SOLUTION FOR HOMEWORK #4

1. Section 7.3

4. Let $x = 4\sin\theta$ for $-\pi/2 \le \theta \le \pi/2$. Then $dx = 4\cos\theta d\theta$ and $\sqrt{16-x^2} = 4\cos\theta$. Thus

$$\begin{aligned}
\int \frac{x^3}{\sqrt{16-x^2}}\,dx &= \int \frac{4^3\sin^3\theta}{4\cos\theta}4\cos\theta\,d\theta = 4^3\int \sin^3\theta\,d\theta \\
&= 4^3\int (1-\cos^2\theta)\sin\theta\,d\theta \\
&\overset{u=\cos\theta}{=} -4^3\int (1-u^2)\,du = -4^3(u-\frac{1}{3}u^3)+C \\
&= -4^3(\cos\theta - \frac{1}{3}\cos^3\theta)+C \\
&= -4^2\sqrt{16-x^2} + \frac{1}{3}(\sqrt{16-x^2})^3 + C,
\end{aligned}$$

where C is a constant. Therefore

$$\begin{aligned}
\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx &= (-4^2\sqrt{16-x^2} + \frac{1}{3}(\sqrt{16-x^2})^3)|_0^{2\sqrt{3}} \\
&= -32 + \frac{8}{3} - (-64 + \frac{64}{3}) = 32 - \frac{56}{3} = \frac{40}{3}.
\end{aligned}$$

6. Let $u = x^2+4$. Then $du = 2x\,dx$. Thus

$$\begin{aligned}
\int x^3\sqrt{x^2+4}\,dx &= \int (u-4)\sqrt{u}\,\frac{du}{2} = \frac{1}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C \\
&= \frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C,
\end{aligned}$$

where C is a constant. Therefore

$$\begin{aligned}
\int_0^2 x^3\sqrt{x^2+4}\,dx &= (\frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2})|_0^2 \\
&= (\frac{1}{5}2^{15/2} - \frac{4}{3}2^{9/2}) - (\frac{1}{5}2^5 - \frac{4}{3}2^3) = \frac{64}{15}(\sqrt{2}+1).
\end{aligned}$$

8. Let $x = a\sec\theta$ for $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$. Then $dx = a\sec\theta\tan\theta d\theta$ and $\sqrt{x^2-a^2} = a\tan\theta$. Thus

$$\begin{aligned}\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx &= \int \frac{a\tan\theta}{(a\sec\theta)^4} a\sec\theta\tan\theta d\theta = \frac{1}{a^2}\int \sin^2\theta\cos\theta d\theta \\ &\overset{u=\sin\theta}{=} \frac{1}{a^2}\int u^2 du = \frac{1}{3a^2}u^3 + C \\ &= \frac{1}{3a^2}\sin^3\theta + C = \frac{1}{3a^2}\left(\frac{\sqrt{x^2-a^2}}{x}\right)^3 + C \\ &= \frac{(x^2-a^2)^{3/2}}{3a^2x^3} + C,\end{aligned}$$

where C is a constant.

12. Let $u = x^2+4$. Then $du = 2x\,dx$ and

$$\int x\sqrt{x^2+4}\,dx = \int \sqrt{u}\frac{du}{2} = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+4)^{3/2} + C,$$

where C is a constant. Thus

$$\int_0^1 x\sqrt{x^2+4}\,dx = \frac{1}{3}(x^2+4)^{3/2}|_0^1 = \frac{1}{3}(5^{3/2}-8).$$

2. Section 7.4

8. Using long divison one gets $r^2 = (r+4)(r-4)+16$. Thus

$$\begin{aligned}\int \frac{r^2}{r+4}\,dr &= \int \frac{(r+4)(r-4)+16}{r+4}\,dr = \int (r-4+\frac{16}{r+4})\,dr \\ &= \frac{1}{2}r^2 - 4r + 16\ln|r+4| + C,\end{aligned}$$

where C is a constant.

12. $x^2+3x+2 = (x+2)(x+1)$. Thus $\frac{x-1}{x^2+3x+2} = \frac{A}{x+2} + \frac{B}{x+1}$ for some scalars A and B. Multiply $(x+2)(x+1)$ on both sides we get $x-1 = A(x+1)+B(x+2)$. Setting $x=-1$ we get $B=-2$. Setting $x=-2$ we get $A=3$. Thus

$$\int \frac{x-1}{x^2+3x+2}\,dx = \int (\frac{3}{x+2} + \frac{-2}{x+1})\,dx = 3\ln|x+2| - 2\ln|x+1| + C,$$

where C is a constant. Therefore

$$\begin{aligned}\int_0^1 \frac{x-1}{x^2+3x+2}\,dx &= (3\ln|x+2| - 2\ln|x+1|)|_0^1 \\ &= (3\ln 3 - 2\ln 2) - (3\ln 2 - 0) = 3\ln 3 - 5\ln 2.\end{aligned}$$

20. $\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ for some scalars A, B, and C. Multiply $(x-3)(x+2)^2$ on both sides we get $x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$. Setting $x = 3$ we get $A = 9/25$. Setting $x = -2$ we get $C = -4/5$. Compare the coefficients of x^2 on both sides we get $1 = A + B$. Thus $B = 16/25$. Therefore

$$\begin{aligned}\int \frac{x^2}{(x-3)(x+2)^2}\,dx &= \int (\frac{9/25}{x-3} + \frac{16/25}{x+2} - \frac{4/5}{(x+2)^2})\,dx \\ &= \frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C,\end{aligned}$$

where C is a constant.

24. Let $u = x + 1$. Then

$$\begin{aligned}\int \frac{x^3}{(x+1)^3}\,dx &= \int \frac{(u-1)^3}{u^3}\,du = \int (1 - \frac{3}{u} + \frac{3}{u^2} - \frac{1}{u^3})\,du \\ &= u - 3\ln|u| - \frac{3}{u} + \frac{1}{2u^2} + C \\ &= x + 1 - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C,\end{aligned}$$

where C is a constant.

26. Since $x^3 + 3x = x(x^2+3)$, $\frac{x^2-x+6}{x^3+3x} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$ for some scalars A, B, and C. Multiply $x(x^2+3)$ on both sides we get $x^2 - x + 6 = A(x^2+3) + (Bx+C)x$. Setting $x = 0$ we get $A = 2$. Compare the coefficients of x^2 on both sides we get $1 = A + B$, and hence $B = -1$. Compare the coefficients of x on both sided we get $-1 = C$. Thus

$$\begin{aligned}\int \frac{x^2-x+6}{x^3+3x}\,dx &= \int (\frac{2}{x} + \frac{-x-1}{x^2+3})\,dx = \int (\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3})\,dx \\ &= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C,\end{aligned}$$

where C is a constant.