FIRST QUALIFYING EXAM
August 22, 2002

There are two sections, Algebra and Analysis. Solve **FOUR** *problems from each section. Do* **NOT** *hand in more than four problems for a section; if you do, only the first four problems will be counted. Do each problem on a separate sheet. Show all work. Be sure to write your person number clearly on* **EACH** *sheet. Remember, your goal is to convince the grader that you know what you are doing.*

Section I. Algebra. Do any four problems.

1. Let V be the vector space of real polynomials of degree ≤ 2. Define a linear transformation $L\colon V \to V$ by

$$L(P(x)) = (-3x + x^2)P''(x) + 3P'(x) + P(x) + 3xP(0),$$

where $P(x) = ax^2 + bx + c$.

a) Find the matrix representations of L and L^{-1} with respect to the basis $x^2, x, 1$ in V.

b) Find a basis for V consisting of eigenvectors for L.

2. The Cayley-Hamilton theorem states that if $p(t)$ is the characteristic polynomial of a square matrix A, then $p(A) = 0$. Prove this theorem for diagonal matrices.

3. Let V be a vector space over a field F and let $T\colon V \to V$ be a linear transformation which is *nilpotent* ($T^k = 0$ for some $k > 0$). Prove that $I + T$ is invertible.

4. Let V be a finite dimensional vector space and $T\colon V \to V$ be a linear transformation. Prove there exists a linear transformation $S\colon V \to V$ such that $TST = T$.

5. Let F be a field and let $P(x) = a_0 + \ldots + a_n x^n$ be an irreducible polynomial of degree $n \geq 2$ in $F[x]$. Let $\alpha = x + \langle P(x) \rangle$, where $\langle P(x) \rangle$ denotes the ideal generated by $P(x)$. Express α^{-1} in $F[x]/\langle P(x) \rangle$ in terms of $\alpha^0, \ldots, \alpha^{n-1}$.

6. Prove the "Second Isomorphism Theorem for Rings:" Let A be a subring of a ring R and let B be an ideal of R. Let $A + B := \{a + b \mid a \in A, B \in B\}$. Prove that $A/(A \cap B)$ is isomorphic to $(A + B)/B$.

7. Show that there is no non-trivial automorphism of the field $\mathbf{Q}$ of rational numbers.

8. Let G be an Abelian group. For each integer n, let $\varphi_n \colon G \to G$ be the homomorphism defined by $\varphi_n(a) := a^n$. Show that φ_n is one-to-one for all positive integers n if and only if the only finite subgroup of G is $\{1\}$.

Section II. Analysis. Do any four problems.

9. a) Prove that if a polynomial with real coefficients has $m \geq 2$ distinct real zeros then its derivative has at least $m - 1$ distinct real zeros.

b) Deduce from a) that a polynomial of degree $n > 0$ has at most n distinct real zeros.

10. Assume f is a real-valued continuous function on the rectangle $\{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$. Let

$$F(x) = \int_c^d f(x, y)dy.$$

Prove that F is a continuous function on $[a, b]$.

11. Let $K_1 \supset K_2 \supset K_3 \supset \ldots$ be a nested collection of compact subsets of R^m. If $\bigcap_{j=1}^{\infty} K_j$ is contained in an open set U of R^m, prove that $K_n \subset U$ for some positive integer n.

12. Assume that $a < b$ are reals, and that $[a, b] \subset P \cup Q$ where $a \in P$, $b \in Q$, and P and Q are both open in $\mathbf{R}$. Prove that $[a, b]$ is connected by showing that $P \cap Q \neq \emptyset$.

13. Let a be a real number and let $\{a_n\}$ be a sequence of real numbers such that every subsequence of $\{a_n\}$ has a subsequence that converges to a. Prove that $\lim_{n \to \infty} a_n = a$.

14. Let f be a monotonic function defined on the closed interval $[a, b]$. Prove that f has at most countably-many discontinuities.