \text{if } \mu(L_0) = \mu(L_\pm)+1 \\ \sqrt{q(v-v^{-1})}\cdot\sqrt{\frac{q}{v-v^{-1}}}\bar{P}_{L_0} & \text{if } \mu(L_0) = \mu(L_\pm)-1, \end{cases}$$

$$\bar{P}_{T_n} = 1,$$

where $\mu(L)$ denotes the number of components of L. After simplification we get

$$v\bar{P}_{L_+} - v^{-1}\bar{P}_{L_-} = \begin{cases} (v-v^{-1})\bar{P}_{L_0} & \text{if } \mu(L_0) = \mu(L_\pm)+1 \\ q\bar{P}_{L_0} & \text{if } \mu(L_0) = \mu(L_\pm)-1 \end{cases} \quad (1)$$

$$\bar{P}_{T_n} = 1. \quad (2)$$

Using the above defining relations for $\bar{P}_L$ one can prove by induction on the dimensions of the resolving tree of L that $\bar{P}_L \in \mathbb{Z}[v^{\pm 1}, q]$. This implies that the limit $Q_L(q) = lim_{v\to 1}\bar{P}_L(v, q)$ exists and $Q_L(q) \in \mathbb{Z}[q]$.

2. Let $\bar{Q}_L(q) = \sum_{k=0}^{\infty} c_k(L)q^k$. Since $c_k(L) = 0$ for $k \geq \mu(L)$, $\bar{Q}_L$ is a polynomial in q. We are going to show that $\bar{Q}_L$ satisfies conditions (1),(2), and hence it is equal to Q_L.

   By definition, $\bar{Q}_{T_n} = 1$ and therefore (2) is satisfied. Let L_+, L_-, L_0 be any skein related triple of oriented links. If $\mu(L_0) = \mu(L_\pm) + 1$ then the specified crossing in $L_\pm$ is a self-intersection of one component and $\bar{Q}_{L_+} = \bar{Q}_{L_-}$. Therefore in this case condition (1) is satisfied.

   Assume now that the specified crossing in $L_\pm$ is an intersection of two different components. In this case we have $\mu(L_0) = \mu(L_\pm) - 1$ and the proof will be completed once we show the following lemma.

Lemma 1 *For any triple of skein related links L_+, L_-, L_0,*

$$c_k(L_+) - c_k(L_-) = \begin{cases} 0 & \text{if } \mu(L_0) = \mu(L_\pm) + 1 \\ c_{k-1}(L_0) & \text{if } \mu(L_0) = \mu(L_\pm) - 1. \end{cases}$$

Proof: If $\mu(L_0) = \mu(L_\pm) + 1$ then the specified crossing in $L_\pm$ is a self-crossing of a component and the statement is obvious. Hence assume that $\mu(L_0) = \mu(L_\pm) - 1$ and that the specified crossing in $L_\pm$ is between the components L_A and L_B, $A, B \in \{1, 2, ..., n\}$, $n = \mu(L_\pm)$. The number $c_k(L_+)$ is by definition a sum of certain products

$$l_{i_1 j_1} \cdot ... \cdot l_{i_k j_k}.$$

Separate the products in which l_{AB} appears from the other products,

$$c_k(L_+) = l_{AB} \cdot \sum l_{i_1 j_1} \cdot \ldots \cdot l_{i_a j_a} \cdot l_{i_{a+1} j_{a+1}} \cdot \ldots \cdot l_{i_{k-1} j_{k-1}} +$$

$$\sum l_{s_1 p_1} \cdot \ldots \cdot l_{s_k p_k},$$

where $i_1, \ldots, i_a \in \{A, B\}$ and $i_{a+1}, \ldots, i_{k-1},\ j_1, \ldots, j_{k-1} \notin \{A, B\}$. Denote the first sum by S_1 and the second one by S_2. Then $c_k(L_+) = l_{AB} S_1 + S_2$.

Observe that if S_1 contains a product

$$l_{i_1 j_1} \cdot \ldots \cdot l_{i_a j_a} \cdot l_{i_{a+1} j_{a+1}} \cdot \ldots \cdot l_{i_{k-1} j_{k-1}}$$

then it also contains the products

$$l_{i'_1 j_1} \cdot \ldots \cdot l_{i'_a j_a} \cdot l_{i_{a+1} j_{a+1}} \cdot \ldots \cdot l_{i_{k-1} j_{k-1}},$$

where $i'_1,\ i'_2,\ldots,\ i'_a$ are arbitrary elements of $\{A,B\}$. By summing all elements of this form we get

$$(l_{Aj_1}+l_{Bj_1})\cdot\ldots\cdot(l_{Aj_a}+l_{Bj_a})\cdot l_{i_{a+1}j_{a+1}}\cdot\ldots\cdot l_{i_{k-1}j_{k-1}}.$$

It is not difficult to conclude that $S_1=c_{k-1}(L_0)$. For $c_k(L_-)$ we have

$$c_k(L_-)=(l_{AB}-1)S_1+S_2,$$

and therefore

$$c_k(L_+)=c_k(L_-)+c_{k-1}(L_0)\quad \text{for } k=1,2\ldots n-1.$$

This completes the proof of Lemma 1 and Theorem 1. □

Not all symmetric polynomials in l_{ij} can be determined by the value of the Homfly-pt polynomial. J. Birman [B] gives an example of two 3-braids whose closures have the same Homfly-pt polynomial but different linking numbers between their components:

$$\gamma_1=\sigma_1^{-2}\sigma_2^3\sigma_1^{-1}\sigma_2^4\sigma_1^{-2}\sigma_2^4\sigma_1^{-1}\sigma_2$$

$$\gamma_2=\sigma_1^{-2}\sigma_2^3\sigma_1^{-1}\sigma_2^4\sigma_1^{-1}\sigma_2\sigma_1^{-2}\sigma_2^4$$

Closures of these braids are three component links $\hat{\sigma}_1,\hat{\sigma}_2$, such that

$$lk_{12}(\hat{\sigma}_1)=3,\quad lk_{13}(\hat{\sigma}_1)=-2,\quad lk_{23}(\hat{\sigma}_1)=2$$

and

$$lk_{12}(\hat{\sigma}_2)=0,\quad lk_{13}(\hat{\sigma}_2)=-1\quad lk_{23}(\hat{\sigma}_2)=4.$$

Therefore

$$c_1(\hat{\sigma}_1)=c_1(\hat{\sigma}_2)=3,\quad c_2(\hat{\sigma}_1)=c_2(\hat{\sigma}_2)=-4$$

and $c_k(\hat{\sigma}_1)=c_k(\hat{\sigma}_2)=0$, for $k\geq 3$, but

$$lk_{12}(\hat{\sigma}_1)\cdot lk_{13}(\hat{\sigma}_1)\cdot lk_{23}(\hat{\sigma}_1)=-12\neq 0=lk_{12}(\hat{\sigma}_2)\cdot lk_{13}(\hat{\sigma}_2)\cdot lk_{23}(\hat{\sigma}_2).$$

Recall that an invariant of oriented links is a Vassiliev invariant of degree d if it vanishes on all singular links with more than d singular points.

Lemma 2 *c_n is a Vassiliev invariant of degree n.*

Proof: The statement is obvious for $n = 0$. Assume now that the statement holds for $n < d$ and let L be a link with $d + 1$ singular points. If we denote the resolutions of these points by $L_{\varepsilon_1,...,\varepsilon_{d+1}}$, where $\varepsilon_1, ..., \varepsilon_{d+1} = \pm 1$ then we need to prove that

$$\sum_{\varepsilon_1,...,\varepsilon_{d+1}=\pm 1} (-1)^{m(v_1,...,v_{d+1})} c_d(L_{\varepsilon_1,...,\varepsilon_{d+1}}) = 0$$

where $m(\varepsilon_1, ..., \varepsilon_{d+1})$ is the number of -1's among $\varepsilon_1, ..., \varepsilon_{d+1}$. If one of the singular points is a self-intersection of a component then the statement is obvious. Otherwise, the left side of the above equation is

$$\sum_{\varepsilon_1,...,\varepsilon_d=\pm 1} (-1)^{m(\varepsilon_1,...,\varepsilon_d)} c_{d-1}(L_{\varepsilon_1,...,\varepsilon_d,0})$$

by Lemma 1. Since this sum vanishes by the inductive assumption, c_d is a Vassiliev invariant of degree d. □

We conclude this note by stating three open problems and one exercise. Let V be a space of rational valued functions in variables $l_{ij}, i, j = 1, 2, ...$ invariant under the action of the symmetric group S_∞ :

$$\sigma \cdot f(l_{ij}\ i, j = 1, 2, ...) = f(l_{\sigma(i)\sigma(j)}\ i, j = 1, 2, ...),\ \sigma \in S_\infty.$$

Each element of V is an invariant of oriented links.

L is called algebraically split link if the components of L can be split into two groups so that the linking number between any two representatives of different groups is 0.

Problem 1 *Do the numbers $c_1(L), c_2(L), ...$ determine whether or not L is an algebraically split link?*

Exercise 1 *Prove that if $c_1(L) = c_2(L) = 0$ the then linking number between any two components of L is* 0.

References

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