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ARITHMETIC IN THE MIDDLE KINGDOM

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In the Cairo Museum, under the numbers 25367 and 25368, are two Egyptian writing-tablets of the usual type, made of wood and covered on both sides with a layer of polished plaster to take the writing. Each measures roughly 18 inches by 10, and each is inscribed in black ink in the hieratic script. The first tablet bears on one side the scanty remains of a letter and a list of servants, and on the other some mathematical calculations: the list is dated in the 28th year of a king whose name is not given. The second tablet bears on one side a list of twenty-seven servants, and some calculations which are continued on the other face. The style of the writing and the names of the servants fix the date of the tablets to the Middle Kingdom, about 2000—1800 B.C. They are said to have been found at Akhmim.

These two tablets were first published by Daressy[1] who supposed them to contain tables or examples of multiplications of whole numbers and fractions, more particularly the fractions $\frac{1}{2}$ and its powers $\frac{1}{4}$, $\frac{1}{8}$, and so on. This explanation was entirely erroneous, as will be seen in the sequel. Möller[2] was the first to observe that among the signs used in the calculations were the now well-known signs for the $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ etc. of the *hekat* or bushel. But he failed to see the drift of the exercises, as is clear from his statement that in them these various parts of the *hekat* were multiplied by one another, a process as abhorrent to the Egyptian sense of units and dimensions as the multiplication of half-an-ounce by a quarter-of-a-pound would be to ours. Attention was called to Möller's error by Sethe[3], but in a manner which leaves little doubt that this acute thinker had failed to see the exact import of the figures. Since that time no one, so far as I know, has occupied himself with the tablets. Rightly understood they form such an admirable commentary on Egyptian mathematical methods that they are well worthy of close study.

Their purpose may be explained in a few words. The Egyptian *hekat* or bushel, a measure of capacity used mainly for measuring grain, was for practical purposes divided by continuous halving, that is to say the parts used in everyday measurements and calculations were the $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$ and $\frac{1}{64}$. Anything smaller than $\frac{1}{64}$ was expressed in terms of a small measure called the *ro*, of which there were 320 in a bushel and consequently 5 in $\frac{1}{64}$ of a bushel, 10 in $\frac{1}{32}$, 20 in $\frac{1}{16}$, 40 in $\frac{1}{8}$, 80 in $\frac{1}{4}$ and 160 in $\frac{1}{2}$ bushel.

Having once fixed on these particular fractions of the bushel for practical use the Egyptians refused to employ any others. Thus they never spoke of one-seventh or one-third of a bushel, but reduced these to terms of the $\frac{1}{2}$, $\frac{1}{4}$ etc., down to $\frac{1}{64}$th, and the small remainders, if any, to the *ro* and its fractions. We behave in a similar manner, for one-seventh of a ton conveys little to most of us until we have reduced it to hundredweights, quarters, pounds and ounces, these being the particular divisions of the ton which we

[1] *Rec. de Trav.*, XXVIII, 62–72.

[2] *Zeitschr. f. äg. Spr.*, 48, 99.

[3] Sethe, *Von Zahlen und Zahlworten*, 74, n. 2. The statement there made that on the tablets "the whole numbers stand for bushels" is not correct.

recognize as separate units. The Egyptian was, however, more methodical than we are, for each of his units was half the next above it, except the *ro*, which was one-fifth of the $\frac{1}{64}$ bushel. All these dimidiated parts, the $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ etc., were just as much real units to him as our pound and ounce are to us: each probably had its special name, and the signs used to represent them were, in later times at least, identified each with a part of the picture of the magic eye of Horus.

The calculations on the tablets are nothing more than the expression of various fractions ($\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{13}$) of the bushel in terms of the recognized divisions, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ etc., and the *ro*. For example, one-eleventh of a bushel is shown to be equivalent to ($\frac{1}{16} + \frac{1}{64}$) bushel + $4\frac{1}{11}$ *ro*, and this was the only correct way of expressing one-eleventh of a bushel in Egyptian.

Before we can follow the working by which this result is reached a word concerning Egyptian multiplication and division is necessary. The Egyptian only multiplied directly by two figures, 2 and 10. The latter was obviously chosen because the numeral system was a decimal one, so that in order to multiply say 76 (𓎉𓎉𓎉𓎉𓎉𓎉𓎉𓏺𓏺𓏺𓏺𓏺𓏺) by 10 all we need to do is to substitute hundred-signs for tens, and ten-signs for units, 𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓎉𓎉𓎉𓎉𓎉𓎉, 760. The figure 2 was chosen simply because it was the lowest digit after 1. All other multiplication was built up on this. To multiply by 3 you multiplied by 2 and added in the original number. To multiply 5 by 13 you did as follows:

$$
\begin{aligned}
—1 \times 5 &= 5\\
2 \times 5 &= 10\\
—4 \times 5 &= 20\\
—8 \times 5 &= 40
\end{aligned}
$$

You next observed that, of the multipliers on the left, 8, 4 and 1 add up to 13, so that to get 13 times 5 all that was necessary was to add the these products in the right-hand column corresponding to the multipliers 1, 4 and 8, *viz.* 40, 20 and 5. It was customary to place a tick against the multipliers so chosen, in order to assist the eye in picking out and adding the correct products on the right.

Division in Egyptian was merely a reversed form of multiplication, for the Egyptian instead of saying divide 77 by 7 said operate on 7 to find 77. Here again 2 and 10 were the only whole numbers used as multipliers.

$$
\begin{aligned}
—1 \times 7 &= 7\\
—2 \times 7 &= 14\\
4 \times 7 &= 28\\
—8 \times 7 &= 56
\end{aligned}
$$

We now observe that in the right-hand column the products 7, 14 and 56 add up to 77; we therefore tick off those lines and add the corresponding multipliers 1, 2 and 8, which give the correct 11.

We are now in a position to follow the working on the tablets. Let us take the example in which $\frac{1}{10}$ of a bushel is to be found. It is as follows:

1	10
10	100
20	200
2	20

1	$(\frac{1}{16}+\frac{1}{32})$ bushel + 2 *ro*
—2	$(\frac{1}{8}+\frac{1}{16})$ „ + 4 „
4	$(\frac{1}{4}+\frac{1}{8}+\frac{1}{64})$ bushel + 3 *ro*
—8	$(\frac{1}{2}+\frac{1}{4}+\frac{1}{32}+\frac{1}{64})$ bushel + 1 *ro*

The first step (bottom of last page) consists in reducing one-tenth of a bushel to *ro*, and since a bushel contains 320 *ro* this is equivalent to dividing 320 by 10, *i.e.* multiplying 10 to find 320. In modern form the first four lines would read:

$$1 \times 10 = 10$$
$$—10 \times 10 = 100$$
$$—20 \times 10 = 200$$
$$—2 \times 10 = 20$$

Looking down the column of products on the right we notice that 100 + 200 + 20 gives the required 320 *ro*, and picking out the multipliers corresponding to these products we find them to be 10 + 20 + 2. In an ordinary multiplication we should at once add these and get 32 *ro*, but that is not done here, for the 10 and 20 *ro* are precisely $\frac{1}{32}$ and $\frac{1}{16}$ of a bushel respectively, and enable us to give our answer in the required form $(\frac{1}{16}+\frac{1}{32})$ bushel + 2 *ro*.

This then is the correct way of expressing one-tenth of a bushel in Egyptian. It now remains to prove the answer. If this is $\frac{1}{10}$ of a bushel then ten times this amount should come to a bushel, and we now set out to multiply our answer by 10. This is done in the last four lines (top of this page). In view of what has been said above the multiplication needs little comment. Each multiplication is by 2. The dimidiated fractions of the bushel lend themselves admirably to this, for $\frac{1}{64}$ becomes $\frac{1}{32}$ and so on. Whenever the *ro* come to more than 5 (*e.g.* in multiplying 3 *ro* or 4 *ro* by 2) the 5 must be taken out and expressed as $\frac{1}{64}$ bushel and the remainder left as *ro*. The 8-line and the 2-line are ticked off, since 8 times + 2 times is 10 times, and the products on the right in these two lines will be found when added[1] to give exactly a bushel.

Slightly more complicated is the following sum, in which one-seventh of a bushel is worked out:

1	7
10	70
20	140
40	280
2	12 (error for 14)
4	24 (error for 28)
$\frac{1}{7}$	1
$\frac{1}{4}+\frac{1}{28}$	2
$\frac{1}{2}+\frac{1}{14}$	4

—1	$(\frac{1}{8}+\frac{1}{64})$ bushel	$+ (1\frac{1}{7}+\frac{1}{14})$[2] *ro*
—2	$(\frac{1}{4}+\frac{1}{32})$ „	$+ (1\frac{1}{4}+\frac{1}{7}+\frac{1}{28})$ *ro*
—4	$(\frac{1}{2}+\frac{1}{16})$ „	$+ (2\frac{1}{2}+\frac{1}{4}+\frac{1}{14}+\frac{1}{28})$[3] *ro*

[1] The dimidiated fractions render such an addition very simple. The Egyptian doubtless did it in his head.

[2] Error for $(\frac{1}{2}+\frac{1}{7}+\frac{1}{14})$.

[3] The $\frac{1}{14}$ is erroneously omitted.

The first step, comprising the first six lines, is the division of 320 *ro*, or one bushel, by 7. Note how the multipliers chosen, 10 *ro*, 20 *ro* etc., are such as can be directly expressed as dimidiated fractions of the bushel. Adding the products 7, 280 and 28, we get 315, and the multipliers corresponding are 1, 40 and 4 (ticks omitted), the sum of which, for a reason which will appear presently, we will write as $5 + 40$. But 315 is 5 short of 320, and so we must still divide this 5 by 7, and add the result to our quotient $5 + 40$. This is done in lines seven to nine. And here another vital point in Egyptian mathematics comes to the fore. The Egyptian never used, and had no notation for, fractions whose numerator was greater than 1, with the sole exception of $\frac{2}{3}$. Thus he could not say, as we should, that $5 \div 7$ was $\frac{5}{7}$. What he did was to multiply 7 to get 5, keeping his trial multipliers always in the form of fractions whose numerators were 1. If this step were conceivable in modern mathematics it would have to be set out as follows:

$$—\tfrac{1}{7} \times 7 = 1$$
$$(\tfrac{1}{4} + \tfrac{1}{28}) \times 7 = 2$$
$$—(\tfrac{1}{2} + \tfrac{1}{14}) \times 7 = 4$$

Here it will be seen that each line is got from the last by doubling. But since the Egyptian may not use and has no notation for $\frac{2}{7}$ he is forced to break it up into the sum of two fractions which he can express, namely $(\frac{1}{4} + \frac{1}{28})$. This he did by reference to his tables: two sets of tables have actually survived in which the fractions whose numerators are 2 and whose denominators are the various odd numbers 3, 5, 7 etc. are split up each into the sum of two or more fractions whose numerators are unity[1]. On doubling again the $(\frac{1}{4} + \frac{1}{28})$ obviously becomes $(\frac{1}{2} + \frac{1}{14})$, thus avoiding the use of the impossible $\frac{4}{7}$. The products on the right in the first and third lines now add up to the required 5, and the corresponding multipliers in these lines must when added give us the quotient when 5 is divided by 7. The result is $(\frac{1}{2} + \frac{1}{7} + \frac{1}{14})$, but the scribe has unfortunately written 1 instead of the $\frac{1}{7}$.

This number of *ro* must now be added on to the original quotient, which was $(5 + 40)$ *ro*, or $(\frac{1}{64} + \frac{1}{8})$ bushel, and we get 320 *ro* divided by $7 = (\frac{1}{8} + \frac{1}{64})$ bushel $+ (\frac{1}{2} + \frac{1}{7} + \frac{1}{14})$ *ro*, which is our answer. Just as in the previous example this is now proved. If it is equivalent to $\frac{1}{7}$ of a bushel we should, if we multiply it by 7, get exactly a bushel. This is done in the last three lines, the multipliers being 1, 2 and 4. These added together give 7, and they are therefore ticked off and the products corresponding to them added and seen to give just a bushel.

In a precisely similar manner the scribe of our tablets has dealt with one-eleventh and one-thirteenth of a bushel. The former he reduces to the form $(\frac{1}{16} + \frac{1}{64})$ bushel $+ 4\frac{1}{11}$ *ro*, and the latter, after an unsuccessful first attempt, he finds quite correctly to be $\frac{1}{16}$ bushel $+ (4\frac{1}{2} + \frac{1}{13} + \frac{1}{26})$ *ro*.

One-third of a bushel is found in quite a different manner, and it is precisely this fact which has misled students of the tablets. The working of this sum is as follows:

1	$\frac{1}{3}$
2	$\frac{2}{3}$
4	$1\frac{1}{3}$

[1] One in the Rhind Math. Pap. and the other in the Kahun fragments.

(One-third of)	$\frac{1}{64}$	bushel	$= 1\frac{2}{3}$ *ro*
„	$\frac{1}{32}$	„	$= 3\frac{1}{3}$ „
„	$\frac{1}{16}$	„	$= \frac{1}{64}$ bushel $+ 1\frac{2}{3}$ *ro*
„	$\frac{1}{8}$	„	$= \frac{1}{32}$ „ $+ 3\frac{1}{3}$ „
„	$\frac{1}{4}$	„	$= (\frac{1}{16} + \frac{1}{64})$ bushel $+ 1\frac{2}{3}$ *ro*
„	$\frac{1}{2}$	„	$= (\frac{1}{8} + \frac{1}{32})$ „ $+ 3\frac{1}{3}$ „
„	—1	„	$= (\frac{1}{4} + \frac{1}{16} + \frac{1}{64})$ „ $+ 1\frac{2}{3}$ „
	—2	„	$= (\frac{1}{2} + \frac{1}{8} + \frac{1}{32})$ „ $+ 3\frac{1}{3}$ „

The procedure here is as follows. In the first three lines one-third of 5 *ro* is taken and found to amount to $1\frac{2}{3}$ *ro*. The Egyptian way of doing this is to multiply $\frac{1}{3}$ by 5, and in modern form these lines would read:

$$—1 \times \tfrac{1}{3} = \tfrac{1}{3}$$
$$2 \times \tfrac{1}{3} = \tfrac{2}{3}$$
$$—4 \times \tfrac{1}{3} = 1\tfrac{1}{3}$$

The addition of lines one and three gives us the required 5 times $\frac{1}{3}$ is $1\frac{2}{3}$. Having obtained the equation $1\frac{2}{3}$ *ro* = one-third of 5 *ro* (or $\frac{1}{64}$ bushel) the rest is easy, for we have only to go on continuously doubling both sides until the one-third of $\frac{1}{64}$ bushel becomes one-third of 1 bushel. By this time the $1\frac{2}{3}$ *ro* has become $(\frac{1}{4} + \frac{1}{16} + \frac{1}{64})$ bushel $+ 1\frac{2}{3}$ *ro*, which is the answer. This is last of all proved by multiplying by 3, *i.e.* adding 2-times to 1-times, and showing by addition that the result is 1 bushel.

Why was one-third treated differently from the other fractions? Herein lies yet one more valuable lesson in Egyptian arithmetic. The Egyptian reckoner, although not too fond of fractions and forced to avoid all but those whose numerator was unity, was an expert in the use of one-third. Two-thirds was the only exception to his rule concerning numerators, and, strange as it may seem to us, he was capable of taking $\frac{2}{3}$ of a number in a single process, which is equivalent to saying that he used the $\frac{2}{3}$-times table and probably knew it off by heart. Stranger still, he obtained one-third of a quantity not by dividing it by 3 but by halving two-thirds of it[1].

In the case before us he saw that no more formidable fractions than thirds of a *ro* would be involved, and no more complicated process than doubling them. Hence he abandoned the usual method of dividing 320 *ro* by 3 in favour of the more simple division of 5 *ro* by 3 followed by continuous doublings.

Truly might it be said that he who has closely studied these two tablets and understood them has little to learn concerning the elementary processes of Egyptian arithmetic.

[1] In this sum he might have found one-third of 5 *ro* in this way instead of multiplying $\frac{1}{3}$ by 5.