A REVIEW OF OYIBO'S GRAND UNIFIED THEOREM

With Realizations of a Hierarchy of Oyibo-Einstein Relativities

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Abstract

Recently Gabriel Oyibo, a Professor of Mathematics at OFAPPIT Institute of Technology, New York, U.S.A. has written two books published by Nova Science Publishers, New York, USA under the single title **Grand Unified Theorem (GUT)** and two different subtitles:

(a) *Representation of the Unified Theory or the Theory of Everything ISBN 1-59033-134-6 (Copyright © 2001)* (referred to as GUT-I), and

(b) Discovery of the Theory of Everything and the Fundamental Building Block of Quantum Theory. ISBN 1-1-59033-835-9 (Copyright © 2004) (referred to as GUT-II)

The analytical framework of the GUT in these two books is reviewed in this article from both mathematical and physical points of view in order to bring out its connection with the usual realization of conformal invariance in *analytical projective space-time geometry* and *dilatation and conformal current conservation* in relativistic quantum field theory. We find, on one hand, that the group of conformal transformations of Oyibo's generic equations of conservation

$$(G_{0n})_t + (G_{1n})_x + (G_{2n})_y + (G_{3n})_z = 0, \quad (n = 0,1,2,3,4). \tag{1}$$

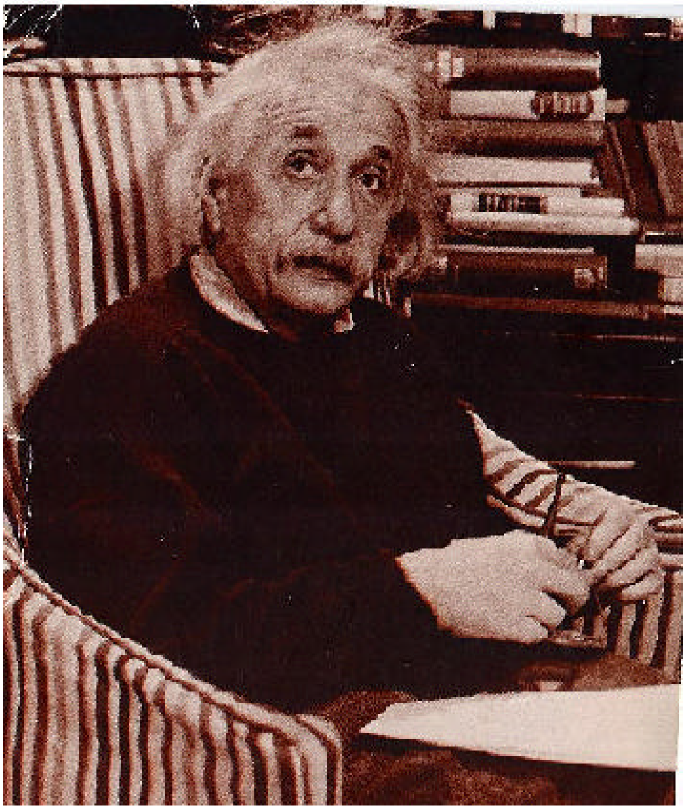
are characterized by the relation $T_n : X_n : Y_n : Z_n = t : x : y : z$ *(n=1,2,3,4)* which is the condition for (t, x, y, z) to be *homogenous* coordinates of a projective space-time geometry, and on the other hand, that Oyibo's generic solutions of Eqs.(1)

$$h_n = g_{n0}t^{n+1} + g_{n1}x^{n+1} + g_{n2}y^{n+1} + g_{n3}z^{n+1}. \tag{2}$$

in terms of the absolute invariants h_n of the subgroup of transformations for the independent coordinate variables, are related in quantum field theory to the conserved dilatation and conformal currents, $D_m = x^n \Theta_{mn}$ and $K_{mn} = (2x_m x_l - g_{ml}x^2)\Theta_n{}^l$,
in such a way that the g_{mn} are determined by the energy-momentum tensor of the quantum field Θ_{mn} by the dual relation $\Theta_{n0} : \Theta_{n1} : \Theta_{n2} : \Theta_{n3} = g_{n0} : g_{n1} : g_{n2} : g_{n3}$, (*n=0,1,2,3)*, from which we infer that $D_0 \equiv h_0$. In other words, the generic solution (h_0) may be interpreted physically and formally as the time-component (D_0) of the conserved dilatation current for the quantum field. An outline of the group of conformal transformations including its relation to gauge symmetry in electrodynamics is given in an appendix (A) for the convenience of physicists not familiar with the analytical mathematics approach on which GUT is based. The *n=2* realization of the generic solutions h_n in terms of "light caustics" formed by refraction of laser beam in a periodically grated glass is related to the usual "extended" relativity principle $(vV = c^2 = V'^2 - v'^2)$), relating subluminal $(v < c)$ and superluminal $(V > c)$ Lorentz frames; and a general procedure (based on Lax pairing

of symmetric and antisymmetric tensors) for constructing realizations for $n=3,4$ is indicated. Moreover, instead of using the ponderomotive force $G_p(G)$ to characterize the generic equation of motion, $dv/dt = G_p(G)$, for the grand unified force field, we have introduced a potential energy function(U_n) for the grand unified force field given by the time-component of a unfied gauge field ($eA_{0n} = U_n$) and determined by the Riccati's equation, $-\partial U_n / \partial r = H(U_n)$, in which $H(U_n)$ is a functional of U_n. This has enabled us, by solving the Riccati's equation, to construct a unified field (Hulthen) potential which reduces to the usual long-range (inverse-square law) of gravitation and electrostatics and to short-range nuclear, and non-potential (contact/overlap) and weak forces in appropriate limits. The Hulthen potential can be used to account for a new representation of the neutron as a compressed hydrogen atom system in terms of which all elements and their isotopes may indeed be viewed as envisaged in GUT, as composed of only normal $H=(p^+e^-)$, and compressed, $n=(p^+,e^-)_{HM}$, systems the latter being described by Santilli's "Hadronic Mechanics" (*HM*). We are, therefore, led to the conclusion that Professor Oyibo's GUT has a sound mathematical and physical basis and is a viable framework for a Grand Unified Field Theory of Everything.

1. INTRODUCTION



Albert Einstein is said to have been led to his 1905 theory of special relativity by a *fancy* he had when he was 15 years old. The fancy which arose from his *experience* with flying kites as a child was a question:

What would happen if I were to travel at the speed of light (in vacuum) and look in a plane mirror?

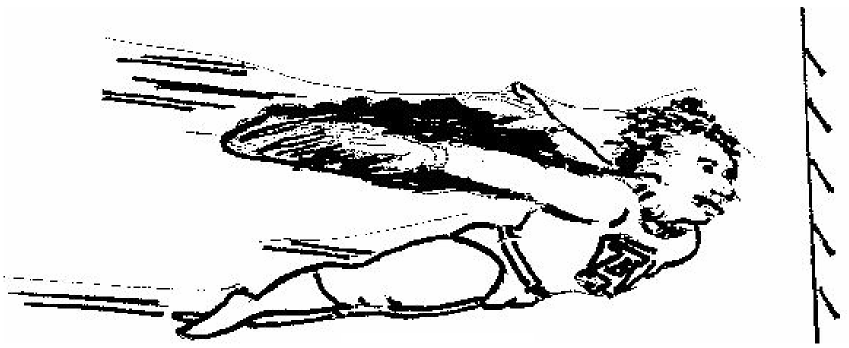


Fig. 1

The answer which his theory of relativity finally produced was that one would see nothing because light from one's aura would not reach the mirror. The most important consequence of this fancy was that time turned out to be a kind of illusion. A person who is moving with speed (v) will experience time dilatation, i.e., the slowing down of time by a factor $1/(1-v^2/c^2)^{\frac{1}{2}}$, $c \approx 3\times10^8 m/s$ being the speed of light. It also led to the twin paradox which states that an astronaut on return to earth after space travel will be younger than his brother on earth who never went on space travel. One consequence of the theory of special relativity was that one would never attain the speed of light, because putting $v^2 = c^2$ in the slowing down factor, $1/(1-v^2/c^2)^{\frac{1}{2}}$, would make it indefinitely large. In other words, Einstein's original fancy turned out to be a fantasy, but another consequence of the theory, that mass (m) and energy (E) should be equivalent in accordance with the formula,

$$E = mc^2, \qquad (1.1)$$

turned out to be true.

Public interest in Albert Einstein's relativity theory had become so great in 1929 that, when he presented to the Prussian Academy his comprehensive theory fusing electromagnetism and gravitation in a single law, the *New York Times* urged him to prepare an explanation of his new work in terms as simple as the subject would allow. Einstein's article entitled, *Field Theories, Old and New,* appeared in the *New York Times* on February 3, 1929. In the article, Einstein distinguished three stages of the relativity theory in the following words[1]:

"The first, special theory of relativity, owes its origin principally to Maxwell's theory of the electromagnetic field. From this, combined with the empirical fact that there does not exist any physically distinguishable state of motion which may be called 'absolute rest' arose a new theory of space and time. It is well known that this theory discarded the absolute character of the conception of simultaneity of two spatially separated events…

"On the other hand, the services rendered by the special theory of relativity to its parent, Maxwell's theory of the electromagnetic field, are less adequately recognized. Up to that time, the electric field and the magnetic field were regarded as existing separately even if a close causal correlation between the two types of field was provided by Maxwell's field equations. But the special theory of relativity showed that this causal correlation corresponds to an essential identity of the two types of field. In fact, the same condition of space, which in one coordinate system appears as a pure magnetic field, appears simultaneously in another coordinate system in relative motion as an electric field and vice versa. Relationships of this kind displaying an identity between different conceptions, which therefore reduce the number of independent hypotheses and concepts of field theory and heighten its logical self-containedness are a characteristic feature of the theory of relativity. For instance, the special theory also indicated the essential identity of the conceptions, inertia mass and energy. This is all generally known and is only mentioned here in order to emphasize the unitary tendency which dominates the whole development of the theory.

"The second stage in the development of the theory of relativity is the so-called general theory of relativity. This theory also starts from a fact of experience which till then had received no satisfactory interpretation: the equality of inertial and gravitational mass, or in other words, the fact known since the days of Galileo and Newton that all bodies fall with equal acceleration in the earth's gravitational field. The theory uses the special relativity theory as its basis and at the same time

modifies it: the recognition that there is no state of motion whatever which is physically privileged - i.e., that not only velocity but also acceleration are without absolute significance – forms the starting point of the theory. It then compels a much more profound modification of the conception of space and time than were involved in the special theory. For even if the special theory forced us to fuse space and time together to an invisible four-dimensional continuum, yet the Euclidean character of the continuum remained essentially intact in this theory. In the general theory of relativity, this hypothesis regarding the Euclidean character of our space-time continuum had to be abandoned and the latter given the structure of a so-called Riemannian space. Before we attempt to understand what these terms mean, let us recall what the theory accomplished.

"It furnished an exact field theory of gravitation and brought the latter into a fully determinate relationship to the metrical properties of the continuum. The theory of gravitation and inertia were fused into an essential entity. The confirmation which this theory has received in recent years through the measurement of the deflection of light rays in a gravitational field and spectroscopic examination of binary stars is well known.

"The characteristics which essentially distinguished the general theory of relativity and even more the new third stage of the theory, the unitary field theory, from other physical theories are the degree of formal speculation, the slender empirical basis, the boldness in the theoretical construction and, finally, the fundamental reliance on the uniformity of the secrets of natural law and their accessibility to the speculative intellect. It is this feature which appears as a weakness to physicists who incline toward realism or positivism, but is especially attractive, nay, fascinating, to the speculative mathematical mind. Meyers in his brilliant studies on the theory of knowledge justly draws a comparison of the intellectual attitude of the relativity theoretician with that of Descartes, or even Hegel, without thereby implying the censure which a physicist would read into this."

In the New York Times article cited above, Einstein made a distinction between Euclidean and Riemannian space, and showed how, by making use of empirically known properties of space, especially the propagation of light, it is possible to show that space-time continuum has a Riemannian metric, and the quantities g_{mn} appertaining to it, determine not only the metric but also the gravitational field.

Accordingly, Einstein found the laws governing the gravitational field in answer to the question: *Which are the simplest mathematical laws to which the Riemannian metric tensor, g_{ab}, can be subjected?* His field laws of gravitation which have proved themselves more accurate than the Newtonian law are[2]:

$$R_{mn} - \tfrac{1}{2} R g_{mn} = -k\Theta_{mn} \qquad (1.2)$$

where R_{mn} is the Ricci's (also called Einstein's) tensor given in terms of the Riemann curvature tensor R_{mabn} for the 4-dimensional space-time geometry by

$$R_{mn} = R_{mabn} g^{ab}, \quad R = R_{mn} g^{mn} \qquad (1.3)$$

is the curvature scalar, Θ_{mn} is the energy momentum tensor, and $k = 8pG/c^4$, G being the gravitational constant and c the speed of light.

Einstein's unsuccessful attempt to *additively* fuse together the *symmetric* tensor (g_{mn}) and the *antisymmetric* tensor (f_{mn}) representing the gravitational field and the electromagnetic field respectively into a "unified field" metric tensor

$$\overline{g}_{mn} = f_{mn} + g_{mn} \qquad (1.4)$$

was plagued by the fact that the algebraic identity, $f_{mn} x^m x^n \equiv 0$, always subtracted the electromagnetic field contribution to the fundamental quadratic form, $ds^2 = \overline{g}_{mn} dx^m dx^n$, characterizing the Riemannian metric tensor, $\overline{g}_{mn}$. However, if one *multiplicatively* fuses the two (symmetric and antisymmetric) matrices $g = \|g_{mn}\|, f = \|f_{mn}\|$ into a new "unified field" matrix, $\|\hat{g}_{mn}\| \equiv \hat{g} = fg$ (called a "Lax pair"[3]) then a *new "unified field" conservation law* can be constructed in the following way. Consider the simplest non-trivial case of a *symmetric* 2×2 matrix (*g(t)*) and an *antisymmetric* 2×2 matrix (*f*), given by:

$$g(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (1.5)$$

Then, by using the fact that the matrix (*U*) defined by

$$U(t) = e^{ft}, \quad U^+(t) = e^{-ft}, \quad U(t)U^+(t) = I$$

is an orthogonal matrix, we see that, under the orthogonal transformation:

$$g(0) \to g(t) = e^{ft} g(0) e^{-ft}, \qquad (1.6)$$

we would be led to a Heisenberg-type equation of motion.

$$\frac{dg}{dt} = fg - gf\,, \qquad (1.7)$$

which has the property that it conserves the eigenvalues of g, and involves the Lie-algebraic product, $fg - gf$, defined by the Lax pairs, fg and gf.

This leads us to the dictum by Werner Heisenberg, one of the founding fathers of quantum physics and the quantum culture, cited by the Chinese-born physicist, Fritjof Capra, in his classic 1984-published book entitled *The Tao of Physics* [4] that:

> ***It is probably true quite generally that, in the history of human thinking, the most fruitful development frequently take place at those points where two different lines of thought meet. These lines may have their roots in quite different parts of human culture, in different times or different cultural environment or different religious traditions: hence if they actually meet, that is, if they are at least so much related to each other that a real interaction can take place, then one may hope that new and interesting developments may follow.***

Accordingly, it is no wonder that by extending the line of thought (of Lax pairing) to an abstract group of *conformal transformations* of an arbitrary function, defined by

$$G(y^1, y^2, \ldots, y^p) \to G(Y^1, Y^2, \ldots, Y^p) = F(y^1, y^2, \ldots, y^p, k) \bullet G(y^1, y^2, \ldots, y^p) \quad (1.8)$$

a real interaction seems to have now occurred between Einstein's relativity theory outlined above and the *Grand Unified Theorem* proposed by a New York-based Nigerian, Professor Gabriel Oyibo, in his two books [5,6] published in 2001 and 2004 by Nova Science Publishers Inc, New York, costing \$240 (≈ N25,000) each, which we now proceed to review.

2. CONFORMAL INVARIANCE IN MATHEMATICS AND PHYSICS

2.0 Preface and Earlier Review

It is stated in the Preface to the first book[5] on Grand Unified Theorem (GUT) sub-titled *Representation of the Unified Theory or the Theory of Everything ISBN 1-59033-134-6 (Copyright © 2001)* (referred to as GUT-I), that the book was based on a paper entitled, *Generalized Mathematical Proof of Einstein's Theory Using A New Group Theory*. In this earlier paper, Professor Oyibo formulated a *Grand Unified Theorem* that provided a mathematical basis for a Grand Unified Field Theory or a Theory of Everything (TOE) in the form of generic solutions to the unified force field from which both the Newtonian and Einsteinian gravitational force fields as well as the electromagnetic, strong and weak force fields seem to be recoverable. The conclusion on p. 25 of GUT-I that *"both momentum and energy equations are conformal invariants of mass conservation equations"* under the group of conformal transformation which was announced earlier[7] in *Our Time Press* of February 1999 has been the subject of an earlier review by Animalu[8] at the Conference on Ordinary Differential Equations organized by the National Mathematical Centre, Abuja in July, 2000.

Because of the highly technical nature of this previous review, the present review is carried out from both mathematical and physical points of view in order to bring out the connection of the analytical framework of GUT with the usual realization of conformal invariance in analytical *projective space-time geometry* as well as discuss how to overcome the constraints on the masses of particles arising from *exact conformal current conservation* in relativistic quantum field theory. For this reason, in order to read GUT-I beyond the Preface to the first chapter dealing with *Conformal Invariance*, it is necessary for the reader to bear in mind a distinction between a (mathematical) *theorem* and a (physical) *theory*, and how the two approaches may meet. Whereas a *theorem* can be proved (mathematically) using established axioms (e.g., the axioms of Euclidean geometry or of abstract groups), a *theory* becomes acceptable (as a physical law of nature) only when its prediction is in agreement with experiment data. For example, the relation between rest mass and energy ($E = mc^2$, c being the speed of light), which Oyibo referred to as "Einstein's law" on p. 21 of GUT-I, was derived by Einstein from his *theory* of special relativity[1,2] . Thus, while Einstein's approach to his Unified Field Theory was based on a generalization of his highly successful special relativity theory, and its invariance under the Lorentz group O(3,1) of transformations of Minksowki 4-dimensional space-time geometry as discussed in Sec. 1, Oyibo's approach to his Grand Unified Theorem appears to have been abstracted from his methodology for solving the Navier Stokes equation in fluid mechanics using invariance of an arbitrary function under a group of conformal transformations.

2.1 GUT Approach to Conformal Invariance

The first problem encountered in reading Chapter 1 of GUT-I on conformal invariance is the lack of direct relationship between Oyibo's definition of "conformal" transformation of an arbitrary function and the usual characterization of conformal invariance or symmetry in analytical projective space-time geometry and in relativistic quantum field theory, which we must now discuss.

In chapter 1 of GUT-I, the conformal invariance of an arbitrary function G given by

$$G = G(Y^1, Y^2, \ldots, Y^p)$$

is defined by a group of transformations given by

$$T_k : Y^i = f^i(y^1, y^2, \ldots, y^p, k),$$

if T_k is the group of the transformations such that

$$G(Y^1, Y^2, \ldots, Y^p) = F(y^1, y^2, \ldots, y^p, k) \bullet G(y^1, y^2, \ldots, y^p) \qquad (2.1)$$

where $F(y^1, y^2, \ldots, y^p, k)$ is a function of y^l and k the single group parameter. However, in the actual proof of the conformal invariance for the conservation equations of the "generic" quantities $G_{mn} = G_{mn}(x, y, z, t, \dot{x}, \dot{y}, \dot{z}, r, m, T, P, \ldots)$ which are arbitrary functions of space and time coordinates *(x,y,z,t)*, velocities ($\dot{x}, \dot{y}, \dot{z}$), density ($r$), fluid or gas viscosity (m), temperature (T), pressure (P), etc, in chapter 3,

$$(G_{0n})_t + (G_{1n})_x + (G_{2n})_y + (G_{3n})_z = 0, \quad (n = 0,1,2,3,4). \qquad (2.2)$$

given explicitly in Eqs.(3.15) to (3.17) of GUT-I, the following relation was obtained at the bottom of Eq.(3.17):

$T_n = J_n t, \quad X_n = J_n x, \quad Y_n = J_n y, \quad Z_n = J_n z$ (J_n being constants) for *n=1,2,3,4*

i.e., $$T_n : X_n : Y_n : Z_n = t : x : y : z \qquad (2.3)$$

This is the condition for the four coordinates (t, x, y, z) to be regarded as *homogenous coordinates* of a 3-dimensional projective space, as discussed in the earlier review by Animalu[8], and consequently, for the usual *geometric principle of duality* [9] to be applicable to GUT. By generalizing the conformal transformation of Eq.(2.1) to the system of partial differential equations of order n given by the following equation

$$G_j(x^1 \ldots x^p, .y^1 \ldots y^q, \frac{\partial^n y^1}{(\partial x^1)^n} \ldots, \frac{\partial^n y^q}{(\partial x^p)^n}) = 0, \qquad (2.4)$$

Oyibo was able to obtain solutions of Eqs.(2.2) in terms of the absolute invariants h_n of the subgroup of transformations for the independent coordinate variables as follows:

$$h_n = g_{n0} t^{n+1} + g_{n1} x^{n+1} + g_{n2} y^{n+1} + g_{n3} z^{n+1}. \qquad (2.5)$$

It is clear that h_n is a function of the local-time and space coordinates (t, x, y, z), and of the "metric" parameters $(g_{n0}, g_{n1}, g_{n2}, g_{n3})$ as well as *n*. However, to provide physical interpretation of these solutions for quantum mechanical systems, it is necessary to fill a gap in Oyibo's presentation, namely how conformal invariance of Eq.(2.4) is related to the characterization of conformal symmetry of a relativistic quantum field.

2.2 Conformal Invariance in Quantum Field Theory

According to the article on *Scale Symmetry* by MIT Professor Roman Jackiw[10] published in *Physics Today* (January 1970), conformal and scale or dilatation symmetry of a quantum field are defined in terms of the dilatation and conformal currents,

$$D_m = x^n \Theta_{mn} \text{ and } K_{mn} = (2x_m x_l - g_{ml} x^2)\Theta_n{}^l \qquad (2.6)$$

by the conservation laws

$$\partial^m D_m = \Theta_m{}^m = 0 \text{ and } \partial^n K_{mn} = 2x_m \Theta_l{}^l = 0. \qquad (2.7)$$

where Θ_{mn} is the energy-momentum tensor and $L = \Theta_m{}^m$ the Lagrangian of the quantum field. Consequently, for exact conformal and scale symmetry, the Lagrangian $L = \Theta_m{}^m$ must vanish (by virtue of the field equations of motion), which constrains all particles to have *zero mass* in the framework of the Lorentz group, O(3,1). An outline of the conformal group of transformations is given in Appendix A.

Now, by comparing the characteristic solutions of Oyibo's generic GUT equations when *n=0*, given by

$$h_0 = g_{00}t + g_{01}x + g_{02}y + g_{03}z \ . \tag{2.8}$$

with the explicit expression for the dilatation current,

$$D_m \equiv x^n \Theta_{mn} = ct\Theta_{m0} + x\Theta_{m1} + y\Theta_{m2} + z\Theta_{m3} \, . \tag{2.9}$$

we infer that if

$$\Theta_{00} : \Theta_{01} : \Theta_{02} : \Theta_{03} = g_{00} : g_{01} : g_{02} : g_{03} \, , \tag{2.10}$$

then $D_0 \equiv h_0$. In other words, h_0 may be interpreted physically as the time-component (D_0) of the dilatation current for a quantum field which is conserved in the limit of exact scale symmetry, and we infer that the relation in Eq.(2.10) is the *dual* of the one in Eq.(2.3) (see, p. 14 of Todd[9]). Note that for a space-time of constant Ricci (or Einstein) tensor ($R_{mn} = 0$), Einstein's field laws of gravitation in Eq.(1.2) reduces to $\frac{1}{2} R g_{mn} = k\Theta_{mn}$ which implies the relation in Eq.(2.10).

However, since conformal invariance is exact only for a *massless* field, like the electromagnetic (photon) field or the neutrino field, a question arises as to whether Oyibo's concluding statement on p. 25 of GUT-I *"that both momentum and energy equations are conformal invariants of mass conservation equations, under the group of transformations in Eq.(3.7)"* does not become trivial for *massive* quantum fields. This brings us to Animalu's letter[11] to *Physics Today* (June 1972) commenting on Professor Roman Jackiw's article [10] on *Scale Symmetry* cited above. In the letter, Animalu pointed out that the constraint of exact scale and conformal symmetry to only massless particles applies only in the framework of the Lorentz group, O(3,1). But if one goes to the larger (conformal) group, O(4,2) which contains O(3,1) as a subgroup, it is quite possible to construct an explicit acceptable scale-invariant wave equation for massive spin-$\frac{1}{2}$ composite particle system, such as the quark triplet field, *q* = (*u,d,s*). To this end, one begins with the most general minimal linear parity-conserving current in the O(4,2) algebra of Dirac matrices[12]

$$J_m = \bar{q}g_m q - i\{\bar{q}M^{-1}(\partial_m q) - (\partial_m \bar{q})M^{-1}q\} \tag{2.11}$$

where

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \tag{2.12}$$

defines the quark masses (m_u, m_d, m_s) assumed to be nonzero. The simplest wave equation that conserves this current is the O(4,2) wave equation

$$ig^m \partial_m q + M^{-1}\partial^m \partial_m q = 0. \tag{2.13}$$

The Lagrangian density may be written $L = \Theta_m{}^m$, where Θ_{mn} is the energy momentum tensor

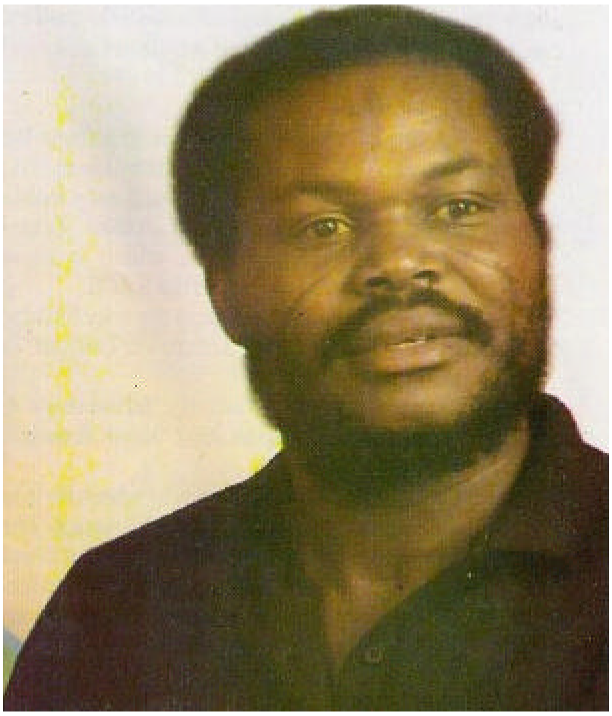
$$\Theta_{mn} = i\tfrac{1}{4}\{[\bar{q}g_m(\partial_n q) - (\partial_n \bar{q})g_m q] + [m \leftrightarrow n]\} - \tfrac{1}{2}\{(\partial_m \bar{q})M^{-1}(\partial_n q) + [m \leftrightarrow n]\} \tag{2.14}$$

(where $[m \leftrightarrow n]$ means that the term is repeated with m and n interchanged). But $L = \Theta_m{}^m = 0$ by virtue of the field equation, and hence the dilatation and conformal currents defined by Eqs.(2.9) are exactly conserved, so that the model has exact conformal symmetry. However, the O(4,2) mass equation for (2.13) is simply,

$$m - m^2 / M = 0 \qquad (2.15)$$

which has two eigenvalues, $m = 0$ and $m = M$. We may associate $m=0$ with strictly massless Goldstone's bosons and $m = M$ with the real world of strongly interacting particles with non-zero rest masses in the quark model. Indeed, if $m_u = m_d$, then chiral $SU(3)\times SU(3)$ symmetry is broken in the manner prescribed by Gell-Mann, Oakes and Renner [13]. We are thus able to obtain a non-trivial conformal-invariant wave equation that fulfils Lorentz-invariance, gauge-invariance, O(4,2) invariance and SU(3)-invariance in the appropriate limit. Equations of type (2.13) include *convective currents*.(the second-order derivative structure) and were first considered in connection with radiation reaction theories by Rosen[14].

3. HIERARCHY OF OYIBO-EINSTEIN RELATIVITIES.



3.1 Hierarchy of Relativities Corresponding to n=0,1,2,3,4.

Oyibo has stated on p.16 of GUT-I that ***"the bottom line of finding a Unified Force Field Theory was the consistency of the practical results obtained using the fundamental components of the methodology in earlier work on the hydrodynamical equations in gas and fluid flows and turbulence as well as the proof of Einstein's theory"***. The consistency of practical results so far lies in the consistent physical/geometrical interpretation of the hierarchy of solutions in Eq.(2.5), for $n = 0$ and $n = 1$, in relation to the homogeneous "point coordinates" system

(t,x,y,z) in Eq.(2.3) and the “metric” parameters, “plane coordinates” or “field coordinates” system $(g_{n0}, g_{n1}, g_{n2}, g_{n3})$ in Eq.(2.10) provided by the usual *geometric principle of duality* (p. 14 of Todd[9]).

It is apparent from reading the section of GUT-I (p.59) and GUT-II (p.63) on whether the solutions in Eq.(2.5) for $n \geq 2$ have anything to do with Fermat’s Last Theorem, that it is very challenging to construct physical realization(s) of the level curves represented by the solutions for $n \geq 2$. We need, however, to tackle this problem to find further evidence of the consistency of the practical results of Oyibo’s approach derived from fluid dynamics

3.2 Construction of a Realization of n=2 Hierarchy.

We may begin the construction of a realization of the *n=2* hierarchy of relativities by posing the following question along Einstein’s fancy depicted earlier:

If Einstein were at the centre of a droplet of liquid moving on a plane with velocity *v*, how would he relate the apparent value of the refractive index (*n=c/v*) of light in vacuum to its value (*n’=c/v’*) in the droplet of liquid in accordance with his special relativity principle?

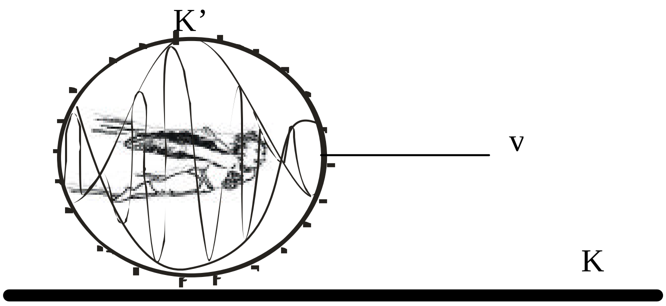


Fig. 2

Let us suppose that the fixed plane plays the part of a tube or of an unprimed coordinate system (K), and the droplet plays the part of the drop of liquid or of the moving coordinate system (K’), while the ray of light plays the part of a moving point whose coordinates are *(ct,x,y,z)* with respect to K and (*ct’,x’,y’,z’*) with respect to K’, as in Einstein’s special relativity analysis of the result of the Fizeau experiment (p. 10 ref. [2]) on propagation of light in a liquid moving inside a tube. In this circumstance, the usual Lorentz transformation may be expressed in the form [15]

$$n'^{-1/2} cdt' = n^{1/2}(cdt - (v/c)dx),\ n'^{-1/2} dx' = n^{1/2}(dx - cdt),\ dy' = dy,\ dz' = dz \qquad (3.1)$$

where *n* (=*c/v*) is the refractive index for motion in K and *n’* (=*c/v’*) the refractive index for motion in K’; and Einstein’s special relativity

$$c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \qquad (3.2)$$

holds, provided that *n’* and *n* are such that

$$1/n' = n - 1/n, \text{ i.e., } 1/n' = (c^2 - v^2)/vc. \qquad (3.3)$$

This implies that the singularity at *v=c, i.e. n = 1,* in special relativity theory is associated with the limit, $n' \to \infty$, which is evidently unphysical for a real liquid.

Assuming that in a real liquid, the variation of the refractive index (*n'*) is due to the dependence of *c* and *v* on temperature (*T*) or time (*t*) say, let us define a differential refractive index in such a way that
$v'/c = 2\dot{c}/\dot{v}$ (where $\dot{c} = dc/dt$ and $\dot{v} = dv/dt$, so that

$$1/2n' = \dot{c}/\dot{v} \equiv dc/dv = (c^2 - v^2)/2vc. \qquad (3.4)$$

This states that dc/dv is determined by the ratio of the "velocity potential", $f = \frac{1}{2}(c^2 - v^2)$, and the "stream function", $S = cv$, for an irrotational flow of a non-viscous incompressible fluid in a 2-dimensional (v,c) – plane defined by the velocity of light (*c)* and the velocity of the droplet (*v*)**.**

Now, by integrating Eq.(3.4), we get

$$c^2 = \frac{v^2}{3} + \frac{c_0^3}{3v}, \text{ or } 3vc^2 = v^3 + c_0^3 \qquad (3.5)$$

where c_0^3 is a constant. Thus, if $c = c_0$ and we put $v = r/t$, this expression takes the form

$$3r(c_0t)^2 = r^3 + (c_0t)^3 \qquad (3.6)$$

which may be regarded as a representation of Eq.(2.5) for *n=2* , i.e.,

$$3r(c_0t)^2 \equiv h_2 = g_{20}t^3 + g_{21}x^3 + g_{22}y^3 + g_{33}z^3 \equiv (c_0t)^3 + r^3. \qquad (3.7)$$

where $r \equiv \sqrt{(x^2 + y^2 + z^2)}$. By virtue of the geometric principle of duality, Eq.(3.6) may be rewritten as done in catastrophe theory[16], in the form

$$X + Yr + r^3 = 0 \qquad (3.8)$$

which means that, for fixed *r*, Eq.(3.8) represents a line in (X,Y)-space which is normal to a parabola parameterized by *r* in such a way that any point on the parabola has coordinates (r, r^2) (p. 78 of ref.[16]) and the envelope of the normal (as r varies) is a semi-cubical parabola (called *canonical cusp catastrophe*) defined by the discriminant equation:

$$27X^2 + Y^3 = 0. \qquad (3.9)$$

This represents the involute of the parabola shown in Fig.(3a). The corresponding involute of an ellipse is an hypocycloid shown in Fig. 3b and can be produced as "light caustic" formed by refraction of a laser beam in a periodically grated piece of glass (p. 271 of ref.[16]) .

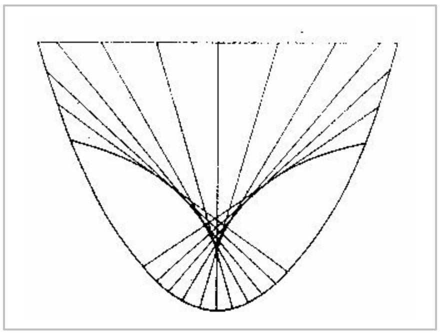


Fig. 3a

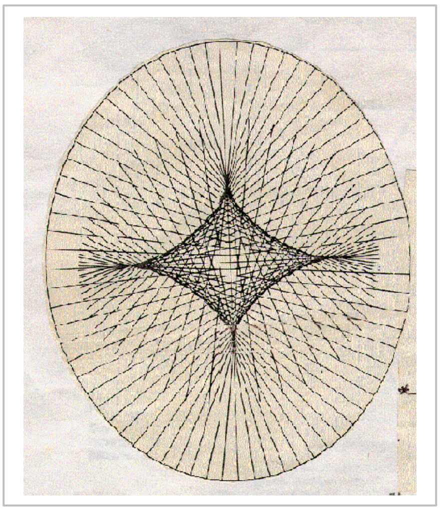


Fig. 3b

This is a practical result corresponding to n = 2 in the hierarchy of Oyibo-Einstein relativities characterized by Eq.(2.5) which we are after. It is, therefore, true, as stated on p. 22 of GUT-I, that:

> *"In the final analysis what establishes the integrity of this analysis is not so much the group laws or group defined parameters but the end results or the final conclusion reached".*

3.3 " Extended" Relativity, God, Genesis and Cosmology

Another important result that follows by associating a "group velocity" V with the droplet of liquid (regarded as a wavepacket) in Fig. 2 is a realization of the usual "extended" relativity principle [17] in the form of the following (dual) transformation law

$$uV = c^2 = V'^2 - u'^2 \qquad (3.10)$$

between subluminal ($u < c$) and superluminal ($V > c$) velocities in (v, V)-space. The relation $uV = c^2$ is the mathematical condition for the pair (v, V) to be inverse points with respect to a hyperbola defined by the following nonlinear transformation in velocity space

$$v = \frac{c^2}{(V'+v')} \equiv c^2 \frac{V'-v'}{(V'^2 - v'^2)}; \quad V = \frac{c^2}{(V'-v')} \equiv c^2 \frac{V'+v'}{(V'^2 - v'^2)} \qquad (3.11a)$$

such that the speed of light is invariant in the form in Eq.(3.10). The inverse of this transformation is

$$v' = \tfrac{1}{2}(V - v), \quad V' = \tfrac{1}{2}(V + v) \qquad (3.11b)$$

Once in possession of such an "extended" relativity principle, one can elucidate the various speculations about the origin of the universe discussed on pp.95-96 of GUT-I. According to Charles Animalu[18], the (Biblical) creation story recorded in the Book of Genesis and the (Big Bang) cosmological theory of evolution can be reconciled if the speed V_{word} (of creation) of the Word of God that spoke the universe into existence is related to the speed of sound $(v_{sound} \approx 300\,m/s)$ and of light $(c \approx 3\times10^8\,m/s)$ of creation as follows:

$$v_{sound} V_{word} = c^2 \qquad (3.12)$$

so that

$$V_{word} \approx c^2/300 = 9\times10^{16}/300 = 3\times10^{14}\,m/s.$$

It would then follow that a created object ("universe") moving with the speed of the Word of God would experience a time dilatation (since the Biblical Age of 7000 years) by an amount

$$1/(1 - V_{word}^2/c^2)^{\frac{1}{2}} \approx \frac{V_{word}}{c} i = \frac{c}{v_{sound}} i = 10^6 i,$$

where the factor $i = \sqrt{-1}$ means that space and time are interchanged in the usual Lorentz transformation involving a superluminal relative speed, V_{word}. This gives the light-years dimension of the universe (according to the Big Bang theory) as $7000\times10^6 = 7\times10^9$ light-years, i.e., 7 billion light-years, (where 1 light-year $= 10^7 \times 3\times10^8\,m = 3\times10^{15}\,m$), which is in reasonable agreement with experimental data.

Note that we may also separate Eq.(3.4) into a pair of coupled differential equations:

$$\frac{dc}{dt} = K(c^2 - v^2), \text{ (Riccati' s equation)}$$
$$\frac{du}{dt} = 2Kcu, \qquad (3.12)$$

which may be rewritten as a (matrix) Riccati equation

$$\begin{pmatrix} dc/dT \\ du/dT \end{pmatrix} = \begin{pmatrix} Kc & -Ku \\ Ku & Kc \end{pmatrix} \begin{pmatrix} c \\ u \end{pmatrix}, \qquad (3.13)$$

where K is a constant and. This implies that the refractive index ($n \equiv c/u$)) could be determined by solving the Riccati equation in (3.12) for $c = c(t)$. We shall return to the importance of the Riccati equation[19] in Sec. 4.2.

3.4 Construction of a Realization of n=3 Hierarchy

Let us now return to the generic arbitrary functions of h_n in Eq.(2.5) which , according to GUT-I p. 59, "could mathematically be considered to be characteristics depicting the so-called level curves for describing the solutions" of the Grand Unified Field Eqs.(2.2), and attempt to construct a realization of the n=3 hierarchy:

$$h_3 = g_{30}(ct)^4 + g_{31}x^4 + g_{32}y^4 + g_{33}z^4. \qquad (3.14)$$

The fact that this is a *curve of fourth order or quartic curve* suggests that it is associated with the interaction of two quadratic forms, two of which are of physical interest, namely:

$$s^2 = +\{(ct)^2 - r^2\}, \quad . \qquad (3.15a)$$

characterizing "Einstein states" of special relativity theory and its dual (as defined in the classical theory of tachyons by Recami[15]),

$$s^2 = -\{(ct)^2 - r^2\} \qquad (3.15b)$$

where $r = \sqrt{x^2 + y^2 + z^2}$. However, the novel ("O yibo") states of interest are obtained from the four linear homogeneous equations:

$$[rg_{mn} + ct(fg)_{mn} + sd_{mn}]w_n = 0 \qquad (3.16a)$$

which involves the usual Minkowski space-time metric tensor g_{mn} and its "Lax pair" $(fg)_{mn}$ with an antisymmetric tensor (f_{mn}) where

$$\|g_{mn}\| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \|f_{mn}\| = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \|(fg)_{mn}\| = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \quad (3.16b)$$

so that Eq.(3.16a) has the explicit form:

$$\begin{pmatrix} r+s & 0 & -ct & 0 \\ 0 & -r+s & 0 & -ct \\ -ct & 0 & -r+s & 0 \\ 0 & ct & 0 & -r+s \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = 0\,. \qquad (3.17)$$

The vanishing of the secular determinant leads to the quartic equation

$$[s^2 - r^2 - (ct)^2][(s-r)^2 + (ct)^2] = 0, \qquad (3.18)$$

from which one obtains the two quadratic relations:

$$s^2 - r^2 - (ct)^2 = 0, \text{ i.e., } s^2 = r^2 + (ct)^2; \qquad (3.19a)$$

$$(s-r)^2 + (ct)^2 = 0, \text{ i.e., } s = r \pm ict \text{ or } |s| = \sqrt{r^2 + (ct)^2} \qquad (3.19b)$$

By putting $s = ct$, we may rewrite (3.19a & b) in the respective forms:

$$s^2 = (ct)^2 + x^2 + y^2 + z^2, \qquad (3.20a)$$

$$c^2(t \pm it)^2 - x^2 - y^2 - z^2 = 0 \qquad (3.20b)$$

and observe that they are identical when $s = ct = 0$. However, by putting

$$c^2(t \pm it)^2 = c^2(t^2 + t^2)e^{\pm 2iq}, q = \tan^{-1}(t/t)$$

and re-defining $\hat{c}^2 = c^2 e^{\pm 2iq}$, Eq.(3.20b) can also be re-written so as to have the same form as (3.15b) i.e.,

$$(\hat{c}t)^2 = -\{(\hat{c}t)^2 - x^2 - y^2 - z^2\}. \qquad (3.20c)$$

We recognize Eq.(3.20b) as a light cone with *complex time*, as proposed heuristically by Jannussis and co-workers[15] while Eq.(3.20c) is the dual of the Minkowski space defined in "extended relativity" by Recami[15], (or *isodual* space in Santilli's terminology [15]) with complex speed of light ($\hat{c}$). And we note that if $(\hat{c}t, \hat{c}t, x, y, z)$ are treated as homogeneous coordinates of a projective space, then the invariance group of Eq.(3.20c) would be the O(3,2)-de Sitter group and its projection on the plane $z = 0$, has the form

$$(\hat{c}t)^2 + (\hat{c}t)^2 - x^2 - y^2 = 0, \qquad (3.20d)$$

which is a ruled quadric in $(\hat{c}t, \hat{c}t, x, y)$-space having real generating lines.

In the corresponding momentum space with *five* homogeneous momentum coordinates, $(p_t, p_x, p_y, p_z, p_m) \equiv (p_0, p_1, p_2, p_3, mc)$, the analogs of Eqs.(3.20a) and (3.20b) are

$$(mc)^2 - p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0 \quad \text{or} \quad mc = \pm\sqrt{p_0^2 + p^2}\,, \qquad (3.21a)$$

$$(mc \pm ip_0)^2 - p_1^2 - p_2^2 - p_3^2 = 0\,, \qquad (3.21b)$$

where $p = \sqrt{p_1^2 + p_2^2 + p_3^2}$. Observe that the difference between Eq.(3.21a) and the usual Einstein's mass-energy-momentum relation,

$$p_0^2 - p^2 = (mc)^2 \text{ or } mc = \pm\sqrt{p_0^2 - p^2}\,.$$

is a mere interchange of mass (mc) and energy (p_0), which is possible because the two are treated on equal footing in 5-momentum space. Moreover, we can re-interpret $\hat{E} \equiv (mc \pm ip_0)c$ in Eq.(3.21b) as the complex energy of a massive particle traveling at the speed of light(c) in a conformal invariant space-time. We are thus lead to Oyibo's conclusion elaborated on p. 28 of GUT-II, *that mass conservation equation is a conformal invariant of energy conservation equation*. Alternatively, by using

$$(mc \pm ip_0)^2 = [(mc)^2 + p_0^2]e^{\pm 2ij} \equiv (m\hat{c})^2 + \hat{p}_0^2$$

where $j = \tan^{-1}(p_0/mc)$, to rewrite Eq.(2.21b) in the form ,

$$(m\hat{c})^2 = -\{\hat{p}_0^2 - p_1^2 - p_2^2 - p_3^2\}, \qquad (3.21c)$$

we obtain the mass-energy-momentum relation of "extended relativity" theory[15], in which $\hat{p}_0 = e^{\pm ij} p_0 \equiv E/\hat{c}$, where $\hat{c} \equiv ce^{\pm ij}$ is a complex velocity of light. Such a complex velocity of light implies that the underlying medium is dispersive.

3.5 Construction of a Realization of n=4 Hierarchy

Let us turn finally to the construction of a realization of the generic arbitrary functions of h_n in Eq.(2.5), for n = 4. We begin by observing that in terms of the following 5×5 matrix representations of the tensors,

$$\|G_{mn}\| \equiv \begin{bmatrix} G_{00} & G_{01} & G_{02} & G_{03} & G_{04} \\ G_{10} & G_{11} & G_{12} & G_{13} & G_{14} \\ G_{20} & G_{21} & G_{22} & G_{23} & G_{24} \\ G_{30} & G_{31} & G_{32} & G_{33} & G_{34} \\ G_{40} & G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix}, \|g_{mn}\| \equiv \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} & g_{04} \\ g_{10} & g_{11} & g_{12} & g_{13} & g_{14} \\ g_{20} & g_{21} & g_{22} & g_{23} & g_{24} \\ g_{30} & g_{31} & g_{32} & g_{33} & g_{34} \\ g_{40} & g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix} \quad (3.22)$$

we may generalize Eq.(2.2) in the form

$$(G_{0n})_t + (G_{1n})_x + (G_{2n})_y + (G_{3n})_z + (G_{4n})_s = 0, \quad (n = 0,1,2,3,4). \quad (3.23a)$$

which reduces to Eq.(2.2) if

$$(G_{n4})_s = 0\,. \quad (3.23b)$$

Because of symmetry ($G_{mn} = G_{nm}$), the only new term included by this generalization is G_{44} ; and an explicit expression for G_{44} used by Oyibo to compute Schwarzschild's solution of Einstein's gravitational field equations is given in Eq.(3.148) on p. 95 of GUT-II.

However, by defining $h_4 \equiv s^5$, Eq.(2.3) may be rewritten (for n=4) as a homogeneous form

$$g_{40}(ct)^5 + g_{41}x^5 + g_{42}y^5 + g_{43}z^5 + g_{44}s^5 = 0, \quad (3.24)$$

where $g_{44} = -1$ and (t, x, y, z, s) are the five *homogeneous* coordinates of a point in a projective space. To construct a realization in the same way as the n=3 case, we notionally separate the 5×5 matrix $\|g_{mn}\|$ into symmetric and antisymmetric parts:

$$g_{mn} = \tfrac{1}{2}(g_{mn} + g_{nm}) + \tfrac{1}{2}(g_{mn} - g_{nm}) \equiv g^S_{mn} + f_{mn} \quad (3.25)$$

in terms of which the appropriate generalization of the set of Eq.(3.16a) takes the form:

$$[rg^S_{mn} + ct(fg^S)_{mn} + sd_{mn}]w_n = 0 \quad , \; m, n = 0,1,2,3,4 \,, \quad (3.26)$$

and the associated secular determinant generates the required quintic curve realization of Eq.(3.24). The result can again be interpreted geometrically in terms of the invariant theory of a configuration of three quadric surfaces in 3-dimensional space.

Some observations may be made about the above methodology (based on invariant theory of quadratic forms in 3- or 4-dimensional projective geometry) for constructing realizations of Oyibo-Einstein relativities for $n > 2$. Firstly, unlike Einstein's (differential geometric) method based on covariance of the metric defined by the fundamental quadric form, $ds^2 = g_{mn}dx^m dx^n$, under coordinate transformation, the projective geometric methodology is non-metrical and based on invariant theory of more than one (interacting) quadratic forms. Secondly, because all equations are homogeneous in the projective-geometric method, *the geometric principle of duality* is applicable. Finally, the emergence of *complex time* in Eq.(3.20b) or equivalently, *complex (time-dependent) speed of light* in Eq.(3.20c) is compatible with the characterization of differential refractive index, ($dc/dv \equiv \dot{c}/\dot{v}$) in Eq.(3.4) in the construction of the n=2 realization of the Oyibo-Einstein relativities. We are, therefore, led to the conclusion that the Oyibo's solutions in Eq.(2.5) of the generic conservation laws of GUT in Eq.(2.2) have geometrical and physical meaning, whose consequences can be explored most directly through the methodology of *projective geometry*.

4. THE UNIFIED FORCE FIELD AND GAUGE PRINCIPLE

4.1 Definition of the Unified Force Field in GUT

On p. 78 of GUT-I, Oyibo admits that "one of the most challenging problems in the field unification problem geometrically is one of finding a common geometric object, entity or platform in which all the force fields could be combined or united. For the individual fields such as gravitational field, it was the metric tensors, for the electromagnetic field, it was the field intensities, etc which served as geometric or intermediary functional objects to describe the fields. However, when they are united, which one geometric object or objects will be proper? This is a difficult question which has frustrated the search for the unified theory for so long". He states on p. 75 of GUT-I that "the Grand Unified Theorem formulations are *geometro-dynamic* in that the universe is considered to be a unified force field in which matter is just a concentration of the field". Accordingly, in Eq.(3.47) of GUT-I, he generated what he called the "unified force field" ((G) by integrating the momentum fluxes, G_{11}, G_{22} and G_{33}, over appropriate elemental areas,

$$G = \int G_{11} dzdy = \int G_{22} dxdz = \int G_{33} dxdy = \overline{F}(h_0) + (1 + f_{1n} t^n) \overline{F}_1(h_n) \qquad (4.1a)$$

where $\overline{F}(h_0)$ is the "wave" component of Force field, f_{1n} is a constant, and $\overline{F}_1(h_n)$ is the generalized "classical", Newtonian or Einsteinian, component of the Force Field. He then separated $\overline{F}(h_0)$ and $\overline{F}_1(h_n)$ notionally into five contributions designated as gravitational, electromagnetic, strong, weak and other (presently unknown) force fields. Finally, in Eq.(3.48) he generated the ponderomotive force equations by relating G to the acceleration that it produces in the generic form

$$\frac{d\vec{v}}{dt} = G_p(G) \qquad (4.1b)$$

where G_p is a functional of G. He then provided what he called "some simplified form of the unified force field solutions", G, for static field in Eqs.(3.173) on p.89 of GUT-I, for n = 0 and 1, to obtain in Eq.(3.179a) on p.90 a static unified force field that produce potential energy of the form

$$U_n(r) = \frac{1}{g_n^{\bar{r}_n} r} \qquad (4.2)$$

g_n being the force field range parameter. From this, he obtained the various force field potentials for gravity, electromagnetic, strong and weak as well as unknown force fields from the proper (experimental) values of g_n.

4.2 Relation to Gauge Principle

One of the challenging tasks in this review is to relate the GUT methodology to other familiar methodologies, especially methodologies based on the gauge principle. To this end, we may replace the "unified force field" (G) by a "unified potential energy field", ($U_n \equiv eA_{0n}$, A_{0n} being the time component of a unified gauge potential, A_{mn} say) and generate the acceleration from the negative gradient of U_n, as follows

$$\frac{d\vec{v}}{dt} = -\frac{\partial U_n}{\partial r} = H(U_n) \qquad (4.3)$$

where $H(U_n)$ is a *functional* of U_n. We now observe that the *1/r*-potential, U_n in Eq.(4.3) is determined by a nonlinear first-order differential equation:

$$\frac{dU_n}{dr} = \frac{U_n^2}{g_n^2} \quad \text{or} \quad \frac{dA_{0n}}{dr} = \frac{A_{0n}^2}{g_n} \qquad (4.4)$$

This is a special case of Riccati's equation[19] which suggests that an obvious step to take in order to achieve a progressive generalization of the potential energy function in Eq.(4.3) to include other non-potential forces, such as forces due to contact/overlap of wavepackets representing extended particles, is to replace Eq.(4.4) by the most general form of the Riccati equation:

$$\frac{dU_n}{dr} = PU_n^2 + QU_n + R \quad \text{or} \quad \frac{\partial A_{0n}}{\partial r} = (1/g_n)A_{0n}^2 + zA_{0n} - k \qquad (4.5)$$

where P,Q,R and V,k are constants (in general, functions of r). Note that the derivative and nonlinear parts, $\{\partial A_{0n}/\partial r - (1/g_n)A_{0n}^2\}$, of Eq.(4.5) may be re-interpreted as appropriate component of the SU(2) Yang-Mills gauge field[20]:

$$F_{mn}^a \equiv \frac{\partial A_n^a}{\partial x^m} - \frac{\partial A_m^a}{\partial x^n} + g_0 e^{abc} A_m^b A_n^c$$

where g_0 is a constant. Moreover, as is well-known [p. 49 of ref.[21]), Schwarzschild's "solution",

$$g_{rr}(r) = [1 - 2m(r)/r]^{-1},\ m(r) \equiv k\int_0^r 4pr^2 r(r)dr\ ,$$

of Einstein's equation for the gravitational field is governed by Riccati's equation of the form in Eq.(4.5) with the following *r*-dependent coefficients, $P,\ Q,\ R$:

$$\frac{dg_{rr}}{dr} - \frac{1}{r}g_{rr} - (8pkrr - \frac{1}{r})g_{rr}^2 = 0 \qquad (4.6)$$

where $ds^2 = g_{00}(r)dr^2 + g_{rr}(r)dr^2 + r^2(dq^2 + \sin^2 q dj^2)$, and $16pkr$ is the scalar curvature, r being the (curvature) energy density.

Furthermore, we can convert the potential A_{0n} into a (Higgs) scalar field, $f \equiv f(r)$, by using the well-known standard transformation that converts the *nonlinear* first-order Riccati Eq.(4.5) into a *linear* second-order differential equation for f (see, p.201 of Piaggio[19]) :

$$A_{0n} = -g_n \frac{d\log(f)}{dr} \equiv -g_n \frac{\left(\frac{df}{dr}\right)}{f} \equiv -g_n \frac{f_1}{f}, \qquad (4.7)$$

where $f_1 \equiv \frac{df}{dr}$. Note that this transformation may be rewritten as a (Weyl-like) gauge principle in the integral form:

$$f = \exp\left(-(1/g_n)\int_0^r A_{0n}dr\right). \qquad (4.8)$$

From (4.7) we find

$$\frac{dA_{0n}}{dr} \equiv -g_n \frac{f_2}{f} + g_n \frac{f_1^2}{f^2}, \qquad (4.9)$$

so that, on substitution in Eq.(4.5), the terms in f_1^2 disappear, and hence, on multiplying the resulting equation through by f/g_n, we obtain a *linear* second-order differential equation:

$$f_2 - zf_1 - (k/g_n)f = 0\ . \qquad (4.10)$$

And if we select $z \equiv -2/r$ and put $m_f^2 = k/g_n$, this equation takes the standard form

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{k}{g_n} f \equiv \left(\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} - m_f^2 \right) f(r) = 0 \qquad (4.11)$$

which is the static limit of the Klein-Gordon equation for a spin-0 scalar field f of mass m_f in units such that $\hbar = c_0 = 1$. It is in this sense that we may agree with the statement on page 53 of GUT-II that "the present work seems to suggest that the wave equation is hidden in the generic conservation laws".

As is well-known, a solution of Eq.(4.11) has the form of a Yukawa potential:

$$f(r) = (-g_n^2 / r)\exp(-m_f r) \qquad (4.12)$$

which characterizes a strong (nuclear) force. However, the primary interest of this result lies in the fact that, having transformed Riccati's equation for A_{0n} into Klein-Gordon equation for Higgs field $f(x)$, we can use standard (Lagrangian field theoretic) techniques to put together the spin-0 field, $f(x)$, a spin-$\frac{1}{2}$ field, $y(x)$, and the remaining components $(A_k, k = 1,2,3)$ and other bosons making up the Yang-Mills-fields, $A_m^a(x)$, to unify electromagnetic, weak and strong interactions in a gauge and conformal invariant model. Typically, in the zero gauge-coupling limit $(A_m^a \equiv 0)$ we have the following conserved current and field equations $(\hbar = c_0 = 1)$:

$$J_m(x) = \bar{y}(x) g_m y(x) - i f^*(x) \overleftrightarrow{\partial}_m f(x) \qquad (4.13a)$$

$$\left(i g_m \partial_m - M \right) y(x) = \Gamma y(x) f(x) \qquad (4.13b)$$

$$\left(\partial^2 + m_f^2 \right) f(x) = -\bar{y}(x) \Gamma y(x) f(x) \qquad (4.13c)$$

where M is the mass of the spinor field $y(x)$ and m_f the mass of the Higgs scalar field $f(x)$. Since the wavefunction $y(x)$ of the spinor field has spin $(\uparrow, \downarrow)$ as dichotomic variable, considerable simplification can be achieved by using spin labels, $y_\uparrow(x)$ and $y_\downarrow(x)$, to approximate the spin-0 Higgs field through a relation (involving m_f for dimensional reasons):

$$f = m_f^{-\frac{1}{2}} y_\uparrow \text{ or } m_f^{-\frac{1}{2}} y_\downarrow . \qquad (4.14)$$

This enables us to abstract from Eq.(4.13a), a current density like the one in Eq.(2.11)

$$J_m(x) = \bar{y}(x) \left(g_m - i m_f^{-1} \overleftrightarrow{\partial}_m \right) y(x) \qquad (4.15a)$$

which is conserved by the following relativistic wave equation:

$$\left(i g_m \partial^m - (m_f + M) - (1/m_f) \partial_m \partial^m \right) y = 0 . \qquad (4.15b)$$

This relativistic wave equation for a spin-$\frac{1}{2}$ field has non-trivial mass eigenvalues in the limit of exact conformal invariance and was used by Animalu[22] in 1971 to determine the mass ratio of quarks consistent with the mass spectra of the observed strongly interacting particles (hadrons = baryons and mesons).

4.3 Relation to Spontaneous Symmetry Breaking

We could, as in Yang-Mills-Higgs gauge theories[20] of spontaneous symmetry breaking, expand the unified force potential to quartic order :

$$\frac{dU}{dr} = KU^4 + PU^2 + R \equiv K(|U|^2 - U_T^2)^2 . \qquad (4.16)$$

This makes the attainment of asymptotic freedom ($dU/dr = 0$) manifest itself in the same way as spontaneous symmetry breaking, as the sketches of dU/dr versus $|U|$ in Fig. 4 suggest.

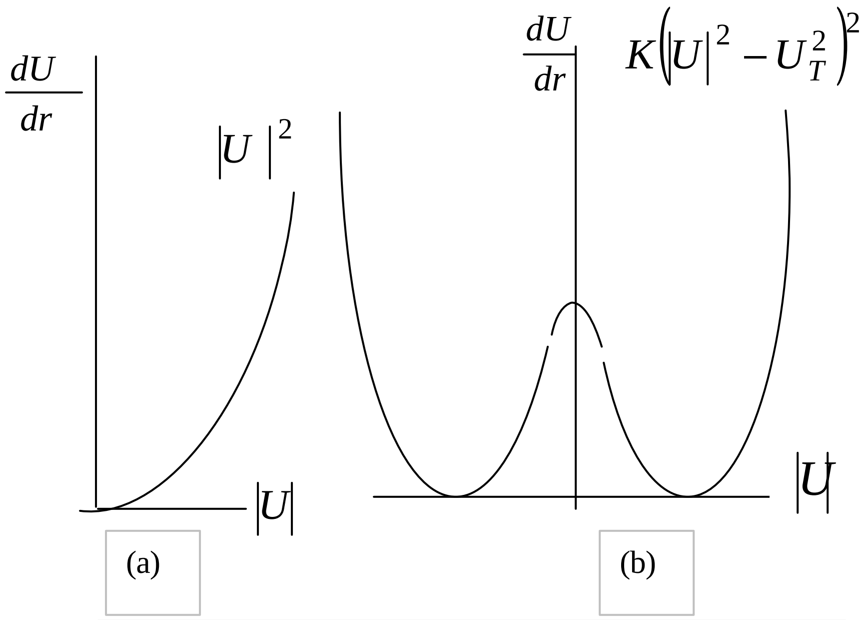


Fig. 4: Graphs of dU/dr versus $|U|$ for (a) Coulomb force in Eq.(4.4) and (b) the generalization in Eq.(4.16).

4.4 Realization of Contact/Overlap Interaction

The objections to the hypothesis by Rutherford[23] in 1920 that a neutral particle, subsequently identified by Chadwick[23] in 1932 as the neutron, could result by compressing hydrogen atom until its negatively charged orbital electron (e^-) touches and neutralizes the positively charged proton (p^+) prompted Santilli[24] to question the applicability of quantum mechanics and Einstein's special relativity to the physical conditions of compression envisaged by Rutherford. Accordingly, Santilli[24] propose in 1978 a generalization of quantum mechanics into a new discipline called Strong Interaction or Hadronic Mechanics, which he subsequently used to account for all the physical characteristics of the neutron as a nonlocal-nonhamiltonian structure of a compressed hydrogen atom system, $n = (p^+ \uparrow, e^- \downarrow)_{HM}$, (called Rutherford-Santilli neutron) in which the binding force resulting from contact/overlap of the wavepackets of p^+ and e^- was represented by a Hulthen potential. The Hulthen potential is thus a candidate for the "other forces" envisaged by Oyibo on p. 9 of GUT-I.

An explicit construction of the Hulthen potential for such a Rutherford-Santilli neutron by Animalu[25] was based on the generalization from (4.4) to (4.5) and an exact solution of the Riccati equation (4.5) when P, Q, R are constants (independent of *r*). As shown in Appendix B, one obtains a general solution of the form:

$$V_H(r) = \frac{\left(\frac{Q-\sqrt{Q^2-4PR}}{2P}\right)\times\frac{\left(V_0+\frac{Q+\sqrt{Q^2-4PR}}{2P}\right)}{\left(V_0+\frac{Q-\sqrt{Q^2-4PR}}{2P}\right)}}{\exp\left(\frac{r\sqrt{Q^2-4PR}}{P}\right)-\frac{\left(V_0+\frac{Q+\sqrt{Q^2-4PR}}{2P}\right)}{\left(V_0+\frac{-Q+\sqrt{Q^2-4PR}}{2P}\right)}} \qquad (4.17)$$

where $V_0 \equiv V_H(0)$ is a constant of integration. This is equivalent to the standard form of a Hulthen potential,

$$V_H(r) = \frac{-Mc_0^2}{\exp[(m_0c_0/\hbar)r]-1} \qquad (4.18)$$

provided that $\left(V_0+\frac{Q+\sqrt{Q^2-4PR}}{2P}\right)\Big/\left(V_0+\frac{-Q+\sqrt{Q^2-4PR}}{2P}\right)=1,$

giving

$$P \neq 0,\ Q = 0,\ R < 0; \qquad (4.19)$$

and hence, since $V_0 \equiv V_H(0) \to \infty$, (in units such that $\hbar = c_0 = 1$)

$$m_0 = 2\sqrt{-\frac{R}{P}},\ M = \sqrt{\frac{-R}{P}}\left(\frac{V_0+\sqrt{\frac{-R}{P}}}{V_0-\sqrt{\frac{-R}{P}}}\right) \to \sqrt{\frac{-R}{P}} \equiv \tfrac{1}{2}m_0 \qquad (4.20)$$

By comparing this potential with an expansion of the Hulthen potential defined in the following way

$$U_H(r) = \frac{-Me^{-m_0r}}{1-e^{-m_0r}} \approx \frac{-Me^{-m_0r}}{m_0r+O(r^2)} \approx \frac{-(M/m_0)e^{-m_0r}}{r} \equiv f(r) \qquad (4.21)$$

one finds $m_f = m_0$ and $g^2 \equiv M/m_f = \frac{1}{2}$ which is in the range of strong nuclear coupling constants. Moreover, one can solve the eigenvalue problem for the compressed H-atom in the form (Animalu[25]):

$$\left(-\frac{1}{2\overline{m}^*}\frac{\hbar^2}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+U_H(r)\right)y_\uparrow(r) = Ey_\uparrow(r) \qquad (4.22)$$

$\overline{m}^*$ being an effective reduced mass of the electron, and find that the minimum size of the potential hole compatible with a bound state characterizing a spin-$\frac{1}{2}$ neutron state $y_\uparrow(r)$ is given by $M = m_p^2/\overline{m}_e = \frac{1}{2}(e^2/\sqrt{2}G_F) = 38GeV/c^2$, where G_F is the Fermi coupling constant for weak interactions and *e* is the electron charge of the proton.

5. THE GAGUT'S QUANTUM THEOREM

The representation of the neutron as a compressed hydrogen atom now serves to elucidate by far the most controversial claim in Oyibo's GUT, namely the "proof" on p.81 of GUT-II that "The concentration of the Unified Force Field, called Matter, has one fundamental building block, which is the hydrogen atom". A close reading shows that Oyibo's proof is based on the existence of not only the normal hydrogen ($H = pe^-$) but also its two isotopes ($H_n = npe^-$) and ($H_{nn} = nnpe^-$) containing one neutron (n) and two neutrons (nn) respectively. For this reason, he was able to represent the Helium as $He = 2H_n$. There is no difficulty in treating only $H = pe^-$ as the only fundamental building block, as long as one can represent the neutron as a compressed (pe^-) system.

6. DISCUSSION AND CONCLUSION

In this paper, I have reviewed the analytical framework of Professor Gabriel Oyibo's GUT from both mathematical and physical points of view. It is apparent that the novelty of the GUT program lies in its foundation on the conformal group of transformations which, unfortunately, is one of the least understood approximate space-time and momentum space symmetries in nature, involving conserved dilatation vector and conformal tensor currents and their related gauge principles and hypothetical particles, like the "dilaton" (which is the celebrated *n=0* "wave component" of his generic solutions). I have, therefore, taken the liberty of elaborating on the group of conformal transformations in Appendix A, in order to eliminate the apparent obstacle in the appreciation of the profound nature of Oyibo's contribution. One can argue that the observed absence of scale invariance (and hence conformal invariance) is due to the *coexistence* of a dilation current $D_m = x_l \Theta^l_m$ and a boson field (dilaton), $S \equiv m_S^{-2} \Theta^l_l$, such that $\partial_m S = -D_m$. Such an argument is currently used to "explain away" the observed absence of magnetic monopole associated with conservation of magnetic charge (the dual of the usual electromagnetic current) by the dual symmetry of Maxwell-Dirac electrodynamics. Coincidentally, the uncertainty in Oyibo's definition of the Unified Force Field in GUT also stems from the lack of connection in the book between the underlying group of conformal transformations and the gauge principle; but, again, I took the liberty of rectifying this in Sec. 4.2 of this review in order to achieve the desired generalization in the form of a soluble nonlinear Riccati's equation which has geometrical origin and meaning (see, E. Goursat[19]).

It is also necessary to underline another important feature of this review, namely the construction of the realizations of the hierarchy of solutions of the generic equations of GUT for *n=2,3,4* through the link between conformal transformation and the Lie-Santilli iso-approach to unification (Animalu[3]) based on Lax pairing of symmetric and antisymmetric tensors and leading to a generalization of the procedure in Sec. 3.5.

We are therefore led to the conclusion that Professor Oyibo's GUT has a sound mathematical and physical basis and is a viable framework for a Grand Unified Field Theory of Everything.

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APPENDIX A

CONFORMAL GROUP OF TRANSFORMATIONS

IN SPACE-TIME GEOMETRY

It is well-known that the group of conformal transformations in space-time and four-momentum space contains, in addition to the (Lorentz) transformations of special relativity:

$$x'_m = \Lambda^n_m x_n + a_m; \qquad p'_m = \Lambda^n_m p_n + b_m \tag{A.1}$$

also the dilatation:

$$x'_m = r x_m; \qquad p'_m = r^{-1} p_m \tag{A.2}$$

and the special conformal transformation:

$$x'_m = (x_m + f_m x^2)/(1 + 2 f^l x_l + f^2 x^2); \;\; p'_m = (p_m + k_m p^2)/(1 + 2k^l p_l + k^2 p^2) \tag{A.3}$$

The special conformal transformations may be built up from two inversions (*I*) and a translation (*T*) in the form *ITI*, as follows:

$$x_m \xrightarrow[I]{} x_m / x^2 \xrightarrow[T]{} x_m / x^2 + f_m \xrightarrow[I]{} \frac{x_m / x^2 + f_m}{\left(x_m / x^2 + f_m\right)^2} \tag{A.4}$$

The conserved dilatation is a scalar for a single particle,

$$D = x^l p_l , \tag{A.5a}$$

but a four-vector for a quantum field

$$D_m = x_l \Theta^l_m , \tag{A.5b}$$

while the conserved conformal current is a four-vector for a single particle

$$K_m = (2x_m x_n - g_{mn} x^2) p^n \tag{A.6a}$$

but a tensor for a quantum field

$$K_{mn} = (2x_m x_l - g_{ml} x^2)\Theta^l_m \tag{A.6b}$$

where Θ_{mn} is the symmetric energy-momentum tensor of the quantum field.

We observe that K_m may, by virtue of (A.6a) and A.5a) be rewritten in the form

$$x^n (M_{mn} + g_{mn} D) = K_m , \tag{A.7a}$$

where $M_{mn} = x_m p_n - x_n p_m$ is the antisymmetric angular momentum tensor. This may be compared with the Maxwell equation in the form

$$\partial^n (F_{mn} + g_{mn} f) = J_m \tag{A.7b}$$

where $f \equiv \partial^r A_r$ is the divergence of the electromagnetic vector potential which vanishes if the Lorentz condition $(\partial^r A_r = 0)$ is obeyed. Thus in terms of the vector potential (A_m), the electromagnetic vector current

$$J_m^{V=}(2\partial_m \partial_l - g_{ml}\partial^2)A^l \qquad \text{(A.8)}$$

is seen to be the momentum space analog of the conformal current.

This analogy makes it necessary to unify conformal transformations in space-time and in momentum space geometries, defining a relativistic (x_m, p_m) – "phase space" analogous to the usual (*qp*)- "phase space" of classical (Hamiltonian) mechanics. Such a unification occurs in an algebraically symmetric manner at short distance, $s = 1/m$, (in units such that $\hbar = c = 1$), i.e. at distances on the order of the Compton wavelength of an elementary particle, as follows. We unify the two equations in (A.3) in the form (treating x_m and p_m as c-numbers)

$$x_m \to x'_m = (x_m - s^2 p_m)/s^2 \quad (s^2 = x^2 \equiv x^l x_l \neq 0) \qquad \text{(A.9a)}$$

$$p_m \to p'_m = (p_m - m^2 x_m)/s^2 \quad (m^2 = p^2 \equiv p^l p_l \neq 0) \qquad \text{(A.9b)}$$

where $s^2 = (1 + 2x^l p_l + x^2 p^2)$. But, because $p_m = m(dx_m/ds)$, we have from $s^2 = x^l x_l$, that $D = x^m p_m = sm$ and hence $D = x^2 p^2 = s^2 m^2 = D^2$, so that $s^2 = (1 - D)^2$.

Now (A.9a) and (A.9b) define an SU(2) group of transformations if the determinant of the transformation is unity, i.e., if

$$s^{-4}(1 - D^2) = +1 \quad \text{i.e.,} \quad (1 - D^2)^2 = (1 - D)^3$$

Accordingly, this gauge symmetry is broken when $D = 1$, i.e., when $sm = 1$, corresponding to short distances. In this circumstance,

$$p'_m = p_m - m^2 x_m = 0$$

is the analog of the gauge-invariant substitution in the electromagnetic case and the singularity at

$$p_m = m^2 x_m \qquad \text{(A.10a)}$$

is the analog of $p_m = eA_m$ in the electromagnetic case. However, the relation (A.10a) has a simple geometrical interpretation, namely that since $m^2 = p^2 \equiv p^l p_l$, it may be rewritten in the form

$$p_m / p^2 = x_m \qquad \text{(A.10b)}$$

which, geometrically speaking, means that x_m is the *inverse point* of p_m with respect to the quadric surface $p^l p_l = m^2$. This is a *new* type of inversion symmetry, unlike parity (space-inversion) and time-reversal (time-inversion). Its significance is that it transforms a *time-like* vector into a *space-like* vector, or slower-than-light particles (tardyons) into faster-than-light particles (tachyons), just as charge conjugation and parity transform particles into their antiparticles. It is this novel feature of conformal transformations that has enriched the geometrical and physical content of Oyibo's GUT (as evident in Secs. 3.3 and 3.4 of this review).

APPENDIX B

SOLUTION OF THE RICCATI EQUATION

In this appendix I present (Charles Animalu's) closed form solution of the Riccati's equation:

$$\frac{dV_H}{dr} = PV_H^2 + QV_H + R,$$

(P, Q, R) being constants. The equation may be expressed and integrated in the form:

$$\int_0^r dr = \int_{V_0}^{V_H} \frac{dV_H}{(V_H - A)(V_H - B)},$$

where

$$A = \frac{-Q + \sqrt{Q^2 - 4PR}}{2P}\,;\; B = \frac{-Q - \sqrt{Q^2 - 4PR}}{2P}$$

Thus, by using the partial fraction expansion

$$\frac{1}{(V_H - A)(V_H - B)} \equiv \frac{A - B}{(V_H - A)} + \frac{B - A}{(V_H - B)}$$

we find

$$\begin{aligned} r &= \int_{V_0}^{V_H} \frac{(A-B)dV_H}{(V_H - A)} + \int_{V_0}^{V_H} \frac{(B-A)dV_H}{(V_H - B)} \\ &= (A-B)\ln\left[\frac{(V_H(r) - A)}{(V_0 - A)}\right] + (B-A)\ln\left[\frac{(V_H(r) - B)}{(V_0 - B)}\right] \\ &= (A-B)\ln\left[\frac{(V_H(r) - A)}{(V_H(r) - B)}\frac{(V_0 - B)}{(V_0 - A)}\right] \end{aligned}$$

i.e., $\exp\left[\frac{r}{A-B}\right] = \frac{(V_H(r) - A)}{(V_H(r) - B)}\frac{(V_0 - B)}{(V_0 - A)},$

$$\left(V_H(r)-B\right)\exp\left[\frac{r}{A-B}\right]=\left(V_H(r)-A\right)\frac{\left(V_0-B\right)}{\left(V_0-A\right)},$$

$$V_H(r)\left(\exp\left[\frac{r}{A-B}\right]+\frac{\left(-V_0+B\right)}{\left(V_0-A\right)}\right)=-B\frac{\left(V_0-B\right)}{\left(V_0-A\right)},$$

$$\therefore\quad V_H(r)=\frac{\left(\dfrac{Q-\sqrt{Q^2-4PR}}{2P}\right)\times\dfrac{\left(V_0+\dfrac{Q+\sqrt{Q^2-4PR}}{2P}\right)}{\left(V_0+\dfrac{Q-\sqrt{Q^2-4PR}}{2P}\right)}}{\exp\left(\dfrac{r\sqrt{Q^2-4PR}}{P}\right)-\dfrac{\left(V_0+\dfrac{Q+\sqrt{Q^2-4PR}}{2P}\right)}{\left(V_0+\dfrac{-Q+\sqrt{Q^2-4PR}}{2P}\right)}}$$

$$\equiv\frac{-M}{\exp(m_0 r)-1}.$$

This is the general result that we are after.