DISTRIBUTIVE PRODUCTS AND THEIR HOMOLOGY

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Abstract. We develop a theory of sets with distributive products (called shelves and multi-shelves) and of their homology. We relate the shelf homology to the rack and quandle homology.

1. Binary distributive operations

Let X be a set. Binary operations $\star : X \times X \to X$ with composition

$$x \star_1 \star_2 y = (x \star_1 y) \star_2 y$$

and the identity $x \star y = x$ form a monoid, $M(X)$.

A collection of operations $\star_i$, $i \in I$ is called *mutually (right) distributive* if

$$(a \star_i b) \star_j c = (a \star_j c) \star_i (b \star_j c)$$

for all $i, j \in I$.

A direct computation gives:

Lemma 1. *If $\{\star_i\}_{i\in I}$ are mutually-distributive then the set of all compositions of $\star_i$'s is mutually distributive. Hence every maximal set of mutually-distributive products is a monoid.*

We say that $\star$ is *(right) self-distributive* or, simply, *distributive*, if

$$(x \star y) \star z = (x \star z) \star (y \star z), \qquad (1)$$

that is, if the one element set $\{\star\}$ is mutually distributive. In this case, $(X, \star)$ is called a *shelf*, [Cr]. (Hence (1) might be called "shelf-distributivity".) We abbreviate $x \star y$ to xy whenever it does not lead to a confusion.

We call a collection of mutually distributive products on a set, $(X, \star_i, i \in I)$, *a multi-shelf.*

Example 2. *Every set X with the one of the following products is a shelf:*
(1) $x \star y = f(x)$, for some $f : X \to X$.
(2) $x \star y = g(y)$ for some $g : X \to X$ such that $g^2 = g$.
(3) Given $A \subset X$, $x \star y = \begin{cases} y & \text{if } y \in A \\ x & \text{otherwise.} \end{cases}$

Example 3. *Let X be a collection of subsets of Ω.*
(1) If X is closed under intersections then $(X, \cap)$ is a shelf.
(2) If X is closed under set subtractions then X with operation of subtraction is a shelf.
(3) $X = 2^\Omega$ with operations $\cap, \cup$, and the "identity" product, $x \star y = y$, is a multi-shelf.

Finally, we have

Example 4 (Free shelves)**.** *Given a set S there is the free shelf $X(S)$ on S. (It is the free magma, $F(S)$, on S, quotiented by the smallest equivalence relation relating two words in $F(S)$, if they are related by the distributive operation $(w_1w_2)w_3 = (w_1w_3)(w_2w_3)$.)*

Proposition 5. *There are precisely 6 different two element shelves up to an isomorphism. Two of them are given by Example 3(1) and (2) for $X = \{\emptyset, \Omega\}$. The other four are of the types described in Example 2 for $X = \{0,1\}$: (a) $f = 0$, (b) $f = Id_X$, (c) $f(x) = 1 - x$, (d) $g = Id_X$.*

Proof. by direct enumeration. □

Other examples of shelves are given by quandles and, more generally, by racks, i.e. shelves X such that for each $x, y \in X$ there is a unique z for each $z \star x = y$. Additionally there are some simple ways of building "bigger" shelves from smaller ones. We leave the proofs of the following to the reader.

Proposition 6. *If $(X_i, \star_i)$ are shelves for $i \in I$ then*
(1) their disjoint union $X = \coprod_{i\in I} X_i$ with

$$x \star x' = \begin{cases} x \star_i x' & \text{iff } x, x' \in X_i \text{ for some } i \\ x & \text{iff } x \in X_i, x' \in X_j, i \neq j \end{cases}$$

is a shelf as well.
(2) $X = \prod_{i\in I} X_i$ with

$$(x_i,\, i \in I) \star (x'_i,\, i \in I) = (x_i \star_i x'_i,\, i \in I)$$

is a shelf as well.

Problem 7. *Find other "natural" families of shelves which are not racks.*

2. The simplicial complex and the homology of a shelf

2.1. **Shelf complex.** Given a shelf X consider a simplicial complex $S(X)$ whose vertices are elements of X and whose simplices of the form

$$\sigma_{x_0,\dots,x_n} = \langle x_n, x_{n-1}x_n, \dots, (\dots((x_0x_1)x_2)\dots x_{n-1})x_n\rangle\,,$$

for $x_0, \dots, x_n \in X$ such that $x_0, x_1x_0, \dots, (\dots((x_nx_{n-1})x_{n-2})\dots x_1)x_0$ are distinct. The distributivity of the product implies that faces of simplices are simplices as well. This is *the shelf complex of X*. Since every simplex of a simplicial complex is determined by its vertices, $\sigma_{x_0,\dots,x_n}$ may denote the same simplex for different n-tuples $(x_0, \dots, x_n)$. We will call the homology of that complex the *simplicial shelf homology of X*.

Example 8. *Consider $X = \{\emptyset, \{a\}, \{b\}\} \subset 2^{\{a,b\}}$ with the set subtraction, as in Example 3. Then $S(X)$ is homeomorphic to the circle.*

Problem 9. *Is every homotopy type of a simplicial complex realized by a simplicial complex of a shelf?*

Alternatively, one can consider a CW-complex, $S'(X)$, whose n-cells, $\sigma_{x_0,\dots,x_n}$, are in 1-1 correspondence with all n-tuples of elements of X.

2.2. **Shelf homology.** Here is another version of homology for shelves:

Let $C_n(X)$ be the free abelian group with basis $\{(x_0,...,x_n) : x_0,...,x_n \in X\}$ and let $d_n : C_n(X) \to C_{n-1}(X)$ be given by

$$d_n(x_0,...,x_n) = \sum_{i=0}^{n}(-1)^i(x_0 \star x_i,...,x_{i-1} \star x_i, x_{i+1},...,x_n).$$

(For example, $d_1(x_0,x_1) = (x_1) - (x_0)$, $d_2(x_0,x_1,x_2) = (x_1,x_2) - (x_0x_1,x_2) + (x_0x_2,x_1x_2)$.) Additionally, we assume that $C_{-1}(X) = \mathbb{Z}$ and $d_0 : C_0(X) = ZX \to C_{-1}(X)$ sends every (x) to 1.

Lemma 10. *$(C_*(X), d_*)$ is a chain complex.*

Proof. Let

$$d_{n,i}(x_0,...,x_n) = (x_0 \star x_i,...,x_{i-1} \star x_i, x_{i+1},...,x_n).$$

Then $d_n = \sum_{i=0}^{n}(-1)^i d_{n,i}$ and it is easy to see that $d_{n-1,i+1}d_{n,j} = d_{n-1,j}d_{n,i}$, for $i \geq j$. Hence, the $(n+1)n$ summands of $d_{n-1}d_n(x_0,...,x_n)$ can be matched in pairs of coinciding terms of opposite signs. □

We call the homology groups of this chain complex, the *(algebraic) shelf homology of X*. Its generalizations and relations to rack and quandle homologies will be explained in Sec 4.

Note that the reduced simplicial chain complex of $S(X)$ is a subcomplex of the (algebraic) chain complex of X. Hence we have a homomorphism from the simplicial to the algebraic shelf homology.

2.3. **H_0 and Left Orbits.** Let $\sim$ be the smallest equivalence relation on X such that $x \sim y \star x$ for all $x, y \in X$. We call the equivalence classes of this relation the *left orbits*. A shelf with a single left orbit is called *connected*.

Remark 11. *(1) The following shelves are connected:*
(a) Racks
(b) The shelves in Examples 2(1),(3).
(c) The shelves in Example 3(1) and (2): (Proof for (1): $x \sim y \cap x = x \cap y \sim y$. Proof for (2): $x - x = \emptyset$. Hence $x \sim \emptyset$ for every $x \in X$.)
(2) The left orbits in Example 2(2) are in bijection with the elements of the image of g.

Remark 12. *Since $x' \sim x$ implies $y' \star x' \sim x' \sim x \sim y \star x$ for every $y, y' \in X$, the product on X descends to a self-distributive product on the set O of the left orbits in X and the natural quotient map $\pi : X \to O$ is a homomorphism. The induced product on O is $[x] \star [y] = [y]$ (as in Example 2).*

Proposition 13. *There is a natural bijection between left orbits of X and the connected components of $S(X)$.*

Proof. left for the reader □

We also have (for algebraic homology):

Theorem 14. *$H_0(X) = \mathbb{Z}^{r-1}$, where r is the number of left orbits of X.*

Proof. $H_0(X)$ is the quotient of the group

$$Z_0(X) = \left\{ \sum_{x \in X} c_x(x) : \sum_{x \in X} c_x = 0 \right\}$$

by its subgroup B_0 generated by $(x)-(yx)$. Let $x_0, ..., x_{r-1}$ be representatives of all left orbits in X. Let $\mathbb{Z}^{r-1} = \langle e_1, ..., e_{r-1} \rangle$. We have $\phi : \mathbb{Z}^{r-1} \to Z_0(X)$ sending e_i to $(x_i) - (x_0)$. Since each generator of $B_0(X, \star)$ is a linear combination of elements of a single orbit, ϕ descends to an embedding $\psi : \mathbb{Z}^{r-1} \hookrightarrow H_0(X)$. It is easy to see that it is onto. □

3. Further properties of homology

Theorem 15. *If one of the conditions holds:*
(1) $x \to x \star y$ is a bijection on X for some y, or
(2) there is $y \in X$ such that $y \star x = y$ for all $x \in X$,
then $H_n(X) = 0$ for all n.

Proof. (1) Since the map $f(x) = x \star y : X \to X$ is a shelf homomorphism, it induces a chain map $C_*(X) \to C_*(X)$ which is chain homotopic to the zero map by the homotopy map $(x_0, ..., x_n) \to (x_0, ..., x_n, y)$. Therefore $Id = f_*(f^{-1})_* = 0$ as endomorphisms of $H_*(X)$ and, hence, $H_*(X)$ vanishes.
(2) If $y \star x = y$ for all $x \in X$ then the identity map on $C_*(X)$ is homotopic to the zero map via the chain homotopy $C_n(X) \to C_{n+1}(X)$ sending $(x_0, ..., x_n)$ to $(y, x_0, ..., x_n)$. □

Theorem 16. *If X is a shelf with $x \star y = y$, c.f. Example 2(2), then $H_n(X) = \mathbb{Z}^{(|X|-1)|X|^n}$ for every n, where $|X|$ is the cardinality of X.*

Proof. Consider the map $s : C_n(X) \to C_{n+1}(X)$ sending $(x_0, ..., x_n)$ to $(x_0, x_0, x_1, ..., x_n)$. Note that $D_n = s(C_{n-1}(X)) \subset C_n(X)$ for $n \geq 1$ and $D_0 = 0$ define a subcomplex D_* of $C_*(X)$. Since $ds + sd$ is the identity on D_n, for $n \geq 1$, the identity map is homotopic to the zero map and, hence, this subcomplex is acyclic. (For $n = 0$ it follows from $D_0 = 0$.) Hence $H_*(X) = H_*(C_*(X)) = H_*(C_*(X)/D_*)$. However, since the differential vanishes on $C_n(X)/D_n$, $H_n(X) = C_n(X)/D_n$ is the free abelian group of rank $|X|^{n+1} - |X|^n$. □

Let A be a subshelf of X. We say that $r : X \to A$ is a *retraction* of X onto A if r is the identity on A and $r(x_1 \star x_2) = r(x_1) \star r(x_2)$ for every $x_1, x_2 \in X$. We say that the retraction is *strong* if additionally $X \star X \subset A$. In this case we call A *a strong retract of* X.

Theorem 17. *If A is a strong retract of X then*

$$H_n(X) = H_n(A) \oplus \mathbb{Z}(X \setminus A) \otimes \bigoplus_{k=1}^{n-1} H_k(A) \otimes \mathbb{Z}X^{n-k-1}.$$

Proof. Let V be the subgroup of $\mathbb{Z}X$ generated by the elements $x - r(x)$ for $x \in X \setminus A$. Since this set is a basis of V, $V = \mathbb{Z}^{|X \setminus A|}$. Let

$$C_{n,k} = \mathbb{Z}A^{n-k} \otimes V \otimes \mathbb{Z}X^{k-1}$$

be considered as a subgroup of $\mathbb{Z}X^{n-k} \otimes \mathbb{Z}X \otimes \mathbb{Z}X^{k-1} = \mathbb{Z}X^n = C_n(X)$, for $1 \leq k \leq n$ and let

$$C_{n,0} = \mathbb{Z}A^n.$$

Note that the chain complex $C_*(X)$ decomposes into a direct sum of subcomplexes

$$C_*(X) = \oplus_{k=0}^{n} C_{*,k}$$

and that the chain complex $C_{*,k}$ is isomorphic to $C_{*-k}(A) \otimes V \otimes \mathbb{Z}X^{k-1}$, for $k \geq 1$, and to $C_*(A)$, for $k = 0$. This implies the statement. □

Corollary 18. *If X is a shelf with $x \star y = g(y)$ for $g : X \to X$ such that $g^2 = f$, c.f. Example 2(2), then $H_n(X) = \mathbb{Z}^{(r-1)|X|^{n-1}}$ for every n, where $|X|$ is the cardinality of X and $r = |f(X)|$.*

Corollary 19. *(1) For finite X, $rk\, H_n(X) = |X - A| \cdot \sum_{k=1}^{n-1} rk\, H_k(A) \cdot |X|^{n-k-1}$.*
(2) If A is acyclic then X is acyclic.
(3) If $rk\, H_k(A) = (c-1)|A|^k$ where c is the number of left orbits in X (eg. if $a * b = g(b)$) then $rk\, H_n(X) = (c-1)|X|^n$.
(4) If we allow any $rk\, H_1(A)$ and $rk\, H_2(A)$ and $R_k(A) = ...$

Note that r above is the number of left orbits in X. Therefore this result is an analogous to [LN, Thm 1.1] for finite racks. Although the above examples suggest that all homology groups of connected shelves vanish, the following example shows that that is not the case.

Example 20. *By direct computation, the shelf of Example 2(3) with $|X| = 4$, $|A| = 2$, $H_n(X)$ is free of ranks $0, 1, 4, 16$ for $n = 0, 1, 2, 3$.*

Denote the rank of an abelian group A by $rk\, A$. The computational evidence suggests the following:

Conjecture 21. *For every shelf X,*
(1) $H_n(X)$ is free abelian for every n.
(2) $rk\, H_{n+1}(X) = |X| \cdot rk\, H_n(X)$ for large n.

As we have noticed before, for every shelf X there is a shelf homomorphism $\pi : X \to O$ onto the shelf of its left orbits. Examples 20 and 22 show that the induced map on homology groups does not have to be neither 1-1 nor onto.

Example 22. *Consider $X = \{1, 2, 3\}$ with multiplication*

$$x \star y = \begin{cases} y & \text{if } y \geq x \\ 1 & \text{otherwise.} \end{cases}$$

One can check easily that it is a shelf with 2 orbits. A direct computation yields $H_2(X) = \mathbb{Z}^2$, $H_3(X) = \mathbb{Z}^6$, $H_2(X) = \mathbb{Z}^{18}$.

4. Multi-shelf, Rack, and Quandle Homologies

Consider a multi-shelf $(X, \star_1, ..., \star_n)$ (i.e. a set with mutually distributive products). Let $d_*^1, ..., d_*^n$ be the induced differentials on $C_*(X)$.

Lemma 10 can be generalized as follows:

Lemma 23. *$d_{n-1}^k d_n^l = -d_{n-1}^l d_n^k$ for any $k, l \in \{1, ..., n\}$.*

The proof is a modification of that of Lemma 10: Let

$$d_{n,i}^k(x_1, ..., x_{n+1}) = (x_1 \star_k x_i, ..., x_{i-1} \star_k x_i, x_{i+1}, ..., x_{n+1}).$$

Then $d_n^k = \sum_{i=1}^{n+1}(-1)^i d_{n,i}^k$ and it is easy to see that $d_{n-1,i+1}^k d_{n,j}^l = d_{n-1,j}^l d_{n,i}^k$, for $i \geq j$. Hence, the $(n+1)n$ summands of $d_{n-1}d_n(x_1,...,x_{n+1})$ can be matched in pairs of coinciding terms of opposite signs. □

Let $d_n = \sum_{k=1}^n c_k d_n^k : C_n(X) \to C_{n-1}(X)$ for some fixed $c_1,...,c_k \in \mathbb{Z}$. By the above lemma, $d_{n-1}d_n = 0$ for all n and, hence, $(C_*(X), d_*)$ is a chain complex. We denote its homology by $H_*(X, \sum_{k=1}^n c_k \star_k)$ and call it the *multi-shelf homology*.

The "identity" product, $a \star_0 b = a$ is mutually distributive with every shelf product $\star$. The groups $H_*(X, \star - \star_0)$ are the *rack homology* groups introduced in [FRS, CJKS2], c.f. [EG, LN]. (The origin of their name stems from the fact that this homology was originally defined for racks only, even though the definition clearly makes sense for all shelves.) It is important to note however that by the standard definition of rack homology, $H_n(X, \star - \star_0)$ is the $n+1$-st rack homology of X. The connections between the shelf homology of X and the simplicial complex of X as well as Theorem 14 justify however, in our opinion, our choice of grading of the shelf homology.

Let $D_n(X) \subset C_n(X)$ be generated by chains $(x_1,...,x_{n+1})$ such that $x_i = x_{i+1}$ for some i. If a rack X is a spindle (that is $x \star x = x$ for all $x \in X$) then $D_*(X)$ is a sub-chain complex of $C_*(X)$ of *degenerate chains*. Therefore one can consider the homology of $C_*(X)/D_*(X)$. The n-th homology of $C_*(X)/D_*(X)$ with the differential $\star - \star_0$ is called the $n+1$-*st quandle homology* of $(X, \star)$,[CKS, LN, Mo, NP1, NP2]. (It is defined for all spindles despite the term "quandle" in its name.) Applications of quandle homology to topology are discussed in [Ca, CEGS, CJKLS, CJKS1, CJKS2, I, M, N, Za].

Here is another interesting example of the multi-product homology:

Example 24. *Let $X = 2^\Omega$ for some set Ω. The following three products are mutually distributive:*

$$x_1 *^1 x_2 = x_1, \quad x_1 *^2 x_2 = x_1 \cap x_2, \quad x_1 *^3 x_2 = x_1 \cup x_2.$$

Let $N = |X| = 2^{|\Omega|}$. Then our computations show the following:

Conjecture 25. $rk\, H_n(X, a_1 \star_1 + a_2 \star_2 + a_3 \star_3) =$

$$\begin{cases} N^n & \text{for } (a_1,a_2,a_3) = (0,0,0) \\ N^{(n-1)} & \text{for } (a_1,a_2,a_3) = c(1,-1,-1),\ c \in \mathbb{Z} \setminus \{0\} \\ 1 & \text{for } a_1,a_2,a_3 \text{ such that } a_1+a_2+a_3 = 0, (a_1,a_2,a_3) \neq (0,0,0) \\ 0 & \text{otherwise} \end{cases}$$

More generally we propose:

Conjecture 26. *For every set $\star_1,...,\star_N$ of mutually distributive products, there is a finite union of hyperplanes H in $\mathbb{Z}^N$ and a sequence $r_1, r_2, ... \in \mathbb{Z}_{\geq 0}$ such that $rk\, H_n(X, a_1 \star_1 + ... + a_N \star_N) = r_n$ for all $(a_1,...,a_N) \in \mathbb{Z}^N \setminus H$ and for all n.*

In other words the above conjecture states that $rk\, H_n(X, a_1 \star_1 + ... + a_N \star_N)$ is independent of $(a_1,...,a_N)$ with a codimension ≥ 1 of possible exceptions.

References

[Ca] J. S. Carter, A Survey of Quandle Ideas, arXiv:1002.4429v2

[CEGS] J.S. Carter, M. Elhamdadi, M. Graña, M. Saito, Cocycle knot invariants from quandle modules and generalized quandle cohomology, *Osaka J. Math.* **42**(2005), 499–541.

[CJKLS] J.S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito, Quandle cohomology and state-sum invariants of knotted curves and surfaces, *Trans. Amer. Math. Soc.* **355** (2003), 3947–3989, arXiv:math/9903135 .

[CJKS1] J.S. Carter, D. Jelsovsky, S. Kamada, M. Saito, Computations of quandle cocycle invariants of knotted curves and surfaces, *Adv. Math.* **157** (2001), no. 1, 36–94.

[CJKS2] J.S. Carter, D. Jelsovsky, S. Kamada, M. Saito, Quandle homology groups, their Betti numbers, and virtual knots, *J. Pure Appl. Algebra* **157** (2001), no. 2-3, 135–155.

[CKS] J.S. Carter, S. Kamada, M. Saito, Geometric interpretations of quandle homology, *J. Knot Theory Ramifications* **10** (2001), no. 3, 345-386.

[Cr] A. Crans, Lie 2-algebras, Ph.D. Dissertation, 2004, UC Riverside, arXive:math.QA/0409602.

[De] Dehornoy XXXXXXXXX

[EG] P. Etingof, M. Graña, On rack cohomology, *J. Pure Appl. Algebra* **177** (2003), no. 1, 49–59.

[FRS] R. Fenn, C. Rourke, B. Snaderson, James bundles, *Proc. London Math. Soc.* (3) **89** (2004), no. 1, 217-240.

[I] A. Inoue, Quandle and hyperbolic volume, arXiv:0812.0425

[LN] R.A. Litherland, S. Nelson, The Betti numbers of some finite racks, *J. Pure Appl. Algebra* **178** (2003), no. 2, 187–202.

[M] S. Matveev, Distributive groupoids in knot theory, *Math. USSR- Sbornik* **47** (1982), 73–83.

[Mo] T. Mochizuki, Some calculations of cohomology groups of finite Alexander quandles, *J. Pure Appl. Algebra* **179** (2003), no. 3, 287–330.

[NP1] M. Niebrzydowski, J.H. Przytycki, Homology of dihedral quandles, *J. Pure Appl. Algebra* **213** (2009), no. 5, 742–755.

[NP2] M. Niebrzydowski, J.H. Przytycki, Homology operations on homology of quandles, arXiv:0907.4732.

[N] M. Niebrzydowski, Three geometric applications of quandle homology, *Topology Appl.* **156** (2009), no. 9, 1729-1738.

[Za] J. Zablow, On relations and homology of the Dehn quandle, *Algebr. Geom. Topol.* **8** (2008), no. 1, 19–51.

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