FIRST QUALIFYING EXAM
January 8, 2003

There are two sections, Algebra and Analysis. Solve **FOUR** *problems from each section. Do* **NOT** *hand in more than four problems for a section; if you do, only the first four problems will be counted. Do each problem on a separate sheet. Show all work. Be sure to write your person number clearly on* **EACH** *sheet. Remember, your goal is to convince the grader that you know what you are doing.*

Section I. Algebra. Do any four problems.

1. Let V and W be vector spaces and let $T\colon V \to W$ be an isomorphism.

a. If $x_1, \dots, x_k$ are linearly independent in V, prove that $Tx_1, \dots, Tx_k$ are linearly independent in W.

b. If $x_1, \dots, x_\ell$ span V, prove that $Tx_1, \dots, Tx_\ell$ span W.

2. Let $\mathcal{P}_\backslash$ denote the vector space of all polynomials with real coefficients and degree $\leq n$, with the usual polynomial addition and scalar multiplication. Prove that the map $T\colon \mathcal{P}_\backslash \to \mathcal{P}_\backslash$ defined by $T(P(x)) = P(x) + P'(x)$ is an isomorphism.

3. If a non-zero vector space V over $\mathbf{R}$ is the linear span of a finite set $S = \{v_1, \dots, v_m\}$, prove that there is a subset B of S such that B is a basis of V.

4. Prove the following version of the Gram-Schmidt Orthogonalization Theorem: If $u_1, \dots, u_m$ are linearly independent vectors in $\mathbf{R}^n$, then there exist uniquely determined orthogonal (perpendicular) vectors $v_1, \dots, v_m$ in $\mathbf{R}^n$ and

uniquely determined scalars $a_{i,j}$ such that

$$\begin{aligned}
u_1 &= v_1 \\
u_2 &= a_{1,2}v_1 + v_2 \\
u_3 &= a_{1,3}v_1 + a_{2,3}v_2 + v_3 \\
\vdots & \quad \vdots \\
u_m &= a_{1,m}v_1 + a_{2,m}v_2 + \ldots + a_{m-1,m}v_{m-1} + v_m.
\end{aligned}$$

5. Let G be a non-trivial finite Abelian group, such that every non-identity element of G has even order. For each integer n, define a mapping $\varphi_n \colon G \to G$ by $\varphi_n(g) = g^n$. Show that φ_n is an automorphism of G if and only if n is odd.

6. Let S be the set consisting of all pairs $v = (v_1, v_2)$ of real numbers, with an associative multiplication given by $v \cdot w = (v_1w_1, v_1w_2 + v_2)$. Let e be the multiplicative identity of S. Let G be the group of all elements in S that have a multiplicative inverse.

(a) Determine the identity e and determine those elements of S that belong to the group G.

(b) Show that $H = \{v \in G \mid v_1 = 1\}$ is a normal subgroup of G.

7. Let R_1 and R_2 be rings with identity, and let $R = R_1 \oplus R_2$ be their direct product. Show that for every ideal $\mathcal{J}$ of R there exist ideals $\mathcal{J}$ of R_i such that $\mathcal{J}$ is of the form $\mathcal{J} = \{(,) \mid \ \in \mathcal{J}$ for $\ = , \}$.

8. Let G be a finite group with center $C = \{a \in G \mid ag = ga \text{ for all } g \in G\}$. Give the proof, using conjugacy classes, of the Class Equation:

$$|G| = |C| + \sum_{i=1}^{k} n_i,$$

where each n_i is an integer greater than 1 and n_i divides $|G|$ for each i.

Section II. Analysis. Do any four problems.

9. Let $f(x, y, z, u, v) = (xy^2 + xzu + yv^2 + 2, u^3yz + 2xv - u^2v^2 - 3)$. Apply the implicit function theorem to show that there is a C^∞ function $g = (g_1, g_2)$ defined on an open neighborhood of $(2, 0, -1)$, with values in $\mathbf{R}^2$, such that $g(2, 0, -1) = (1, 1)$ and $f(x, y, z, g_1(x, y, z), g_2(x, y, z)) = (0, 0)$.

Also find $\partial g_1/\partial x$ and $\partial g_2/\partial x$ at $(2, 0, -1)$.

10. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers such that $|a_n - a_{n+1}| < 1/2^n$ for all n. Prove that $\{a_n\}_{n=1}^\infty$ is a convergent sequence.

11. Find f and g, non-negative continuous functions defined on $(0, \infty)$, that satisfy the following:

$$\int_0^\infty f(x)dx < \infty, \text{ but } \int_0^\infty f(x)^2 dx = \infty$$
$$\int_0^\infty g(x)dx = \infty, \text{ but } \int_0^\infty g(x)^2 dx < \infty$$

12. Let C be a compact set in $\mathbf{R}^n$, and let S be a subset of C such that for any x and y in S, $\|x - y\| \geq 1$. Prove that S is finite.

13. Let $\mathcal{U} = \{\mathcal{U}_\alpha \mid \alpha \in \mathcal{A}\}$ be an open cover of a subset S in $\mathbf{R}^n$. If $\{a_n\}_{n=1}^\infty$ is a sequence in S such that $\lim_{n\to\infty} a_n$ is in U_α for some $\alpha \in A$, prove that there exist $\alpha_1, \ldots, \alpha_k \in A$ such that $a_n \in U_{\alpha_1} \cup \ldots \cup U_{\alpha_k}$ for all n.

14. Determine whether the function $f(x) = x \sin x$ is uniformly continuous on the half line $\{x \in \mathbf{R} \mid x \geq 0\}$.