FIRST QUALIFYING EXAM

January 7, 2004

There are two sections, Algebra and Analysis. Solve **FOUR** *problems from each section. Do* **NOT** *hand in more than four problems for a section; if you do, only the first four problems will be counted. Do each problem on a separate sheet. Show all work. Be sure to write your person number clearly on* **EACH** *sheet. Remember, your goal is to convince the grader that you know what you are doing.*

Section I. Algebra. Do any four problems.

1. Let V be an n-dimensional vector space over a field F and let $T\colon V \to V$ be a linear transformation. If k is the dimension of the null space of T, and r is the dimension of the range of T, prove that $k + r = n$.

2. Let G be an additive subgroup of $\mathbf{R}$ and let a, b, c, d be elements of G. Suppose $a + b = c + d$. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Find the eigenvalues of A and show that each eigenvalue is an element of G.

3. Suppose V is a real vector space of dimension at least two, and T is a linear transformation on V. Then there is a subspace W of dimension 1 or 2 such that T maps W into itself.

4. Let $\mathcal{P}$ denote the linear space of all real polynomials of degree ≤ 2. Let a and b be distinct real numbers and define a map $\langle\,,\rangle$ from $\mathcal{P} \times \mathcal{P}$ to the real numbers $\mathbf{R}$ by $\langle P(x), Q(x)\rangle = P(a)Q(a) + P(b)Q(b)$. Prove that $\langle\,,\rangle$ is not an inner product on $\mathcal{P}$.

5. Determine the number of group elements of each possible order in the group S_4.

6. Show that an integral domain with the property that every strictly decreasing chain of ideals $\mathcal{I} \supset \mathcal{I} \supset \ldots$ must be finite in length is a field.

7. Let p be an odd prime and let $\mathbf{Z}_p^*$ denote the group $\mathbf{Z}_p - \{0\}$ under multiplication mod p.

i) Prove that the subgroup of squares in $\mathbf{Z}_p^*$ has index 2.

ii) Prove that the product of two non-squares is a square.

8. Let H and K be normal subgroups of a group G. If $G = \{hk \mid h \in H, k \in K\}$, prove that $G/(H \cap K)$ is isomorphic to the direct product of G/H and G/K. Note: $hk \neq kh$ is possible for $h \in H$, $k \in K$, but $kH = Hk$ and $hK = Kh$ is always true by normality.

Section II. Analysis. Do any four problems.

9. Let $f_n(x) = nxe^{-nx^2}$, for $x \in [0,1]$. Show whether or not

i) $f_n(x)$ converges for each $x \in [0,1]$.

ii) $f_n(x)$ converges uniformly on $[0,1]$.

10. Let

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Prove that the partial derivatives of f exist at $(0,0)$, but that $f(x,y)$ is not differentiable at $(0,0)$.

11. Let C be a compact subset of $\mathbf{R}^n$, and let S be a subset of C such that for any x and y in S, $\|x - y\| \geq 1$. Prove that S is finite.

12. Let $f\colon \mathbf{R} \to \mathbf{R}$ be uniformly continuous, with $f(x) \geq 0$ for all x. If $\int_0^\infty f(x)dx < \infty$, show that $\lim_{x\to\infty} f(x) = 0$.

13. Assume $f\colon \mathbf{R} \to \mathbf{R}$ is continuous and locally one-one, i.e., for each $x \in \mathbf{R}$, there exists $\varepsilon_x > 0$ such that f is one-one on $(x - \varepsilon_x, x + \varepsilon_x)$. Prove that f is one-one on $\mathbf{R}$.

14. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Define $\liminf a_n$ and $\limsup a_n$ by

$$\liminf a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k$$
$$\limsup a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k$$

Show that for any $\varepsilon > 0$, there exists N such that $n \geq N$ implies $a_n \leq \limsup a_n + \varepsilon$. Show that if $\liminf a_n = \limsup a_n = a$, then $\lim_{n\to\infty} a_n = a$.