FIRST QUALIFYING EXAM

August 21, 2003

There are two sections, Algebra and Analysis. Solve **FOUR** *problems from each section. Do* **NOT** *hand in more than four problems for a section; if you do, only the first four problems will be counted. Do each problem on a separate sheet. Show all work. Be sure to write your person number clearly on* **EACH** *sheet. Remember, your goal is to convince the grader that you know what you are doing.*

Section I. Algebra. Do any four problems.

1. Let

$$A := \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

2. A matrix A is *idempotent* if $A^2 = A$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $b \neq 0$. Find a function f from $\mathbf{R}^2$ to $\mathbf{R}$ and a function g from $\mathbf{R}$ to $\mathbf{R}$ such that A is idempotent if and only if $c = f(a, b)$ and $d = g(a)$.

3. Let V be the vector space of all polynomials with real coefficients and degree $\leq n$. Let $P(x)$ be a polynomial in V of degree exactly n. Show that $P(x), P'(x), \ldots, P^{(n)}(x)$ form a basis for V.

4. Let V be a finite-dimensional vector space over the real numbers. Prove that a linear transformation $T \colon V \to V$ is one-to-one if and only if it is onto.

5. i) Give an example of a group G and elements a and b of G such that a and b commute but the order of ab is not the least common multiple of the orders of a and b.

ii) Let a and b be elements of a group G such that a and b commute. Prove that if the order of a is p^r and the order of b is q^s where p and q are distinct primes and r and s are positive integers, then the order of ab is $p^r q^s$.

6. Let R be a commutative ring with 1. An element r of R is *nilpotent* if $a^n = 0$ for some positive integer n. Prove that the set $\mathfrak{I}$ of all nilpotent elements in R is an ideal in R. Prove that the quotient ring $R/\mathfrak{I}$ has no nonzero nilpotent elements.

7. Let $S(x)$ be the polynomial $x^3 + x^2 + 1$ in $\mathbf{Z}_2[x]$. Define a ring homomorphism from $\mathbf{Z}_2[x] \to \mathbf{Z}_2[x]/\langle S(x)\rangle$ by $\varphi(P(x)) = P(x+1) + \langle S(x)\rangle$. Find a generator for the kernel of φ. Write your answer as a polynomial with $\mathbf{Z}_2$ coefficients and prove that you have found a generator.

8. (a) Prove that if A and B are normal subgroups of a group G and $A \cap B = \{e\}$ where e is the identity of G, then $ab = ba$ for all $a \in A$ and $b \in B$.

(b) Use part (a) to prove that if A and B are normal subgroups of a group G such that $A \cdot B = G$ and $A \cap B = \{e\}$, then G is isomorphic to the direct product of A and B.

Section II. Analysis. Do any four problems.

9. Does

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

exist? If so, what is its value? Prove your answers.

10. Let $f\colon \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) := \begin{cases} x^2, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

Prove that f is differentiable at $x = 0$ and discontinuous when $x \neq 0$.

11. Let $f\colon (a,b) \to \mathbf{R}$ be a differentiable function on an open interval (a,b), with $|f'(x)| \leq M$ for all $x \in (a,b)$. Prove that f is uniformly continuous on (a,b).

12. Assume f is a continuous function from the reals into the reals such that for every x and y, $|f(x) - f(y)| \geq |x - y|$. Prove that the range of f consists of all the reals.

13. Let $f\colon [a,b] \to \mathbf{R}$ be bounded. Assume that f is continuous on the half-open interval $(a,b]$. Show directly, using a standard definition of Riemann integrability, that f is integrable on $[a,b]$.

14. Let f be a continuous function on a closed interval $I = [a,b]$ where $f(a) < 0$ and $f(b) > 0$. Let $E = \{x \in I \mid f(x) < 0\}$, and let $u = \sup E$. Prove that $f(u) = 0$.