NULL ACTIONS AND RIM NON-OPEN EXTENSIONS OF STRONGLY PROXIMAL ACTIONS

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Abstract. Answering a question of Glasner, we show that any finitely generated nonabelian free group has a minimal null action which is a RIM non-open extension of an effective strongly proximal action.

1. Introduction

In this work we consider continuous actions of a countably infinite discrete group Γ on a compact metrizable space X. Such an action is called *tame* [7, 18] if the induced Γ-action on the space $C(X)$ of all complex-valued continuous functions on X contains no isomorphic dynamical copy of $\ell^1(\Gamma)$, i.e. for any $f \in C(X)$ there is no constant $C > 0$ such that

$$C\|g\|_1 \leq \|\sum_{s\in\Gamma} g(s)(sf)\|_\infty \leq C^{-1}\|g\|_1$$

for all $g \in \ell^1(\Gamma)$, where $(sf)(x) = f(s^{-1}x)$ for all $x \in X$. Tameness can also be described in terms of the Ellis enveloping semigroup $E(X, \Gamma)$ of the action $\Gamma \curvearrowright X$, which is the closure of the image of Γ in the product space X^X. Indeed, the following conditions are equivalent [2, 7, 10, 12]:

1. $\Gamma \curvearrowright X$ is tame,
2. $E(X, \Gamma)$ is a separate Fréchet compact space, hence with cardinality at most $2^{\aleph_0}$,
3. $E(X, \Gamma)$ does not contain a homeomorphic copy of the Stone-Čech compactification of $\mathbb{N}$,
4. every element of $E(X, \Gamma)$ is a Baire class 1 function from X to itself.

Actually the equivalence between (2) and (3) is a dynamical version of the Bourgain-Fremlin-Talagrand dichotomy theorem [7, 10].

In [9] Glasner established a structure theorem for tame minimal actions, extending the results for abelian Γ in [8, 14, 16]. The action $\Gamma \curvearrowright X$ is called *minimal* if there is no proper nonempty closed Γ-invariant subset of X. It is called *strongly proximal* if

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for the induced Γ-action on the compact space $\mathscr{M}(X)$ of all Borel probability measures on X, every nonempty closed Γ-invariant subset intersects with X [3]. Given another continuous action of Γ on a compact metrizable space Y, a Γ-equivariant continuous surjective map $\pi : X \to Y$ is called a *factor map* or *extension*. The extension π is said to be *strongly proximal* if for every $y \in Y$ and every $\mu \in \mathscr{M}(X)$ with support contained in $\pi^{-1}(y)$, the orbit closure of μ in $\mathscr{M}(X)$ intersects with X [3]. A *relative invariant measure* (RIM) for π is a continuous Γ-equivariant map from Y to $\mathscr{M}(X)$ such that the support of the image of each $y \in Y$ is contained in $\pi^{-1}(y)$ [4]. The extension π is called *point-distal* if there is a point $x \in X$ with dense orbit such that for any $x \neq x' \in X$ with $\pi(x) = \pi(x')$ the orbit closure of (x, x') does not intersect with the diagonal [24, VI.4.1]. It is called *almost one-to-one* if there is some $x \in X$ with dense orbit in X such that $\pi^{-1}(\pi(x)) = \{x\}$ [24, IV.6.1]. It is called *equicontinuous* or *isometric* if, given a compatible metric ρ on X, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, x' \in X$ with $\pi(x) = \pi(x')$ and $\rho(x, x') < \delta$ one has $\rho(sx, sx') < \varepsilon$ for all $s \in \Gamma$ [24, V.2.1]. Then Glasner's structure theorem states as follows: for every tame minimal action $\Gamma \curvearrowright X$ there is a commutative diagram of minimal Γ-actions

$$\begin{tikzcd}
& \tilde{X} \arrow[dl, "\eta"'] \arrow[dd, "\pi"'] & X^* \arrow[l, "\theta^*"'] \arrow[d, "\iota"] \\
X & & Z \arrow[d, "\sigma"] \\
& Y & Y^* \arrow[l, "\theta"]
\end{tikzcd}$$

such that $\Gamma \curvearrowright \tilde{X}$ is tame, η is a strongly proximal extension, $\Gamma \curvearrowright Y$ is a strongly proximal action, π is a point-distal extension with a unique RIM, θ, θ^* and ι are almost one-to-one extensions, and σ is an isometric extension.

The map π is open exactly when θ and θ^* are trivial. In such case the above diagram reduces to

$$\begin{tikzcd}
& \tilde{X} \arrow[dl, "\eta"'] \arrow[d, "\iota"] \arrow[dd, bend left=60, "\pi"] \\
X & Z \arrow[d, "\sigma"] \\
& Y
\end{tikzcd}$$

This leads Glasner to ask the following question [9, Problem 5.5]

Problem 1.1. Let $\Gamma \curvearrowright \tilde{X}$ be a tame minimal action, and let $\Gamma \curvearrowright Y$ be a strongly proximal action. If $\pi : \tilde{X} \to Y$ is a RIM extension, then must π be open?

When Γ is amenable, every minimal strongly proximal action is the trivial action on a singleton [5, Theorem III.3.1]. Thus Problem 1.1 has affirmative answer in this case, even without assuming $\Gamma \curvearrowright \tilde{X}$ to be tame.

Our goal in this paper is to answer Problem 1.1 negatively. In fact we shall construct counterexample for a weaker statement. Recall that for a sequence $\mathfrak{s} = \{s_n\}_{n\in\mathbb{N}}$ in Γ, the *sequence topological entropy* of an action $\Gamma \curvearrowright X$ with respect to $\mathfrak{s}$ and a finite open cover $\mathcal{U}$ of X is defined as

$$\mathrm{h}_{\mathrm{top}}(X, \mathcal{U}; \mathfrak{s}) = \limsup_{n\to\infty} \frac{1}{n} \log N(\bigvee_{i=1}^{n} s_i^{-1}\mathcal{U}),$$

where $N(\bigvee_{i=1}^{n} s_i^{-1}\mathcal{U})$ denotes the minimal number of elements of $\bigvee_{i=1}^{n} s_i^{-1}\mathcal{U}$ needed to cover X. The action $\Gamma \curvearrowright X$ is called *null* if $\mathrm{h}_{\mathrm{top}}(X, \mathcal{U}; \mathfrak{s}) = 0$ for all $\mathfrak{s}$ and $\mathcal{U}$ [13, 15]. It is known that null actions are tame (see Section 2). An action $\Gamma \curvearrowright Y$ is called *effective* if for any distinct s, t in Γ one has $sy \neq ty$ for some $y \in Y$. Our main result is

Theorem 1.2. *For any nonabelian finitely generated free group Γ, there are a null (hence tame) minimal action $\Gamma \curvearrowright \tilde{X}$, an effective strongly proximal action $\Gamma \curvearrowright Y$, and a point-distal non-open extension $\tilde{X} \to Y$ with a unique RIM.*

There are two ingredients in our construction. McMahon [20, Example 3.2.(1)] constructed examples of RIM non-open extension $\Gamma \curvearrowright \tilde{X}$ of minimal equicontinuous actions $\Gamma \curvearrowright Y$ for $\Gamma = G \times \mathbb{Z}$, where G is any dense countable subgroup of the p-adic integer group $\mathbb{Z}_p$. These examples do not provide counterexamples for Problem 1.1 for three reasons. The first is that Γ in these examples is abelian. The second is that $\Gamma \curvearrowright Y$ is equicontinuous instead of strongly proximal. The third is that it is not clear whether $\Gamma \curvearrowright \tilde{X}$ in these examples are tame or not. Among these difficulties, the third one is most difficult. To prove Theorem 1.2 we modify McMahon's construction to handle these three difficulties. Our second ingredient is the combinatorial independence developed in [16]. It enables us to turn the question of checking tameness or nullness to a combinatorial problem. The latter is still nontrivial but manageable.

This paper is organized as follows. We recall the basic of combinatorial independence in Section 2. McMahon's construction is recalled in Section 3. As a showcase of our technique, we construct some null minimal actions for every residually finite group in Section 4. Theorem 1.2 is proved in Section 5.

Throughout this paper Γ will be a countably infinite group with identity element e_Γ. All Γ-actions are assumed to be continuous actions on compact metrizable spaces unless specified otherwise. For each compact metrizable space X, we denote by $\mathscr{M}(X)$ the space of all Borel probability measures on X, equipped with the weak*-topology.

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2. Combinatorial independence

In this section we recall the combinatorial independence description of tameness and nullness [16, 17]. Let Γ act on a compact metrizable space X continuously.

Let (A_1, A_2) be a pair of subsets of X. We say that a set $M \subseteq \Gamma$ is an *independence set* for (A_1, A_2) if the collection $\{(s^{-1}A_1, s^{-1}A_2) : s \in M\}$ is independent in the sense that $\bigcap_{s \in F} s^{-1}A_{\omega(s)} \neq \emptyset$ for every nonempty finite set $F \subseteq M$ and $\omega \in \{1, 2\}^F$.

We say that a pair $(x_1, x_2) \in X^2$ is an *IT-pair* if for every product neighborhood $U_1 \times U_2$ of (x_1, x_2) the pair (U_1, U_2) has an infinite independence set.

The following is the combinatorial independence characterization of tameness [16, Proposition 6.4] [17, Proposition 8.14]. It's based on the proof of Rosenthal's ℓ_1 theorem [21, 22].

Proposition 2.1. *The action $\Gamma \curvearrowright X$ is tame if and only if X has no non-diagonal IT-pairs.*

We say that a pair $(x_1, x_2) \in X^2$ is an *IN-pair* if for every product neighborhood $U_1 \times U_2$ of (x_1, x_2) the pair (U_1, U_2) has arbitrarily large finite independence sets. We denote by $\mathrm{IN}_2(X, \Gamma)$ the set of all IN-pairs.

The following summarizes the basic properties of IN-pairs and gives the combinatorial independence characterization of nullness [16, Proposition 5.4].

Proposition 2.2. *The following are true:*

1. *Let (A_1, A_2) be a pair of closed subsets of X which has arbitrarily large finite independence sets. Then there exists an IN-pair (x_1, x_2) with $x_j \in A_j$ for $j = 1, 2$.*
2. *The action $\Gamma \curvearrowright X$ is null if and only if X has no non-diagonal IN-pairs.*
3. *$\mathrm{IN}_2(X, \Gamma)$ is a closed Γ-invariant subset of X^2.*
4. *Let $\pi : (X, \Gamma) \to (Y, \Gamma)$ be a factor map. Then $\pi^2(\mathrm{IN}_2(X, \Gamma)) = \mathrm{IN}_2(Y, \Gamma)$.*

It follows from Propositions 2.1 and 2.2 that null actions are tame. We remark that there are minimal tame nonnull subshifts for $\mathbb{Z}$ [16, Section 11].

3. McMahon's construction

We recall McMahon's construction of RIM extensions in [19, 20]. See [24, VI.6.5] for details.

Let Γ act on a compact metrizable space X continuously and assume that the action is minimal and X consists of more than one orbit. Let x_1 be a point of X such that the stabilizer group $\{s \in \Gamma : sx_1 = x_1\}$ of x_1 is trivial. Also let $f : X \setminus \{x_1\} \to \{1, -1\}$ be a continuous function such that it cannot be extended to X continuously. Then Γ has a minimal continuous action on some compact metrizable space X_f with the following properties:

1. there is a continuous Γ-equivariant surjective map $\pi_f : X_f \to X$ such that $\pi_f^{-1}(x)$ consists of one point if $x \in X \setminus \Gamma x_1$ and two points if $x \in \Gamma x_1$,

(2) the function $f \circ \pi_f$ on $X_f \setminus \pi_f^{-1}(x_1)$ extends to a continuous function $\tilde{f} : X_f \to \{1, -1\}$.

Furthermore, as an extension of $\Gamma \curvearrowright X$, the action $\Gamma \curvearrowright X_f$ is unique up to conjugacy. Indeed, X_f is constructed as follows. Take a point $x_0 \in X \setminus \Gamma x_1$ and denote by Z the Stone-Čech compactification of Γx_0 equipped with the relative topology from $\Gamma x_0 \subseteq X$. Then the induced Γ-action on Z is minimal and $f \circ \pi_Z|_{\Gamma x_0}$ extends uniquely to a continuous function $f_Z : Z \to \{1, -1\}$, where $\pi_Z : Z \to X$ is the factor map extending the embedding $\Gamma x_0 \hookrightarrow X$. Define an equivalence relation on Z by $z_1 \sim z_2$ if $f_Z(sz_1) = f_Z(sz_2)$ for all $s \in \Gamma$ and $\pi_Z(z_1) = \pi_Z(z_2)$. This equivalent relation is Γ-invariant and closed. Then X_f is defined as the quotient space $Z/\sim$. Though Z is possibly not metrizable, X_f is. Since $z_1 \sim z_2$ implies that $\pi_Z(z_1) = \pi_Z(z_2)$, there is a unique continuous map $\pi_f : X_f \to X$ such that the composition map $Z \to X_f \xrightarrow{\pi_f} X$ is equal to π_Z. As both $Z \to X_f$ and π_Z are Γ-equivariant, so is π_f. Since $z_1 \sim z_2$ implies that $f_Z(z_1) = f_Z(z_2)$, there is also a unique continuous function $\tilde{f} : X_f \to \{1, -1\}$ such that the composition function $Z \to X_f \xrightarrow{\tilde{f}} \{1, -1\}$ is equal to f_Z. Then $f \circ \pi_f = \tilde{f}$ on the image of Γx_0 under the quotient map $Z \to X_f$. As this image is dense in X_f and both $f \circ \pi_f$ and the restriction of $\tilde{f}$ are continuous on $X_f \setminus \pi_f^{-1}(x_1)$, one has $f \circ \pi_f = \tilde{f}$ on $X_f \setminus \pi_f^{-1}(x_1)$. We remark that another way of defining X_f is to let it be the Gelfand spectrum of the Γ-invariant C^*-subalgebra of $\ell^\infty(\Gamma)$ generated by functions of the form $t \mapsto g(tx_0)$ for $t \in \Gamma$ and $g \in C(X)$ or $g = f$, where $\ell^\infty(\Gamma)$ is equipped with the supremum norm and pointwise multiplication, addition and conjugation and Γ acts on $\ell^\infty(\Gamma)$ via $(sh)(t) = h(s^{-1}t)$ for all $s, t \in \Gamma$ and $h \in \ell^\infty(X)$.

We take an excursion to discuss when $\Gamma \curvearrowright X_f$ is null. Set $X_{f,+} = \tilde{f}^{-1}(1)$ and $X_{f,-} = \tilde{f}^{-1}(-1)$. Also set $X_+ = f^{-1}(1)$ and $X_- = f^{-1}(-1)$. The following result tells us how to characterize nullness of $\Gamma \curvearrowright X_f$ from information on $\Gamma \curvearrowright X$ and f.

Proposition 3.1. *The action $\Gamma \curvearrowright X_f$ is null if and only if $\Gamma \curvearrowright X$ is null and the pair (X_+, X_-) does not have arbitrarily large finite independence sets.*

Note that as the sets X_+ and X_- are not closed, the condition "not having arbitrarily large finite independence sets" is not automatically satisfied if $\Gamma \curvearrowright X$ is null.

Proposition 3.1 follows from Lemmas 3.2 and 3.3 below.

Lemma 3.2. *The action $\Gamma \curvearrowright X_f$ is null if and only if $\Gamma \curvearrowright X$ is null and the pair $(X_{f,+}, X_{f,-})$ does not have arbitrarily large finite independence sets.*

Proof. Assume that $\Gamma \curvearrowright X_f$ is null. Since $X_{f,+}$ and $X_{f,-}$ are disjoint closed subsets of X_f, by (1) and (2) of Proposition 2.2 the pair $(X_{f,+}, X_{f,-})$ does not have arbitrarily large finite independence sets. Also from (2) and (4) of Proposition 2.2 we know that factors of null actions are null. Thus $\Gamma \curvearrowright X$ is null. This proves the "only if" part.

Assume that $\Gamma \curvearrowright X_f$ is nonnull and $\Gamma \curvearrowright X$ is null. By (2) of Proposition 2.2 there is an IN-pair (z_1, z_2) in X_f^2 with $z_1 \neq z_2$. By (4) of Proposition 2.2 the pair $(\pi_f(z_1), \pi_f(z_2))$ is an IN-pair in X^2. Since $\Gamma \curvearrowright X$ is null, by (2) of Proposition 2.2 the IN-pairs in X^2 must be diagonal ones. Thus $\pi_f(z_1) = \pi_f(z_2)$. Then $\pi_f(z_1) \in \Gamma x_1$. Say, $\pi_f(z_1) = sx_1$ for some $s \in \Gamma$. Then $\pi_f(s^{-1}z_1) = \pi_f(s^{-1}z_2) = x_1$, and hence $\{s^{-1}z_1, s^{-1}z_2\} = \pi_f^{-1}(x_1)$. By (3) of Proposition 2.2 the pair $(s^{-1}z_1, s^{-1}z_2)$ is also an IN-pair. Note that $X_{f,+}$ and $X_{f,-}$ are both closed and open subsets of X_f, and each of them contains exactly one of $s^{-1}z_1$ and $s^{-1}z_2$. Thus the pair $(X_{f,+}, X_{f,-})$ has arbitrarily large finite independence sets. This proves the "if" part. $\square$

Lemma 3.3. *The pair $(X_{f,+}, X_{f,-})$ has arbitrarily large finite independence sets if and only if the pair (X_+, X_-) has arbitrarily large finite independence sets.*

Proof. Since $\pi_f^{-1}(X_+) \subseteq X_{f,+}$ and $\pi_f^{-1}(X_-) \subseteq X_{f,-}$, the "if" part is trivial.

Assume that $(X_{f,+}, X_{f,-})$ has an independence set M with cardinality $2m$ for some positive integer m. Then for any map $\omega : M \to \{+,-\}$ one has $\bigcap_{s\in M} s^{-1}X_{f,\omega(s)} \neq \emptyset$. For each such ω, fix a point $z_\omega \in \bigcap_{s\in M} s^{-1}X_{f,\omega(s)}$. Then for each such ω and $s \in M$, one has $sz_\omega \in X_{f,\omega(s)} \subseteq \pi_f^{-1}(X_{\omega(s)}) \cup \pi_f^{-1}(x_1)$. List the elements of M as $s_1, s_2, \dots, s_{2m}$. Denote by A the set of integers $1 \leq n \leq m$ such that $s_{2n-1}z_\omega \in \pi_f^{-1}(x_1)$ for some ω.

Let $n \in A$ and take one ω_n such that $s_{2n-1}z_{\omega_n} \in \pi_f^{-1}(x_1)$. Then $s_{2n-1}\pi_f(z_{\omega_n}) = x_1$. Since the stabilizer group of x_1 is trivial, we have

$$\pi_f(s_{2n}z_{\omega_n}) = s_{2n}\pi_f(z_{\omega_n}) = s_{2n}s_{2n-1}^{-1}x_1 \neq x_1.$$

If $s_{2n}s_{2n-1}^{-1}x_1 \in X_+$, then $s_{2n}z_{\omega_n} \in X_{f,+}$ and hence $\omega_n(s_{2n}) = +$. Similarly, if $s_{2n}s_{2n-1}^{-1}x_1 \in X_-$, then $s_{2n}z_{\omega_n} \in X_{f,-}$ and hence $\omega_n(s_{2n}) = -$. Thus for any map $\omega : M \to \{+,-\}$, if $\omega(s_{2n}) \neq \omega_n(s_{2n})$, then $s_{2n-1}z_\omega \in \pi_f^{-1}(X_{\omega(s_{2n-1})})$ and hence $s_{2n-1}\pi_f(z_\omega) \in X_{\omega(s_{2n-1})}$.

Set $M' = \{s_{2n-1} : n = 1, \dots, m\}$. For each map $\omega' : M' \to \{+,-\}$, extend it to a map $\omega : M \to \{+,-\}$ such that $\omega(s_{2n}) \neq \omega_n(s_{2n})$ for all $n \in A$. By the above we have $s_{2n-1}\pi_f(z_\omega) \in X_{\omega(s_{2n-1})} = X_{\omega'(s_{2n-1})}$ for all $n = 1, \dots, m$. Therefore $\bigcap_{s\in M'} s^{-1}X_{\omega'(s)} \neq \emptyset$, i.e. M' is an independence set for (X_+, X_-). This proves the "only if" part. $\square$

Remark 3.4. The analogues of Proposition 3.1 and Lemmas 3.2 and 3.3 for tameness all hold with similar proofs.

We come back to the construction of McMahon. Denote by $\mathscr{M}^*(X)$ the set of *nonatomic* μ in $\mathscr{M}(X)$, i.e. $\mu(\{x\}) = 0$ for every $x \in X$. Also denote by π_{f*} the continuous surjective map $\mathscr{M}(X_f) \to \mathscr{M}(X)$ induced by π_f.

Lemma 3.5. *For each $\mu \in \mathscr{M}^*(X)$, the set $(\pi_{f*})^{-1}(\mu)$ consists of a single point, which we denote by μ_f. The map $\mathscr{M}^*(X) \to \mathscr{M}(X_f)$ sending μ to μ_f is continuous.*

Proof. Let $\nu \in \mathscr{M}(X_f)$ with $\pi_{f*}(\nu) = \mu$. Since μ is nonatomic, we have

$$\nu(\pi_f^{-1}(\Gamma x_1)) = \mu(\Gamma x_1) = 0.$$

Restrict π_f to $\pi_f^{-1}(X \setminus \Gamma x_1)$ we get a map $\psi : \pi_f^{-1}(X \setminus \Gamma x_1) \to X \setminus \Gamma x_1$. We claim that ψ is a homeomorphism. Since ψ is continuous and bijective, it suffices to show that ψ^{-1} is continuous. Let $(x_j)_{j\in J}$ be a net in $X \setminus \Gamma x_1$ converging to some $x_\infty \in X \setminus \Gamma x_1$. We just need to show that $\psi^{-1}(x_j) \to \psi^{-1}(x_\infty)$ as $j \to \infty$. Since X_f is compact, passing to a subnet if necessary, we may assume that $\psi^{-1}(x_j)$ converges to some $z \in X_f$ as $j \to \infty$. Then $x_j = \pi_f(\psi^{-1}(x_j))$ converges to $\pi_f(z)$, and hence $\pi_f(z) = x_\infty$. Thus $z = \psi^{-1}(x_\infty)$, and we conclude that $\psi^{-1}(x_j) \to \psi^{-1}(x_\infty)$ as desired. This proves our claim.

Since ψ is a homeomorphism, it is a Borel isomorphism. Then for any Borel set $A \subseteq X_f$, we have

$$\nu(A) = \nu(A \setminus \pi_f^{-1}(\Gamma x_1)) = \mu(\psi(A \setminus \pi_f^{-1}(\Gamma x_1))) = \mu(\pi_f(A) \setminus \Gamma x_1).$$

Therefore ν is unique.

Denote by φ the map $\mathscr{M}^*(X) \to \mathscr{M}(X_f)$ sending μ to μ_f. We shall show that φ is continuous. Let $\{\mu_j\}_{j\in J}$ be a net in $\mathscr{M}^*(X)$ converging to some $\mu_\infty \in \mathscr{M}^*(X)$. We just need to show that $\varphi(\mu_j) \to \varphi(\mu_\infty)$ as $j \to \infty$. Since $\mathscr{M}(X_f)$ is compact, passing to a subnet if necessary, we may assume that $\varphi(\mu_j)$ converges to some $\nu \in \mathscr{M}(X_f)$ as $j \to \infty$. Then $\mu_j = \pi_{f*}(\varphi(\mu_j))$ converges to $\pi_{f*}(\nu)$, and hence $\pi_{f*}(\nu) = \mu_\infty$. Thus $\nu = \varphi(\mu_\infty)$, and we conclude that $\varphi(\mu_j) \to \varphi(\mu_\infty)$ as desired. Therefore φ is continuous. □

Now assume that $\Gamma \curvearrowright Y$ and $\Gamma \curvearrowright Z$ are continuous actions on compact metrizable spaces such that $X = Y \times Z$ and the action $\Gamma \curvearrowright X$ is the product action $\Gamma \curvearrowright Y \times Z$. Also assume that there is some Γ-invariant nonatomic $\mu_Z \in \mathscr{M}(Z)$. Denote by π_Y the projection $X \to Y$. For each $y \in Y$, we have the nonatomic measure $\delta_y \times \mu_Z \in \mathscr{M}(X)$, where δ_y denotes the point mass at y. Then $y \mapsto \delta_y \times \mu_Z$ is a RIM for the extension π_Y. Thus by Lemma 3.5 the map $y \mapsto (\delta_y \times \mu_Z)_f$ is a RIM for the extension $\pi_Y \circ \pi_f$.

In [19, 20] McMahon took $\Gamma = G \times \mathbb{Z}$ for G being any dense countable subgroup of the p-adic integer group $\mathbb{Z}_p$, and $Y = Z = \mathbb{Z}_p$. His actions $\Gamma \curvearrowright Y$ and $\Gamma \curvearrowright Z$ are the ones factoring through the shift actions $G \curvearrowright Y$ and $\mathbb{Z} \curvearrowright Z$ via treating $\mathbb{Z}$ as a dense subgroup of $\mathbb{Z}_p$ naturally. His measure μ_Z is the normalized Haar measure of Z. Taking suitable choices of f, he showed that the extension $\pi_Y \circ \pi_f$ could be either open or non-open.

4. Residually finite groups

As a warm up, in this section we apply the construction in Section 3 to residually finite groups, and show that sometimes it yields null actions. Though the results of this section will not be used for the proof of Theorem 1.2, the method in this section will be used in Section 5 in a much more complicated way.

Let Γ be a countably infinite residually finite group with identity element e_Γ. This means that there is a strictly decreasing sequence $\{\Gamma_n\}$ of finite-index normal subgroups of Γ such that $\bigcap_{n\in\mathbb{N}}\Gamma_n = \{e_\Gamma\}$.

Denote by X the inverse limit $\varprojlim_{n\to\infty}\Gamma/\Gamma_n$, which is the subset of $\prod_{n\in\mathbb{N}}\Gamma/\Gamma_n$ consisting of $(x_n)_{n\in\mathbb{N}}$ satisfying $\pi_{n,n+1}(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$. Here $\pi_{n,n+1}$ denotes the natural homomorphism $\Gamma/\Gamma_{n+1} \to \Gamma/\Gamma_n$. This is a compact metrizable totally disconnected group. Denote by π_n the quotient map $X \to \Gamma/\Gamma_n$. The group Γ is naturally a subgroup of X, and hence has a natural left shift action on X. Clearly the action $\Gamma \curvearrowright X$ is minimal and free in the sense that every point of X has trivial stabilizer group. Note that X is a Cantor set, so it contains more than one orbit.

Lemma 4.1. *The action $\Gamma \curvearrowright X$ is null.*

Proof. The compact metrizable group X has a translation-invariant metric, which is invariant under the Γ-action. It follows from (2) of Proposition 2.2 that $\Gamma \curvearrowright X$ is null. $\square$

For each $n \geq 2$ take $\gamma_n \in \Gamma_{n-1} \setminus \Gamma_n$. Set $C_n = \pi_n^{-1}(\gamma_n\Gamma_n)$, which is a closed and open subset of X. The sets C_n for $n \geq 2$ are pairwise disjoint, $e_\Gamma \notin C_n$, and $C_n \to \{e_\Gamma\}$ as $n \to \infty$ in the sense that for every neighborhood U of e_Γ in X one has $C_n \subseteq U$ for all large enough n. Set $X_+ = \bigcup_{n\geq 2} C_n$ and $X_- = X \setminus (X_+ \cup \{e_\Gamma\})$.

Lemma 4.2. *Each independence set $M \subseteq \Gamma$ for (X_+, X_-) has cardinality at most* 5.

We leave the proof of Lemma 4.2 to the end of this section. Consider the function $f : X \setminus \{e_\Gamma\} \to \{1, -1\}$ defined by $f(x) = 1$ if $x \in X_+$ and $f(x) = -1$ if $x \in X_-$. Since $C_n \to \{e_\Gamma\}$ as $n \to \infty$, the function f is continuous. Assume further that $[\Gamma_n : \Gamma_{n+1}] > 2$ for all n. Then every neighborhood of e_Γ in X intersects with both X_+ and X_-. Thus f cannot be extended to X continuously. Then we can apply McMahon's construction in Section 3 to obtain the minimal action $\Gamma \curvearrowright X_f$ and the fact map $\pi_f : X_f \to X$. From Proposition 3.1 we conclude

Theorem 4.3. *The action $\Gamma \curvearrowright X_f$ is null.*

Remark 4.4. Fix distinct prime numbers $p, q \geq 3$. For $\Gamma = \mathbb{Z}$, we can take $\Gamma_n = p^n q^n \mathbb{Z}$. Then X is the product of the p-adic integer group $\mathbb{Z}_p$ and the q-adic integer group $\mathbb{Z}_q$. Denote by π_p the projection $X \to \mathbb{Z}_p$. As the normalized Haar measure of $\mathbb{Z}_q$ is nonatomic, from the discussion at the end of Section 3 we know that the extension $\pi_p \circ \pi_f$ has a RIM. The proofs of Lemmas 5.2 and 5.4 in the next section also work in this case to show that $\pi_p \circ \pi_f$ has a unique RIM and is not open.

To prove Lemma 4.2, we need to make some preparation.

Lemma 4.5. *Let $s_1, s_2 \in \Gamma$ and $x, y \in X$ such that $s_1 x \in C_{n_1}$ and $s_2 x \in C_{n_2}$ with $n_1 < n_2$, and $s_1 y \notin C_{n_1}$ and $s_2 y \in C_{m_2}$. Then $m_2 \leq n_1$.*

Proof. Set $t = s_1 s_2^{-1}$. Then $tC_{n_2} \cap C_{n_1} \neq \emptyset$, and hence

$$t\Gamma_{n_1} = t\gamma_{n_2}\Gamma_{n_1} = \gamma_{n_1}\Gamma_{n_1}.$$

That is, $t \in \gamma_{n_1}\Gamma_{n_1}$. If $m_2 > n_1$, then

$$\pi_{n_1}(s_1 y) = \pi_{n_1}(ts_2 y) = t\pi_{n_1}(s_2 y) = t\Gamma_{n_1} = \gamma_{n_1}\Gamma_{n_1},$$

and hence $s_1 y \in C_{n_1}$, which is a contradiction. Therefore $m_2 \leq n_1$. □

Lemma 4.6. *Let $s_1, s_2, s_3 \in \Gamma$ and $x \in X$ such that $s_1 x \in C_{n_1}, s_2 x \in C_{n_2}$ and $s_3 x \in C_{n_3}$ with $n_1 < \min(n_2, n_3)$. Then there is no $y \in X$ satisfying $s_1 y, s_2 y \notin X_+$ and $s_3 y \in X_+$.*

Proof. For each $i = 2, 3$ we have $s_i \pi_{n_i}(x) = \gamma_{n_i}\Gamma_{n_i}$, and hence $s_i \pi_{n_1}(x) = \Gamma_{n_1}$. It follows that $s_2\Gamma_{n_1} = s_3\Gamma_{n_1}$.

Suppose that for some $y \in X$ we have $s_1 y, s_2 y \notin X_+$ and $s_3 y \in X_+$. Say, $s_3 y \in C_{m_3}$. Applying Lemma 4.5 to $s_1, s_3 \in \Gamma$ and $x, y \in X$ we have $m_3 \leq n_1$. Thus $s_2\Gamma_{m_3} = s_3\Gamma_{m_3}$, and hence $s_2\pi_{m_3}(y) = s_3\pi_{m_3}(y)$. Since $s_2 y \notin C_{m_3}$ and $s_3 y \in C_{m_3}$, we have $s_2\pi_{m_3}(y) \neq \gamma_{m_3}\Gamma_{m_3}$ and $s_3\pi_{m_3}(y) = \gamma_{m_3}\Gamma_{m_3}$, which is a contradiction. □

We are ready to prove Lemma 4.2.

Proof of Lemma 4.2. Assume that (X_+, X_-) has an independence set M with cardinality 6. For each map $\omega : M \to \{+, -\}$, fix a point $x_\omega \in \bigcap_{s \in M} s^{-1} X_{\omega(s)}$. For any such ω and any $s \in \omega^{-1}(+)$, the point sx_ω lies in C_n for a unique $n \geq 2$. Denote this n by $g(\omega, s)$. By Lemma 4.6 for each such ω there is a set $B_\omega \subseteq \omega^{-1}(+)$ with cardinality at most 1 such that the function $s \mapsto g(\omega, s)$ is constant on $\omega^{-1}(+) \setminus B_\omega$.

Take distinct $s_1, s_2 \in M$. Define a map $\omega : M \to \{+, -\}$ by $\omega(s_1) = \omega(s_2) = -$ and $\omega(s) = +$ for all $s \in M \setminus \{s_1, s_2\}$. Then $\omega^{-1}(+) \setminus B_\omega$ has cardinality at least 3. Say, $g(\omega, s) = n_1$ for all $s \in \omega^{-1}(+) \setminus B_\omega$.

Take $s_3 \in \omega^{-1}(+) \setminus B_\omega$. Define a map $\tilde{\omega} : M \to \{+, -\}$ by $\tilde{\omega}(s_3) = -$ and $\tilde{\omega}(s) = +$ for all $s \in M \setminus \{s_3\}$. Say, $g(\tilde{\omega}, s) = n_2$ for all $s \in \tilde{\omega}^{-1}(+) \setminus B_{\tilde{\omega}}$. Then $\tilde{\omega}^{-1}(+) \setminus B_{\tilde{\omega}}$ has nonempty intersection with both $\{s_1, s_2\}$ and $\omega^{-1}(+) \setminus (B_\omega \cup \{s_3\})$. Without loss of generality, we may assume $s_1 \in \tilde{\omega}^{-1}(+) \setminus B_{\tilde{\omega}}$. Take $s_4 \in (\tilde{\omega}^{-1}(+) \setminus B_{\tilde{\omega}}) \cap (\omega^{-1}(+) \setminus (B_\omega \cup \{s_3\}))$.

Now we have s_1, s_3, s_4 pairwise distinct. We also have $s_3 x_\omega, s_4 x_\omega \in C_{n_1}$, $s_1 x_{\tilde{\omega}}, s_4 x_{\tilde{\omega}} \in C_{n_2}$, and $s_1 x_\omega, s_3 x_{\tilde{\omega}} \notin X_+$.

From $s_3 x_\omega, s_4 x_\omega \in C_{n_1}$ and $s_1 x_\omega \notin C_{n_1}$ we have

$$s_3\pi_{n_1}(x_\omega) = \gamma_{n_1}\Gamma_{n_1} = s_4\pi_{n_1}(x_\omega) \neq s_1\pi_{n_1}(x_\omega),$$

and hence $s_3\Gamma_{n_1} = s_4\Gamma_{n_1} \neq s_1\Gamma_{n_1}$. Similarly, from $s_1 x_{\tilde{\omega}}, s_4 x_{\tilde{\omega}} \in C_{n_2}$ and $s_3 x_{\tilde{\omega}} \notin C_{n_2}$ we have $s_1\Gamma_{n_2} = s_4\Gamma_{n_2} \neq s_3\Gamma_{n_2}$.

If $n_1 \geq n_2$, then from $s_3\Gamma_{n_1} = s_4\Gamma_{n_1}$ we have $s_3\Gamma_{n_2} = s_4\Gamma_{n_2}$, which is a contradiction. If $n_2 \geq n_1$, then from $s_1\Gamma_{n_2} = s_4\Gamma_{n_2}$ we have $s_1\Gamma_{n_1} = s_4\Gamma_{n_1}$, which is also a contradiction. $\square$

5. Free groups

In this section we prove Theorem 1.2.

Let $r \geq 2$ and $\Gamma = F_r$ be the free group with r generators $S = \{a, b, a_3, \dots, a_r\}$.

Denote by Y the Gromov boundary of Γ. This is the set of all infinite reduced words in $S \cup S^{-1}$, i.e. the set of elements $x = (x_n)_{n\in\mathbb{N}}$ in $(S \cup S^{-1})^{\mathbb{N}}$ satisfying $x_{n+1} \neq x_n^{-1}$ for all $n \in \mathbb{N}$. It is a closed subset of $(S \cup S^{-1})^{\mathbb{N}}$, hence compact metrizable. The group Γ acts on Y continuously by concatenation and cancellation. Clearly this action is minimal and effective. By [3, page 161] this action is also strongly proximal.

It is well known that Γ is residually finite [23, Corollary C-1.126]. Let $\{\Gamma_n\}$ be a strictly decreasing sequence of finite-index normal subgroups of Γ with $\bigcap_{n\in\mathbb{N}} \Gamma_n = \{e_\Gamma\}$. Set $Z = \varprojlim_{n\to\infty} \Gamma/\Gamma_n$ and $X = Y \times Z$. As in Section 4, Γ is naturally a subgroup of the compact metrizable totally disconnected group Z and has a natural left shift action on Z. This action $\Gamma \curvearrowright Z$ is minimal and free. Then the product action $\Gamma \curvearrowright X$ is also free. Denote by μ_Z the normalized Haar measure on Z, which is nonatomic and Γ-invariant. For each $n \in \mathbb{N}$, denote by π_n the natural homomorphism $Z \to \Gamma/\Gamma_n$.

Lemma 5.1. *The product action $\Gamma \curvearrowright X$ is minimal.*

Proof. An action $\Gamma \curvearrowright Z'$ is called *weakly non-contractible* if it is minimal and there is some $\mu \in \mathscr{M}(Z')$ with support Z' such that the orbit closure of μ in $\mathscr{M}(Z')$ is minimal. Every weakly non-contractible action $\Gamma \curvearrowright Z'$ is disjoint from every minimal strongly proximal action $\Gamma \curvearrowright Y'$ in the sense that the product action $\Gamma \curvearrowright Y' \times Z'$ is minimal [3, Theorem 6.1]. Since μ_Z has support Z and is Γ-invariant, we know that $\Gamma \curvearrowright Z$ is weakly non-contractible. Therefore the product action $\Gamma \curvearrowright Y \times Z$ is minimal. $\square$

For each nontrivial $s \in \Gamma$, we say that $y \in Y$ *starts with* s if $y = sy'$ for some $y' \in Y$ such that the last letter of (the reduced form of) s is different from the inverse of the first letter of y'. Similarly, we shall talk about $t \in \Gamma$ starting or ending with s. For any nontrivial $s \in \Gamma$, denote by V_s the set of elements in Y starting with s. Denote by a^∞ the element in Y taking constant value a.

For each $n \geq 2$ take $\gamma_n \in \Gamma_{n-1} \setminus \Gamma_n$, and set $u_n = a^n b a^{-n} b^{-1}$, $D_n = V_{u_n}$ and $C_n = \pi_n^{-1}(\gamma_n\Gamma_n)$. Then $D_n \times C_n$ is a closed and open subset of X. The sets $D_n \times C_n$ are pairwise disjoint, $(a^\infty, e_\Gamma) \notin D_n \times C_n$, and $D_n \times C_n \to \{(a^\infty, e_\Gamma)\}$ as $n \to \infty$ in the sense that for every neighborhood U of (a^∞, e_Γ) in X one has $D_n \times C_n \subseteq U$ for all large enough $n \in \mathbb{N}$. Set $X_+ = \bigcup_{n\geq 2}(D_n \times C_n)$ and $X_- = X \setminus (X_+ \cup \{(a^\infty, e_\Gamma)\})$. Then every neighborhood of (a^∞, e_Γ) in X intersects with both X_+ and X_-.

Define a function $f : X \setminus \{(a^\infty, e_\Gamma)\} \to \{1, -1\}$ by $f(x) = 1$ if $x \in X_+$ and $f(x) = -1$ if $x \in X_-$. Since $D_n \times C_n \to \{(a^\infty, e_\Gamma)\}$ as $n \to \infty$, the function f is continuous. Since every neighborhood of (a^∞, e_Γ) in X intersects with both X_+ and X_-, the function f cannot be extended to X continuously. Therefore we can apply McMahon's construction in Section 3 to obtain the minimal action $\Gamma \curvearrowright X_f$ and the factor map $\pi_f : X_f \to X$.

Denote by π_Y the projection $X \to Y$.

Lemma 5.2. *The extension $\pi_Y \circ \pi_f$ has a unique RIM.*

Proof. From our discussion at the end of Section 3 and using the notation there, we know that $y \mapsto \delta_y \times \mu_Z$ is a RIM for π_Y and $y \mapsto (\delta_y \times \mu_Z)_f$ is a RIM for $\pi_Y \circ \pi_f$.

An extension $\pi' : X' \to Y'$ between continuous actions $\Gamma \curvearrowright X'$ and $\Gamma \curvearrowright Y'$ is called a *group extension* if there is a compact metrizable group Z' acting continuously on X' denoted by $(x', z') \to x'z'$ for $x' \in X'$ and $z' \in Z'$ such that $s(x'z') = (sx')z'$ for all $s \in \Gamma, x' \in X', z' \in Z'$, and $(\pi')^{-1}(\pi'(x')) = x'Z'$ for all $x' \in X'$. Every group extension between minimal actions has a unique RIM [4, Corollary 3.7].

Clearly π_Y is a group extension. By Lemma 5.1 the action $\Gamma \curvearrowright X$ is minimal. Therefore π_Y has a unique RIM, which must be $y \mapsto \delta_y \times \mu_Z$.

Let $y \mapsto \mu_y$ be a RIM for $\pi_Y \circ \pi_f$. Then $y \mapsto \pi_{f*}(\mu_y)$ is a RIM for π_Y. Thus $\pi_{f*}(\mu_y) = \delta_y \times \mu_Z$ for every $y \in Y$. By Lemma 3.5 we get $\mu_y = (\delta_y \times \mu_Z)_f$ for every $y \in Y$. Therefore $\pi_Y \circ \pi_f$ has a unique RIM. □

Lemma 5.3. *The extension $\pi_Y \circ \pi_f$ is point-distal.*

Proof. Denote by π_Z the projection $X \to Z$.

Let $\tilde{x} \in \pi_f^{-1}(X \setminus \Gamma(a^\infty, e_\Gamma))$ and $\tilde{x}' \in X_f$ with $\pi_Y \circ \pi_f(\tilde{x}) = \pi_Y \circ \pi_f(\tilde{x}')$ such that the orbit closure of $(\tilde{x}, \tilde{x}')$ in X_f^2 intersects with the diagonal. Then the orbit closure of $(\pi_Z \circ \pi_f(\tilde{x}), \pi_Z \circ \pi_f(\tilde{x}'))$ in Z^2 intersects with the diagonal. Note that the compact metrizable group Z has a translation-invariant metric, which is then invariant under the Γ-action. It follows that $\pi_Z \circ \pi_f(\tilde{x}) = \pi_Z \circ \pi_f(\tilde{x}')$. Therefore $\pi_f(\tilde{x}) = \pi_f(\tilde{x'})$, and hence $\tilde{x} = \tilde{x}'$. Since $\Gamma \curvearrowright X$ is minimal by Lemma 5.1, this means that every point in $\pi_f^{-1}(X \setminus \Gamma(a^\infty, e_\Gamma))$ witnesses the definition of point-distality. □

We remark that Glasner showed that every RIM extension between tame minimal actions is point-distal [9, Theorem 4.4]. Thus Lemma 5.3 also follows directly once we show later that $\Gamma \curvearrowright X_f$ is null.

Lemma 5.4. *The map $\pi_Y \circ \pi_f$ is not open.*

Proof. Denote by $\tilde{f}$ the continuous extension of $f \circ \pi_f : \pi_f^{-1}(X \setminus \{(a^\infty, e_\Gamma)\}) \to \{1, -1\}$ to X_f. Set $X_{f,+} = \tilde{f}^{-1}(1)$. Then $X_{f,+}$ is an open subset of X_f. But

$$\pi_Y \circ \pi_f(X_{f,+}) = \pi_Y(\{(a^\infty, e_\Gamma)\} \cup X_+) = \{a^\infty\} \cup \bigcup_{n \geq 2} D_n$$

is not open. Therefore $\pi_Y \circ \pi_f$ is not open. □

We are left to show that $\Gamma \curvearrowright X_f$ is null, which is also the most technical part. For this purpose we need to make some preparation.

Lemma 5.5. *Let $t \in \Gamma$ and $y \in Y$. Assume that t does not start with u_n. Then $ty \in D_n$ if and only if y starts with $t^{-1}u_n$.*

Proof. Since t does not start with u_n, the element $t^{-1}u_n$ ends with b^{-1}.

Suppose that y starts with $t^{-1}u_n$. Then $y = t^{-1}u_n y'$ for some $y' \in Y$ not starting with b. We have $ty = u_n y'$. Since y' does not start with b, the point $u_n y'$ starts with u_n. Thus $ty \in D_n$. This proves the "if" part.

Now suppose that $ty \in D_n$. Then $ty = u_n y'$ for some $y' \in Y$ not starting with b. We have $y = t^{-1}u_n y'$. Since $t^{-1}u_n$ ends with b^{-1} and y' does not start with b, $t^{-1}u_n y'$ starts with $t^{-1}u_n$. This proves the "only if" part. $\square$

Lemma 5.6. *Let $t \in \Gamma$ and $y \in Y$. Assume that t starts with u_n. Then $ty \in D_n$ if and only if y does not start with $t^{-1}u_n b$.*

Proof. We have $t = u_n s$ for some $s \in \Gamma$ not starting with b. Then $ty \notin D_n$ if and only if y starts with $(b^{-1}s)^{-1} = t^{-1}u_n b$. $\square$

Lemma 5.7. *Let $t \in \Gamma$ be nontrivial such that $tD_{n_2} \cap D_{n_1} \neq \emptyset$. Then one of the following 3 situations must hold:*

(1) t ends with $u_{n_2}^{-1} = ba^{n_2}b^{-1}a^{-n_2}$,
(2) t starts with $u_{n_1} = a^{n_1}ba^{-n_1}b^{-1}$,
(3) $n_2 \neq n_1$ and $t = u_{n_1}u_{n_2}^{-1} = a^{n_1}ba^{n_2-n_1}b^{-1}a^{-n_2}$.

Proof. Assume that (1) and (2) do not hold. Take $y \in D_{n_2}$ with $ty \in D_{n_1}$. Then y starts with u_{n_2}. Since t does not start with u_{n_1}, we know that $t^{-1}u_{n_1}$ ends with b^{-1} and by Lemma 5.5 the element y also starts with $t^{-1}u_{n_1}$. Then either u_{n_2} starts with $t^{-1}u_{n_1}$ or $t^{-1}u_{n_1}$ starts with u_{n_2}. Since $t^{-1}u_{n_1}$ ends with b^{-1}, if u_{n_2} starts with $t^{-1}u_{n_1}$, then we must have $u_{n_2} = t^{-1}u_{n_1}$, i.e. (3) holds. If $t^{-1}u_{n_1}$ starts with u_{n_2}, since t^{-1} does not start with u_{n_2}, then it is easy to see that we still have $t^{-1}u_{n_1} = u_{n_2}$. $\square$

Lemma 5.8. *Let $s_1, s_2 \in \Gamma$ and $z, z' \in Z$ such that $s_1 z, s_2 z \in C_n$, and $s_2 z' \in C_m$. Then $m = n$ if and only if $s_1 z' \in C_n$.*

Proof. Set $t = s_1 s_2^{-1}$. Then $tC_n \cap C_n \neq \emptyset$, and hence $t\gamma_n\Gamma_n = \gamma_n\Gamma_n$. It follows that $t \in \Gamma_n$.

Assume that $m = n$. Then $s_1 z' = t(s_2 z') \in tC_n = C_n$. This proves the "only if" part.

Conversely, assume that $s_1 z' \in C_n$. Then $s_2 z' = t^{-1}(s_1 z') \in t^{-1}C_n = C_n$, and hence $m = n$. This proves the "if" part. $\square$

Note that for each $n \in \mathbb{N}$, Γ_n is torsion-free. Since $\bigcap_{m\in\mathbb{N}} \Gamma_m = \{e_\Gamma\}$, for each $n \in \mathbb{N}$, when m is large enough, Γ_n/Γ_{n+m} has nontrivial elements with order strictly bigger than 2. Thus replacing $\{\Gamma_n\}$ by a suitable subsequence, we may choose

$\gamma_n \in \Gamma_{n-1} \setminus \Gamma_n$ such that $\gamma_n^2 \notin \Gamma_n$ for every $n \geq 2$, which we shall assume from now on.

Lemma 5.9. *Let $s_1, s_2 \in \Gamma$ and $z, z' \in Z$ such that $s_1 z \in C_{n_1}$ and $s_2 z \in C_{n_2}$ with $n_1 < n_2$, and $s_1 z' \in C_{m_1}$ and $s_2 z' \in C_{m_2}$. Then $s_1 s_2^{-1} \in \gamma_{n_1}\Gamma_{n_1}$, and one of the following two situations must hold:*

(1) $m_1 = n_1 < m_2$,
(2) $m_1 = m_2 < n_1$.

Proof. Set $t = s_1 s_2^{-1}$. Then $tC_{n_2} \cap C_{n_1} \neq \emptyset$, and hence

$$t\Gamma_{n_1} = t\gamma_{n_2}\Gamma_{n_1} = \gamma_{n_1}\Gamma_{n_1}.$$

That is, $t \in \gamma_{n_1}\Gamma_{n_1}$. We consider three cases.

Consider first the case $m_1 < m_2$. Similarly we have $t \in \gamma_{m_1}\Gamma_{m_1}$. Then $t \in C_{n_1} \cap C_{m_1}$, and hence $m_1 = n_1$.

Next we consider the case $m_1 = m_2$. Then $tC_{m_1} \cap C_{m_1} \neq \emptyset$, and hence $t \in \Gamma_{m_1}$, which implies $m_1 < n_1$.

Finally we consider the case $m_1 > m_2$. Similarly we have $t^{-1} \in \gamma_{m_2}\Gamma_{m_2}$. Then $t \in \gamma_{n_1}\Gamma_{n_1} \cap \gamma_{m_2}^{-1}\Gamma_{m_2}$, and hence $n_1 = m_2$. It follows that $\gamma_{n_1}\Gamma_{n_1} = \gamma_{n_1}^{-1}\Gamma_{n_1}$, and hence $\gamma_{n_1}^2 \in \Gamma_{n_1}$, which is a contradiction to our assumption. Therefore the case $m_1 > m_2$ does not happen. ☐

Lemma 5.10. *Let $s_1, s_2 \in \Gamma$ be distinct such that $s_1^{-1}X_+ \cap s_2^{-1}X_+ \neq \emptyset$. Set $t = s_1 s_2^{-1}$. Then at least one of the following 10 situations must hold:*

(1) Type (A1) for (s_1, s_2): $t = u_m v u_{n_2}^{-1}$ for some $2 \leq m < n_1 < n_2$ determined by t and some nontrivial $v \in \Gamma$ not starting with b and not ending with b^{-1}; for any $x'' = (y'', z'') \in X$ with $s_2 x'' \in D_{k_2} \times C_{k_2}$, we have $s_1 x'' \in X_+$ if and only if either $k_2 = m$, in which case $s_1 x'' \in D_m \times C_m$, or $k_2 = n_2$ and $s_2 y''$ starts with $t^{-1}u_{n_1}$, in which case $s_1 x'' \in D_{n_1} \times C_{n_1}$.
(2) Type (A2) for (s_1, s_2): $t = u_{n_1} v u_m^{-1}$ for some $2 \leq m < n_1$ determined by t and some nontrivial $v \in \Gamma$ not starting with b and not ending with b^{-1}; for any $x'' = (y'', z'') \in X$ with $s_2 x'' \in D_{k_2} \times C_{k_2}$, we have $s_1 x'' \in X_+$ if and only if either $k_2 > n_1$, in which case $s_1 x'' \in D_{n_1} \times C_{n_1}$, or $k_2 = m$ and $s_2 y''$ starts with $t^{-1}u_m$, in which case $s_1 x'' \in D_m \times C_m$.
(3) Type (B1) for (s_1, s_2): $t = u_{n_1} v u_m^{-1}$ for some $2 \leq n_1 < m$ determined by t and some nontrivial $v \in \Gamma$ not starting with b and not ending with b^{-1}; for any $x'' = (y'', z'') \in X$ with $s_2 x'' \in D_{k_2} \times C_{k_2}$, we have $s_1 x'' \in X_+$ if and only if $s_1 x'' \in D_{n_1} \times C_{n_1}$ if and only if either $m \neq k_2 > n_1$, or $k_2 = m$ and $s_2 y''$ does not start with $t^{-1}u_{n_1}b$.
(4) Type (B2) for (s_1, s_2): $t = u_{n_1} v$ for some $n_1 \geq 2$ determined by t and some $v \in \Gamma$ not starting with b and not ending with u_m^{-1} for any $m > n_1$; for any $x'' = (y'', z'') \in X$ with $s_2 x'' \in D_{k_2} \times C_{k_2}$, we have $s_1 x'' \in X_+$ if and only if $k_2 > n_1$ and $t \neq u_{n_1} a^{-k_2}$, in which case $s_1 x'' \in D_{n_1} \times C_{n_1}$.

(5) *Type (B3) for (s_1, s_2): $t = u_{n_1}u_{n_2}^{-1}$ for some $2 \leq n_1 < n_2$ determined by t; for any $x'' = (y'', z'') \in X$ with $s_2x'' \in D_{k_2} \times C_{k_2}$, we have $s_1x'' \in X_+$ if and only if $k_2 = n_2$, in which case $s_1x'' \in D_{n_1} \times C_{n_1}$.*

(6) *Type (B4) for (s_1, s_2): $t = vu_{n_2}^{-1}$ for some $2 \leq n_1 < n_2$ determined by t and some $v \in \Gamma$ not ending with b^{-1} and not starting with u_{n_1}; for any $x'' = (y'', z'') \in X$ with $s_2x'' \in D_{k_2} \times C_{k_2}$, we have $s_1x'' \in X_+$ if and only if $k_2 = n_2$ and s_2y'' starts with $t^{-1}u_{n_1}$, in which case $s_1x'' \in D_{n_1} \times C_{n_1}$.*

(7) *Type (C1) for (s_1, s_2): $t = u_{n_1}vu_{n_2}^{-1}$ for some distinct $n_1, n_2 \geq 2$ determined by t and some nontrivial $v \in \Gamma$ not starting with b and not ending with b^{-1}; for any $x'' = (y'', z'') \in X$ with $s_2x'' \in D_{k_2} \times C_{k_2}$, we have $s_1x'' \in X_+$ if and only if either $k_2 = n_1$, in which case $s_1x'' \in D_{n_1} \times C_{n_1}$, or $k_2 = n_2$ and s_2y'' starts with $t^{-1}u_{n_2}$, in which case $s_1x'' \in D_{n_2} \times C_{n_2}$.*

(8) *Type (C2) for (s_1, s_2): $t = u_mvu_m^{-1}$ for some $m \geq 2$ determined by t and some nontrivial $v \in \Gamma$ not starting with b and not ending with b^{-1}; for any $x'' = (y'', z'') \in X$ with $s_2x'' \in D_{k_2} \times C_{k_2}$, we have $s_1x'' \in X_+$ if and only if $k_2 = m$ and s_2y'' does not start with $t^{-1}u_mb$, in which case $s_1x'' \in D_m \times C_m$.*

(9) *Type (C3) for (s_1, s_2): $t = vu_m^{-1}$ for some $m \geq 2$ determined by t and some $v \in \Gamma$ not starting with u_m and not ending with b^{-1}; for any $x'' = (y'', z'') \in X$ with $s_2x'' \in D_{k_2} \times C_{k_2}$, we have $s_1x'' \in X_+$ if and only if $k_2 = m$ and s_2y'' starts with $t^{-1}u_m$, in which case $s_1x'' \in D_m \times C_m$.*

(10) *The pair (s_2, s_1) has one of the above types.*

Proof. We consider first the case that there exist $x = (y, z), x' = (y', z') \in X$ such that $s_1x \in D_{n_1} \times C_{n_1}$, $s_2x \in D_{n_2} \times C_{n_2}$ and $s_1x', s_2x' \in D_m \times C_m$ with $n_1 \neq n_2$. By symmetry we may assume that $n_1 < n_2$. By Lemma 5.9 we have $t \in \gamma_{n_1}\Gamma_{n_1} \subseteq \Gamma_{n_1-1}$ and $m < n_1$. Since $tD_m \cap D_m \neq \emptyset$, by Lemma 5.7 either t starts with u_m or t ends with u_m^{-1}. We separate the case into two subcases.

Consider first the subcase t ends with $u_{n_2}^{-1}$. Then t must start with u_m. Thus $t = u_mvu_{n_2}^{-1}$ for some nontrivial $v \in \Gamma$ not starting with b and not ending with b^{-1}. We say that (s_1, s_2) has type (A1). Clearly m and n_2 are determined by t. Also n_1 is determined by t as $t \in C_{n_1}$. Let $x'' = (y'', z'') \in X$ such that $s_2x'' \in D_{k_2} \times C_{k_2}$. If $k_2 \neq n_2$, then $s_1y'' = t(s_2y'') \in D_m$. Thus if $k_2 \neq n_2$ and $s_1x'' \in X_+$ then $s_1x'' \in D_m \times C_m$ and hence $k_2 = m$ by Lemma 5.9. Conversely, if $k_2 = m$, then by Lemma 5.8 we do have $s_1x'' \in D_m \times C_m$. If $k_2 = n_2$ and $s_1x'' \in X_+$, then by Lemma 5.9 we have $s_1x'' \in D_{n_1} \times C_{n_1}$ and hence by Lemma 5.5 the point s_2y'' must start with $t^{-1}u_{n_1} = u_{n_2}v^{-1}u_m^{-1}u_{n_1}$. Conversely, if $k_2 = n_2$ and s_2y'' starts with $t^{-1}u_{n_1}$, then by Lemma 5.5 we do have $s_1x'' \in D_{n_1} \times C_{n_1}$.

Next consider the subcase t does not end with $u_{n_2}^{-1}$. Since $tD_{n_2} \cap D_{n_1} \neq \emptyset$, by Lemma 5.7 either $t = u_{n_1}u_{n_2}^{-1}$ or t starts with u_{n_1}. Then t cannot start with u_m, and hence t ends with u_m^{-1}. It follows that t starts with u_{n_1}. Thus $t = u_{n_1}vu_m^{-1}$ for some nontrivial $v \in \Gamma$ not starting with b and not ending with b^{-1}. We say that (s_1, s_2) has type (A2). Clearly m and n_1 are determined by t. Let $x'' = (y'', z'') \in X$ such that $s_2x'' \in D_{k_2} \times C_{k_2}$. If $k_2 \neq m$, then $s_1y'' = t(s_2y'') \in D_{n_1}$. Thus if $k_2 \neq m$ and $s_1x'' \in X_+$, then $s_1x'' \in D_{n_1} \times C_{n_1}$ and hence $k_2 > n_1$ by Lemma 5.9. Conversely, if $k_2 > n_1$, then we do have $s_1x'' \in D_{n_1} \times C_{n_1}$. If $k_2 = m$ and $s_1x'' \in X_+$, then by Lemma 5.9 we have $s_1x'' \in D_m \times C_m$ and hence by Lemma 5.5 the point s_2y'' starts with $t^{-1}u_m = u_m v^{-1} u_{n_1}^{-1} u_m$. Conversely, if $k_2 = m$ and s_2y'' starts with $t^{-1}u_m$, then by Lemma 5.5 we do have $s_1x'' \in D_m \times C_m$.

Now consider the case there exist $x = (y, z) \in X$ such that $s_1x \in D_{n_1} \times C_{n_1}$, $s_2x \in D_{n_2} \times C_{n_2}$ with $n_1 \neq n_2$, but there is no $x' \in X$ such that $s_1x', s_2x' \in D_m \times C_m$. By symmetry we may assume that $n_1 < n_2$. By Lemma 5.9 we have $t \in \gamma_{n_1}\Gamma_{n_1}$, and for any $x' = (y', z') \in X$ such that $s_1x' \in D_{m_1} \times C_{m_1}$ and $s_2x' \in D_{m_2} \times C_{m_2}$ we have

$$m_1 = n_1 < m_2. \tag{1}$$

We separate the case into four subcases.

We consider first the subcase that there are some $x^\sharp = (y^\sharp, z^\sharp), x' = (y', z') \in X$ such that $s_1x^\sharp, s_1x' \in D_{n_1} \times C_{n_1}$ and $s_2x^\sharp \in D_{m_2} \times C_{m_2}$ and $s_2x' \in D_m \times C_m$ with $n_1 < \min(m_2, m)$ and $m_2 \neq m$, and t ends with u_m^{-1}. Since $tD_{m_2} \cap D_{n_1} \neq \emptyset$, by Lemma 5.7 the element t starts with u_{n_1}. Thus $t = u_{n_1} v u_m^{-1}$ for some nontrivial $v \in \Gamma$ not starting with b and not ending with b^{-1}. We say (s_1, s_2) has type (B1). Clearly n_1 and m are determined by t. Let $x'' = (y'', z'') \in X$ such that $s_2x'' \in D_{k_2} \times C_{k_2}$. If $s_1x'' \in X_+$, then by (1) we have $k_2 > n_1$ and $s_1x'' \in D_{n_1} \times C_{n_1}$, and hence by Lemma 5.6 the point s_2y'' does not start with $t^{-1}u_{n_1}b = u_m v^{-1} b$. Conversely, if $k_2 > n_1$ and s_2y'' does not start with $t^{-1}u_{n_1}b$, then by Lemma 5.6 we do have $s_1x'' \in D_{n_1} \times C_{n_1}$. In particular, if $m \neq k_2 > n_1$, then $s_1x'' \in D_{n_1} \times C_{n_1}$.

Next consider the subcase that for any $x' \in X$ such that $s_1x' \in D_{n_1} \times C_{n_1}$ and $s_2x' \in D_{m_2} \times C_{m_2}$ the element t does not end with $u_{m_2}^{-1}$ and we do have some $x' = (y', z') \in X$ such that $s_1x' \in D_{n_1} \times C_{n_1}$ and $s_2x' \in D_{m_2} \times C_{m_2}$ with $m_2 \neq n_2$. Since $tD_{m_2} \cap D_{n_1}$ and $tD_{n_2} \cap D_{n_1}$ are nonempty, by Lemma 5.7 the element t must start with u_{n_1}. Say, $t = u_{n_1} v$ for some $v \in \Gamma$ not starting with b. We say (s_1, s_2) has type (B2). Clearly n_1 is determined by t. Note that v cannot end with u_k^{-1} for any $k > n_1$, otherwise we can easily find some $\tilde{x} \in X$ such that $s_1\tilde{x} \in D_{n_1} \times C_{n_1}$ and $s_2\tilde{x} \in D_k \times C_k$ contradicting to our assumption. Let $x'' = (y'', z'') \in X$ such that $s_2x'' \in D_{k_2} \times C_{k_2}$. If $s_1x'' \in X_+$, then by (1) we have $k_2 > n_1$ and $s_1x'' \in D_{n_1} \times C_{n_1}$, and hence by Lemma 5.6 the point s_2y'' does not start with $t^{-1}u_{n_1}b = v^{-1}b$, which implies that $t \neq u_{n_1} a^{-k_2}$. Conversely, if $k_2 > n_1$ and $t \neq u_{n_1} a^{-k_2}$, then s_2y'' does not start with $t^{-1}u_{n_1}b = v^{-1}b$ and hence by Lemma 5.6 we do have $s_1x'' \in D_{n_1} \times C_{n_1}$.

Next consider the subcase that for any $\tilde{x} \in X$ such that $s_1\tilde{x} \in D_{n_1} \times C_{n_1}$ and $s_2\tilde{x} \in D_m \times C_m$ we have $m = n_2$. If t starts with u_{n_1}, then for any large enough $m > n_1$ and any $\tilde{x} \in X$ with $s_2\tilde{x} \in D_m \times C_m$ one has $s_1\tilde{x} \in D_{n_1} \times C_{n_1}$, which contradicts our assumption. Since $tD_{n_2} \cap D_{n_1}$ is nonempty, by Lemma 5.7 either $t = u_{n_1} u_{n_2}^{-1}$ or t ends with $u_{n_2}^{-1}$. Suppose that $t = u_{n_1} u_{n_2}^{-1}$. We say (s_1, s_2) has type (B3). Clearly n_1 and n_2 are determined by t. Let $x'' = (y'', z'') \in X$ with $s_2x'' \in D_{k_2} \times C_{k_2}$. If $s_1x'' \in X_+$, then by (1) and our assumption $k_2 = n_2$ and

$s_1x'' \in D_{n_1} \times C_{n_1}$. Conversely, if $k_2 = n_2$, then by Lemma 5.5 we do have $s_1x'' \in D_{n_1} \times C_{n_1}$. Now suppose that t ends with $u_{n_2}^{-1}$ instead. Then $t = vu_{n_2}^{-1}$ for some $v \in \Gamma$ not ending with b^{-1} and not starting with u_{n_1}. We say (s_1, s_2) has type (B4). Clearly n_2 is determined by t. Also n_1 is determined by t as $t \in C_{n_1}$. Let $x'' = (y'', z'') \in X$ with $s_2x'' \in D_{k_2} \times C_{k_2}$. If $s_1x'' \in X_+$, then by (1) and our assumption $k_2 = n_2$ and $s_1x'' \in D_{n_1} \times C_{n_1}$, and hence by Lemma 5.5 the point s_2y'' starts with $t^{-1}u_{n_1} = u_{n_2}v^{-1}u_{n_1}$. Conversely, if $k_2 = n_2$ and s_2y'' starts with $t^{-1}u_{n_1}$, then by Lemma 5.5 we do have $s_1x'' \in D_{n_1} \times C_{n_1}$.

Finally consider the case there is no $x \in X$ such that $s_1x \in D_{n_1} \times C_{n_1}$, $s_2x \in D_{n_2} \times C_{n_2}$ with $n_1 \neq n_2$. We separate it into three subcases.

Consider first the subcase that there are $x, x' \in X$ such that $s_1x, s_2x \in D_{n_2} \times C_{n_2}$ and $s_1x', s_2x' \in D_{n_1} \times C_{n_1}$ with $n_1 \neq n_2$. Since $tD_{n_2} \cap D_{n_2}$ and $tD_{n_1} \cap D_{n_1}$ are nonempty, by Lemma 5.7 the element t must end with either $u_{n_1}^{-1}$ or $u_{n_2}^{-1}$. Without loss of generality, assume that t ends with $u_{n_2}^{-1}$. As $tD_{n_1} \cap D_{n_1}$ is nonempty, Lemma 5.7 implies that t starts with u_{n_1}. Thus $t = u_{n_1}vu_{n_2}^{-1}$ for some nontrivial $v \in \Gamma$ not starting with b and not ending with b^{-1}. We say (s_1, s_2) has type (C1). Clearly n_1 and n_2 are determined by t. Let $x'' = (y'', z'') \in X$ such that $s_2x'' \in D_{k_2} \times C_{k_2}$. If $k_2 \neq n_2$, then $s_1y'' = t(s_2y'') \in D_{n_1}$. Thus if $k_2 \neq n_2$ and $s_1x'' \in X_+$ then $s_1x'' \in D_{n_1} \times C_{n_1}$ and hence $k_2 = n_1$ by Lemma 5.8. Conversely, if $k_2 = n_1$, then by Lemma 5.8 we do have $s_1z'' \in C_{n_1}$ and hence $s_1x'' \in D_{n_1} \times C_{n_1}$. If $k_2 = n_2$, then by Lemma 5.8 we have $s_1z'' \in C_{n_2}$. Thus if $k_2 = n_2$ and $s_1x'' \in X_+$, then $s_1x'' \in D_{n_2} \times C_{n_2}$ and hence by Lemma 5.5 the point s_2y'' must start with $t^{-1}u_{n_2} = u_{n_2}v^{-1}u_{n_1}^{-1}u_{n_2}$. Conversely, if $k_2 = n_2$ and s_2y'' starts with $t^{-1}u_{n_2}$, then by Lemma 5.5 we do have $s_1x'' \in D_{n_2} \times C_{n_2}$.

Next consider the subcase that there is some $m \geq 2$ such that for any $x = (y, z) \in X$ such that $s_1x \in D_{n_1} \times C_{n_1}$, $s_2x \in D_{n_2} \times C_{n_2}$ one has $n_1 = n_2 = m$. Since $tD_m \cap D_m$ is nonempty, by Lemma 5.7 either t starts with u_m or t ends with u_m^{-1}. Note that t starts with u_m exactly when t^{-1} ends with u_m^{-1}. By symmetry we may assume that t ends with u_m^{-1}. Suppose that t also starts with u_m. Then $t = u_mvu_m^{-1}$ for some nontrivial $v \in \Gamma$ not starting with b and not ending with b^{-1}. We say that (s_1, s_2) has type (C2). In this case m is determined by t. Let $x'' = (y'', z'') \in X$ such that $s_2x'' \in D_{k_2} \times C_{k_2}$. If $s_1x'' \in X_+$, then by our assumption $k_2 = m$ and $s_1x'' \in D_m \times C_m$, and hence by Lemma 5.6 the point s_2y'' does not start with $t^{-1}u_mb = u_mv^{-1}b$. Conversely, if $k_2 = m$ and s_2y'' does not start with $t^{-1}u_mb$, then by Lemmas 5.6 and 5.8 we do have $s_1x'' \in D_m \times C_m$. Now suppose instead that t does not start with u_m. Then $t = vu_m^{-1}$ for some $v \in \Gamma$ not starting with u_m and not ending with b^{-1}. We say that (s_1, s_2) has type (C3). In this case m is also determined by t. Let $x'' = (y'', z'') \in X$ such that $s_2x'' \in D_{k_2} \times C_{k_2}$. If $s_1x'' \in X_+$, then by our assumption $k_2 = m$ and $s_1x'' \in D_m \times C_m$, and hence by Lemma 5.5 the point s_2y'' starts with $t^{-1}u_m = u_mv^{-1}u_m$. Conversely, if $k_2 = m$ and s_2y'' starts with $t^{-1}u_m$, then by Lemmas 5.5 and 5.8 we do have $s_1x'' \in D_m \times C_m$. $\square$

In the next 9 lemmas we give an upper bound for the size of an independence set $M \subseteq \Gamma$ for (X_+, X_-) when every pair in M (with respect to some linear order of M) has a fixed type in Lemma 5.10.

For each finite independence set $M \subseteq \Gamma$ for (X_+, X_-) and each map $\omega : M \to \{+,-\}$, fix a point $x_\omega = (y_\omega, z_\omega) \in \bigcap_{s\in M} s^{-1}X_{\omega(s)}$. For any such ω and any $s \in \omega^{-1}(+)$, the point sx_ω lies in $D_n \times C_n$ for a unique $n \geq 2$. Denote this n by $g(\omega, s)$.

Lemma 5.11. *Let $M \subseteq \Gamma$ be a finite independence set for (X_+, X_-) with cardinality ℓ. List the elements of M as $s'_1, \dots, s'_\ell$. Assume that (s'_i, s'_j) has type (A1) for all $1 \leq i < j \leq \ell$. Then $\ell < n_{A1} := 21$.*

Proof. Assume that $\ell \geq 21$.

We claim that there are some map $\omega_0 : M \to \{+,-\}$ and a set $A \subseteq M$ with cardinality 5 such that $\omega_0 = +$ on A and the map $s \mapsto g(\omega_0, s)$ on A is injective. Let ω' be the map $M \to \{+\}$. Then either there is some subset A of M with cardinality 5 such that the map $s \mapsto g(\omega', s)$ on A is injective or there is some subset B of M with cardinality 6 such that $g(\omega', s) = m$ for some $m \geq 2$ and all $s \in B$. In the first situation we can take $\omega_0 = \omega'$. Thus assume that $B \subseteq M$ has cardinality 6 and $g(\omega', s) = m$ for all $s \in B$. List the elements of B as $\theta_1, \dots, \theta_6$ such that (θ_i, θ_j) has type (A1) for all $1 \leq i < j \leq 6$. Take $\omega'' : M \to \{+,-\}$ such that $\omega''(\theta_1) = -$ and $\omega''(\theta_i) = +$ for all $2 \leq i \leq 6$. For any $2 \leq i \leq 6$, since (θ_1, θ_i) has type (A1) and $\omega''(\theta_1) = -$, we have $g(\omega'', \theta_i) \neq m$. For any $2 \leq i < j \leq 6$, since (θ_i, θ_j) has type (A1), we have $g(\omega'', \theta_i) < g(\omega'', \theta_j)$. Then we can take $A = \{\theta_2, \dots, \theta_6\}$ and $\omega_0 = \omega''$. This proves our claim.

List the elements of A as $s_1, \dots, s_5$ such that (s_i, s_j) has type (A1) for all $1 \leq i < j \leq 5$. Set $n_i = g(\omega_0, s_i)$. Then $n_1 < n_2 < \cdots < n_5$. Now $s_5 y_{\omega_0}$ starts with $\xi_i := s_5 s_i^{-1} u_{n_i}$ for $i = 2, 3, 4$. Write $\{2, 3, 4\}$ as $\{i_1, i_2, i_3\}$ such that the length of ξ_{i_3} is no less than those of ξ_{i_2} and ξ_{i_1}.

For $k = 1, 2$, take $\omega_k : M \to \{+,-\}$ such that $\omega_k(s_{i_k}) = -$ and $\omega_k(s) = +$ for all $s \in A \setminus \{s_{i_k}\}$. We claim that $g(\omega_k, s_5) < n_1$. Suppose that $g(\omega_k, s_5) \geq n_1$ instead. Since (s_1, s_5) has type (A1), we have $g(\omega_k, s_5) = n_5$. As (s_{i_3}, s_5) has type (A1), the element $s_5 y_{\omega_k}$ starts with ξ_{i_3} and hence starts with ξ_{i_k}. Since (s_{i_k}, s_5) has type (A1), we get $s_{i_k} x_{\omega_k} \in X_+$, which is a contradiction to $\omega_k(s_{i_k}) = -$. This proves our claim.

Since (s_{i_1}, s_5) has type (A1) and $g(\omega_2, s_5) < n_1 < n_{i_1}$, we get $g(\omega_2, s_{i_1}) = g(\omega_2, s_5)$. As (s_1, s_5) has type (A1) and $g(\omega_1, s_5), g(\omega_2, s_5) < n_1$, we have $g(\omega_1, s_5) = g(\omega_2, s_5)$. Since (s_{i_1}, s_5) has type (A1) and $g(\omega_1, s_5) = g(\omega_2, s_5) = g(\omega_2, s_{i_1}) < n_{i_1}$, we get $s_{i_1} x_{\omega_1} \in X_+$, which is a contradiction to $\omega_1(s_{i_1}) = -$. □

Lemma 5.12. *Let $M \subseteq \Gamma$ be a finite independence set for (X_+, X_-) with cardinality ℓ. List the elements of M as $s'_1, \dots, s'_\ell$. Assume that (s'_i, s'_j) has type (A2) for all $1 \leq i < j \leq \ell$. Then $\ell < n_{A2} := 13$.*

Proof. Assume that $\ell \geq 13$.

We claim that there are some map $\omega_0 : M \to \{+, -\}$ and a set $A \subseteq M$ with cardinality 4 such that $\omega_0 = +$ on A and the map $s \mapsto g(\omega_0, s)$ on A is injective. Let ω' be the map $M \to \{+\}$. Then either there is some subset A of M with cardinality 4 such that the map $s \mapsto g(\omega', s)$ on A is injective or there is some subset B of M with cardinality 5 such that $g(\omega', s) = k$ for some $k \geq 2$ and all $s \in B$. In the first situation we can take $\omega_0 = \omega'$. Thus assume that $B \subseteq M$ has cardinality 5 and $g(\omega', s) = k$ for all $s \in B$. Take $\theta_5 \in B$ such that (θ, θ_5) has type (A2) for all $\theta \in B \setminus \{\theta_5\}$. Note that $\theta_5 y_{\omega'}$ starts with $\zeta_\theta := \theta_5\theta^{-1}u_k$ for every $\theta \in B \setminus \{\theta_5\}$. Take $\theta_1 \in B \setminus \{\theta_5\}$ such that the length of ζ_{θ_1} is no bigger than that of ζ_θ for all $\theta \in B \setminus \{\theta_1, \theta_5\}$. Take $\omega'' : M \to \{+, -\}$ such that $\omega''(\theta_1) = -$ and $\omega''(\theta) = +$ for all $\theta \in B \setminus \{\theta_1\}$. For any $\theta \in B \setminus \{\theta_1, \theta_5\}$, since (θ, θ_5) has type (A2), either $k < g(\omega'', \theta) < g(\omega'', \theta_5)$ or $k = g(\omega'', \theta) = g(\omega'', \theta_5)$ and $\theta_5 y_{\omega''}$ starts with ζ_θ. In the latter situation $\theta_5 y_{\omega''}$ also starts with ζ_{θ_1}, and hence $\theta_1 x_{\omega''} \in X_+$, which contradicts $\omega''(\theta_1) = -$. Thus $k < g(\omega'', \theta) < g(\omega'', \theta_5)$ for every $\theta \in B \setminus \{\theta_1, \theta_5\}$. For any distinct $\theta, \theta' \in B \setminus \{\theta_1, \theta_5\}$, since either (θ, θ') or (θ', θ) has type (A2), we have $g(\omega'', \theta) \neq g(\omega'', \theta')$. Then we can take $A = B \setminus \{\theta_1\}$ and $\omega_0 = \omega''$. This proves our claim.

List the elements of A as $s_1, \ldots, s_4$ such that (s_i, s_j) has type (A2) for all $1 \leq i < j \leq 4$. Set $n_i = g(\omega_0, s_i)$. Then $n_1 < n_2 < n_3 < n_4$.

Take $\omega_1 : M \to \{+, -\}$ such that $\omega_1(s_1) = -$ and $\omega_1(s) = +$ for all $s \in A \setminus \{s_1\}$. Since (s_1, s_4) has type (A2), we have $g(\omega_1, s_4) \leq n_1$. Set $m = g(\omega_1, s_4)$. For $i = 2, 3$, since (s_i, s_4) has type (A2) and $g(\omega_1, s_4) \leq n_1 < n_i$, we get that $g(\omega_1, s_i) = m$ and $s_4 y_{\omega_1}$ starts with $\xi_i := s_4 s_i^{-1} u_m$. Write $\{2, 3\}$ as $\{i_1, i_2\}$ such that the length of ξ_{i_2} is no less than that of ξ_{i_1}.

Take $\omega_2 : M \to \{+, -\}$ such that $\omega_2(s_1) = \omega_2(s_{i_1}) = -$ and $\omega_2(s_{i_2}) = \omega_2(s_4) = +$. Since (s_1, s_4) has type (A2), we have $g(\omega_2, s_4) \leq n_1$. As (s_{i_2}, s_4) has type (A2) and $g(\omega_2, s_4) \leq n_1 < n_{i_2}$, we have $g(\omega_2, s_4) = m$ and $s_4 y_{\omega_2}$ starts with ξ_{i_2}. Then $s_4 y_{\omega_2}$ also starts with ξ_{i_1}. Since (s_{i_1}, s_4) has type (A2) and $g(\omega_2, s_4) = m = g(\omega_1, s_{i_1})$, we get that $s_{i_1} x_{\omega_2} \in X_+$, which contradicts $\omega_2(s_{i_1}) = -$. □

Lemma 5.13. *Let $M \subseteq \Gamma$ be a finite independence set for (X_+, X_-) with cardinality ℓ. List the elements of M as $s_1, \ldots, s_\ell$. Assume that (s_i, s_j) has type (B1) for all $1 \leq i < j \leq \ell$. Then $\ell < n_{B1} := 4$.*

Proof. Assume that $\ell = 4$.

Let ω_0 be the map $M \to \{+\}$. Set $n_i = g(\omega_0, s_i)$. Then $n_1 < n_2 < n_3 < n_4$.

Define $\omega_1 : M \to \{+, -\}$ by $\omega_1(s_1) = \omega_1(s_2) = -$ and $\omega_1(s_3) = \omega_1(s_4) = +$. Since (s_3, s_4) has type (B1), we have $g(\omega_1, s_4) > n_3$. Set $m = g(\omega_1, s_4)$. For $i = 1, 2$, since (s_i, s_4) has type (B1) and $g(\omega_1, s_4) > n_3 > n_i$, we get that $s_4 y_{\omega_1}$ starts with $\xi_i := s_4 s_i^{-1} u_{n_i} b$. Write $\{1, 2\}$ as $\{i_1, i_2\}$ such that the length of ξ_{i_2} is no less than that of ξ_{i_1}.

Define $\omega_2 : M \to \{+, -\}$ by $\omega_2(s_{i_2}) = -$ and $\omega_2(s) = +$ for all $s \in M \setminus \{s_{i_2}\}$. Since (s_3, s_4) has type (B1), we have $g(\omega_2, s_4) > n_3$. As (s_{i_2}, s_4) has type (B1), we have $g(\omega_2, s_4) = m$ and $s_4 y_{\omega_2}$ starts with ξ_{i_2}. Then $s_4 y_{\omega_2}$ also starts with ξ_{i_1}. Since (s_{i_1}, s_4) has type (B1), we get that $s_{i_1} x_{\omega_2} \notin X_+$, which contradicts $\omega_2(s_{i_1}) = +$. □

Lemma 5.14. *Let $M \subseteq \Gamma$ be a finite independence set for (X_+, X_-) with cardinality ℓ. List the elements of M as $s_1, \dots, s_\ell$. Assume that (s_i, s_j) has type (B2) for all $1 \le i < j \le \ell$. Then $\ell < n_{B2} := 4$.*

Proof. Assume that $\ell = 4$.

Let ω_0 be the map $M \to \{+\}$. Set $n_i = g(\omega_0, s_i)$. Then $n_1 < n_2 < n_3 < n_4$.

Define $\omega_1 : M \to \{+,-\}$ by $\omega_1(s_1) = \omega_1(s_2) = -$ and $\omega_1(s_3) = \omega_1(s_4) = +$. Since (s_3, s_4) has type (B2), we have $g(\omega_1, s_4) > n_3$. Set $m = g(\omega_1, s_4)$. For $i = 1, 2$, since (s_i, s_4) has type (B2) and $g(\omega_1, s_4) > n_3 > n_i$, we get that $s_i s_4^{-1} = u_{n_i} a^{-m}$.

Define $\omega_2 : M \to \{+,-\}$ by $\omega_2(s_2) = -$ and $\omega_2(s) = +$ for all $s \in M \setminus \{s_2\}$. Since (s_3, s_4) has type (B2), we have $g(\omega_2, s_4) > n_3$. Set $m' = g(\omega_2, s_4)$. As (s_2, s_4) has type (B2) and $g(\omega_2, s_4) > n_3 > n_2$, we have $s_2 s_4^{-1} = u_{n_2} a^{-m'}$, and hence $m = m'$. Then $s_1 s_4^{-1} = u_{n_1} a^{-m'}$. Since (s_1, s_4) has type (B2) and $s_1 s_4^{-1} = u_{n_1} a^{-m'}$, we get that $s_1 x_{\omega_2} \notin X_+$, which contradicts $\omega_2(s_1) = +$. □

Lemma 5.15. *Let $M \subseteq \Gamma$ be a finite independence set for (X_+, X_-) with cardinality ℓ. List the elements of M as $s_1, \dots, s_\ell$. Assume that (s_i, s_j) has type (B3) for all $1 \le i < j \le \ell$. Then $\ell < n_{B3} := 3$.*

Proof. Assume that $\ell = 3$.

Let ω_0 be the map $M \to \{+\}$. Set $n_i = g(\omega_0, s_i)$. Then $n_1 < n_2 < n_3$.

Define $\omega_1 : M \to \{+,-\}$ by $\omega_1(s_1) = -$ and $\omega_1(s_2) = \omega_1(s_3) = +$. Since (s_2, s_3) has type (B3), we have $g(\omega_1, s_3) = g(\omega_0, s_3)$. As (s_1, s_3) has type (B3) and $g(\omega_1, s_3) = g(\omega_0, s_3)$, we get that $s_1 x_{\omega_1} \in X_+$, which contradicts $\omega_1(s_1) = -$. □

Lemma 5.16. *Let $M \subseteq \Gamma$ be a finite independence set for (X_+, X_-) with cardinality ℓ. List the elements of M as $s_1, \dots, s_\ell$. Assume that (s_i, s_j) has type (B4) for all $1 \le i < j \le \ell$. Then $\ell < n_{B4} := 3$.*

Proof. Assume that $\ell = 3$.

Let ω_0 be the map $M \to \{+\}$. Set $n_i = g(\omega_0, s_i)$. Then $n_1 < n_2 < n_3$. For $i = 1, 2$, since (s_i, s_3) has type (B4), we get that $s_3 y_{\omega_0}$ starts with $\xi_i := s_3 s_i^{-1} u_{n_i}$. Write $\{1, 2\}$ as $\{i_1, i_2\}$ such that the length of ξ_{i_2} is no less than that of ξ_{i_1}.

Define $\omega_1 : M \to \{+,-\}$ by $\omega_1(s_{i_1}) = -$ and $\omega_1(s_{i_2}) = \omega_1(s_3) = +$. Since (s_{i_2}, s_3) has type (B4), we have that $g(\omega_1, s_3) = g(\omega_0, s_3)$ and $s_3 y_{\omega_1}$ starts with ξ_{i_2}. Then $s_3 y_{\omega_1}$ also starts with ξ_{i_1}. Since (s_{i_1}, s_3) has type (B4), we get that $s_{i_1} x_{\omega_1} \in X_+$, which contradicts $\omega_1(s_{i_1}) = -$. □

Lemma 5.17. *Let $M \subseteq \Gamma$ be a finite independence set for (X_+, X_-) with cardinality ℓ. List the elements of M as $s'_1, \dots, s'_\ell$. Assume that (s'_i, s'_j) has type (C1) for all $1 \le i < j \le \ell$. Then $\ell < n_{C1} := 6$.*

Proof. Assume that $\ell = 6$.

Let ω_0 be the map $M \to \{+\}$. Then there is some $m \geq 2$ such that $g(\omega_0, s) = m$ for all $s \in M$. Denote by A the set of $s \in M \setminus \{s_6'\}$ such that $sx \in D_m \times C_m$ whenever $x \in X$ and $s_6'x \in D_m \times C_m$. Set $B = M \setminus (\{s_6'\} \cup A)$. Then either $|A| \geq 3$ or $|B| \geq 3$.

Consider first the case $|A| \geq 3$. Take distinct points $s_1, s_2, s_3 \in A$. Define a map $\omega' : M \to \{+,-\}$ by $\omega'(s_3) = -$ and $\omega'(s) = +$ for all $s \in M \setminus \{s_3\}$. Then there is some $n \geq 2$ such that $g(\omega', s) = n$ for all $s \in M \setminus \{s_3\}$. Since $s_3 \in A$, we have $n \neq m$. For $i = 1, 2$, since (s_i, s_6') has type (C1), we get that $s_6' y_{\omega'}$ starts with $\xi_i := s_6' s_i^{-1} u_n$. Write $\{1,2\}$ as $\{i_1, i_2\}$ such that the length of ξ_{i_2} is no less than that of ξ_{i_1}. Take a map $\omega_1 : M \to \{+,-\}$ such that $\omega_1(s_3) = \omega_1(s_{i_1}) = -$ and $\omega_1(s_{i_2}) = \omega(s_6') = +$. As (s_{i_2}, s_6') has type (C1), we know that $g(\omega_1, s_{i_2}) = g(\omega_1, s_6')$ must be either n or m. As $\omega_1(s_3) = -$, we have $g(\omega_1, s_{i_2}) = g(\omega_1, s_6') = n$, and hence $s_6' y_{\omega_1}$ starts with ξ_{i_2}. Then $s_6' y_{\omega_1}$ also starts with ξ_{i_1}. Since (s_{i_1}, s_6') has type (C1), it follows that $s_{i_1} x_{\omega_1} \in X_+$, which contradicts $\omega_1(s_{i_1}) = -$.

Next consider the case $|B| \geq 3$. Take distinct points $s_1, s_2, s_3 \in B$. For any $i = 1, 2$, we have that $s_6' y_{\omega_0}$ starts with $\xi_i := s_6' s_i^{-1} u_m$. Write $\{1,2\}$ as $\{i_1, i_2\}$ such that the length of ξ_{i_2} is no less than that of ξ_{i_1}. Consider any map $\omega' : M \to \{+,-\}$ satisfying that $\omega'(s_{i_1}) = -$ and $\omega'(s_{i_2}) = \omega'(s_6') = +$. Since (s_{i_2}, s_6') has type (C1), if $g(\omega', s_{i_2}) = g(\omega', s_6') = m$, then $s_6' y_{\omega'}$ starts with ξ_{i_2} and hence starts with ξ_{i_1}, which implies that $s_{i_1} x_{\omega'} \in X_+$, a contradiction. Therefore $g(\omega', s_{i_2}) = g(\omega', s_6')$ is different from m, and hence does not depend on the choice of ω'. Take two maps $\omega_1, \omega_2 : M \to \{+,-\}$ satisfying the conditions for ω' such that $\omega_1(s_3) = +$ while $\omega_2(s_3) = -$. Set $n = g(\omega_1, s_{i_2}) = g(\omega_1, s_6') = g(\omega_1, s_3)$. Then $n \neq m$, and $n = g(\omega_2, s_{i_2}) = g(\omega_2, s_6')$. Since (s_3, s_6') has type (C1) and $s_3 \notin A$, for any $x \in X$ with $s_6' x \in D_n \times C_n$ we have $s_3 x \in D_n \times C_n$. In particular, $s_3 x_{\omega_2} \in D_n \times C_n$, which contradicts $\omega_2(s_3) = -$. $\square$

Lemma 5.18. *Let $M \subseteq \Gamma$ be a finite independence set for (X_+, X_-) with cardinality ℓ. List the elements of M as $s_1, \ldots, s_\ell$. Assume that (s_i, s_j) has type (C2) for all $1 \leq i < j \leq \ell$. Then $\ell < n_{C2} := 4$.*

Proof. Assume that $\ell = 4$.

Let ω_0 be the map $M \to \{+\}$. Then there is some $m \geq 2$ such that $g(\omega_0, s) = m$ for all $s \in M$.

Define $\omega_1 : M \to \{+,-\}$ by $\omega_1(s_1) = \omega(s_2) = -$ and $\omega_1(s_3) = \omega_1(s_4) = +$. Since (s_3, s_4) has type (C2), we have $g(\omega_1, s_4) = m$. For $i = 1, 2$, as (s_i, s_4) has type (C2), we know that $s_4 y_{\omega_1}$ starts with $\xi_i := s_4 s_i^{-1} u_m b$. Write $\{1,2\}$ as $\{i_1, i_2\}$ such that the length of ξ_{i_2} is no less than that of ξ_{i_1}.

Consider any map $\omega_2 : M \to \{+,-\}$ satisfying $\omega_2(s_{i_2}) = -$ and $\omega_2(s_{i_1}) = \omega_2(s_4) = +$. Since (s_{i_1}, s_4) has type (C2), we have that $g(\omega_2, s_4) = m$ and $s_4 y_{\omega_2}$ does not start with ξ_{i_1}. Then $s_4 y_{\omega_2}$ does not start with ξ_{i_2}. As (s_{i_2}, s_4) has type (C2), we get that $s_{i_2} x_{\omega_2} \in X_+$, which contradicts $\omega_2(s_{i_2}) = -$. $\square$

Lemma 5.19. *Let $M \subseteq \Gamma$ be a finite independence set for (X_+, X_-) with cardinality ℓ. List the elements of M as $s_1, \dots, s_\ell$. Assume that (s_i, s_j) has type (C3) for all $1 \le i < j \le \ell$. Then $\ell < n_{C3} := 3$.*

Proof. Assume that $\ell = 3$.

Let ω_0 be the map $M \to \{+\}$. Then there is some $m \ge 2$ such that $g(\omega_0, s) = m$ for all $s \in M$. For $i = 1, 2$, since (s_i, s_3) has type (C3), we get that $s_3 y_{\omega_0}$ starts with $\xi_i := s_3 s_i^{-1} u_m$. Write $\{1, 2\}$ as $\{i_1, i_2\}$ such that the length of ξ_{i_2} is no less than that of ξ_{i_1}.

Define $\omega_1 : M \to \{+, -\}$ by $\omega_1(s_{i_1}) = -$ and $\omega_1(s_{i_2}) = \omega_1(s_3) = +$. Since (s_{i_2}, s_3) has type (C3), we have that $g(\omega_1, s_3) = m$ and $s_3 y_{\omega_1}$ starts with ξ_{i_2}. Then $s_3 y_{\omega_1}$ starts with ξ_{i_1}. As (s_{i_1}, s_3) has type (C3), we get that $s_{i_1} x_{\omega_1} \in X_+$, which contradicts $\omega_1(s_{i_1}) = -$. □

According to the Ramsey theorem [1, page 183], given any natural numbers k and $c_1, \dots, c_k$ there is a natural number n such that for any graph $\mathcal{G}$ with n vertices and exactly one (unoriented) edge between any two distinct vertices, if we color the edges of $\mathcal{G}$ into k colors, then there are some $1 \le i \le k$ and a set A of the vertices of $\mathcal{G}$ with cardinality c_i such that the edge between any two distinct vertices in A has the i-th color. The smallest such number n is denoted by $R_k(c_1, \dots, c_k)$.

Lemma 5.20. *Each independence set $M \subseteq \Gamma$ for (X_+, X_-) has cardinality strictly less than*

$$R_{18}(n_{A1}, n_{A1}, n_{A2}, n_{A2}, n_{B_1}, n_{B_1}, n_{B2}, n_{B2}, n_{B3}, n_{B3}, n_{B4}, n_{B4}, n_{C1}, n_{C1}, n_{C2}, n_{C2}, n_{C3}, n_{C3}).$$

Proof. We may assume that M is finite. Consider the graph $\mathcal{G}$ with vertex set M and one unoriented edge between any two distinct vertices. We shall color the edges of $\mathcal{G}$ with 18 colors as follows. Fix a linear order on M. For any $s < s'$ in M, we color the edge between s and s' with color $*$ (resp. $*'$) if the pair (s, s') (resp. (s', s)) has type $*$, where $*$ is one of the 9 types (A1), ..., (C3) in Lemma 5.10. If certain edge can be colored in more than one way, take any choice. If $|M|$ is no less than the above Ramsey number, then for certain type $*$, there is a subset A of M with cardinality n_* such that either the pair (s, s') has type $*$ for all $s < s'$ in A or the pair (s', s) has type $*$ for all $s < s'$ in A, which is impossible by Lemmas 5.11-5.19. □

By [16, Corollary 12.3] the action $\Gamma \curvearrowright Y$ is null. From Lemma 4.1 we know that $\Gamma \curvearrowright Z$ is null. It is also clear from (2) and (4) of Proposition 2.2 that the class of null actions is closed under taking products. Thus the product action $\Gamma \curvearrowright X$ is null. Then from Proposition 3.1 and Lemma 5.20 we conclude that the action $\Gamma \curvearrowright X_f$ is null. This finishes the proof of Theorem 1.2.

To end this section, we discuss one observation pointed out to us by Glasner. A minimal action $\Gamma \curvearrowright \mathcal{X}$ is called *strictly SPI* [6, page 127] if there are an ordinal τ, an action $\Gamma \curvearrowright \mathcal{X}_\alpha$ and a factor map $\pi_\alpha : \mathcal{X} \to \mathcal{X}_\alpha$ for each ordinal $\alpha \le \tau$ such that π_τ is the identity map of $\mathcal{X}$, $\mathcal{X}_0$ is a singleton, π_β factors through π_α for all $\beta < \alpha \le \tau$, for each ordinal $\alpha < \tau$ the factor map $\mathcal{X}_{\alpha+1} \to \mathcal{X}_\alpha$ is either strongly proximal or isometric, and for each limit ordinal $\alpha \leq \tau$ the map π_α is the inverse limit of π_β for $\beta < \alpha$. In such case, one has a canonical choice of these $\Gamma \curvearrowright \mathcal{X}_\alpha$, called the *canonical SPI tower* of $\Gamma \curvearrowright \mathcal{X}$, defined inductively by taking $\Gamma \curvearrowright X_0$ to be the action on a singleton, and for each limit ordinal α taking $\Gamma \curvearrowright \mathcal{X}_\alpha$ to be the inverse limit of $\Gamma \curvearrowright \mathcal{X}_\beta$ for all $\beta < \alpha$, taking $\Gamma \curvearrowright \mathcal{X}_{\alpha+2n+1}$ to be the largest strongly proximal factor of $\mathcal{X} \to \mathcal{X}_{\alpha+2n}$ (i.e. one has factor maps $\mathcal{X} \to \mathcal{X}_{\alpha+2n+1} \to \mathcal{X}_{\alpha+2n}$, the extension $\mathcal{X}_{\alpha+2n+1} \to \mathcal{X}_{\alpha+2n}$ is strongly proximal, and for any factor maps $\mathcal{X} \to \mathcal{X}' \to \mathcal{X}_{\alpha+2n}$ with $\mathcal{X}' \to \mathcal{X}_{\alpha+2n}$ strongly proximal the map $\mathcal{X} \to \mathcal{X}'$ factors through $\mathcal{X} \to \mathcal{X}_{\alpha+2n+1}$) for all integers $n \geq 0$, and taking $\Gamma \curvearrowright \mathcal{X}_{\alpha+2n+2}$ to be the largest isometric factor of $\mathcal{X} \to \mathcal{X}_{\alpha+2n+1}$ (i.e. one has factor maps $\mathcal{X} \to \mathcal{X}_{\alpha+2n+2} \to \mathcal{X}_{\alpha+2n+1}$, the extension $\mathcal{X}_{\alpha+2n+2} \to \mathcal{X}_{\alpha+2n+1}$ is isometric, and for any factor maps $\mathcal{X} \to \mathcal{X}' \to \mathcal{X}_{\alpha+2n+1}$ with $\mathcal{X}' \to \mathcal{X}_{\alpha+2n+1}$ isometric the map $\mathcal{X} \to \mathcal{X}'$ factors through $\mathcal{X} \to \mathcal{X}_{\alpha+2n+2}$) for all integers $n \geq 0$. This inductive process stops at some ordinal τ when the map $\mathcal{X} \to \mathcal{X}_\tau$ is a homeomorphism. We are grateful to Glasner for showing us the following result.

Proposition 5.21. *The action $\Gamma \curvearrowright X_f$ is strictly SPI, and $X_f \to X \to Y \to \{pt\}$ is its canonical SPI tower, where $\Gamma \curvearrowright \{pt\}$ is the trivial action on a singleton.*

Proof. We know that $\Gamma \curvearrowright Y$ is strongly proximal. Let $\Gamma \curvearrowright Y^*$ be the largest strongly proximal factor of $\Gamma \curvearrowright X_f$. Then we have factor maps $X_f \to Y^* \xrightarrow{\vartheta} Y$. The proof of Lemma 5.1 shows that the product action $\Gamma \curvearrowright Y^* \times Z$ is minimal, and hence it is a factor of $\Gamma \curvearrowright X_f$. We now have the following diagram:

$$\begin{tikzcd} X_f \arrow[d] \arrow[dd, bend left, "\pi_f"] \\ Y^* \times Z \arrow[d, "\vartheta\times\mathrm{id}_Z"'] \\ Y \times Z \end{tikzcd}$$

By construction, the map π_f has the property that $\pi_f^{-1}(y, z)$ is a singleton on the complement of a countable set. If for some $y \in Y$ the set $\vartheta^{-1}(y)$ is not a singleton, then for each $z \in Z$ we have that $(\vartheta \times \mathrm{id}_Z)^{-1}(y, z) = \vartheta^{-1}(y) \times \{z\}$ is not a singleton as well, which is a contradiction since Z is uncountable. Thus ϑ is injective, and hence $\Gamma \curvearrowright Y$ is the largest strongly proximal factor of $\Gamma \curvearrowright X_f$.

The extension $X = Y \times Z \to Y$ is clearly isometric. The extension π_f is almost one-to-one, hence has no nontrivial isometric factor. It follows that $\Gamma \curvearrowright X$ is the largest isometric factor of $X_f \to Y$.

Finally, since π_f is almost one-to-one, it is strongly proximal. $\square$

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