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A GAUGE-INVARIANT RELATIVISTIC THEORY OF THE RUTHERFORD-SANTILLI NEUTRON

A.O.E. Animalu
Institute for Basic Research, 4A Limpopo Street, F.H.A. Maitama, Abuja.
& Dept. of Physics & Astronomy, University of Nigeria, Nsukka, Nigeria
(e-address: nascience@aol.com, ibr32@aol.com)

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Abstract

We present a gauge-invariant relativistic reconstruction of our earlier non-relativistic theory of the *Rutherford-Santilli neutron,* defined as the structure of the neutron as a bound state of one proton and one electron based on hadronic mechanics (HM), $n = (e^- \downarrow - p^+ \uparrow)_{HM}$, but reformulated as a superfluid phase of a compressed H-atom in the form, $n = (e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$, which characterizes a Cooper-like pairing $(e^- \downarrow, \bar{e}^0 \uparrow)$ of the electron e^- and a new massive neutral spin-$\frac{1}{2}$ particle e^0 in $1s^2$ configuration around the proton p^+. After a review of the non-relativistic theory, its gauge invariance is reformulated in such a way that the transition from $e^- - p^+$ Coulomb interaction potential, $V_C \equiv eA_0(r)$, to $\bar{e}^0 - p^+$ Hulthen interaction potential, $V_H \equiv e\hat{A}_0(r) = -M/[e^{m_0 r} - 1]$, is described by a general *nonlinear* first-order differential (Riccati's) equation for $\hat{A}_0(r)$ whose exact solution determines the ratio of the two masses (M, m_0) defining V_H. Finally a gauge-invariant relativistic theory is constructed for the $n = (e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$ system consisting of the usual Dirac equation for $e^- - p^+$ relative motion and an iso-dirac equation for $p^+ - \bar{e}^0$ relative motion (derived from conservation of an iso-current density) whose exact solution for the mass of e^0 is used (without adjustable constants) to predict the masses of the neutron and the other members of the S(3) baryon octet, $\Lambda, \Sigma^{\pm,0}, \Xi^{-,0}$. Good agreement between the predicted mass ratios and experimental data is obtained.

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1. INTRODUCTION

In a recent paper[1], we have presented a non-relativistic theory of the *Rutherford-Santilli neutron*, defined by Santilli[2] as the structure of the neutron as a bound state of one proton and one electron based on "hadronic mechanics" (HM), $n = (p^+ \uparrow, e^- \downarrow)_{HM}$ [see, Fig. 1(a)]. However, we recharacterized the structure in a new form $n = (e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$ [see Fig. 1(b)] as a superfluid phase of a compressed H-atom formed by Cooper-like pairing $(e^- \downarrow, \bar{e}^0 \uparrow)$ of the electron e^- and a new massive neutral spin-$\frac{1}{2}$ particle e^0 (called spinion in superconductivity theory[3]) in a closed $(1s^2)$ shell configuration around the proton (p^+) "trigger".

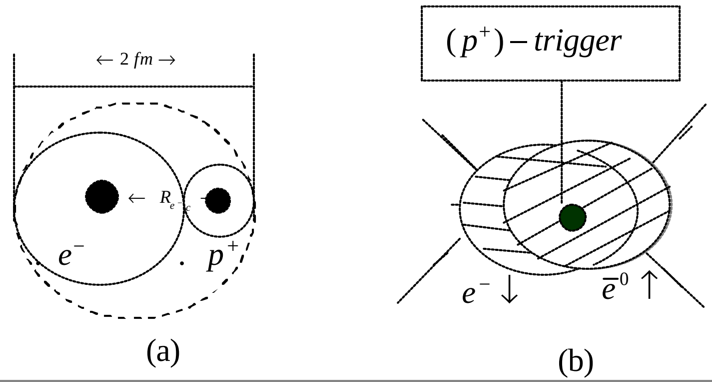

Fig.1: (a) Structure of the neutron[2], $n = (p^+ \uparrow, e^- \downarrow)_{HM}$, when the sum of the classical electric charge radii of e^- and p^+ is equal to 1 *fm* under compression; and (b) the new structure[1], $n = (e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$, showing mutual overlapping of the wavepackets of the pair $(e^- \downarrow, \bar{e}^0 \uparrow)$ around the proton (p^+).

The non-relativistic theory of the $n = (e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$ system in Fig.1(b) will be reviewed in Sec. 2 in order to underscore the features that make it equivalent to the $n = (p^+ \uparrow, e^- \downarrow)_{HM}$ system in Fig. 1(a) and also determine the signature of the new particle e^0 due to the expected isotopic lifting of unitary symmetry, $SU(3) \to \hat{S}U(3)$ required by hadronic mechanics. This will involve an explicit construction of the isounit of hadronic mechanics relating fractionally-charged SU(3) quarks to integrally-charged iso-particles in general, leptons in particular. We shall see that since the isotopic lifting $SU(3) \to \hat{S}U(3)$, from fractionally charged quarks to integrally charged iso-particles is not unique, the real problem is to distinguish between physical and mathematical constituents in a Bohr-like model of structure which would reflect the well-established parallelism between molecular forces derivable from normal (e^-, p^+) system and nuclear forces derived from the compressed system (as indicated in Fig.2).

Moreover, the justification for the introduction of the new $n = (e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$ system lies in its similarity to our earlier non-relativistic theory[4] of Cooper pairing of electrons, $CP = (e^- \downarrow, e^- \uparrow)_{HM}$, around a copper ion (Cu^{z+}) "trigger" in the cuprate superconductors, in the form $CP = (e^- \downarrow - Cu^{z+} - e^- \uparrow)_{HM}$. This approach led in ref.[4] to a prediction of the critical temperatures for the superconducting phase transition in good agreement with experimental data. Subsequently, the basic wave equations of the theory were verified by Animalu and Santilli[5] and found to be consistent with the axioms of hadronic mechanics. However, since a number of adjustable parameters are introduced in the non-relativistic formulation, and in order to eliminate these adjustable parameters (so as to make the theory more predictive) we intend, in this paper, to take the next step of constructing a gauge-invariant non-relativistic theory of the Rutherford-Santilli neutron in Sec. 3 and then generalizing it to a gauge-invariant relativistic theory in Sec. 4.

As the problem of distinguishing between physical and mathematical hadronic constituents within a Bohr-like model of structure lies in finding rigorous analytical tool(s) for non-perturbative treatment of the transition from the static Coulomb (electromagnetic) $e^- - p^+$ binding potential energy, $V_C \equiv eA_0(r) = -e^2 / r$ in the normal H-atom to a strong

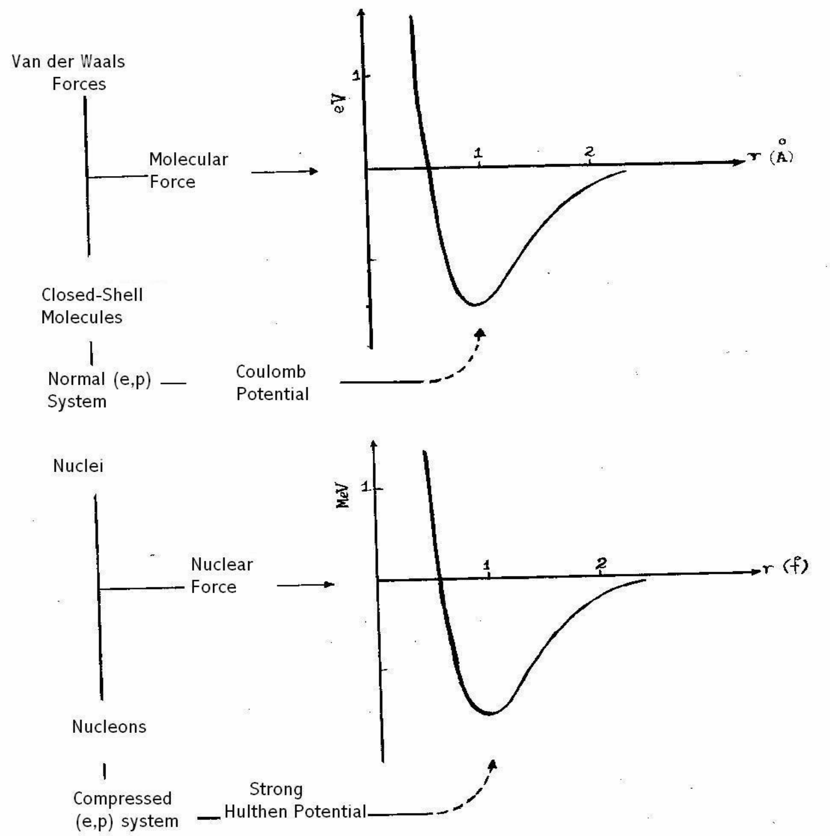

Fig. 2: Parallelism between molecular and nuclear forces appropriately scaled in energy.

binding (Hulthen) potential energy, $V_H \equiv e\hat{A}_0(r) = -Mc_0^2/[e^{(m_0 c_0/\hbar)r} - 1]$ in the compressed H-atom, the objective of the gauge invariance principle in Sec. 3 is to find an expression for the progressive generalization ($A_0 \to \hat{A}_0$) of the usual time-component (A_0) of the electromagnetic potential (A_m).This will be achieved by exploiting, on one hand, the analogy in ref.[1] between the Birkhoffian function $B(\dot{q}, \dot{p})$ of hadronic mechanics and the Gibbs function of classical thermodynamics, and on the other hand, the relativistic generalization of the Ginzburg-Landau equation[6] by Higgs[7] in the framework of Yang-Mills-Higgs gauge theory[7], to express the *effective force,* $(\dot{p} = -\partial V/\partial q)$ *as a functional of the effective potential energy V.* This will lead in Sec. 3 to a *non-linear* first-order differential (Riccati's)equation[8] for the effective (Hulthen) potential energy whose exact solution will prescribe a precise relation between the masses (M, m_0) defining V_H.

Another problem which will arise in the gauge-invariant relativistic theory in Sec. 4 is how, starting from the usual Dirac equation for the relative $e^- - p^+$ motion characterized by a spin $-\frac{1}{2}$ field $\psi(x)$ of mass *M*, to construct an exactly soluble iso-Dirac equation for relative $p^+ - \bar{e}^0$ motion. This will be solved by deriving the required iso-dirac equation from conservation of an iso-current density, $\hat{J}_\mu(x) = \hat{\bar{\psi}}(x)\left(\gamma_\mu - im_\phi^{-1}\overleftrightarrow{\partial}_\mu\right)\hat{\psi}(x)$, that includes a "convective current"[9], generated via a gauge principle that relates the Hulthen potential to a (Higgs) scalar field, of mass $m_\phi = m_0$. The solution of the iso-dirac equation in Sec. 4 for the e^0 mass will be used to predict the masses of the neutron and other members of the SU(3) baryon octet. Finally, in Sec. 5, we shall discuss the results and draw the attendant conclusions.

2. REVIEW OF THE NON-RELATIVISTIC THEORY AND REPRESENTATION OF ISO-QUARKS AS LEPTONS

2.1 Review of the Non-Relativistic theory

Let us now proceed to review the non-relativistic theory of the Rutherford-Santilli neutron presented in ref.[1] in order to systematize notations as well as make this paper self-contained. The non-relativistic theory of the $(e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$ system is based on a pair of non-relativistic wave equations (in Nambu representation) :

$$\hat{H}\hat{\Psi} \equiv HS\hat{T}\hat{\Psi} \equiv \left(\frac{p^2}{2\overline{m}*} - \frac{e^2}{r}\right)\hat{\tau}_3 = \hat{E}\hat{\Psi}, \quad \hat{\Psi} = \begin{bmatrix} \psi_\uparrow \\ \hat{\psi}^*_\downarrow \end{bmatrix}, \quad (2.1)$$

where, $\hat{t}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -S\hat{T} \end{pmatrix}$, and $S \Rightarrow$ scale transformation of space-time coordinates: $(c_0 t, \vec{r}) \to (b_4 c_0 t, b\vec{r})$, such that isorelativistic transformation law holds in the form:

$$c_0^2 b_4^2 dt'^2 - b^2(dx'^2 + dy'^2 + dz'^2) = c_0^2 b_4^2 dt^2 - b^2(dx^2 + dy^2 + dz^2)$$

(b_4, b) being parameters representing the effects of external pressure and temperature, and c_0 the speed of light in vacuum and $\overline{m}^* = \overline{m} b^2$ is an effective reduced mass $(\overline{m}^{-1} = m_{e^-}^{-1} + m_p^{-1})$ of the electron. Explicitly, in Eq.(2.1)

$$\hat{T} = 1 - \left|\hat{y}_\uparrow^*\right\rangle\left\langle\hat{y}_\uparrow\right|, \qquad (\hat{T}^2 = \hat{T}) \tag{2.2}$$

is the isotopic lifting operator of Hadronic Mechanics introduced in ref.[4], where $\left\langle \hat{y}_\uparrow^* \middle| \hat{y}_\uparrow \right\rangle = 1$ but $\left\langle \hat{y}_\uparrow^* \middle| y_\downarrow \right\rangle \equiv Z \neq 0$, so that $\left\langle y_\downarrow^* \middle| \hat{T} \middle| y_\downarrow \right\rangle = 1 - Z$, while $\left\langle \hat{y}_\uparrow^* \middle| \hat{T} \middle| \hat{y}_\uparrow \right\rangle = 0$, and hence the charge on the particle $y_\downarrow$ is depleted by an amount Z while the charge on the particle $\hat{y}_\uparrow$ appears to vanish altogether. For this reason, $y_\downarrow$ was identified in ref.[1] with a down-spin electron $(e^- \downarrow)$, while $\hat{y}_\uparrow$ was identified with an up-spin neutral massive particle $(e^0 \uparrow)$, (called *spinion* in superconductivity theory[3]). Accordingly, (2.1) leads to the pair of s-wave equations

$$H y_\downarrow(r) \equiv \left(-\frac{1}{2\overline{m}} \frac{\hbar^2}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + V_C \right) y_\downarrow(r) = E_\downarrow y_\downarrow(r) \tag{2.3}$$

for the relative $e^- \downarrow - p^+$ motion, and

$$HS\hat{T}\hat{y}_\uparrow^*(r) \equiv \left(-\frac{b^{-2}}{2\overline{m}} \frac{\hbar^2}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + V_H(r) \right) \hat{y}_\uparrow^*(r) = \hat{E}_\uparrow \hat{y}_\uparrow^*(r) \ . \tag{2.4}$$

for the relative $p^+ - \overline{e}^0 \uparrow$ motion. Here $V_C = -e^2 / r$, is the usual $e^- \downarrow - p^+$ Coulomb interaction potential energy in the normal H-atom and $V_H(r)$ is the effective $p^+ - \overline{e}^0 \uparrow$ (Hulthen) interaction potential energy defined as a parameterization of the renormalized Coulomb potential and the non-potential term due to the overlapping of $y_\downarrow$ and $\hat{y}_\uparrow^*$:

$$-\frac{e^2}{br}-|E_{\downarrow}|\left|\langle\hat{y}^*_{\uparrow}|y_{\downarrow}\rangle^*\right|\frac{y_{\downarrow}(r)}{\hat{y}^*_{\uparrow}(r)}\to-\frac{Mc_0^2}{\exp[(m_0c_0/\hbar)r]-1}\equiv V_H(r)\,. \quad (2.5)$$

It was also stated in ref.[1] that the total energy of the structure model, $n=(e^-\downarrow, p^+)$, given in ref.[2]:

$$\begin{aligned} E_T^S &= (m_p+m_{e^-}b^2b_4^2)c_0^2+\hat{E}_{kin\uparrow}-\hat{E}^0_{\uparrow}] \\ &\equiv (m_p+m_{e^-}b_4^2b^2)c_0^2+\hat{B}_{\uparrow} \end{aligned} \quad (2.6)$$

would be equal to the total energy of $n=(e^-\downarrow - p^+ - \bar{e}^0\uparrow)_{HM}$ given in ref.[1]:

$$\begin{aligned} E_T &= [m_{e^-}c_o^2+E_{kin\downarrow}-E^0_{\downarrow}]+[(m_p+m_{e^-}b^2b_4^2)c_0^2+\hat{E}_{kin\uparrow}-\hat{E}^0_{\uparrow}] \\ &\equiv [m_{e^-}c_o^2+B_{\downarrow}]+[(m_p+m_{e^-}b^2b_4^2)c_0^2+\hat{B}_{\uparrow}] \end{aligned} \quad (2.7)$$

where $m_{e^-}=m_{e^-\downarrow}=m_{e^0\uparrow}$, $(m_{e^-\downarrow}, m_{e^0\uparrow}$ being the rest masses of $e^-\downarrow$ and $e^0\uparrow$), under two conditions. The first condition arises from the fortuitous cancellation of the reduced rest-mass of the electron $(\overline{m}\approx m_{e^-})$ by its non-relativistic binding energy $(|E^0_{\downarrow}|=e^2/2R)$ in the configuration shown in Fig. 1(a), at $e^- - p^+$ separation of order

$$R=1fm\approx\hbar/m_pc_0\approx e^2/2m_{e^-}c_0^2\,.$$

The second is either that the kinetic energy $E_{kin\downarrow}$ of the electron $e^-\downarrow$ should *vanish* upon the contact of its classical electric charge sphere with that of the proton at rest, or else that its large value, $E_{kin\downarrow}=(m_p^2/m_{e^-})c_0^2\equiv Mc_0^2$, should be *absorbed* in the Hulthen potential in Eq.(2.5). In this paper, we choose to absorb $E_{kin\downarrow}$ in the Hulthen potential, so that Eq.(2.4) may now be rewritten as a non-relativistic wave equation for motion of the neutral massive spin-$\frac{1}{2}$ particle $\bar{e}^0$ relative to p^+ in the more explicit form:

$$\begin{aligned} &\left(-\frac{b^{-2}}{2m_e}\frac{\hbar^2}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+V_H(r)\right)\hat{y}^*_{\uparrow}(r)=\hat{E}_{\uparrow}\hat{y}^*_{\uparrow}(r)\,, \\ &V_H(r)=-E_{kin\downarrow}/\exp[(m_0c_0/\hbar)r]-1], \\ &E_{kin\downarrow}\equiv(m_p^2/m_e)\,c_0^2=Mc_0^2 \end{aligned} \quad (2.8)$$

Finally, with this value of M, the model meets the requirement of having a minimum size of the Hulthen potential hole for a bound state to be formed. Moreover, the rest mass of the proton is given quite accurately by the equation:

$$m_p = (m_p^2 / m_e) e^{-1/3a_0}, \quad (a_0 = e^2 / \hbar c_0 = 1/137), \qquad (2.9)$$

which is analogous to the expression derived in ref.[4] for the critical temperature of the superconducting phase transition based on a similar structure model for Cooper pairing, $CP = (e^- \downarrow - Cu^{z+} - e^- \uparrow)_{HM}$, in the cuprate superconductors.

However, even with the equality of the total energies of $n = (e^- \downarrow, p^+)$ and $n = (e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$, the dependence of both models on a rather large number of empirical parameters (M, m_0, b, b_4, B) makes it is necessary to review how the signatures of e^- and e^0 could be derived from those of the SU(3) quarks in any characterization of the transition from the normal to the compressed system based on hadronic mechanics..

2.2 Representation of Iso-Quarks as Leptons

A generalization of the unitary symmetry group, SU(3), to the iso-unitary symmetry group, $\hat{S}U(3)$, is required by hadronic mechanics. This requirement will now be reviewed for the purpose of selecting from several possibilities an isotopic element for the lifting $SU(3) \to \hat{S}U(3)$ incorporating the projection operator property $(\hat{T}^2 = \hat{T})$ of the *non-local* isotopic lifting operator defined in Eq.(2.2). We begin by recalling that the conventional Gell-Mann-Zweig[10] quarks (*u,d,s*) are defined by the conventional Lie-algebraic structure of the SU(3) generators,

$$l_i l_j - l_j l_i = 2i f_{ijk} l_k, \ (i, j, k = 0,1,...,8) \qquad (2.10a)$$

with fractional baryon number (B_q) and electric charges (e Q_q),

$$B_q = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad Q_q \equiv \frac{1}{2}(l_3 + l_8/\sqrt{3}) = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \qquad (2.10b)$$

A question arises, therefore, as to whether one can select an isotopic element (G), so that the isotopic lifting $SU(3) \to \hat{S}U(3)$ defined by a Lie-isotopic generalization of Eq.(2.10a) in the following way,

$$\hat{l}_i G \hat{l}_j - \hat{l}_j G \hat{l}_i = 2i\hat{f}_{ijk} G \hat{l}_k, \quad (\hat{l}_k = G^+ l_k G) \qquad (2.11a)$$

will characterize iso-quarks corresponding to the physical lepton triplet (n, e^-, m^-) with integral fermion number B_l and electric charges (eQ_l), i.e.,

$$B_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_l \equiv \frac{1}{2}(\hat{l}_3 + \hat{l}_8/\sqrt{3}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \qquad (2.11b)$$

An affirmative answer to this question follows if we use our experience with the procedure for isotopic lifting of the SU(2) algebra of Cooper pair creation and annihilation operators in superconductors (see, ref.[11]) to select an isotopic element (G) incorporating the projection operator $\hat{T}$ in the following way:

$$G = \begin{pmatrix} \sqrt{\frac{3}{2}}\left(1-\hat{T}\right) & 0 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & -\sqrt{3} & 0 \end{pmatrix}, G^+ = \begin{pmatrix} \sqrt{\frac{3}{2}}\left(1-\hat{T}\right) & 0 & 0 \\ 0 & 0 & -\sqrt{3} \\ 0 & \sqrt{3} & 0 \end{pmatrix}. \qquad (2.12)$$

We verify directly that this leads to the desired result,

$$\hat{Q}_q \equiv \frac{1}{2}(\hat{l}_3 + \hat{l}_8/\sqrt{3}) = G^+\left(\frac{1}{2}(l_3 + l_8/\sqrt{3})\right)G$$

$$\equiv \begin{pmatrix} \sqrt{\frac{3}{2}}\left(1-\hat{T}\right) & 0 & 0 \\ 0 & 0 & -\sqrt{3} \\ 0 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{3}{2}}\left(1-\hat{T}\right) & 0 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & -\sqrt{3} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \left(1-\hat{T}\right)^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \left(1-\hat{T}\right) & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \equiv Q_l, \qquad (2.13)$$

provided that we select the eigenvalue ($\hat{T} = 1$) of the projection operator $\hat{T}$, corresponding to the normal neutrino state. All other iso-quark states characterize mutations of the neutrino. In particular, when $\hat{T} = 0$, we obtain the integral iso-quark charges derived by Santilli [12], based on a different selection of a (symmetric matrix) isotopic element

$g = diag(g_{11}, g_{22}, g_{33})$ such that $\hat{I}_k$ for $k = 3, 8$, and hence the electric charge $\hat{Q}$, have the forms:

$$\hat{I}_3 = \begin{pmatrix} g_{11}g_{22}^2 & 0 & 0 \\ 0 & -g_{22}g_{11}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{I}_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} g_{11}g_{22}^2 & 0 & 0 \\ 0 & g_{22}g_{11}^2 & 0 \\ 0 & 0 & \dfrac{-2g_{11}^2 g_{22}^2}{g_{33}^2} \end{pmatrix}$$

$$\hat{Q} = \frac{1}{2}(\hat{I}_3 + \frac{1}{\sqrt{3}}\hat{I}_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad (2.14)$$

when $g_{11} = \sqrt[3]{6},\ g_{22} = \dfrac{3}{\sqrt[3]{12}},\ g_{33} = \dfrac{3}{\sqrt[3]{4}}$. This spectrum of integral charges are comparable with those of the Han-Nambu quarks[13].

At this juncture, it is pertinent to recall that Barut's construction of hadron multiplets[14] was based on rewriting the equality of the quark and lepton (fermion number and electric charge) projection operators earlier observed by Animalu[15]

$$B_l + Q_l = B_q + Q_q. \qquad (2.15)$$

in the form

$$(B_l - \tfrac{2}{3}B_l) + (Q_l + \tfrac{2}{3}B_l) = B_q + Q_q$$

to find that

$$B_l - \tfrac{2}{3}B_l = \tfrac{1}{3}B_l = B_q \quad \text{and} \quad Q_l + \tfrac{2}{3}B_l = Q_q. \qquad (2.16)$$

From this, Barut concluded that *quarks are obtained from leptons by shifting 2/3 of the fermionic charge to the electric charge*. It is now apparent that such a shifting is indeed achieved through the isotopic lifting transformation of hadronic mechanics described above, and leads to identification of iso-quarks with the leptons. However, whereas the quarks obey the Gell-Mann-Nishijima relation $Q_q = I_3 + \frac{1}{2}Y_q$ with

$$I_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y_q = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}, \qquad (2.17)$$

the iso-quarks, i.e. the leptons, must obey the appropriate relation required by isotopic lifting. In particular, contrary to Barut's proposal to keep the same I_3 assignments for leptons, we find

$$\hat{I}_3 = G^+ I_3 G = \begin{pmatrix} \frac{3}{4}\left(1-\hat{T}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}, \tag{2.18}$$

which indicates a mutation of the usual isospin assignment to the lepton triplet; and in like manner,

$$\hat{Y}_q = G^+ Y_q G = \begin{pmatrix} \frac{1}{2}(1-\hat{T}) & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & +1 \end{pmatrix} \tag{2.19}$$

is different from Barut's assignment, but as expected, we recover

$$\hat{Q}_q \equiv \hat{I}_3 + \tfrac{1}{2}\hat{Y}_q = Q_l\,. \tag{2.20}$$

Consequently, while our neutron structure model $n = (e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$ is compatible with iso-quark theory, it is fundamentally different from (but analogous to) Barut's model of the neutron $(n \sim pe^-\bar{n})$ as far as identification of $\bar{e}^0$ with the anti-neutrino $\bar{n}$ is concerned. In this sense, we are inclined to agree with Santilli[private communication] that he "does not accept the existence of neutrinos because he has shown that, when hadrons are represented as extended objects, all physical laws, including that on the conservation of energy and angular momentum can be verified without any need for conjecturing the existence of neutrinos", (see, http://www.neutronstructure.org). It is necessary, therefore, to probe more deeply into the nature of the new particle e^0 by reconstructing the non-relativistic theory in a gauge-invariant framework.

3. GAUGE-INVARIANT RECONSTRUCTION OF THE NON-RELATIVISTIC THEORY

We have stated in Sec. 1 that the real problem of distinguishing between physical and mathematical hadronic constituents within a Bohr-like model of structure lies in finding rigorous analytical tool(s) for non-perturbative treatment of the transition from electromagnetic (Coulomb) binding forces in normal atomic systems to strong (hadronic) binding forces in compressed atomic systems. To this end, we proceed, in this section, to reconstruct the non-relativistic theory of the $n=(e^{-}\downarrow - p^{+} - \bar{e}^{0}\uparrow)_{HM}$ system by highlighting its gauge invariance in such a way that the progressive generalization of the Coulomb potential to the Hulthen potential can be described by a class of *nonlinear* first-order differential (Riccati's) equations[8] whose exact solutions will yield analytical expressions of general interest and enable us to relate some of the unknown parameters in Eq.(2.7) to each other.

We begin by rewriting the conventional Schrodinger equation for the normal H-atom in the form:

$$\left(-\frac{1}{2\overline{m}}\frac{\hbar^2}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+(e/c_o)A_0\right)y_{\downarrow}(r)=E_{\downarrow}y_{\downarrow}(r)\,. \qquad (3.1a)$$

This makes evident the fact that the static Coulomb $(e^{-}-p^{+})$ interaction is determined by the time-component (A_0) of the electromagnetic four-vector potential $(A_m, m=0,1,2,3)$, $V_C(r)\equiv(e/c_o)A_0(r)=-e^2/r$. Consequently, if we express the interparticle Coulomb force $(-dV_C/dr=-e^2/r^2)$ as a functional of the potential energy V_C, by eliminating explicit r-dependence between dV_C/dr and V_C, the result would be a non-linear first-order differential equation:

$$\frac{dV_C}{dr}=\frac{V_C^2}{e^2}, \quad \text{or} \quad \frac{\partial A_0}{\partial r}=(1/e)A_0^2\,. \qquad (3.1b)$$

This is a special case of Riccati's equation[8] which suggests that an obvious step to take in order to achieve a progressive generalization of the Coulomb potential to the Hulthen potential in the isoschrodinger equation for the compressed system,

$$\left(-\frac{b^{-2}}{2\overline{m}}\frac{\hbar^2}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+V_H(r)\right)\hat{y}^{*}_{\uparrow}(r)=E_{\uparrow}\hat{y}^{*}_{\uparrow}(r) \qquad (3.2a)$$

is to replace Eq.(3.1b) by the most general form of the Riccati's equation

$$\frac{dV_H}{dr}=PV_H^2+QV_H+R, \text{ or } \frac{\partial\hat{A}_0}{\partial r}=(1/e)\hat{A}_0^2+z\hat{A}_0-k \qquad (3.2b)$$

K, P, R and z ,k being constants (in general, functions of r). Note that the derivative and nonlinear parts, $\{\partial A_0/\partial r-(1/e)A_0^2\}$, of Eq.(3.2b) may be re-interpreted as appropriate component of the SU(2) Yang-Mills gauge field[7],

$$F_{mn}^{a} \equiv \frac{\partial A_{n}^{a}}{\partial x^{m}} - \frac{\partial A_{m}^{a}}{\partial x^{n}} + g_{0} e^{abc} A_{m}^{b} A_{n}^{c} \qquad (3.3)$$

where g_0 is a coupling constant. Note also that the (matrix) Riccati's equation has an underlying Lie-admissible algebraic structure[16] which makes (3.2b) an explicit unequivocal realization of the iso-mathematical structure of hadronic mechanics. Eqs.(3.2a) and (3.2b) define, therefore, the generalization we are after.

An alternative way of looking at the generalization lies in the analogy between, on one hand, the Birkhoffian function $B(\dot{p},\dot{q})$ of hadronic mechanics and Gibbs free energy of classical thermodynamics discussed in ref.[1], and on the other hand, Higgs' generalization of the Ginzburg-Landau equation for the order parameter in superconductvity theory[6] in the framework of Yang-Mills-Higgs gauge theory[7]. We recall, as discussed in ref.[1], that an extension of the analogy between classical thermodynamics and classical mechanics leads to a correspondence between the Gibbs free energy of classical thermodynamics (as well as its thermodynamic differential relations):

$$\begin{gathered} G(P,T) = U + PV - TS,\, dG = VdP - SdT; \\ V = \partial G/\partial P,\, S = -\partial G/\partial T \end{gathered} \qquad (3.4)$$

and the Birkhoffian function of Santilli's hadronic mechanics (as well as the associated canonical equations of motion):

$$\begin{gathered} B(\dot{p},\dot{q}) = H - \dot{q}p + q\dot{p}, \quad dB = qd\dot{p} - pd\dot{q}; \\ q = \partial B/\partial\dot{p},\, p = -\partial B/\partial\dot{q}. \end{gathered} \qquad (3.5)$$

Therefore, besides the addition of $(-\dot{q}p + q\dot{p})$ as external term to the Hamiltonian, $H(q,p)$, to get the Birkhoffian, the Birkhoffian function, $B(\dot{p},\dot{q})$, differs further from $H(q,p)$ in the fact that it is a function of the velocity $(\dot{q})$ and of the rate of change of momentum $(\dot{p})$ equivalent to the force $(-dV/dq)$. Consequently, for the purpose of prescribing the nature of the second-order phase transition from the normal H-atom state to its compressed (superfluid) state via a progressive generalization of the Coulomb interparticle force represented by $\dot{p} = -dV/dr$, Eq.(3.2b) is a statement of a general hypothesis (cf the Ginzburg-Landau expansion of the free energy as a functional of the order parameter[6]) that *the effective* *interparticle force is a functional of the potential energy*. Consequently, a further generalization of Eq.(3.2b) to include higher order terms is conceivable: for example, we could, as in Higgs' generalization of the Ginzburg-Landau theory in the framework of Yang-Mills-Higgs gauge theories of spontaneous symmetry breaking expand the force to quartic order :

$$\frac{dV}{dr} = KV^4 + PV^2 + R \equiv K(|V|^2 - V_T^2)^2 . \qquad (3.6)$$

This would make the attainment of asymptotic freedom ($dV/dr = 0$) manifest itself in the same way as spontaneous symmetry breaking, as the sketches of dV/dr versus $|V|$ in Fig. 3 suggest.

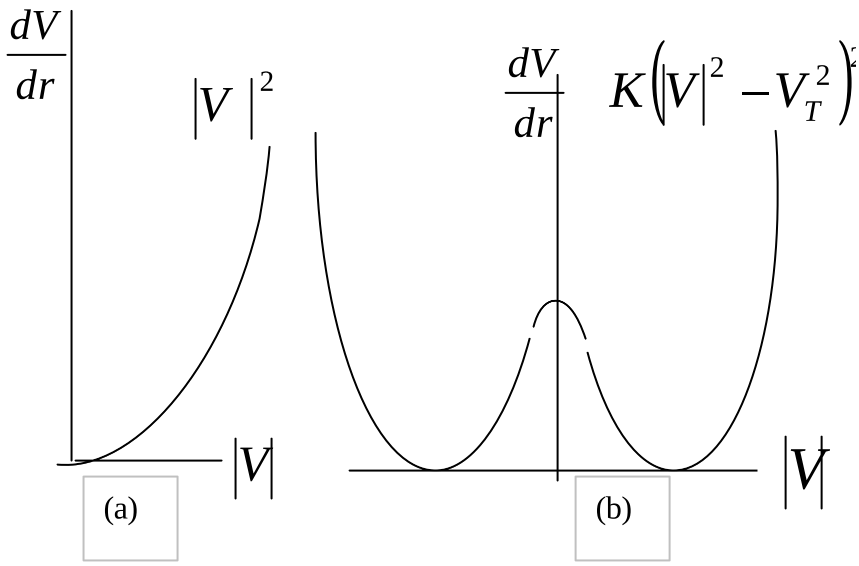

Fig. 3: Graphs of dV/dr versus V for (a) Coulomb force in Eq.(3.1b) and (b) the generalization in Eq.(3.6).

Let us now return to analytically examine the physical content of the generalization from (3.1b) to (3.2b), by observing that when K, P, R are constants (independent of r), the Riccati equation (3.2b) can be integrated *exactly* (as shown in Appendix A) to find a general solution of the form:

$$V_H(r) = \frac{\left(\frac{Q-\sqrt{Q^2-4PR}}{2P}\right)\times\frac{\left(V_0+\frac{Q+\sqrt{Q^2-4PR}}{2P}\right)}{\left(V_0+\frac{Q-\sqrt{Q^2-4PR}}{2P}\right)}}{\exp\left(\frac{r\sqrt{Q^2-4PR}}{P}\right)-\frac{\left(V_0+\frac{Q+\sqrt{Q^2-4PR}}{2P}\right)}{\left(V_0+\frac{-Q+\sqrt{Q^2-4PR}}{2P}\right)}} \quad (3.7)$$

where $V_0 \equiv V_H(0)$ is a constant of integration. This is equivalent to the standard form of the Hulthen potential,

$$V_H(r) = \frac{-Mc_0^2}{\exp[(m_0 c_0/\hbar)r]-1} \quad (3.8)$$

provided that

$$\left(V_0+\frac{Q+\sqrt{Q^2-4PR}}{2P}\right)\Bigg/\left(V_0+\frac{-Q+\sqrt{Q^2-4PR}}{2P}\right)=1,$$

giving

$$P\neq 0,\ Q=0,\ R<0; \quad (3.9)$$

and hence, since $V_0 \equiv V_H(0)\to\infty$, (in units such that $\hbar = c_0 = 1$)

$$m_0 = 2\sqrt{-\frac{R}{P}},\ M=\sqrt{\frac{-R}{P}}\left(\frac{V_0+\sqrt{\frac{-R}{P}}}{V_0-\sqrt{\frac{-R}{P}}}\right)\to\sqrt{\frac{-R}{P}}\equiv\tfrac{1}{2}m_0 \quad (3.10)$$

The above formal derivation of the Hulthen potential is, by virtue of the fact that it is able to predict the important ratio, $M/m_0=\frac{1}{2}$, evidently more satisfactory than the *ad hoc* prescription of the Hulthen potential via the parameterization in Eq.(2.5) even though the two approaches complement each other. The significance of this ratio will emerge in the relativistic theory to which we now turn.

4. CONSTRUCTION AND VERIFICATION OF A GAUGE INVARIANT RELATIVISTIC THEORY

4.1 Construction

Technically, as shown in 1996 by Santilli[17] for the $n = (e^- \downarrow - p^+ \uparrow)_{HM}$ system, the construction of an iso-dirac equation for a compressed H-atom would require the introduction of the isotopies of the Minkowski spacetime, of the Poincare symmetry and of special relativity, with the consequential additional empirical parameters (for an up-to-date summary, see http://www.neutronstructure.org/part6.htm).

However, because of the reformulation of the Hulthen potential for the non-relativistic theory in the form of a first-order nonlinear differential (Riccati's) equation, some novel physical insight would be gained in this section by constructing an *exactly soluble* gauge-invariant relativistic wave equation for the $n = (e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$ system involving only the two parameters (M, m_0, with $M / m_0 = \frac{1}{2}$) in the Hulthen potential. This can be done rather simply by using a Lagrangian formulation in which the electromagnetic interaction is introduced via an isotopic lifting transformation of the conventional gauge-invariant interaction term, $J^\mu(x) A_\mu(x) \;\rightarrow\; \hat{J}^\mu(x) \hat{A}_\mu(x)$, relating the relative $e^- - p^+$ motion in an external electromagnetic field $A_\mu(x)$ to the relative ($\bar{e}^0 - p^+$) motion in an external effective field, $\hat{A}_\mu(x)$. The required relativistic and iso-relativistic wave equations would then be those that conserve the respective current and iso-current densities,

$$J^\mu(x) = \bar{\psi}(x) \gamma_\mu \psi(x) \text{ and } \hat{J}^\mu(x) = \hat{\bar{\psi}}(x) \left(\gamma_\mu - i m_f^{-1} \overleftrightarrow{\partial}_\mu \right) \hat{\psi}(x),$$

which we now proceed to construct.

We start with the conventional relativistic (Dirac) equation for the relative $e^- - p^+$ motion in an external electromagnetic field:

$$\left[i\hbar g^{m}\left(\partial_{m} - \frac{ie}{\hbar c_0} A_m \right) + \overline{m} c_0^2 \right] y = 0 \qquad (4.1a)$$

where the time-component (A_0) of the electromagnetic four-vector potential A_m representing the static ($e^- - p^+$) Coulomb interaction satisfies, as before, the simplest form of Riccati's equation:

$$\frac{\partial A_0}{\partial r} - (1/e)A_0^2 = 0\,. \qquad (4.1b)$$

For the relative $\overline{e}^0 - p^+$ motion, we naturally presume that the time-component ($\hat{A}_0$) of the effective (Hulthen) potential ($\hat{A}_m$) is determined, as before, by the complete form of the Riccati equation:

$$\frac{\partial \hat{A}_0}{\partial r} = (1/e)\hat{A}_0^2 + z\hat{A}_0 - k \qquad (4.2)$$

$z\,, k$ being constants (in general, functions of r). However, we now convert the Hulthen potential $\hat{A}_0$ into a (Higgs) scalar field, $f \equiv f(r)$, by using the well-known standard transformation that converts the *nonlinear* first-order Riccati Eq.(4.2b) into a *linear* second-order differential equation for f (see, p.201 of Piaggio[8]) :

$$\hat{A}_0 = -e\frac{d\log(f)}{dr} \equiv -e\frac{\left(\frac{df}{dr}\right)}{f} \equiv \text{-e}\frac{f_1}{f}, \qquad (4.3a)$$

where $f_1 \equiv \dfrac{df}{dr}$. Note that this transformation may be rewritten as a (Weyl-like) gauge principle in the integral form:

$$f = \exp\left(-(1/e)\int_0^r \hat{A}_0 dr \right). \qquad (4.3b)$$

From (4.3a) we find

$$\frac{d\hat{A}_0}{dr} \equiv -e\frac{f_2}{f} + e\frac{f_1^2}{f^2}\,, \qquad (4.4)$$

so that, on substitution in Eq.(4.2), the terms in f_1^2 disappear, and hence, on multiplying the resulting equation through by f/e, we obtain a *linear* second-order differential equation:

$$f_2 - z f_1 - (k/e) f = 0\,. \qquad (4.5)$$

In addition, if we select $z \equiv -2/r$ and put $m_f^2 = k/e$, this equation takes the standard form

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{k}{e} f \equiv \left(\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} - m_f^2 \right) f(r) = 0 \qquad (4.6)$$

which is the static limit of the Klein-Gordon equation for a spin-0 scalar field f of mass m_f in units such that $\hbar = c_0 = 1$.

As is well-known, a solution of Eq.(4.6) has the form of a Yukawa potential:

$$f(r) = (-g^2 / r)\exp(-m_f r) \qquad (4.7)$$

which characterizes a strong (nuclear) force. By comparing this potential with an expansion of the Hulthen potential defined by Eq.(3.8) in the following way

$$V_H(r) = \frac{-Me^{-m_0 r}}{1-e^{-m_0 r}} \approx \frac{-Me^{-m_0 r}}{m_0 r + O(r^2)} \approx \frac{-(M/m_0)e^{-m_0 r}}{r} \equiv f(r) \qquad (4.8)$$

we find $m_f = m_0$ and $g^2 \equiv M/m_f = \frac{1}{2}$ which is in the range of strong nuclear coupling constants.

Our interest now lies in the fact that, having transformed Riccati's equation for $\hat{A}_0$ into Klein-Gordon equation for Higgs field $f(x)$, we can use standard (Lagrangian field theoretic) techniques to put together the spin-0 field, $f(x)$, the spin-$\frac{1}{2}$ e^0 field, $\hat{y}(x)$, and the remaining components $(\hat{A}_k, k = 1,2,3)$ and other bosons making up the Yang-Mills-fields, $\hat{A}_m^a(x)$, to unify electromagnetic, weak and strong interactions in a gauge-invariant model. Then in the zero gauge-coupling limit $(A_m^a \equiv 0)$ of interest for the compressed H-atom problem, we have the following conserved current and field equations ($\hbar = c_0 = 1$):

$$\hat{J}_m(x) = \hat{\bar{y}}(x) g_m \hat{y}(x) - i f^*(x) \overleftrightarrow{\partial}_m f(x) \qquad (4.9a)$$

$$\left(i g_m \partial_m - M\right)\hat{y}(x) = \Gamma \hat{y}(x) f(x) \qquad (4.9b)$$

$$\left(\partial^2 + m_f^2\right) f(x) = -\hat{\bar{y}}(x) \Gamma \hat{y}(x) f(x) \qquad (4.9c)$$

where M is the mass of the spinor field $\hat{y}(x)$ and m_f the mass of the Higgs scalar field $f(x)$. Since the wavefunction $\hat{y}(x)$ of e^0 has spin

$(\uparrow, \downarrow)$ as dichotomic variable, considerable simplification can be achieved by using spin labels, $\hat{y}_{\uparrow}(x)$ and $\hat{y}_{\downarrow}(x)$, to approximate the spin-0 Higgs field through a relation (involving m_f for dimensional reasons):

$$f = m_f^{-\frac{1}{2}} y_{\uparrow} \text{ or } m_f^{-\frac{1}{2}} y_{\downarrow}. \qquad (4.10)$$

This enables us to abstract from Eq.(4.9a), the iso-current density

$$\hat{J}_m(x) = \bar{\hat{y}}(x)\left(g_m - i m_f^{-1} \vec{\partial}_m\right)\hat{y}(x) \qquad (4.11a)$$

which is conserved by the following relativistic wave equation:

$$\left(i g_m \partial^m - (m_f + M) - (1/m_f)\partial_m \partial^m\right)\hat{y} = 0 . \qquad (4.11b)$$

This is the final gauge-invariant iso-relativistic wave equation for e^0 that we are after. Note that it contains only the two parameters (M, m_f) defining V_H.

4.2 Prediction of the Neutron Mass

We observe here that the contribution to the iso-current density in Eq.(4.11a) from the Higgs field, called *convective current*, is responsible for the anomalous magnetic moment of the electron in the "heavy electron" theory developed by Barut, Cordero and Ghirardi[9]. Moreover, because the convective current gives rise to the second-order derivative (non-renormalizable) structure, Eq.(4.11b) leads to two mass values:

$$m_j = \left[(p_j^0)^2 - (\vec{p}_j)^2\right]^{\frac{1}{2}}, \; j = 1, \; 2 \qquad (4.12a)$$

given by the two roots of the quadratic mass equation:

$$m_f^{-1} m^2 + m - (m_f + M) = 0, \qquad (4.12b)$$

$m^2 + m_f m - m_f(m_f + M) = 0$. Hence (since $m_f / 2 = M$), the two masses may be rewritten (with $m_1 \equiv m_+, m_2 \equiv m_-$) as

$$m_{\pm} = M(-1 \pm \sqrt{7}), \text{ or } \frac{m_+}{M} = 1.65, \; \frac{m_-}{M} = -3.65 \qquad (4.12c)$$

This has the significance that since $M \equiv m_p^2 / m_{e^-} = K m_{e^-}$ (because $m_p = 2 m_{e^-} / a_0$), we may similarly express $m_{\pm}$ using the same scale in the forms: $m_+ = K \hat{m}_{e^0}^+$ and $m_- = K \hat{m}_{e^0}^-$ so that in place of (4.12c) we have

$$\hat{m}_{e^0}^+ / m_{e^-} = 1.65, \; \hat{m}_{e^0}^- / m_{e^-} = -3.65 \qquad (4.13)$$

Thus, for the $n = (e^- \downarrow - p^+ - \bar{e}^0 \uparrow)_{HM}$ system, we *predict* the mass

$$m_n = m_p + m_{e^-} + \hat{m}^+_{e^0} + \hat{B} \equiv m_p + m_{e^-} + 1.65 m_{e^-} + \hat{B}$$
$$\equiv m_p + m_{e^-} + 1.53 m_{e^-}\ ,\ (\text{with } \hat{B} = -0.12 m_{e^-}). \qquad (4.14)$$

which is consistent with the experimental value of the neutron rest mass. The small empirically determined binding energy ($\hat{B} = -0.12 m_{e^-}$) is also consistent with Eq.(4.11b) being indicative of a system that is "asymptotically free".

The beauty of having a theory with a minimum of free parameters lies, however, in its predictive power which, in this case, arises from the physical significance of the negative mass ratio in Eq.(4.12c) to which we now turn.

4.3 Prediction of the Masses of the SU(3) Baryon octet

Following our earlier convective current approach to the determination of the mass ratio of quarks in composite systems (Animalu[9]), let us replace the mass parameters, (M, m_f) in Eq.(4.11b) by diagonal SU(3)matrices as follows:

$$M = \left(\sqrt{\frac{3}{2}}\, l_0\right) m_d \equiv \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_d \end{pmatrix},$$
$$m_f = \left(-\sqrt{3}\, l_8\right) m_d \equiv \begin{pmatrix} -m_d & 0 & 0 \\ 0 & -m_d & 0 \\ 0 & 0 & 2m_d \end{pmatrix}, \qquad (4.15a)$$

so that

$$\frac{1}{m_f} \equiv \left(\frac{-1}{m_d\sqrt{3}}\right) l_8^{-1} \equiv \left(\frac{-1}{2m_d}\right)\left(\sqrt{\frac{3}{2}}\right)\left(l_0 + \sqrt{2}\, l_8\right)$$
$$\equiv \begin{pmatrix} -1/m_d & 0 & 0 \\ 0 & -1/m_d & 0 \\ 0 & 0 & 1/2m_d \end{pmatrix}. \qquad (4.15b)$$

This makes Eq.(4.11b) to simulate the Gell-Mann, Oakes and Reener[18] model of chiral SU(3)xSU(3) gauge symmetry breaking down to the SU(3) of strong interactions without mixing isotopic spin doublets, so that it reduces to two distinct equations:

$$\left(ig_m\partial^m + m_d^{-1}\partial_m\partial^m\right)\hat{y} = 0 \qquad (4.16a)$$

with two mass eigenvalues, $m = m_d$ or 0; and

$$\left(ig_m\partial^m - 3m_d - (2m_d)^{-1}\partial_m\partial^m\right)\hat{y} = 0 \qquad (4.16b)$$

with two mass eigenvalues determined by the roots of the equation

$$(2m_d)^{-1}m^2 + m - 3m_d = 0,$$

which is the same as Eq.(4.12b) giving

$$m_\pm = m_d(\pm\sqrt{7} - 1), \quad \text{i.e.,} \quad m_+ = 1.65m_d, \quad m_- = -3.65m_d, \text{ or}$$

$$\frac{m_+}{m_d} = 1.65 \ , \ \frac{|m_-|}{m_d} = 3.65 \qquad (4.17)$$

This means that in the *additive* constituent quark model of the spin $\frac{1}{2}$ baryon octet (to which the neutron belongs) (with $m_u = m_d$), we determine from the observed masses of the neutron $m_n = 3m_d = 939.6$, and the other members of the octet,

$$m_{\Lambda'} = 2m_d + m_s = 1142 \approx \tfrac{1}{2}(m_\Lambda + m_{\Sigma^0}) = \tfrac{1}{2}(1116 + 1192)$$

the two mass ratios:

$$\frac{m_s}{m_d} = 1.65, \ \frac{m_{\Lambda'}}{m_d} = 3.65, \qquad (4.18)$$

which are in agreement with those in Eq.(4.17). In other words, the predicted mass ratios of the members of the SU(3) baryon octet are in agreement with experiment, in accordance with the equal-spacing rule, when splitting due to electromagnetic interactions are ignored. It is also noteworthy that the second mass ratio in Eq.(4.18) is related to the mass of the strange particle, Λ, as an excited state of the neutron described by Eq.(4.16b) which can be rewritten in terms of the neutron mass ($3m_d \equiv m_n$) in the form:

$$\left(ig_m\partial^m - m_n - (\tfrac{2}{3}m_n)^{-1}\partial_m\partial^m\right)\hat{y}_n = 0 \ . \qquad (4.19)$$

The last term gives rise to the magnetic transitions discussed by Barut *et al*[9]; and Eq.(4.19) forces on us the conclusion that the neutral spin-$\frac{1}{2}$

particle, e^0, with wavefunction, $\hat{y}_n$, constructed in the gauge-invariant relativistic theory has the same signature as the neutron itself ($e^0 \sim n$).

5. DISCUSSION AND CONCLUSION

In this paper, we have reformulated the non-relativistic theory of the Rutherford-Santilli neutron presented in ref.[1] as a gauge-invariant relativistic theory, establishing in the process, a fruitful linkage between the parameters (M, m_f) of the Hulthen potential and unified gauge theories in a Bohr-like model of hadronic structure. Our central objective at each stage leading to the gauge-invariant relativistic theory has been to construct an *exactly soluble* (non-perturbative) physical model of hadronic structure that would provide a rigorous framework involving a minimum number of empirical parameters for confronting theoretical results with experimental data. The achievement of the above objective was made possible by the *representation of the effective interaparticle force in the compressed H-atom as a functional of the potential energy,* so that rather than introduce an *ad hoc* potential energy for the compressed system, the potential energy is determined by the solutions of a *nonlinear first order (Riccati's) differential equation*, which can also be transformed, *albeit* non-unitarily, into a *linear second-order differential equation*. This is a natural route from hadronic mechanics to the so-called "nonlinear conservation laws" (see, Strang[19]) leading to solitons, instantons, etc. Moreover, as is well-known [p. 49 of ref.[20]), Schwarzschild's celebrated "solution", $g_{rr}(r) = [1 - 2m(r)/r]^{-1}$, $m(r) \equiv k\int_0^r 4pr^2 r(r)dr$, of Einstein's equation for the gravitational field is also governed by Riccati's equation,

$$\frac{dg_{rr}}{dr} - \frac{1}{r}g_{rr} - (8pkrr - \frac{1}{r})g_{rr}^2 = 0 \qquad (5.1)$$

where $ds^2 = g_{00}(r)dr^2 + g_{rr}(r)dr^2 + r^2(dq^2 + \sin^2 q dj^2)$, and $16pkr$ is the scalar curvature, r being the (curvature) energy density.

We conclude and recommend, therefore, that the study of the Rutherford-Santilli neutron should be intensified from both theoretical and experimental points of view, in order to reap the potential benefits of having an exactly soluble model of the internal structure of the strongly interacting particles (hadrons) for testing various contending models and for verification of "hadronic mechanics" laws of physics.

Acknowledgement

I wish to thank Professor R.M. Santilli for stimulating ideas and suggestions, and for sustaining my interest in the problem solved in this paper for close to two decades. I also thanks Dr. C. N. Animalu for finding the form of the exact solution of the Riccati equation in Apendix A.

APPENDIX A

SOLUTION OF THE RICCATI EQUATION

In this appendix we wish to find (in a closed form) the solution of the Riccati's equation:

$$\frac{dV_H}{dr} = PV_H^2 + QV_H + R,$$

(P, Q, R) being constants. The equation may be expressed and integrated in the form:

$$\int_0^r dr = \int_{V_0}^{V_H} \frac{dV_H}{(V_H - A)(V_H - B)},$$

where

$$A = \frac{-Q + \sqrt{Q^2 - 4PR}}{2P}; B = \frac{-Q - \sqrt{Q^2 - 4PR}}{2P}$$

Thus, by using the partial fraction expansion

$$\frac{1}{(V_H - A)(V_H - B)} \equiv \frac{A - B}{(V_H - A)} + \frac{B - A}{(V_H - B)}$$

we find

$$r=\int_{V_0}^{V_H}\frac{(A-B)dV_H}{(V_H-A)}+\int_{V_0}^{V_H}\frac{(B-A)dV_H}{(V_H-B)}$$

$$=(A-B)\ln\left[\frac{(V_H(r)-A)}{(V_0-A)}\right]+(B-A)\ln\left[\frac{(V_H(r)-B)}{(V_0-B)}\right]$$

$$=(A-B)\ln\left[\frac{(V_H(r)-A)}{(V_H(r)-B)}\frac{(V_0-B)}{(V_0-A)}\right]$$

i.e., $\exp\left[\frac{r}{A-B}\right]=\frac{(V_H(r)-A)}{(V_H(r)-B)}\frac{(V_0-B)}{(V_0-A)}$,

$$(V_H(r)-B)\exp\left[\frac{r}{A-B}\right]=(V_H(r)-A)\frac{(V_0-B)}{(V_0-A)},$$

$$V_H(r)\left(\exp\left[\frac{r}{A-B}\right]+\frac{(-V_0+B)}{(V_0-A)}\right)=-B\frac{(V_0-B)}{(V_0-A)},$$

$$\therefore\quad V_H(r)=\frac{\left(\frac{Q-\sqrt{Q^2-4PR}}{2P}\right)\times\frac{\left(V_0+\frac{Q+\sqrt{Q^2-4PR}}{2P}\right)}{\left(V_0+\frac{Q-\sqrt{Q^2-4PR}}{2P}\right)}}{\exp\left(\frac{r\sqrt{Q^2-4PR}}{P}\right)-\frac{\left(V_0+\frac{Q+\sqrt{Q^2-4PR}}{2P}\right)}{\left(V_0+\frac{-Q+\sqrt{Q^2-4PR}}{2P}\right)}}$$

$$\equiv\frac{-M}{\exp(m_0r)-1}.$$

This it the general result that we are after.

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