ENTROPY, PRODUCTS, AND BOUNDED ORBIT EQUIVALENCE

DAVID KERR AND HANFENG LI

Abstract. We prove that if two topologically free and entropy regular actions of countable sofic groups on compact metrizable spaces are continuously orbit equivalent, and each group either (i) contains a w-normal amenable subgroup which is neither locally finite nor virtually cyclic, or (ii) is a non-locally-finite product of two infinite groups, then the actions have the same sofic topological entropy. This fact is then used to show that if two free uniquely ergodic and entropy regular probability-measure-preserving actions of such groups are boundedly orbit equivalent then the actions have the same sofic measure entropy. Our arguments are based on a relativization of property SC to sofic approximations and yield more general entropy inequalities.

Contents

1. Introduction

At first glance it may seem that dynamical entropy and orbit equivalence should have little to do with one another. One is a conjugacy invariant that is tailor-made for the hairsplitting

Date: October 20, 2021.

job of distinguishing Bernoulli shifts, all of which have the same spectral theory, while the other is a coarse relation between group actions whose tendency to nullify asymptotic behaviour is most devastating in the setting of amenable groups, where entropy finds its classical home [26, 27, 41, 10, 34]. One registers information while the other threatens to destroy it.

This brutal disparity can however be honed so as to bring the two concepts into frequent and sometimes surprising alignment. Indeed entropy turns out to be sensitive in meaningful ways to the various kinds of restrictions that one may naturally impose on an orbit equivalence, its role as an invariant remaining intact in some cases but completely neutralized in others. The history of this relationship traces back several decades and in its original thrust encompasses the work of Vershik on actions of locally finite groups [42, 43], the Ornstein isomorphism machinery for Bernoulli shifts [32], the theory of Kakutani equivalence [19, 11, 33, 9], and Kammeyer and Rudolph's general theory of restricted orbit equivalence for p.m.p. actions of countable amenable groups that all of this inspired [37, 17, 18] (see Chapter 1 of [18] for a genealogy). In a somewhat different vein from these lines of investigation, Rudolph and Weiss later proved in [38] that, for a free p.m.p. action of a countable amenable group, the conditional entropy with respect to a prescribed invariant sub-σ-algebra $\mathscr{S}$ is preserved under every $\mathscr{S}$-measurable orbit equivalence. As Rudolph and Weiss demonstrated in the application to completely positive entropy that motivated their paper, this crisp expression of complementarity between entropy and orbit equivalence, when combined with the Ornstein–Weiss theorem [34], turns out to be very useful as a tool for lifting results from $\mathbb{Z}$-actions to actions of general countable amenable groups. More recently Austin has shown, for free p.m.p. actions of finitely generated amenable groups, that entropy is an invariant of bounded and integrable orbit equivalence, and that there is an entropy scaling formula for stable versions of these equivalences [3]. It is interesting to note that Austin makes use of both the theory of Kakutani equivalence (to handle the virtually cyclic case, which his approach requires him to treat separately) and the Rudolph–Weiss theorem (in a reduction-to-$\mathbb{Z}$ argument which, ironically, forms part of the verification of the non-virtually-cyclic case).

The basic geometric idea at play in Austin's work when the group is not virtually cyclic is the possibility of finding, within suitable connected Følner subsets of the group, a connected subgraph which is sparse but at the same time dense at a specified coarse scale. By recasting this sparse connectivity as a condition on the action that we called *property SC* and circumventing the "derandomization" of [3] with its reliance on the Rudolph–Weiss technique, we established in [25] the following extension beyond the amenable setting: if G is a countable group containing a w-normal amenable subgroup which is neither locally finite nor virtually cyclic, H is a countable group, and $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ are free p.m.p. actions which are Shannon orbit equivalent (i.e., the cocycle partitions all have finite Shannon entropy), then the maximum sofic measure entropies of the actions satisfy

$$h_\nu(H \curvearrowright Y) \geq h_\mu(G \curvearrowright X). \tag{1}$$

One property shared by the groups G in this theorem is that their first ℓ^2-Betti number vanishes, which in the nonamenable world can be roughly intuited as an expression of anti-freeness, and indeed our approach breaks down for free groups (see Section 3.5 of [25]). In what is surely not a coincidence, groups whose Bernoulli actions are cocycle superrigid also have vanishing first ℓ^2-Betti number [35], and it has been speculated that these two properties are equivalent in the nonamenable realm (curiously, however, Bernoulli cocycle or orbit equivalence superrigidity remains generally unknown for wreath products of the form $\mathbb{Z} \wr H$ with H nonamenable, which satisfy the hypotheses on G above).

Given that nonamenable products of countably infinite groups form a standard class of examples within the circle of ideas around superrigidity, cost one, and vanishing first ℓ^2-Betti number, and in particular are known to satisfy Bernoulli cocycle superrigidity by a theorem of Popa [36], it is natural to wonder whether the entropy inequality (1) holds if G is instead assumed to be such a product. In [25] we demonstrated, in analogy with Gaboriau's result on cost for products of equivalence relations [13], that product actions of non-locally-finite product groups, when equipped with an arbitrary invariant probability measure, satisfy property SC, which is sufficient for establishing (1). However, such actions always have maximum sofic entropy zero or $-\infty$. One of the main questions motivating the present paper is whether one can remove this product structure hypothesis on the action.

To this end we establish Theorem A below, which gives the conclusion for bounded orbit equivalence (i.e., orbit equivalence with finite cocycle partitions, as explained in Section 2.3) and uniquely ergodic actions. We say that a p.m.p. action $G \curvearrowright (X, \mu)$ is *uniquely ergodic* if the only G-invariant mean on $L^\infty(X, \mu)$ is integration with respect to μ, i.e., the induced action of G on the spectrum of $L^\infty(X, \mu)$ is uniquely ergodic in the usual sense of topological dynamics. When μ is atomless, unique ergodicity forces the acting group to be nonamenable [39, Theorem 2.4]. In fact an ergodic p.m.p. action $G \curvearrowright (X, \mu)$ is uniquely ergodic if and only if the restriction of the Koopman representation to $L^2(X, \mu) \ominus \mathbb{C}1$ does not weakly contain the trivial representation [39, Proposition 2.3]. It follows that if G is nonamenable then unique ergodicity holds whenever the restriction of the Koopman representation to the orthogonal complement of the constants is a direct sum of copies of the left regular representation, and in particular when the action has completely positive entropy [15, Corollary 1.2][40, Corollary 1.7], and thus occurs in the following examples:

(i) Bernoulli actions $G \curvearrowright (X^G, \mu^G)$, where (X, μ) is a standard probability space and $(gx)_h = x_{g^{-1}h}$ for all $g, h \in G$ and $x \in X^G$ (see Section 2.3.1 of [24]),

(ii) algebraic actions of the form $G \curvearrowright (\widehat{(\mathbb{Z}G)^n/(\mathbb{Z}G)^n A}, \mu)$ where $A \in M_n(\mathbb{Z}G)$ is invertible as an operator on $\ell^2(G)^{\oplus n}$ and μ is the normalized Haar measure [16, Corollary 1.5].

Moreover, if G has property (T) then all of its ergodic p.m.p. actions are uniquely ergodic [39, Theorem 2.5].

As above $h_\mu(\cdot)$ denotes the maximum sofic measure entropy, and we write $\underline{h}_\mu(\cdot)$ for the infimum sofic measure entropy (see Section 2.6).

Theorem A. Let G and Γ be countably infinite sofic groups at least one of which is not locally finite, and let H be a countable group. Let $G \times \Gamma \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ be free p.m.p. actions which are boundedly orbit equivalent. Suppose that the action of H is uniquely ergodic. Then

$$h_\nu(H \curvearrowright Y) \geq \underline{h}_\mu(G \times \Gamma \curvearrowright X).$$

Theorem A is a direct consequence of Theorem 5.2 and Proposition 3.15. When combined with Theorem A of [25] it yields the following Theorem B. We say that an action is *entropy regular* if its maximum and infimum sofic entropies are equal, i.e., the sofic entropy does not depend on the choice of sofic approximation sequence. Entropy regularity for a p.m.p. action is known to hold in the following situations:

(i) the group is amenable, in which case the sofic measure entropy is equal to the amenable measure entropy [23, 6],
(ii) the action is Bernoulli [5, 22],
(iii) the action is an algebraic action of the form $G \curvearrowright (\widehat{(\mathbb{Z}G)^n/(\mathbb{Z}G)^n A}, \mu)$ where $A \in M_n(\mathbb{Z}G)$ is injective as an operator on $\ell^2(G)^{\oplus n}$ and μ is the normalized Haar measure [14],
(iv) the action is a shift action $G \curvearrowright \{1, \dots, n\}^G$ equipped with a Gibbs measure satisfying one of various uniqueness conditions [1, 2].

For the definition of w-normality see the paragraph before Theorem 3.12.

Theorem B. Let G and H be countable sofic groups each of which either

(i) contains a w-normal amenable subgroup which is neither locally finite nor virtually cyclic, or
(ii) is a product of two countably infinite sofic groups at least one of which is not locally finite.

Let $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ be free p.m.p. actions which are uniquely ergodic and entropy regular, and suppose that they are boundedly orbit equivalent. Then

$$h_\nu(H \curvearrowright Y) = h_\mu(G \curvearrowright X).$$

We note that, by a theorem of Belinskaya [4], if two ergodic p.m.p. $\mathbb{Z}$-actions are integrably orbit equivalent, and in particular if they are boundedly orbit equivalent, then they are measure conjugate up to an automorphism of $\mathbb{Z}$ (what is referred to as "flip conjugacy"). On the other hand, a bounded orbit equivalence between ergodic p.m.p. $\mathbb{Z}^d$-actions for $d \geq 2$ can scramble local asymptotic data to the point of scuttling properties like mixing and completely positive entropy, as Fieldsteel and Friedman demonstrated in [12].

Our strategy for proving Theorem A is to localize property SC to sofic approximations, yielding what we call "property sofic SC" for a group or an action, or more generally "property $\mathscr{S}$-SC" where $\mathscr{S}$ is a collection of sofic approximations for the group in question (see Section 3.1). The advantage of this localization is that the action itself need not have a product structure, only the sofic approximation used to model it. This accounts for the appearance of the infimum sofic entropy in Theorem A, in contrast to (1), but as noted above many actions of interest are known to be entropy regular, in which case one does in fact get (1). The trade-off in using property sofic SC is its natural and frustratingly stubborn compatibility with the point-map formulation of sofic entropy, which is a kind of dualization of the homomorphism picture adopted in [25] and requires the choice of a topological model. This has put us into the situation of not being able to control the empirical distribution of microstates except under the hypothesis of unique ergodicity, when the variational principle makes such control unnecessary for the purpose of computing the entropy, and even then we have had to restrict the hypothesis on the orbit equivalence from Shannon to bounded.

Given that we are adhering to the point-map picture with its use of topological models, it makes sense to isolate as much of the argument as possible to the purely topological framework, which also has its own independent interest. Accordingly we establish the following theorem, the second part of which goes into proving Theorem A via Theorem 5.2. It is a direct consequence of Theorems 4.1 and 3.12 and Proposition 3.15. Here $h(\cdot)$ denotes the maximum sofic topological entropy and $\underline{h}(\cdot)$ the infimum sofic topological entropy (see Section 2.5).

Theorem C. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free continuous actions of countable sofic groups on compact metrizable spaces, and suppose that they are continuously orbit equivalent. If G contains a w-normal amenable subgroup which is neither locally finite nor virtually cyclic then

$$h(H \curvearrowright Y) \geq h(G \curvearrowright X),$$

while if G is a product of two countably infinite groups at least one of which is not locally finite then

$$h(H \curvearrowright Y) \geq \underline{h}(G \curvearrowright X).$$

If the actions of G and H above are genuinely free or if G and H are torsion-free, then using the variational principle (Theorem 10.35 in [24]) and (in the case of torsion-free G and H) the main result of [30] one can also derive the first part of the above theorem from Theorem A of [25], or from [3] if G and H are in addition amenable and finitely generated.

In parallel with the p.m.p. setting, we define a continuous action of a countable sofic group on a compact metrizable space to be *entropy regular* if its maximum and infimum sofic topological entropies are equal, and note that this occurs in the following situations:

(i) the group is amenable, in which case the sofic topological entropy is equal to the amenable topological entropy [23],

(ii) the action is a shift action $G \curvearrowright X^G$ where X is a compact metrizable space [24, Proposition 10.28],

(iii) the action is an algebraic action of the form $G \curvearrowright \widehat{(\mathbb{Z}G)^n/(\mathbb{Z}G)^n A}$ where $A \in M_n(\mathbb{Z}G)$ is injective as an operator on $\ell^2(G)^{\oplus n}$ [14].

From Theorem C we immediatety obtain:

Theorem D. Let G and H be countable sofic groups each of which either

(a) contains a w-normal amenable subgroup which is neither locally finite nor virtually cyclic, or

(b) is a product of two countably infinite sofic groups at least one of which is not locally finite.

Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free and entropy regular continuous actions on compact metrizable spaces, and suppose that they are continuously orbit equivalent. Then

$$h(H \curvearrowright Y) = h(G \curvearrowright X).$$

It was shown in [7, 8] that the finite-base shift actions of a finitely generated group satisfy continuous cocycle superrigidity if and only if the group has one end (a property that the groups in Theorem D possess when they are finitely generated—see Example 1 in [7]). As observed in [7], this implies, in conjunction with a theorem from [28], that if a finitely generated group is torsion-free and amenable then each of its shift actions with finite base is continuous orbit equivalence superrigid. Whether such superrigidity ever occurs in the nonamenable setting appears however to be unknown.

We begin the main body of the paper in Section 2 by setting up general notation and reviewing terminology concerning continuous and bounded orbit equivalence and sofic entropy. In Section 3.1 we define properties $\mathscr{S}$-SC and sofic SC for groups, p.m.p. actions, and continuous actions on compact metrizable spaces. In Section 3.2 we determine that a countable group fails to have property sofic SC if it is locally finite or finitely generated and virtually free. In Section 3.3 we verify that, for free p.m.p. actions, property SC implies property sofic SC, and then use this in conjunction with [25] to show that (i) for countable amenable groups property sofic SC is equivalent to the group being neither locally finite nor virtually cyclic, and (ii) if a countable group has a w-normal subgroup which is amenable but neither locally finite nor virtually cyclic then the group has property sofic SC. In Section 3.4 we prove that if a w-normal subgroup has property sofic SC then so does the ambient group, while in Section 3.5 we determine that the product of two countably infinite groups has property $\mathscr{S}$-SC, where $\mathscr{S}$ is the collection of product sofic approximations, if and only if at least one of the factors is not locally finite. Sections 3.6 and 3.7 show property sofic SC to be an invariant of continuous orbit equivalence for topologically free continuous actions on compact metrizable spaces and of bounded orbit equivalence for free p.m.p. actions. Section 4 is devoted to the proof of Theorem 4.1, which together with Theorem 3.12 and Proposition 3.15 gives Theorem C. Finally, in Section 5 we establish Theorem 5.2, which together with Proposition 3.15 yields Theorem A.

Acknowledgements. The first author was partially supported by NSF grant DMS-1800633 and was affiliated with Texas A&M University during the course of the work. Preliminary stages were carried out during his six-month stay at the ENS de Lyon in 2017-2018, during which time he held ENS and CNRS visiting professorships and was supported by Labex MILYON/ANR-10-LABX-0070. The second author was partially supported by NSF grants DMS-1600717 and DMS-1900746. We thank the referee for helpful comments and suggestions.

2. Preliminaries

2.1. **Basic notation and terminology.** Throughout the paper G and H are countable discrete groups, with identity elements e_G and e_H. We write $\mathcal{F}(G)$ for the collection of all nonempty finite subsets of G, and $\overline{\mathcal{F}}(G)$ for the collection of symmetric finite subsets of G containing e_G. For a nonempty finite set V, the algebra of all subsets of V is denoted by $\mathbb{P}_V$, the group of all permutations of V by $\mathrm{Sym}(V)$, and the uniform probability measure on V by m.

Given a property P, a group is said to be *virtually P* if it has a subgroup of finite index with property P, and *locally P* if each of its finitely generated subgroups has property P.

A *standard probability space* is a standard Borel space (i.e., a Polish space with its Borel σ-algebra) equipped with a probability measure. Partitions of such a space are always understood to be Borel. A *p.m.p. (probability-measure preserving) action* of G is an action $G \curvearrowright (X,\mu)$ of G on a standard probability space by measure-preserving transformations. Such an action is *free* if the set X_0 of all $x \in X$ such that $sx \neq x$ for all $s \in G \setminus \{e_G\}$ has measure one. Two p.m.p. actions $G \curvearrowright (X,\mu)$ and $G \curvearrowright (Y,\nu)$ are *measure conjugate* if there exist G-invariant conull sets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ and a G-equivariant measure isomorphism $X_0 \to Y_0$.

A continuous action $G \curvearrowright X$ on a compact metrizable space is said to be *topologically free* if the G_δ set of all $x \in X$ such that $sx \neq x$ for all $s \in G \setminus \{e_G\}$ is dense. It is *uniquely ergodic* if there is a unique G-invariant Borel probability measure on X. By Gelfand theory this is equivalent to the existence of a unique G-invariant state (i.e., unital positive linear functional) for the induced action of G on the C*-algebra $C(X)$ of continuous functions on X given by $(gf)(x) = f(g^{-1}x)$ for all $g \in G$, $f \in C(X)$, and $x \in X$.

A p.m.p. action $G \curvearrowright (X,\mu)$ is *uniquely ergodic* if there is a unique state (or *mean* as it is also called in this setting) on $L^\infty(X,\mu)$ which is invariant for the action of G given by

$(gf)(x) = f(g^{-1}x)$ for all $g \in G$, $f \in L^\infty(X,\mu)$, and $x \in X$. Again by Gelfand theory, this is equivalent to the unique ergodicity, in the topological-dynamical sense above, of the induced action of G on the spectrum of $L^\infty(X,\mu)$.

2.2. **Continuous orbit equivalence.** We say that two continuous actions $G \curvearrowright X$ and $H \curvearrowright Y$ on compact metrizable spaces are *continuously orbit equivalent* if there exist a homeomorphism $\Phi : X \to Y$ and continuous maps $\kappa : G \times X \to H$ and $\lambda : H \times Y \to G$ such that

$$\Phi(gx) = \kappa(g,x)\Phi(x),$$
$$\Phi^{-1}(ty) = \lambda(t,y)\Phi^{-1}(y)$$

for all $g \in G$, $x \in X$, $t \in H$, and $y \in Y$. Such a Φ is called a *continuous orbit equivalence.*

If the action $H \curvearrowright Y$ is topologically free then the continuity of Φ implies that the map κ is uniquely determined by the first line of the above display and satisfies the *cocycle identity*

$$\kappa(fg,x) = \kappa(f,gx)\kappa(g,x)$$

for $f,g \in G$ and $x \in X$. In the case that both $G \curvearrowright X$ and $H \curvearrowright Y$ are topologically free we have

$$\lambda(\kappa(g,x),\Phi(x)) = g$$

for all $g \in G$ and $x \in X$, and λ is uniquely determined by this identity.

2.3. **Bounded orbit equivalence.** Two free p.m.p. actions $G \curvearrowright (X,\mu)$ and $H \curvearrowright (Y,\nu)$ are *orbit equivalent* if there exist a G-invariant conull set $X_0 \subseteq X$, an H-invariant conull set $Y_0 \subseteq Y$, and a measure isomorphism $\Psi : X_0 \to Y_0$ such that $\Psi(Gx) = H\Psi(x)$ for all $x \in X_0$. Such a Ψ is called an *orbit equivalence.* Associated to Ψ are the cocycles $\kappa : G \times X_0 \to H$ and $\lambda : H \times Y_0 \to G$ determined (up to null sets, in accord with our definition of freeness) by

$$\Psi(gx) = \kappa(g,x)\Psi(x),$$
$$\Psi^{-1}(ty) = \lambda(t,y)\Psi^{-1}(y)$$

for all $g \in G$, $x \in X_0$, $t \in H$, and $y \in Y_0$. We say that the cocycle κ is *bounded* if $\kappa(g,X_0)$ is finite for every $g \in G$, and define boundedness for λ likewise. If X_0, Y_0, and Ψ can be chosen so that κ and λ are both bounded, then we say that the actions are *boundedly orbit equivalent*, and refer to Ψ as a *bounded orbit equivalence.*

2.4. **Sofic approximations.** Given a nonempty finite set V we define on V^V the normalized Hamming distance

$$\rho_{\mathrm{Hamm}}(T,S) = \frac{1}{|V|}|\{v \in V : Tv \neq Sv\}|.$$

A *sofic approximation* for G is a (not necessarily multiplicative) map $\sigma : G \to \mathrm{Sym}(V)$ for some nonempty finite set V. Given a finite set $F \subseteq G$ and a $\delta > 0$, we say that such a σ is an (F,δ)-*approximation* if

(i) $\rho_{\mathrm{Hamm}}(\sigma_{st},\sigma_s\sigma_t) \leq \delta$ for all $s,t \in F$, and
(ii) $\rho_{\mathrm{Hamm}}(\sigma_s,\sigma_t) \geq 1-\delta$ for all distinct $s,t \in F$.

A *sofic approximation sequence* for G is a sequence $\Sigma = \{\sigma_k : G \to \mathrm{Sym}(V_k)\}_{k=1}^\infty$ of sofic approximations for G such that for every finite set $F \subseteq G$ and $\delta > 0$ there exists a $k_0 \in \mathbb{N}$ such that σ_k is an (F, δ)-approximation for every $k \geq k_0$. A sofic approximation $\sigma : G \to \mathrm{Sym}(V)$ is said to be *good enough* if it is an (F, δ)-approximation for some finite set $F \subseteq G$ and $\delta > 0$ and this condition is sufficient for the purpose at hand.

The group G is *sofic* if it admits a sofic approximation sequence, which is the case for instance if G is amenable or residually finite. It is not known whether nonsofic groups exist.

Given a sofic approximation $\sigma : G \to \mathrm{Sym}(V)$ and a set $A \subseteq G$, we define an *A-path* to be a finite tuple $(v_0, v_1, \dots, v_n)$ of points in V such that for every $i = 1, \dots, n$ there is a $g \in A$ for which $v_i = \sigma_g v_{i-1}$. The integer n is the *length* of the path, the points $v_0, \dots, v_n$ its *vertices*, and v_0 and v_n its *endpoints*. When $n = 1$ we also speak of an *A-edge*. For $r \in \mathbb{N}$, we say that a set $W \subseteq V$ is *(A, r)-separated* if $\sigma_{A^r} v \cap \sigma_{A^r} w = \emptyset$ for all distinct $v, w \in W$.

2.5. **Sofic topological entropy.** Let $G \curvearrowright X$ be a continuous action on a compact metrizable space. Let d be a compatible metric on X. Let F be a finite subset of G and $\delta > 0$. Let $\sigma : G \to \mathrm{Sym}(V)$ be a sofic approximation for G. On the set of maps $V \to X$ define the pseudometrics

$$d_2(\varphi, \psi) = \left(\frac{1}{|V|} \sum_{v \in V} d(\varphi(v), \psi(v))^2 \right)^{1/2},$$
$$d_\infty(\varphi, \psi) = \max_{v \in V} d(\varphi(v), \psi(v)).$$

Define $\mathrm{Map}_d(F, \delta, \sigma)$ to be the set of all maps $\varphi : V \to X$ such that $d_2(\varphi\sigma_g, g\varphi) \leq \delta$ for all $g \in F$. For a pseudometric space (Ω, ρ) and $\varepsilon > 0$ we write $N_\varepsilon(\Omega, \rho)$ for the maximum cardinality of a subset Ω_0 of Ω which is (ρ, ε)-separated in the sense that $\rho(\omega_1, \omega_2) \geq \varepsilon$ for all distinct $\omega_1, \omega_2 \in \Omega_0$.

Let $\Sigma = \{\sigma_k : G \to \mathrm{Sym}(V_k)\}_{k=1}^\infty$ be a sofic approximation sequence for G. For $\varepsilon > 0$ we set

$$h^\varepsilon_{\Sigma,\infty}(G \curvearrowright X) = \inf_F \inf_{\delta > 0} \limsup_{k \to \infty} \frac{1}{|V_k|} \log N_\varepsilon(\mathrm{Map}_d(F, \delta, \sigma_k), d_\infty),$$
$$h^\varepsilon_{\Sigma,2}(G \curvearrowright X) = \inf_F \inf_{\delta > 0} \limsup_{k \to \infty} \frac{1}{|V_k|} \log N_\varepsilon(\mathrm{Map}_d(F, \delta, \sigma_k), d_2),$$

where the first infimum in each case is over all finite sets $F \subseteq G$. The *sofic topological entropy* of the action $G \curvearrowright X$ with respect to Σ is then defined by

$$h_\Sigma(G \curvearrowright X) = \sup_{\varepsilon > 0} h^\varepsilon_{\Sigma,\infty}(G \curvearrowright X).$$

This quantity does not depend on the choice of compatible metric d, as is readily seen, and by Proposition 10.23 of [24] we can also compute it using separation with respect to d_2, i.e.,

$$h_\Sigma(G \curvearrowright X) = \sup_{\varepsilon > 0} h^\varepsilon_{\Sigma,2}(G \curvearrowright X).$$

We define the *maximum* and *infimum sofic topological entropies* of $G \curvearrowright X$ by

$$h(G \curvearrowright X) = \max_\Sigma h_\Sigma(G \curvearrowright X),$$
$$\underline{h}(G \curvearrowright X) = \inf_\Sigma h_\Sigma(G \curvearrowright X),$$

where Σ ranges in each case over all sofic approximation sequences for G (when G is nonsofic we interpret these quantities to be $-\infty$). It is a straightforward exercise to show that the maximum does indeed exist (we do not know however whether the infimum is always realized). Note that $-\infty$ is a possible value for $h_\Sigma(G \curvearrowright X)$, and so if it occurs for some Σ then $\underline{h}(G \curvearrowright X) = -\infty$, and if it occurs for all Σ then $h(G \curvearrowright X) = -\infty$. The action $G \curvearrowright X$ is *entropy regular* if its maximum and infimum sofic topological entropies are equal, i.e., the sofic topological entropy does not depend on the choice of sofic approximation sequence.

2.6. **Sofic measure entropy.** Let $G \curvearrowright (X, \mu)$ be a p.m.p. action. Let $\mathscr{C}$ be a finite Borel partition of X, F a finite subset of G containing e_G, and $\delta > 0$. Write $\mathrm{alg}(\mathscr{C})$ for the algebra generated by $\mathscr{C}$, which consists of all unions of members of $\mathscr{C}$, and write $\mathscr{C}_F$ for the join $\bigvee_{s\in F} s\mathscr{C}$. Let $\sigma : G \to \mathrm{Sym}(V)$ be a sofic approximation for G. Write $\mathrm{Hom}_\mu(\mathscr{C}, F, \delta, \sigma)$ for the set of all homomorphisms $\varphi : \mathrm{alg}(\mathscr{C}_F) \to \mathbb{P}_V$ satisfying

(i) $\sum_{A\in\mathscr{C}} \mathrm{m}(\sigma_g\varphi(A)\Delta\varphi(gA)) < \delta$ for all $g \in F$, and
(ii) $\sum_{A\in\mathscr{C}_F} |\mathrm{m}(\varphi(A)) - \mu(A)| < \delta$.

For a finite Borel partition $\mathscr{P} \leq \mathscr{C}$ we write $|\mathrm{Hom}_\mu(\mathscr{C}, F, \delta, \sigma)|_{\mathscr{P}}$ for the cardinality of the set of restrictions of elements of $\mathrm{Hom}_\mu(\mathscr{C}, F, \delta, \sigma)$ to $\mathscr{P}$.

Given a sofic approximation sequence $\Sigma = \{\sigma_k : G \to \mathrm{Sym}(V_k)\}_{k=1}^\infty$ for G, we define the *sofic measure entropy* of the action $G \curvearrowright (X, \mu)$ with respect to Σ by

$$h_{\Sigma,\mu}(G \curvearrowright X) = \sup_{\mathscr{P}} \inf_{\mathscr{C}\geq\mathscr{P}} \inf_F \inf_{\delta>0} \limsup_{k\to\infty} \frac{1}{|V_k|} \log |\mathrm{Hom}_\mu(\mathscr{C}, F, \delta, \sigma_k)|_{\mathscr{P}},$$

where the supremum is over all finite partitions $\mathscr{P}$ of X, the first infimum is over all finite partitions $\mathscr{C}$ of X refining $\mathscr{P}$, and the second infimum is over all finite sets $F \subseteq G$ containing e_G.

As in the topological case, one can check that there is a maximum among the quantities $h_{\Sigma,\mu}(G \curvearrowright X)$ over all sofic approximation sequences Σ for G, where $-\infty$ is included as a possible value. The *maximum* and *infimum sofic measure entropies* of $G \curvearrowright (X, \mu)$ are then defined by

$$h_\mu(G \curvearrowright X) = \max_\Sigma h_{\Sigma,\mu}(G \curvearrowright X),$$
$$\underline{h}_\mu(G \curvearrowright X) = \inf_\Sigma h_{\Sigma,\mu}(G \curvearrowright X),$$

where Σ ranges in each case over all sofic approximation sequences for G. When G is nonsofic these quantities are interpreted to be $-\infty$. The action is *entropy regular* if its maximum and infimum sofic measure entropies are equal, i.e., the sofic measure entropy does not depend on the choice of sofic approximation sequence.

3. Properties $\mathscr{S}$-SC and sofic SC

3.1. **Definitions of properties $\mathscr{S}$-SC and sofic SC.** Write $\mathscr{S}_G$ for the collection of all sofic approximations for G.

Let $\mathscr{S}$ be any collection of sofic approximations for G.

Definition 3.1. We say that the group G has *property $\mathscr{S}$-SC* (or *property sofic SC* if $\mathscr{S} = \mathscr{S}_G$) if for any function $\Upsilon : \mathcal{F}(G) \to [0, \infty)$ there exists an $S \in \overline{\mathcal{F}}(G)$ such that for any $T \in \overline{\mathcal{F}}(G)$ there are $C, n \in \mathbb{N}$, and $S_1, \ldots, S_n \in \overline{\mathcal{F}}(G)$ such that for every good enough sofic approximation

$\pi : G \to \mathrm{Sym}(V)$ in $\mathscr{S}$ there are subsets W and $\mathcal{V}_j$ of V for $1 \leq j \leq n$ satisfying the following conditions:

(i) $\sum_{j=1}^n \Upsilon(S_j)\mathrm{m}(\mathcal{V}_j) \leq 1$,
(ii) $\bigcup_{g\in S} \pi_g W = V$,
(iii) if $w_1, w_2 \in W$ satisfy $\pi_g w_1 = w_2$ for some $g \in T$ then w_1 and w_2 are connected by a path of length at most C in which each edge is of the form $(v, \pi_h v)$ for some $1 \leq j \leq n$, $h \in S_j$, and $v \in \mathcal{V}_j$ with $\pi_h v \in \mathcal{V}_j$.

Definition 3.2. We say that a continuous action $G \curvearrowright X$ on a compact metrizable space X with compatible metric d has *property $\mathscr{S}$-SC* (or *property sofic SC* if $\mathscr{S} = \mathscr{S}_G$) if for any function $\Upsilon : \mathcal{F}(G) \to [0,\infty)$ there exists an $S \in \overline{\mathcal{F}}(G)$ such that for any $T \in \overline{\mathcal{F}}(G)$ there are $C, n \in \mathbb{N}$, $S_1, \dots, S_n \in \overline{\mathcal{F}}(G)$, $F^\sharp \in \mathcal{F}(G)$, and $\delta^\sharp > 0$ such that for every good enough sofic approximation $\pi : G \to \mathrm{Sym}(V)$ in $\mathscr{S}$ with $\mathrm{Map}_d(F^\sharp, \delta^\sharp, \pi) \neq \emptyset$ there are W and $\mathcal{V}_j$ for $1 \leq j \leq n$ as in Definition 3.1. By [24, Lemma 10.24] this does not depend on the choice of d.

Definition 3.3. We say that a p.m.p. action $G \curvearrowright (X,\mu)$ has *property $\mathscr{S}$-SC* (or *property sofic SC* if $\mathscr{S} = \mathscr{S}_G$) if for any function $\Upsilon : \mathcal{F}(G) \to [0,\infty)$ there exists an $S \in \overline{\mathcal{F}}(G)$ such that for any $T \in \overline{\mathcal{F}}(G)$ there are $C, n \in \mathbb{N}$, $S_1, \dots, S_n \in \overline{\mathcal{F}}(G)$, a finite Borel partition $\mathscr{C}^\sharp$ of X, an $F^\sharp \in \mathcal{F}(G)$ containing e_G, and a $\delta^\sharp > 0$ such that for every good enough sofic approximation $\pi : G \to \mathrm{Sym}(V)$ in $\mathscr{S}$ with $\mathrm{Hom}_\mu(\mathscr{C}^\sharp, F^\sharp, \delta^\sharp, \pi) \neq \emptyset$ there are W and $\mathcal{V}_j$ for $1 \leq j \leq n$ as in Definition 3.1.

The following proposition shows that, when G is finitely generated, in Definition 3.1 we can fix $n = 1$ and take S_1 to be any symmetric finite generating subset of G containing e_G, but with the price that $\bigcup_{g\in S} \pi_g W$ is only most of V instead of the whole of V.

Proposition 3.4. *Suppose that G is finitely generated. Let A be a generating set for G in $\overline{\mathcal{F}}(G)$. Then G has property $\mathscr{S}$-SC if and only if for any $\varepsilon > 0$ there exists an $S \in \overline{\mathcal{F}}(G)$ such that for any $T \in \overline{\mathcal{F}}(G)$ and $\delta > 0$ there is a $C \in \mathbb{N}$ such that for any good enough sofic approximation $\pi : G \to \mathrm{Sym}(V)$ in $\mathscr{S}$ there are subsets W and $\mathcal{V}$ of V satisfying the following conditions:*

(i) $\mathrm{m}(\mathcal{V}) \leq \varepsilon$,
(ii) $\mathrm{m}(\bigcup_{g\in S} \pi_g W) \geq 1 - \delta$,
(iii) if $w_1, w_2 \in W$ satisfy $\pi_g w_1 = w_2$ for some $g \in T$ then w_1 and w_2 are connected by an A-path of length at most C whose vertices all lie in $\mathcal{V}$.

Proof. Denote by ℓ_A the word length function on G associated to A.

Suppose first that G has property $\mathscr{S}$-SC. Let $\varepsilon > 0$. Define $\Upsilon : \mathcal{F}(G) \to [0,\infty)$ by $\Upsilon(F) = \varepsilon^{-1}|A|^{\max_{g\in F} \ell_A(g)}$. Then there is an $S \in \overline{\mathcal{F}}(G)$ witnessing property $\mathscr{S}$-SC. Let $T \in \overline{\mathcal{F}}(G)$ and $\delta > 0$. Then we have $C, n, S_1, \dots, S_n$ as given by Definition 3.1. Set $m = \max_{1\leq j\leq n} \max_{g\in S_j} \ell_A(g)$. Let $\pi : G \to \mathrm{Sym}(V)$ be a good enough sofic approximation for G in $\mathscr{S}$. Then we have W and $\mathcal{V}_1, \dots, \mathcal{V}_n$ satisfying conditions (i)-(iii) in Definition 3.1. Denote by V' the set of all $v \in V$ satisfying $\pi_{gh} v = \pi_g \pi_h v$ for all $g, h \in A^{mC}$. When π is a good enough sofic approximation, we have $\mathrm{m}(V \setminus V') \leq \delta/|S|$. Set $W' = W \cap V'$. For each $1 \leq j \leq n$, set $m_j = \max_{g\in S_j} \ell_A(g)$ and $\mathcal{V}_j^\dagger = \bigcup_{g\in A^{m_j}} \pi_g \mathcal{V}_j$. Set $\mathcal{V} = \bigcup_{j=1}^n \mathcal{V}_j^\dagger$. Then

$$\mathrm{m}(\mathcal{V}) \leq \sum_{j=1}^n \mathrm{m}(\mathcal{V}_j^\dagger) \leq \sum_{j=1}^n |A^{m_j}| \cdot \mathrm{m}(\mathcal{V}_j) \leq \varepsilon \sum_{j=1}^n \Upsilon(S_j)\mathrm{m}(\mathcal{V}_j) \leq \varepsilon,$$

verifying condition (i) in the proposition statement. Note also that

$$\mathrm{m}\biggl(\bigcup_{g\in S}\pi_g W'\biggr) \geq \mathrm{m}\biggl(\bigcup_{g\in S}\pi_g W\biggr) - |S|\cdot \mathrm{m}(V\setminus V') \geq 1-\delta,$$

which verifies condition (ii) in the proposition statement. Let $g \in T$ and $w_1, w_2 \in W'$ be such that $\pi_g w_1 = w_2$. Then w_1 and w_2 are connected by a path of length at most C in which each edge is an S_j-edge with both endpoints in $\mathcal{V}_j$ for some $1 \leq j \leq n$. It is easily checked that the endpoints of such an edge are connected by an A-path of length at most m_j with all vertices in $\mathcal{V}_j^\dagger$. Thus w_1 and w_2 are connected by an A-path of length at most Cm with all vertices in $\mathcal{V}$, verifying condition (iii) in the proposition statement. This proves the "only if" part.

To prove the "if" part, suppose that G satisfies the condition in the statement of the proposition. Let Υ be a function $\mathcal{F}(G) \to [0,\infty)$. Take $0 < \varepsilon < 1/(2\Upsilon(A))$. Then we have an S as in the statement of the proposition. Let $T \in \overline{\mathcal{F}}(G)$. Take $0 < \delta < 1/(6|T|\Upsilon(T))$. Then we have a C as in the statement of the proposition. Set $n = 2$, $S_1 = A$, and $S_2 = T$. Let $\pi : G \to \mathrm{Sym}(V)$ be a good enough sofic approximation for G in $\mathscr{S}$. Then we have W and $\mathcal{V}$ as in the statement of the proposition. Set $W' = W \cup \pi_{e_G}^{-1}(V \setminus \bigcup_{g\in S}\pi_g W)$. Then $\bigcup_{g\in S}\pi_g W' = V$, verifying condition (ii) in Definition 3.1. Set $\mathcal{V}_1 = \mathcal{V}$ and $\mathcal{V}_2 = \bigcup_{g\in T}((W'\setminus W)\cup \pi_g(W'\setminus W)\cup \pi_g^{-1}(W'\setminus W))$. Then

$$\mathrm{m}(\mathcal{V}_2) \leq (2|T|+1)\mathrm{m}(W'\setminus W) \leq 3|T|\delta,$$

and hence

$$\Upsilon(S_1)\mathrm{m}(\mathcal{V}_1) + \Upsilon(S_2)\mathrm{m}(\mathcal{V}_2) \leq \Upsilon(A)\varepsilon + 3\Upsilon(T)|T|\delta < \frac{1}{2} + \frac{1}{2} = 1,$$

which verifies condition (i) in Definition 3.1. Let $g \in T$ and $w_1, w_2 \in W'$ be such that $\pi_g w_1 = w_2$. If $w_1 \notin W$ or $w_2 \notin W$, then (w_1, w_2) is an S_2-edge with both endpoints in $\mathcal{V}_2$. If $w_1, w_2 \in W$, then w_1 and w_2 are connected by an S_1-path of length at most C such that all vertices of this path lie in $\mathcal{V} = \mathcal{V}_1$, yielding condition (iii) in Definition 3.1. ☐

3.2. **Groups without property sofic SC.** Let $\mathscr{S}$ be a collection of sofic approximations for G which contains arbitrarily good sofic approximations (or, equivalently, which contains a sofic approximation sequence). In Propositions 3.6 and 3.7 we identify two classes of groups which fail to have property $\mathscr{S}$-SC, and in particular fail to have property sofic SC.

Lemma 3.5. *Suppose that G is finite. Then G does not have property $\mathscr{S}$-SC.*

Proof. Suppose to the contrary that G has property $\mathscr{S}$-SC. Define $\Upsilon : \mathcal{F}(G) \to [0,\infty)$ by $\Upsilon(F) = 4|G|$ for all $F \in \mathcal{F}(G)$. Then there is some $S \in \overline{\mathcal{F}}(G)$ satisfying the conditions in Definition 3.1. Put $T = \{e_G\}$. Then there are $C, n \in \mathbb{N}$ and $S_1, \dots, S_n \in \overline{\mathcal{F}}(G)$ satisfying the conditions in Definition 3.1.

Let $\pi : G \to \mathrm{Sym}(V)$ be a good enough sofic approximation in $\mathscr{S}$ so that there are subsets W and $\mathcal{V}_1, \dots, \mathcal{V}_n$ of V satisfying conditions (i)-(iii) in Definition 3.1 and also so that $\mathrm{m}(U) \leq 1/(4|G|)$ where U consists of all $v \in V$ satisfying $\pi_{e_G} v \neq v$. Seeing that $W \subseteq U \cup \bigcup_{j=1}^n \mathcal{V}_j$, we have

$$\mathrm{m}(W) \leq \mathrm{m}(U) + \mathrm{m}\biggl(\bigcup_{j=1}^n \mathcal{V}_j\biggr) \leq \frac{1}{4|G|} + \sum_{j=1}^n \mathrm{m}(\mathcal{V}_j) = \frac{1}{4|G|} + \frac{1}{4|G|}\sum_{j=1}^n \Upsilon(S_j)\mathrm{m}(\mathcal{V}_j) \leq \frac{1}{2|G|}.$$

Thus

$$1 = \mathrm{m}\Big(\bigcup_{g\in S} \pi_g W\Big) \leq |S|\mathrm{m}(W) \leq |G|\mathrm{m}(W) \leq \frac{1}{2},$$

a contradiction. □

Proposition 3.6. *Suppose that G is locally finite. Then G does not have property $\mathscr{S}$-SC.*

Proof. Suppose to the contrary that G has property $\mathscr{S}$-SC. Then G must be infinite by Lemma 3.5. Take a strictly increasing sequence $\{G_k\}$ of finite subgroups of G such that $G = \bigcup_{k\in\mathbb{N}} G_k$. For each $F \in \mathcal{F}(G)$, denote by $\Phi(F)$ the smallest $k \in \mathbb{N}$ satisfying $F \subseteq G_k$. Define $\Upsilon : \mathcal{F}(G) \to [0, \infty)$ by $\Upsilon(F) = 3|G_{\Phi(F)}|$. Then there is some $S \in \overline{\mathcal{F}}(G)$ satisfying the conditions in Definition 3.1. Put $m = \Phi(S)$ and $T = G_{m+1} \in \overline{\mathcal{F}}(G)$. Then there are $C, n \in \mathbb{N}$ and $S_1, \dots, S_n \in \overline{\mathcal{F}}(G)$ satisfying the conditions in Definition 3.1. Put $M = \max\{\max_{1\leq j\leq n} \Phi(S_j), m + 1\}$.

Let $\pi : G \to \mathrm{Sym}(V)$ be a good enough sofic approximation in $\mathscr{S}$ so that there is a set $V_1 \subseteq V$ satisfying the following conditions:

(i) $\pi_g\pi_h v = \pi_{gh}v$ for all $g, h \in G_M$ and $v \in V_1$,
(ii) $\pi_g v \neq \pi_h v$ for all $v \in V_1$ and distinct $g, h \in G_M$,
(iii) $\pi_g V_1 = V_1$ for all $g \in G_M$,
(iv) $\mathrm{m}(V_1) \geq 1/2$.

Then G_M acts on V_1 via π. Denote by $\mathscr{P}$ the partition of V_1 into G_{m+1}-orbits.

By assumption, when π is a good enough sofic approximation we can find subsets W and $\mathcal{V}_1, \dots, \mathcal{V}_n$ of V satisfying conditions (i)-(iii) in Definition 3.1. Note that $V = \bigcup_{g\in S} \pi_g W = \bigcup_{g\in G_m} \pi_g W$, which implies that for every member P of $\mathscr{P}$ the intersection $P\cap W$ is not contained in a single G_m-orbit.

Set $\mathcal{V} = \bigcup_{j=1}^{n} \pi_{G_{\Phi(S_j)}}(\mathcal{V}_j \cap V_1)$. Then

$$\mathrm{m}(\mathcal{V}) \leq \sum_{1\leq j\leq n} |G_{\Phi(S_j)}|\mathrm{m}(\mathcal{V}_j \cap V_1) \leq \sum_{1\leq j\leq n} |G_{\Phi(S_j)}|\mathrm{m}(\mathcal{V}_j) = \frac{1}{3}\sum_{1\leq j\leq n} \Upsilon(S_j)\mathrm{m}(\mathcal{V}_j) \leq \frac{1}{3}. \tag{2}$$

Now let $P \in \mathscr{P}$ and $w_1 \in P \cap W$. Then we can find some $w_2 \in P \cap W$ such that w_1 and w_2 are in different G_m-orbits. We have $w_2 = \pi_t w_1$ for some $t \in G_{m+1} \setminus G_m = T \setminus G_m$. Thus we can find some $1 \leq l \leq C$, $1 \leq j_1, \dots, j_l \leq n$, $v_k \in \mathcal{V}_{j_k}$ for $1 \leq k \leq l$, and $g_k \in S_{j_k}$ for $1 \leq k \leq l$ such that, setting $v_0 = w_1$, we have $\pi_{g_k} v_{k-1} = v_k$ for all $1 \leq k \leq l$ and $v_l = w_2$. Then

$$w_2 = v_l = \pi_{g_l} \dots \pi_{g_1} v_0 = \pi_{g_l\dots g_1} w_1,$$

and hence $g_l \cdots g_1 = t$. It follows that the elements $g_1, \dots, g_l$ cannot all lie in G_m. Denote by i the smallest k satisfying $g_k \notin G_m$. Then $v_i \in \pi_{G_{\Phi(S_{j_i})}} w_1$, and hence $w_1 \in \pi_{G_{\Phi(S_{j_i})}} v_i$. Consequently,

$$\pi_{G_m} w_1 \subseteq \pi_{G_m}\pi_{G_{\Phi(S_{j_i})}} v_i = \pi_{G_{\Phi(S_{j_i})}} v_i \subseteq \pi_{G_{\Phi(S_{j_i})}}(\mathcal{V}_{j_i} \cap V_1) \subseteq \mathcal{V}.$$

Therefore $V_1 = \bigcup \mathscr{P} = \pi_{G_m}(W \cap (\bigcup \mathscr{P})) \subseteq \mathcal{V}$, whence $\mathrm{m}(\mathcal{V}) \geq \mathrm{m}(V_1) \geq 1/2$, contradicting (2). □

Proposition 3.7. *Suppose that G is finitely generated and virtually free. Then G does not have property $\mathscr{S}$-SC.*

Proof. By Lemma 3.5 we may assume that G is infinite. Take a free subgroup G_1 of G with finite index. Then G_1 is nontrivial and, by Schreier's lemma, finitely generated. Take free generators $a_1, \ldots, a_r$ for G_1. Set $A = \{a_1, \ldots, a_r, a_1^{-1}, \ldots, a_r^{-1}, e_G\}$. Denote by ℓ the word length function on G_1 associated to $a_1, \ldots, a_r, a_1^{-1}, \ldots, a_r^{-1}$. For each $n \in \mathbb{N}$ denote by B_n the set of elements g in G_1 satisfying $\ell(g) \leq n$. Take a subset H of G containing e_G such that G is the disjoint union of the sets hG_1 for $h \in H$. Set $D = H \cup A$. For each $F \in \mathcal{F}(G)$, denote by $\Psi(F)$ the smallest $n \in \mathbb{N}$ satisfying $F \subseteq HB_n$, and set $F' = HB_{\Psi(F)}$.

For any $g \in G$ and $h \in H$ we can write gh uniquely as bd with $b \in H$ and $d \in G_1$, and using this factorization we set R to be the maximum value of $\ell(d)$ over all $g \in D$ and $h \in H$. Define $\Upsilon : \mathcal{F}(G) \to [0, \infty)$ by $\Upsilon(F) = 3|HB_R| \cdot |F'|$.

Suppose that G has property $\mathscr{S}$-SC. Then there is some $S \in \overline{\mathcal{F}}(G)$ satisfying the conditions in Definition 3.1. Put $m = \Psi(S)$, $N = m + 1$, and $T = S\{a_1^{2N}, e_G, a_1^{-2N}\}S \in \overline{\mathcal{F}}(G)$. Then there are $C, n \in \mathbb{N}$ and $S_1, \ldots, S_n \in \overline{\mathcal{F}}(G)$ satisfying the conditions in Definition 3.1. Put $C' = C \max_{1 \leq j \leq n}(1 + \Psi(S_j)) \in \mathbb{N}$ and $U = ((HA)^{C'}HB_{2N}) \cup ((HA)^{C'}HB_{2N})^{-1} \in \overline{\mathcal{F}}(G)$.

Let $\pi : G \to \mathrm{Sym}(V)$ be a good enough sofic approximation in $\mathscr{S}$ so that there are subsets W and $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of V satisfying conditions (i)-(iii) in Definition 3.1 and a set $V' \subseteq V$ satisfying the following conditions:

(i) $\pi_g \pi_h v = \pi_{gh} v$ for all $g, h \in U^{10}$ and $v \in V'$,
(ii) $\pi_g v \neq \pi_h v$ for all $v \in V'$ and distinct $g, h \in U$,
(iii) $\mathrm{m}(V') \geq 1/2$.

For each $1 \leq j \leq n$, set $\mathcal{V}_j' = \bigcup_{g \in S_j'} \pi_g \mathcal{V}_j$. Set $\mathcal{V} = \bigcup_{j=1}^n \mathcal{V}_j'$.

Let $v \in V'$. We have $\pi_{a_1^N} v = \pi_{h_1} w_1$ and $\pi_{a_1^{-N}} v = \pi_{h_2} w_2$ for some $h_1, h_2 \in S$ and $w_1, w_2 \in W$. Then

$$\pi_{h_2^{-1} a_1^{-2N} h_1} w_1 = \pi_{h_2}^{-1} \pi_{a_1^{-N}} \pi_{a_1^N}^{-1} \pi_{h_1} w_1 = \pi_{h_2}^{-1} \pi_{a_1^{-N}} v = w_2.$$

By assumption we can find a path from w_1 to w_2 of length at most C in which each edge is an S_j-edge with both endpoints in $\mathcal{V}_j$ for some $1 \leq j \leq n$. Replacing each such edge by a D-path of length at most $1 + \Psi(S_j)$ and with all vertices in $\mathcal{V}_j'$, we find a D-path from w_1 to w_2 of length at most C' such that all vertices are in $\mathcal{V}$. Thus we get some $1 \leq l \leq C'$, $v_k \in \mathcal{V}$ for $1 \leq k \leq l$, and $g_k \in D$ for $1 \leq k \leq l$ such that, setting $v_0 = w_1$, we have $\pi_{g_k} v_{k-1} = v_k$ for all $1 \leq k \leq l$ and $v_l = w_2$. Then

$$\pi_{h_2^{-1} a_1^{-N}} v = w_2 = \pi_{g_l} \ldots \pi_{g_1} w_1 = \pi_{g_l \ldots g_1 h_1^{-1} a_1^N} v.$$

Since $h_2^{-1} a_1^{-N}$ and $g_l \ldots g_1 h_1^{-1} a_1^N$ belong to U, we conclude that $h_2^{-1} a_1^{-N} = g_l \ldots g_1 h_1^{-1} a_1^N$. Set $t_j = g_j \ldots g_1 h_1^{-1} a_1^N \in U$ for $0 \leq j \leq l$. We can write each t_j uniquely as $b_j d_j$ for some $b_j \in H$ and $d_j \in G_1$. Then we have

$$\ell(d_j d_{j-1}^{-1}) \leq R$$

for all $1 \leq j \leq l$. Consider the path p in G_1 from d_0 to d_l defined by concatenating the geodesic from d_{j-1} to d_j for all $1 \leq j \leq l$, where we endow G_1 with the right invariant metric induced from ℓ. Note that as reduced words d_0 and d_l end with a_1 and a_1^{-1} respectively. Thus p passes through e_G. It follows that there is some $1 \leq i \leq l$ with $\ell(d_i) \leq R$. Then $t_i \in HB_R$, whence

$$v = \pi_{t_i}^{-1} v_i = \pi_{t_i^{-1}} v_i \in \bigcup_{g \in (HB_R)^{-1}} \pi_g \mathcal{V}.$$

Therefore $V' \subseteq \bigcup_{g\in(HB_R)^{-1}} \pi_g \mathcal{V}$.

Now we get

$$\frac{1}{2} \le \mathrm{m}(V') \le \mathrm{m}\bigg(\bigcup_{g\in(HB_R)^{-1}} \pi_g\mathcal{V}\bigg) \le |HB_R|\mathrm{m}(\mathcal{V})$$
$$\le |HB_R|\sum_{j=1}^{n}|S_j'|\mathrm{m}(\mathcal{V}_j) = \frac{1}{3}\sum_{j=1}^{n}\Upsilon(S_j)\mathrm{m}(\mathcal{V}_j) \le \frac{1}{3},$$

a contradiction. □

3.3. **Groups with property sofic SC.** In Theorems 3.11 and 3.12 below we will identify classes of groups that have property sofic SC. This will rely on results from [25] that we can access via the connection to property SC established in Proposition 3.10.

Definition 3.8. Let $\mathfrak{Y}$ be a class of free p.m.p. actions of a fixed infinite G. We say that $\mathfrak{Y}$ has *property SC* if for any function $\Upsilon : \mathcal{F}(G) \to [0,\infty)$ there exists an $S \in \overline{\mathcal{F}}(G)$ such that for any $T \in \overline{\mathcal{F}}(G)$ there are $C, n \in \mathbb{N}$, and $S_1, \dots, S_n \in \overline{\mathcal{F}}(G)$ so that for any $G \curvearrowright (X,\mu)$ in $\mathfrak{Y}$ there are Borel subsets W and $\mathcal{V}_j$ of X for $1 \le j \le n$ satisfying the following conditions:

(i) $\sum_{j=1}^{n}\Upsilon(S_j)\mu(\mathcal{V}_j) \le 1$,
(ii) $SW = X$,
(iii) if $w_1, w_2 \in W$ satisfy $gw_1 = w_2$ for some $g \in T$ then w_1 and w_2 are connected by a path of length at most C in which each edge is an S_j-edge with both endpoints in $\mathcal{V}_j$ for some $1 \le j \le n$.

We say that a p.m.p. action $G \curvearrowright (X,\mu)$ has *property SC* if the singleton class containing it has property SC. We say that G itself has *property SC* if the class of all free p.m.p actions $G \curvearrowright (X,\mu)$ has property SC (note that freeness implies atomlessness of the measure since G is infinite).

Remark 3.9. When $\mathfrak{Y}$ consists of either a single free p.m.p. action or all free p.m.p. actions of a fixed G, the existence of the bound C is automatic, as explained in the paragraph following Proposition 3.5 in [25].

Proposition 3.10. *Suppose that G is infinite and sofic. Let $G \curvearrowright (X,\mu)$ be a free p.m.p. action with property SC. Then the action has property sofic SC.*

Proof. We may assume, by passing to a suitable G-invariant conull subset of X, that the action of G is genuinely free. Let Υ be a function $\mathcal{F}(G) \to [0,\infty)$. Since $G \curvearrowright (X,\mu)$ has property SC, using the function 2Υ we find an $S \in \overline{\mathcal{F}}(G)$ such that for any $T \in \overline{\mathcal{F}}(G)$ there are $C, n \in \mathbb{N}$, $S_1, \dots, S_n \in \overline{\mathcal{F}}(G)$, and Borel subsets W and $\mathcal{V}_k$ of X for $1 \le k \le n$ satisfying the following conditions:

(i) $2\sum_{k=1}^{n}\Upsilon(S_k)\mu(\mathcal{V}_k) \le 1$,
(ii) $SW = X$,
(iii) if $w_1, w_2 \in W$ satisfy $gw_1 = w_2$ for some $g \in T$ then w_1 and w_2 are connected by a path of length at most C in which each edge is an S_k-edge with both endpoints in $\mathcal{V}_k$ for some $1 \le k \le n$.

Let $T \in \overline{\mathcal{F}}(G)$. Then we have C, n, S_k for $1 \leq k \leq n$, and W and $\mathcal{V}_k$ for $1 \leq k \leq n$ as above. We now verify conditions (i)-(iii) in Definition 3.1 as referenced in Definition 3.3.

Let $g \in T$. For each $x \in W \cap g^{-1}W$, we can find $g_1, \dots, g_l \in G$ for some $1 \leq l \leq C$ such that $g = g_l g_{l-1} \cdots g_1$ and for each $1 \leq j \leq l$ one has $g_j \in S_{k_j}$ and $g_{j-1} \cdots g_1 x, g_j g_{j-1} \cdots g_1 x \in \mathcal{V}_{k_j}$ for some $1 \leq k_j \leq n$. Then we can find a finite Borel partition $\mathscr{C}_g$ of $W \cap g^{-1}W$ such that

$$|\mathscr{C}_g| \leq Cn^C \Big(\max_{1 \leq k \leq n} |S_k| \Big)^C$$

and for each $A \in \mathscr{C}_g$ we can choose the same $l, g_1, \dots, g_l, k_1, \dots, k_l$ for all $x \in A$. Then for all $1 \leq j \leq l$ the sets $g_{j-1} \cdots g_1 A$ and $g_j g_{j-1} \cdots g_1 A$ are contained in $\mathcal{V}_{k_j}$.

Denote by $\mathscr{C}^\sharp$ the finite partition of X generated by $W, \mathcal{V}_1, \dots, \mathcal{V}_n$ and $\mathscr{C}_g$ for $g \in T$. Set $F^\sharp = (T \cup S \cup \bigcup_{k=1}^n S_k)^{100C} \in \mathcal{F}(G)$. Set $D = |T| C^2 n^C (\max_{1 \leq k \leq n} |S_k|)^C > 0$, and take $\delta > 0$ with $3\delta |T| \Upsilon(T) \leq 1/4$. Take

$$0 < \delta^\sharp \leq \min \left\{ \left(4 \sum_{k=1}^n \Upsilon(S_k) \right)^{-1}, \delta/(|S|(|T| + D + 2)) \right\}.$$

Let $\pi : G \to \mathrm{Sym}(V)$ be a sofic approximation for G with $\mathrm{Hom}_\mu(\mathscr{C}^\sharp, F^\sharp, \delta^\sharp, \pi) \neq \emptyset$ which is good enough so that $\mathrm{m}(V_{F^\sharp}) > 1 - \delta^\sharp$, where $V_{F^\sharp}$ denotes the set of all $v \in V$ satisfying $\pi_{gh} v = \pi_g \pi_h v$ for all $g, h \in F^\sharp$ and $\pi_g v \neq \pi_h v$ for all distinct $g, h \in F^\sharp$. Take $\varphi \in \mathrm{Hom}_\mu(\mathscr{C}^\sharp, F^\sharp, \delta^\sharp, \pi)$. Then φ is an algebra homomorphism $\mathrm{alg}(\mathscr{C}^\sharp_{F^\sharp}) \to \mathbb{P}_V$ satisfying

(i) $\sum_{A \in \mathscr{C}^\sharp} \mathrm{m}(\pi_g \varphi(A) \Delta \varphi(gA)) \leq \delta^\sharp$ for all $g \in F^\sharp$, and
(ii) $\sum_{A \in \mathscr{C}^\sharp_{F^\sharp}} |\mathrm{m}(\varphi(A)) - \mu(A)| \leq \delta^\sharp$.

Let $g \in T$ and $A \in \mathscr{C}_g$. Then we have $l, g_1, \dots, g_l, k_1, \dots, k_l$ as above. Denote by $W''_{g,A}$ the set $\bigcup_{1 \leq j \leq l} (\pi^{-1}_{g_j g_{j-1} \cdots g_1} (\varphi(g_j g_{j-1} \cdots g_1 A)) \Delta \varphi(A))$. Then

$$\begin{aligned} \mathrm{m}(W''_{g,A}) &\leq \sum_{j=1}^l \mathrm{m}(\pi^{-1}_{g_j g_{j-1} \cdots g_1} (\varphi(g_j g_{j-1} \cdots g_1 A)) \Delta \varphi(A)) \\ &= \sum_{j=1}^l \mathrm{m}(\varphi(g_j g_{j-1} \cdots g_1 A) \Delta \pi_{g_j g_{j-1} \cdots g_1} \varphi(A)) \leq C\delta^\sharp. \end{aligned}$$

Also set $W''_g = \bigcup_{A \in \mathscr{C}_g} W''_{g,A}$ and $W'' = \bigcup_{g \in T} W''_g$. Then

$$\mathrm{m}(W''_g) \leq |\mathscr{C}_g| C \delta^\sharp \leq C^2 n^C \Big(\max_{1 \leq k \leq n} |S_k| \Big)^C \delta^\sharp$$

and

$$\mathrm{m}(W'') \leq |T| C^2 n^C \Big(\max_{1 \leq k \leq n} |S_k| \Big)^C \delta^\sharp = D\delta^\sharp.$$

Set $W' = \bigcup_{g \in T} (\pi_g^{-1}(\varphi(W) \cap V_{F^\sharp}) \setminus \varphi(g^{-1}W))$. Note that

$$W' = \bigcup_{g \in T} (\pi_{g^{-1}}(\varphi(W) \cap V_{F^\sharp}) \setminus \varphi(g^{-1}W)) \subseteq \bigcup_{g \in T} (\pi_{g^{-1}} \varphi(W) \setminus \varphi(g^{-1}W)),$$

and hence

$$\mathrm{m}(W') \leq \sum_{g\in T} \mathrm{m}(\pi_{g^{-1}}\varphi(W)\Delta\varphi(g^{-1}W)) \leq |T|\delta^\sharp.$$

Set $W^* = (\varphi(W)\cap V_{F^\sharp})\setminus(W'\cup W'')$, and $W^\dagger = W^*\cup\pi_{e_G}^{-1}(V\setminus\bigcup_{g\in S}\pi_g W^*)$. Then $\bigcup_{g\in S}\pi_g W^\dagger = V$, verifying condition (ii) in Definition 3.1.

We have

$$\begin{aligned}
\mathrm{m}\biggl(\bigcup_{g\in S}\pi_g W^*\biggr) &\geq \mathrm{m}\biggl(\bigcup_{g\in S}\pi_g\varphi(W)\biggr) - |S|\mathrm{m}(W'\cup W''\cup(V\setminus V_{F^\sharp})) \\
&\geq \mathrm{m}\biggl(\varphi\biggl(\bigcup_{g\in S} gW\biggr)\biggr) - \mathrm{m}\biggl(\biggl(\bigcup_{g\in S}\pi_g\varphi(W)\biggr)\Delta\varphi\biggl(\bigcup_{g\in S} gW\biggr)\biggr) \\
&\qquad\qquad - |S|(|T|\delta^\sharp + D\delta^\sharp + \delta^\sharp) \\
&= 1 - \mathrm{m}\biggl(\biggl(\bigcup_{g\in S}\pi_g\varphi(W)\biggr)\Delta\bigcup_{g\in S}\varphi(gW)\biggr) - |S|(|T|+D+1)\delta^\sharp \\
&\geq 1 - \sum_{g\in S}\mathrm{m}(\pi_g\varphi(W)\Delta\varphi(gW)) - |S|(|T|+D+1)\delta^\sharp \\
&\geq 1 - |S|\delta^\sharp - |S|(|T|+D+1)\delta^\sharp \geq 1-\delta,
\end{aligned}$$

and hence

$$\mathrm{m}(W^\dagger\setminus W^*) \leq \mathrm{m}\biggl(V\setminus\bigcup_{g\in S}\pi_g W^*\biggr) \leq \delta.$$

Put $\mathcal{V}_k^\dagger = \varphi(\mathcal{V}_k)$ for $1\leq k\leq n$, $S_{n+1} = T \in \overline{\mathcal{F}}(G)$, and

$$\mathcal{V}_{n+1}^\dagger = \bigcup_{g\in T}((W^\dagger\setminus W^*)\cup\pi_g(W^\dagger\setminus W^*)\cup\pi_g^{-1}(W^\dagger\setminus W^*)).$$

Then

$$\mathrm{m}(\mathcal{V}_{n+1}^\dagger) \leq (2|T|+1)\mathrm{m}(W^\dagger\setminus W^*) \leq 3\delta|T|,$$

and hence

$$\sum_{k=1}^{n+1}\Upsilon(S_k)\mathrm{m}(\mathcal{V}_k^\dagger) \leq 3\delta|T|\Upsilon(T) + \sum_{k=1}^{n}\Upsilon(S_k)(\mu(\mathcal{V}_k)+\delta^\sharp) \leq \frac{1}{4}+\frac{1}{2}+\delta^\sharp\sum_{k=1}^{n}\Upsilon(S_k) \leq 1,$$

verifying condition (i) in Definition 3.1.

Let $g\in T$ and $w_1, w_2 \in W^\dagger$ with $\pi_g w_1 = w_2$. If $w_1\notin W^*$ or $w_2\notin W^*$, then (w_1,w_2) is an S_{n+1}-edge with both endpoints in $\mathcal{V}_{n+1}^\dagger$. Thus we may assume that $w_1, w_2\in W^*$. Then $w_1 = \pi_g^{-1}w_2 \in \pi_g^{-1}(\varphi(W)\cap V_{F^\sharp})$. Since $w_1\notin W'$, we get $w_1\in\varphi(g^{-1}W)$. Thus

$$w_1 \in \varphi(W)\cap\varphi(g^{-1}W) = \varphi(W\cap g^{-1}W) = \varphi\biggl(\bigcup_{A\in\mathscr{C}_g} A\biggr) = \bigcup_{A\in\mathscr{C}_g}\varphi(A).$$

We have $w_1\in\varphi(A)$ for some $A\in\mathscr{C}_g$. Let $l, g_1,\dots,g_l,k_1,\dots,k_l$ be as above for this A. Then for all $1\leq j\leq l$ the sets $g_{j-1}\dots g_1 A$ and $g_j g_{j-1}\dots g_1 A$ are contained in $\mathcal{V}_{k_j}$. Since $w_1\notin W''_{g,A}$, we have $\pi_{g_j g_{j-1}\dots g_1}w_1 \in \varphi(g_j g_{j-1}\dots g_1 A)$ for all $1\leq j\leq l$. Thus $\pi_{g_{j-1}\dots g_1}w_1, \pi_{g_j g_{j-1}\dots g_1}w_1 \in$

$\varphi(\mathcal{V}_{k_j}) = \mathcal{V}^\dagger_{k_j}$ for all $1 \leq j \leq l$. Therefore w_1 and w_2 are connected by a path of length l in which each edge is an S_k-edge with both endpoints in $\mathcal{V}^\dagger_k$ for some $1 \leq k \leq n$, verifying condition (iii) in Definition 3.1. □

Theorem 3.11. *Consider the following conditions for an infinite G:*

(i) G has property SC,
(ii) every free p.m.p. action $G \curvearrowright (X, \mu)$ has property SC,
(iii) there exists a nontrivial Bernoulli action of G with property SC,
(iv) there exists a nontrivial Bernoulli action of G with property sofic SC,
(v) G has property sofic SC,
(vi) G is neither locally finite nor finitely generated and virtually free.

We have (i)⇔(ii)⇔(iii)⇒(iv)⇔(v)⇒(vi). Moreover, when G is amenable all of these conditions are equivalent.

Proof. The equivalence of (i), (ii), and (iii) is the content of Proposition 3.5 of [25]. For (iii)⇒(iv) apply Proposition 3.10. The implication (iv)⇒(v) follows from the fact that nontrivial Bernoulli actions have positive sofic entropy with respect to every sofic approximation sequence [5, 22, 21], while (v)⇒(iv) follows from the definitions and (v)⇒(vi) from Propositions 3.6 and 3.7.

In the case that G is amenable, Proposition 3.28 of [25] asserts that (vi)⇔(i), which gives us the equivalence of all of the conditions. □

A subgroup G_0 of G is said to be *w-normal* in G if there are a countable ordinal γ and a subgroup G_λ of G for each ordinal $0 \leq \lambda \leq \gamma$ such that

(i) for any $\lambda < \lambda' \leq \gamma$ one has $G_\lambda \subseteq G_{\lambda'}$,
(ii) $G = G_\gamma$,
(iii) for each $\lambda < \gamma$ the group G_λ is normal in $G_{\lambda+1}$,
(iv) for each limit ordinal $\lambda' \leq \gamma$ one has $G_{\lambda'} = \bigcup_{\lambda<\lambda'} G_\lambda$.

In conjunction with Theorem 3.11 above, Theorem 3.29 of [25] yields the following.

Theorem 3.12. *Suppose that G has a w-normal subgroup G_0 which is amenable but neither locally finite nor virtually cyclic. Then G has property sofic SC.*

3.4. **W-normal subgroups and property sofic SC.** The proof of the following lemma applies some of the ideas from Section 8.1 of [3] to the sofic framework.

Lemma 3.13. *Suppose that G is finitely generated and not virtually cyclic, and let A be a generating set for G in $\overline{\mathcal{F}}(G)$. Then there is a constant $b > 0$ such that given any*

(i) group H containing G as a subgroup,
(ii) finite subset F of H, and
(iii) $r, M \in \mathbb{N}$ and $\delta > 0$

one can find, for any good enough sofic approximation $\pi : H \to \mathrm{Sym}(V)$ for H, sets $Z \subseteq \mathcal{V} \subseteq V_F$, where V_F denotes the set of all $v \in V$ satisfying $\pi_{gh} = \pi_g\pi_h v$ for all $g, h \in F$ and $\pi_g v \neq \pi_h v$ for all distinct $g, h \in F$, such that $\left|\bigcup_{g\in A^{2r}} \pi_g\mathcal{V}\right|/|V| \geq 1-\delta$, $|\mathcal{V}| \leq b|V|/r$, $|Z| \leq |V|/M$, and every point of $\mathcal{V}$ is connected to some point of Z by an A-path of length at most $2M$ with all vertices in $\mathcal{V}$.

Proof. Since G is not virtually cyclic, there exists a $c > 0$ such that $|A^n| \geq cn^2$ for all $n \in \mathbb{N}$ (Corollary 3.5 of [29]). Set $b = 5/c$. Let H, F, r, M, δ, and π be as in the lemma statement. Set $N = |A|^{3M}$. Take $k \in \mathbb{N}$ such that $|A^{kr}| \geq N|A^{2r}|$.

Denote by V' the set of all $v \in V$ satisfying $\pi_{gh}v = \pi_g\pi_h v$ for all $g, h \in (F \cup A)^{100(M+r)}$ and $\pi_g v \neq \pi_h v$ for all distinct $g, h \in (F \cup A)^{100(M+r)}$. Denote by V'' the set of all $v \in V$ satisfying $\pi_{gh}v = \pi_g\pi_h v$ for all $g, h \in (F \cup A)^{(200+k)(M+r)}$ and $\pi_g v \neq \pi_h v$ for all distinct $g, h \in (F \cup A)^{(200+k)(M+r)}$. Then $\pi_g V'' \subseteq V'$ for every $g \in A^{(k+6)r}$. Assuming that π is a good enough sofic approximation, we have $|V''|/|V| \geq 1 - \delta$. Take a maximal (A, r)-separated subset W of V'', and also take a maximal (A, r)-separated subset W' of V' containing W. Then we have $\bigcup_{g \in A^{2r}} \pi_g W \supseteq V''$, and hence

$$\frac{1}{|V|}\left|\bigcup_{g \in A^{2r}} \pi_g W\right| \geq \frac{|V''|}{|V|} \geq 1 - \delta.$$

Let $w \in W$. Set $T_w = W' \cap \pi_{A^{(k+2)r}} w$. Note that $\pi_{A^{kr}} w \subseteq V' \subseteq \pi_{A^{2r}} W'$. For each $g \in A^{kr}$ we have $\pi_g w \in \pi_{A^{2r}} z$ for some $z \in W'$. Then $z \in \pi_{A^{2r}} \pi_g w \subseteq \pi_{A^{(k+2)r}} w$, and hence $z \in T_w$. Thus

$$\pi_{A^{kr}} w \subseteq \pi_{A^{2r}} T_w.$$

Therefore

$$|T_w| \geq \frac{|A^{kr}|}{|A^{2r}|} \geq N.$$

Set $T = \bigcup_{w \in W} T_w \subseteq W'$. We have

$$|A^r||W'| = \left|\bigsqcup_{w \in W'} \pi_{A^r} w\right| \leq |V|$$

whence

$$|T| \leq |W'| \leq \frac{|V|}{|A^r|} \leq \frac{|V|}{cr^2}. \tag{3}$$

Let $w \in W$. Let (T_w, E_w) be the graph whose edges are those pairs of vertices which can be joined by an A-path of length at most $4r + 1$, and let us show that it is connected. It is enough to demonstrate that a given $v \in T_w$ is connected to w by a path in (T_w, E_w). Choose a shortest A-path from w to v. For each vertex z in this path contained in $\pi_{A^{kr}} w$, the fact that $z \in \pi_{A^{kr}} w \subseteq \pi_{A^{2r}} T_w$ means that we can connect z to some $u_z \in T_w$ by an A-path p_z of length at most $2r$. By inserting p_z and its reverse at z, we construct an A-path from w to v in which points of T_w appear in every interval of length $4r + 1$. Therefore v is connected to w by some path in (T_w, E_w), showing that (T_w, E_w) is connected.

Consider the graph (T, E) whose edges are those pairs of vertices which can be joined by an A-path of length at most $4r + 1$. From the above, every connected component of this graph has at least N points. Starting with (T, E), we recursively build a sequence of graphs with vertex set T by removing one edge at each stage so as to destroy some cycle at that stage, until there are no more cycles left and we arrive at a subgraph (T, E') such that (T, E) and (T, E') have the same connected components and each connected component of (T, E') is a tree.

For each pair (v, w) in E', we choose an A-path in $\pi_{A^{4r}} T$ joining v to w of length at most $4r + 1$. Denote by $\mathcal{V}$ the collection of all vertices which appear in one of these paths. Then

$\mathcal{V} \subseteq \pi_{A^{4r}} T \subseteq V' \subseteq V_F$. Note that each A-connected component of $\mathcal{V}$ has at least N points, and $W \subseteq T \subseteq \mathcal{V}$. Thus

$$\frac{1}{|V|}\bigg|\bigcup_{g\in A^{2r}} \pi_g \mathcal{V}\bigg| \geq \frac{1}{|V|}\bigg|\bigcup_{g\in A^{2r}} \pi_g W\bigg| \geq 1-\delta.$$

Moreover, using (3) we have

$$|\mathcal{V}| \leq |T| + 4r|E'| \leq (4r+1)|T| \leq 5r\cdot\frac{|V|}{cr^2} = \frac{b|V|}{r}.$$

Let $\mathcal{C}$ be an A-connected component of $\mathcal{V}$. Denote by $(\mathcal{C}, E_{\mathcal{C}})$ the graph whose edges are the pairs $(w, v) \in \mathcal{C}^2$ such that $\pi_g w = v$ for some $g \in A$. Then $(\mathcal{C}, E_{\mathcal{C}})$ is connected. Endow $\mathcal{C}$ with the geodesic distance ρ induced from $E_{\mathcal{C}}$. Take a maximal subset $Z_{\mathcal{C}}$ of $\mathcal{C}$ which is (ρ, M)-separated in the sense that the M-balls $\{v \in \mathcal{C} : \rho(v, z) \leq M\}$ for $z \in Z_{\mathcal{C}}$ are pairwise disjoint. Then $Z_{\mathcal{C}}$ is $(\rho, 2M)$-spanning in $\mathcal{C}$, i.e., every point of $\mathcal{C}$ is connected to some point of $Z_{\mathcal{C}}$ by an A-path of length at most $2M$ with all vertices in $\mathcal{C}$. Since $|\mathcal{C}| \geq N = |A|^{3M} > |A|^{2M}$, we have $|Z_{\mathcal{C}}| \geq 2$. Then $|\mathcal{C} \cap \pi_{A^M} z| \geq M$ for every $z \in Z_{\mathcal{C}}$. Since the sets $\mathcal{C} \cap \pi_{A^M} z$ for $z \in Z_{\mathcal{C}}$ are pairwise disjoint, we get

$$|Z_{\mathcal{C}}|M \leq \sum_{z\in Z_{\mathcal{C}}} |\mathcal{C} \cap \pi_{A^M} z| \leq |\mathcal{C}|.$$

Denote by Z the union of the sets $Z_{\mathcal{C}}$ where $\mathcal{C}$ runs over all A-connected components of $\mathcal{V}$. Then every point of $\mathcal{V}$ is connected to some point of Z by an A-path of length at most $2M$ with all vertices in $\mathcal{V}$, and $|Z|/|V| \leq 1/M$. □

For the definition of w-normality, see the paragraph before Theorem 3.12.

Proposition 3.14. *Suppose that G has a w-normal subgroup $G^\flat$ with property sofic SC. Then G has property sofic SC.*

Proof. Suppose first that $G^\flat$ is locally virtually cyclic. Then $G^\flat$ is amenable and, by Lemma 3.5 and Theorem 3.11, neither locally finite nor virtually cyclic. It follows by Theorem 3.12 that G has property sofic SC, as desired.

Suppose now that $G^\flat$ is not locally virtually cyclic. In this case we will first carry out the argument under the assumption that $G^\flat$ is normal in G. Take a finitely generated subgroup G_0 of $G^\flat$ such that G_0 is not virtually cyclic. Take an $S_1 \in \overline{\mathcal{F}}(G_0)$ generating G_0. Let $b > 0$ be as given by Lemma 3.13 for the group G_0 and generating set S_1.

Let Υ be a function $\mathcal{F}(G) \to [0, \infty)$. Choose an $r \in \mathbb{N}$ large enough so that

$$3b\Upsilon(S_1) \leq r. \tag{4}$$

Set $S = S_1^{2r} \in \overline{\mathcal{F}}(G_0)$.

Consider the restriction of 3Υ to $\mathcal{F}(G^\flat)$. Since $G^\flat$ has property sofic SC, there exists an $S^\flat \in \overline{\mathcal{F}}(G^\flat)$ such that for any $T^\flat \in \overline{\mathcal{F}}(G^\flat)$ there are $C^\flat, n^\flat \in \mathbb{N}$ and $S_1^\flat, \dots, S_{n^\flat}^\flat \in \overline{\mathcal{F}}(G^\flat)$ such that for any good enough sofic approximation $\pi : G \to \mathrm{Sym}(V)$ for G there are subsets $W^\flat$ and $\mathcal{V}_k^\flat$ of V for $1 \leq k \leq n^\flat$ satisfying the following conditions:

(i) $\sum_{k=1}^{n^\flat} 3\Upsilon(S_k^\flat)\mathrm{m}(\mathcal{V}_k^\flat) \leq 1$,
(ii) $\bigcup_{g\in S^\flat} \pi_g W^\flat = V$,

(iii) if $w_1, w_2 \in W^\flat$ satisfy $\pi_g w_1 = w_2$ for some $g \in T^\flat$ then w_1 and w_2 are connected by a path of length at most $C^\flat$ in which each edge is an $S_k^\flat$-edge with both endpoints in $\mathcal{V}_k^\flat$ for some $1 \leq k \leq n^\flat$.

Let $T \in \overline{\mathcal{F}}(G)$. Set

$$S_2 = S^\flat T S^\flat \in \overline{\mathcal{F}}(G).$$

Take an $M \in \mathbb{N}$ large enough so that

$$M \geq 12\Upsilon(S_2)|S_2|. \tag{5}$$

Set $T^\flat = \bigcup_{g \in T}(S^\flat S_1^{2M} g S_1^{2M} g^{-1} S^\flat \cup S^\flat g S_1^{2M} g^{-1} S_1^{2M} S^\flat) \in \overline{\mathcal{F}}(G^\flat)$. Then we have $C^\flat$, $n^\flat$, and $S_k^\flat$ for $1 \leq k \leq n^\flat$ as above. Set

$$C = 4M + 2 + C^\flat \in \mathbb{N},$$

and $F = (S_1 \cup T \cup S^\flat \cup \bigcup_{k=1}^{n^\flat} S_k^\flat)^{100MCr} \in \mathcal{F}(G)$.

Now let $\pi : G \to \mathrm{Sym}(V)$ be a good enough sofic approximation for G. By Lemma 3.13 we can find sets $Z \subseteq \mathcal{V}_1 \subseteq V_F$, where V_F denotes the set of $v \in V$ satisfying $\pi_{gh} = \pi_g \pi_h v$ for all $g, h \in F$ and $\pi_g v \neq \pi_h v$ for all distinct $g, h \in F$, such that $\mathrm{m}\big(\bigcup_{g \in S} \pi_g \mathcal{V}_1\big) \geq 1 - 1/M$, $\mathrm{m}(\mathcal{V}_1) \leq b/r$, $\mathrm{m}(Z) \leq 1/M$, and every point of $\mathcal{V}_1$ is connected to some point of Z by an S_1-path of length at most $2M$ with all vertices in $\mathcal{V}_1$. Note that

$$\Upsilon(S_1)\mathrm{m}(\mathcal{V}_1) \leq \Upsilon(S_1)\frac{b}{r} \overset{(4)}{\leq} \frac{1}{3}.$$

Set $W = \mathcal{V}_1 \cup \pi_{e_G}^{-1}(V \setminus \bigcup_{g \in S} \pi_g \mathcal{V}_1) \subseteq V$. Then $\bigcup_{g \in S} \pi_g W = V$, which verifies condition (ii) in Definition 3.1.

Set $\mathcal{V}_2 = \big(\bigcup_{g \in T}((W \setminus \mathcal{V}_1) \cup \pi_g(W \setminus \mathcal{V}_1) \cup \pi_g^{-1}(W \setminus \mathcal{V}_1))\big) \cup \bigcup_{g \in S_2} \pi_g Z \subseteq V$. Then

$$\mathrm{m}(\mathcal{V}_2) \leq 3|T|\mathrm{m}(W \setminus \mathcal{V}_1) + |S_2|\mathrm{m}(Z) \leq 3|S_2|\mathrm{m}(V \setminus \bigcup_{g \in S} \pi_g \mathcal{V}_1) + |S_2|/M \leq 4|S_2|/M,$$

and hence

$$\Upsilon(S_2)\mathrm{m}(\mathcal{V}_2) \leq 4\Upsilon(S_2)|S_2|/M \overset{(5)}{\leq} \frac{1}{3}.$$

Assuming that π is a good enough sofic approximation for G, we have $W^\flat$ and $\mathcal{V}_k^\flat$ for $1 \leq k \leq n^\flat$ as above, in which case

$$\sum_{k=1}^{n^\flat} \Upsilon(S_k^\flat)\mathrm{m}(\mathcal{V}_k^\flat) \leq \frac{1}{3}.$$

Putting the above estimates together we get

$$\Upsilon(S_1)\mathrm{m}(\mathcal{V}_1) + \Upsilon(S_2)\mathrm{m}(\mathcal{V}_2) + \sum_{k=1}^{n^\flat} \Upsilon(S_k^\flat)\mathrm{m}(\mathcal{V}_k^\flat) \leq 1,$$

which verifies condition (i) in Definition 3.1.

Let $g \in T$ and $w_1, w_2 \in W$ be such that $\pi_g w_1 = w_2$. If either $w_1 \in W \setminus \mathcal{V}_1$ or $w_2 \in W \setminus \mathcal{V}_1$, then (w_1, w_2) is an S_2-edge with both endpoints in $\mathcal{V}_2$. Thus we may assume that $w_1, w_2 \in \mathcal{V}_1$. For $i = 1, 2$, we can connect w_i to some $z_i \in Z$ by an S_1-path of length at most $2M$ with all vertices in $\mathcal{V}_1$. Then $w_i = \pi_{t_i} z_i$ for some $t_i \in S_1^{2M}$. We have $\pi_g z_1 = \pi_{a_1} u_1$ for some $u_1 \in W^\flat$

and $a_1 \in S^\flat$, and $z_2 = \pi_{a_2} u_2$ for some $u_2 \in W^\flat$ and $a_2 \in S^\flat$. Note that $a_1^{-1} g$ and a_2^{-1} are both in S_2. Since $\pi_{a_1^{-1} g} z_1 = u_1$, the pair (z_1, u_1) is an S_2-edge with both endpoints in $\mathcal{V}_2$. Also, since $\pi_{a_2^{-1}} z_2 = u_2$ the pair (w_2, u_2) is an S_2-edge with both endpoints in $\mathcal{V}_2$. Note that

$$\begin{aligned}
\pi_{a_2^{-1} t_2^{-1} g t_1 g^{-1} a_1} u_1 &= \pi_{a_2^{-1}} \pi_{t_2^{-1}} \pi_g \pi_{t_1} \pi_{g^{-1}} \pi_{a_1} u_1 \\
&= \pi_{a_2^{-1}} \pi_{t_2^{-1}} \pi_g \pi_{t_1} z_1 \\
&= \pi_{a_2^{-1}} \pi_{t_2^{-1}} \pi_g w_1 \\
&= \pi_{a_2^{-1}} \pi_{t_2^{-1}} w_2 \\
&= \pi_{a_2^{-1}} z_2 \\
&= u_2.
\end{aligned}$$

Since $a_2^{-1} t_2^{-1} g t_1 g^{-1} a_1 \in S^\flat S_1^{2M} g S_1^{2M} g^{-1} S^\flat \subseteq T^\flat$, this means that $u_2 \in \pi_{T^\flat} u_1$. Then u_1 and u_2 are connected by a path of length at most $C^\flat$ in which each edge is an $S_k^\flat$-edge with both endpoints in $\mathcal{V}_k^\flat$ for some $1 \leq k \leq n^\flat$. Therefore w_1 and w_2 are connected by a path of length at most $4M + 2 + C^\flat = C$ in which each edge is either an S_j-edge with both endpoints in $\mathcal{V}_j$ for some $1 \leq j \leq 2$ or an $S_k^\flat$-edge with both endpoints in $\mathcal{V}_k^\flat$ for some $1 \leq k \leq n^\flat$, verifying condition (iii) in Definition 3.1.

Notice that the set S used in the above verification of property sofic SC for G is contained in G_0, which can be any non-virtually-cyclic finitely generated subgroup of $G^\flat$, and only depends on the restriction of Υ to $\overline{\mathcal{F}}(G_0)$. This has the consequence that if $G_1, G_2, \dots$ is a sequence of countable groups such that G_n is a normal subgroup of G_{n+1} for each n and G_1 is not locally virtually cyclic and has property sofic SC then the group $\bigcup_{n=1}^\infty G_n$ has property sofic SC. Indeed we can fix a finitely generated subgroup G_1' of G_1 which is not virtually cyclic and apply the above argument recursively taking $G_0 = G_1'$, $G^\flat = G_n$, and $G = G_{n+1}$ at the nth stage to deduce that G_{n+1} has property sofic SC, and if the function Υ is taken at each stage to be the restriction of a prescribed function $\mathcal{F}(\bigcup_{n=1}^\infty G_n) \to [0, \infty)$ then we can use the same set S for all n, showing that $\bigcup_{n=1}^\infty G_n$ has property sofic SC. It follows by ordinal well-ordering that if $G^\flat$ is merely assumed to be w-normal in G then we can still conclude that G has property sofic SC. □

3.5. **Product groups.** Let G and H be countable groups. Let $\pi : G \to \mathrm{Sym}(V)$ and $\sigma : H \to \mathrm{Sym}(W)$ be sofic approximations. The *product sofic approximation* $\pi \times \sigma : G \times H \to \mathrm{Sym}(V \times W)$ is defined by

$$(\pi \times \sigma)_{(g,h)}(v, w) = (\pi_g(v), \sigma_h(w))$$

for all $g \in G$, $h \in H$, $v \in V$, and $w \in W$. Note that if $\{\pi_k\}$ and $\{\sigma_k\}$ are sofic approximation sequences for G and H, respectively, then $\{\pi_k \times \sigma_k\}$ is a sofic approximation sequence for $G \times H$.

Proposition 3.15. *Let G and H be countably infinite groups. Let $\mathscr{S}$ be the collection of product sofic approximations for $G \times H$. Then $G \times H$ has property $\mathscr{S}$-SC if and only if at least one of G and H is not locally finite.*

Proof. If G and H are both locally finite then $G \times H$ is locally finite and hence does not have property $\mathscr{S}$-SC by Proposition 3.6. Suppose then that at least one of G and H is not locally finite. Take two nontrivial Bernoulli actions $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$. By Proposition 3.32

of [25] the p.m.p. action $G \times H \curvearrowright (X \times Y, \mu \times \nu)$ given by $(g,h)(x,y) = (gx, hy)$ for all $g \in G$, $h \in H$, $x \in X$, and $y \in Y$ has property SC, and hence has property sofic SC by Proposition 3.10.

By [5, 21], for every finite partition $\mathscr{C}$ of X, $F \in \mathfrak{F}(G)$ containing e_G, and $\delta > 0$ one has $\mathrm{Hom}_\mu(\mathscr{C}, F, \delta, \pi) \neq \emptyset$ for every sufficiently good sofic approximation π for G, and for every finite partition $\mathscr{D}$ of Y, $L \in \mathfrak{F}(H)$ containing e_H, and $\delta > 0$ one has $\mathrm{Hom}_\nu(\mathscr{D}, L, \delta, \sigma) \neq \emptyset$ for every sufficiently good sofic approximation σ for H. Given such sofic approximations $\pi : G \to \mathrm{Sym}(V)$ and $\sigma : H \to \mathrm{Sym}(W)$ and $\varphi \in \mathrm{Hom}_\mu(\mathscr{C}, F, \delta, \pi)$ and $\psi \in \mathrm{Hom}_\nu(\mathscr{D}, L, \delta, \sigma)$ we have a homomorphism $\zeta : \mathrm{alg}(\mathscr{C}_F \times \mathscr{D}_L) = \mathrm{alg}((\mathscr{C} \times \mathscr{D})_{F\times L}) \to \mathbb{P}_{V\times W}$ determined by $\zeta(C \times D) = \varphi(C) \times \psi(D)$ for $C \in \mathscr{C}_F$ and $D \in \mathscr{D}_L$, and one can readily verify that ζ belongs to $\mathrm{Hom}_{\mu\times\nu}(\mathscr{C} \times \mathscr{D}, F \times L, 2\delta, \pi \times \sigma)$, showing that this set of homomorphisms is nonempty. Since the algebra of subsets of $X \times Y$ generated by products of finite partitions is dense in the σ-algebra with respect to the pseudometric $d(A,B) = (\mu \times \nu)(A \Delta B)$, it follows by a simple approximation argument that for every finite partition $\mathscr{E}$ of $X \times Y$, finite set $e_{G\times H} \in K \subseteq G \times H$, and $\delta > 0$ one has $\mathrm{Hom}_{\mu\times\nu}(\mathscr{E}, K, \delta, \pi \times \sigma) \neq \emptyset$ for all good enough sofic approximations $\pi : G \to \mathrm{Sym}(V)$ and $\sigma : H \to \mathrm{Sym}(W)$. Since the action $G \times H \curvearrowright (X \times Y, \mu \times \nu)$ has property sofic SC, it follows that $G \times H$ has property $\mathscr{S}$-SC. $\square$

3.6. **Property sofic SC under continuous orbit equivalence.**

Proposition 3.16. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free continuous actions on compact metrizable spaces which are continuously orbit equivalent. Suppose that $G \curvearrowright X$ has property sofic SC. Then $H \curvearrowright Y$ has property sofic SC.*

To prove this proposition we may assume that $X = Y$ and that the identity map of X provides a continuous orbit equivalence between the actions $G \curvearrowright X$ and $H \curvearrowright X$. Let $\kappa : G \times X \to H$ and $\lambda : H \times X \to G$ be the associated cocycles.

The actions of G and H generate an action $G * H \curvearrowright X$ of their free product via the canonical embeddings of G and H into $G * H$. Since the actions of G and H are topologically free, we can find a G-invariant dense G_δ set $W_1 \subseteq X$ on which G acts freely and an H-invariant dense G_δ set $W_2 \subseteq X$ on which H acts freely. Set $X_0 = \bigcap_{s\in G*H} s(W_1 \cap W_2)$. Then X_0 is a $G * H$-invariant dense G_δ subset of X on which both G and H act freely.

Fix a compatible metric d on X which gives X diameter no bigger than 1. For each $g \in G$ there is an $\eta_g > 0$ such that for any $x, y \in X$ with $d(x,y) \leq \eta_g$ one has $\kappa(g,x) = \kappa(g,y)$, and likewise for each $s \in H$ there is an $\eta_s > 0$ such that for any $x, y \in X$ with $d(x,y) \leq \eta_s$ one has $\lambda(s,x) = \lambda(s,y)$. We put $\eta_F = \min_{g\in F} \eta_g > 0$ for a nonempty finite set $F \subseteq G$, and $\eta_L = \min_{s\in L} \eta_s > 0$ for a nonempty finite set $L \subseteq H$.

We will need the following lemma, which will also be of use in the proof of Theorem 4.1.

Lemma 3.17. *Let $L \in \overline{\mathfrak{F}}(H)$, and $0 < \tau < 1$. Set $F = \lambda(L^2, X) \in \overline{\mathfrak{F}}(G)$ and*

$$\tau' = \min\left\{\eta_{L^2}\tau^{1/2}/(8|F|)^{1/2}, \tau/(22|F|^2)\right\} > 0.$$

Let $\pi : G \to \mathrm{Sym}(V)$ be an (F, τ')-approximation for G. Let $\varphi \in \mathrm{Map}_d(F, \tau', \pi)$ be such that $\varphi(V) \subseteq X_0$. Define $\sigma' : H \to V^V$ by

$$\sigma'_t v = \pi_{\lambda(t,\varphi(v))} v$$

for $t \in H$ and $v \in V$. Then there is an (L, τ)-approximation $\sigma : H \to \mathrm{Sym}(V)$ for H such that $\rho_{\mathrm{Hamm}}(\sigma_t, \sigma'_t) \leq \tau$ for all $t \in L^2$.

Proof. Denote by V_F the set of all $v \in V$ satisfying $\pi_g\pi_h v = \pi_{gh}v$ for all $g,h \in F$ and $\pi_g v \neq \pi_h v$ for all distinct $g,h \in F$. Then

$$\mathrm{m}(V \setminus V_F) \leq 2|F|^2\tau' \leq \frac{\tau}{11}.$$

Denote by V_φ the set of all $v \in V$ satisfying $d(\varphi(\pi_g v), g\varphi(v)) \leq \eta_{L^2}$ for all $g \in F$. Then

$$\mathrm{m}(V \setminus V_\varphi) \leq |F|\bigg(\frac{\tau'}{\eta_{L^2}}\bigg)^2 \leq \frac{\tau}{8}.$$

For all $s,t \in L^2$ and $v \in V_F \cap V_\varphi$, since $\lambda(t,\varphi(v)) \in F$ we have

$$d(\varphi(\pi_{\lambda(t,\varphi(v))}v), \lambda(t,\varphi(v))\varphi(v)) \leq \eta_{L^2}$$

and hence

$$\lambda(s,\varphi(\pi_{\lambda(t,\varphi(v))}v)) = \lambda(s,\lambda(t,\varphi(v))\varphi(v)) = \lambda(s,t\varphi(v)),$$

which yields

$$\begin{aligned}\sigma'_s\sigma'_t v = \pi_{\lambda(s,\varphi(\sigma'_t v))}\sigma'_t v &= \pi_{\lambda(s,\varphi(\pi_{\lambda(t,\varphi(v))}v))}\pi_{\lambda(t,\varphi(v))}v \\ &= \pi_{\lambda(s,t\varphi(v))}\pi_{\lambda(t,\varphi(v))}v \\ &= \pi_{\lambda(s,t\varphi(v))\lambda(t,\varphi(v))}v \\ &= \pi_{\lambda(st,\varphi(v))}v \\ &= \sigma'_{st}v,\end{aligned}$$

so that

$$\rho_{\mathrm{Hamm}}(\sigma'_s\sigma'_t, \sigma'_{st}) \leq \mathrm{m}(V \setminus (V_F \cap V_\varphi)) \leq \frac{\tau}{11} + \frac{\tau}{8} = \frac{19\tau}{88}. \tag{6}$$

Note that $\sigma'_{e_H} = \pi_{e_G}$. For each $t \in H$ choose a $\sigma_t \in \mathrm{Sym}(V)$ such that $\sigma_t v = \sigma'_t v$ for all $v \in V$ satisfying $\sigma'_{t^{-1}}\sigma'_t v = v$. For each $t \in L^2$, taking $s = t^{-1}$ in (6) we conclude that

$$\begin{aligned}\rho_{\mathrm{Hamm}}(\sigma_t, \sigma'_t) &\leq \rho_{\mathrm{Hamm}}(\sigma'_{t^{-1}}\sigma'_t, \mathrm{id}) \\ &\leq \rho_{\mathrm{Hamm}}(\sigma'_{t^{-1}}\sigma'_t, \sigma'_{e_H}) + \rho_{\mathrm{Hamm}}(\sigma'_{e_H}, \mathrm{id}) \\ &\leq \frac{19\tau}{88} + \rho_{\mathrm{Hamm}}(\pi_{e_G}, \mathrm{id}) \\ &\leq \frac{19\tau}{88} + \tau' \\ &\leq \frac{19\tau}{88} + \frac{\tau}{22} = \frac{23\tau}{88},\end{aligned}$$

which in particular shows that $\rho_{\mathrm{Hamm}}(\sigma_t, \sigma'_t) < \tau$. For all $s,t \in L$ we then have

$$\begin{aligned}\rho_{\mathrm{Hamm}}(\sigma_s\sigma_t, \sigma_{st}) &\leq \rho_{\mathrm{Hamm}}(\sigma_s, \sigma'_s) + \rho_{\mathrm{Hamm}}(\sigma_t, \sigma'_t) + \rho_{\mathrm{Hamm}}(\sigma'_s\sigma'_t, \sigma'_{st}) + \rho_{\mathrm{Hamm}}(\sigma_{st}, \sigma'_{st}) \\ &\leq \frac{23\tau}{88} + \frac{23\tau}{88} + \frac{19\tau}{88} + \frac{23\tau}{88} = \tau.\end{aligned}$$

For all distinct $s,t \in L^2$, since $\varphi(V) \subseteq X_0$ we have $\sigma'_s v \neq \sigma'_t v$ for all $v \in V_F$ and hence

$$\begin{aligned}\rho_{\mathrm{Hamm}}(\sigma_s, \sigma_t) &\geq \rho_{\mathrm{Hamm}}(\sigma'_s, \sigma'_t) - \rho_{\mathrm{Hamm}}(\sigma_s, \sigma'_s) - \rho_{\mathrm{Hamm}}(\sigma_t, \sigma'_t) \\ &\geq \mathrm{m}(V_F) - \frac{23\tau}{88} - \frac{23\tau}{88}\end{aligned}$$

$$\geq 1 - \frac{\tau}{11} - \frac{23\tau}{44} > 1 - \tau. \qquad \square$$

Proof of Proposition 3.16. Let Υ_H be a function $\mathcal{F}(H) \to [0, \infty)$. Define the function $\Upsilon_G : \mathcal{F}(G) \to [0, \infty)$ by $\Upsilon_G(F) = 2\Upsilon_H(\kappa(F, X))$.

Since the action $G \curvearrowright X$ has property sofic SC, there exists an $S_G \in \overline{\mathcal{F}}(G)$ such that for any $T_G \in \overline{\mathcal{F}}(G)$ there are $C_G, n_G \in \mathbb{N}$, $S_{G,1}, \dots, S_{G,n_G} \in \overline{\mathcal{F}}(G)$, $L_G \in \overline{\mathcal{F}}(G)$, and $0 < \tau_G < 1$ such that, for any (L_G, τ_G)-approximation $\pi : G \to \mathrm{Sym}(V)$ for G with $\mathrm{Map}_d(L_G, \tau_G, \pi) \neq \emptyset$, there are subsets W_G and $\mathcal{V}_{G,j}$ of V for $1 \leq j \leq n_G$ satisfying the following conditions:

(i) $\sum_{j=1}^{n_G} \Upsilon_G(S_{G,j}) \mathrm{m}(\mathcal{V}_{G,j}) \leq 1$,

(ii) $\bigcup_{g \in S_G} \pi_g W_G = V$,

(iii) if $w_1, w_2 \in W_G$ satisfy $\pi_g w_1 = w_2$ for some $g \in T_G$ then w_1 and w_2 are connected by a path of length at most C_G in which each edge is an $S_{G,j}$-edge with both endpoints in $\mathcal{V}_{G,j}$ for some $1 \leq j \leq n_G$.

Set $S_H = \kappa(S_G, X) \in \overline{\mathcal{F}}(H)$.

Let $T_H \in \overline{\mathcal{F}}(H)$. Set $T_G = \lambda(T_H, X) \in \overline{\mathcal{F}}(G)$. Then we have C_G, n_G, $S_{G,j}$ for $1 \leq j \leq n_G$, L_G, and τ_G as above. Set $C_H = C_G$, $n_H = n_G + 1$, $S_{H,j} = \kappa(S_{G,j}, X) \in \overline{\mathcal{F}}(H)$ for $1 \leq j \leq n_G$, and $S_{H,n_H} = T_H \in \overline{\mathcal{F}}(H)$. Take $0 < \delta_H < 1/(6\Upsilon_H(T_H)|T_H|)$. Also, set $A = L_G \cup S_G \cup T_G \cup \bigcup_{j=1}^{n_G} S_{G,j} \in \mathcal{F}(G)$, $L_H = \kappa(A^{2(100+C_G)}, X) \in \mathcal{F}(H)$, and

$$\tilde{\tau}_G = \min\left\{(\tau_G/2)^2, \delta_H/(4|S_G| \cdot |A|^{2(100+C_G)})\right\} > 0,$$
$$\tau_H = \min\left\{\eta_{A^{2(100+C_G)}} \tilde{\tau}_G^{1/2}/(8|L_H|)^{1/2}, \tilde{\tau}_G/(22|L_H|^2), \tau_G/(2|L_H|^{1/2})\right\} > 0.$$

Let $\sigma : H \to \mathrm{Sym}(V)$ be an (L_H, τ_H)-approximation for H with $\mathrm{Map}_d(L_H, \tau_H/2, \sigma) \neq \emptyset$. Choose a $\varphi \in \mathrm{Map}_d(L_H, \tau_H/2, \sigma)$. Since X_0 is dense in X, by perturbing φ if necessary we may assume that $\varphi \in \mathrm{Map}_d(L_H, \tau_H, \sigma)$ and $\varphi(V) \subseteq X_0$. Define $\pi' : G \to V^V$ by

$$\pi'_g v = \sigma_{\kappa(g, \varphi(v))} v$$

for all $v \in V$ and $g \in G$. By Lemma 3.17 there is an $(A^{100+C_G}, \tilde{\tau}_G)$-approximation $\pi : G \to \mathrm{Sym}(V)$ such that $\rho_{\mathrm{Hamm}}(\pi_g, \pi'_g) \leq \tilde{\tau}_G$ for all $g \in A^{100+C_G}$. For each $g \in L_G \subseteq A^{100+C_G}$ we have

$$\begin{aligned}
d_2(g\varphi, \varphi\pi_g) &\leq d_2(g\varphi, \varphi\pi'_g) + d_2(\varphi\pi'_g, \varphi\pi_g) \\
&\leq \left(\frac{1}{|V|}\sum_{v \in V} d(\kappa(g, \varphi(v))\varphi(v), \varphi(\sigma_{\kappa(g,\varphi(v))} v))^2\right)^{1/2} + \tilde{\tau}_G^{1/2} \\
&\leq \left(\frac{1}{|V|}\sum_{v \in V}\sum_{t \in \kappa(g,X)} d(t\varphi(v), \varphi(\sigma_t v))^2\right)^{1/2} + \frac{\tau_G}{2} \\
&\leq (\tau_H^2 |L_H|)^{1/2} + \frac{\tau_G}{2} \\
&\leq \tau_G.
\end{aligned}$$

Thus $\varphi \in \mathrm{Map}_d(L_G, \tau_G, \pi)$. Then we have W_G and $\mathcal{V}_{G,j}$ for $1 \leq j \leq n_G$ as above.

We now verify conditions (i)-(iii) in Definition 3.1 as referenced in Definition 3.2. Denote by V_1 the set of all $v \in V$ satisfying $\pi_{g_1 g_2} v = \pi_{g_1}\pi_{g_2} v$ for all $g_1, g_2 \in A^{100+C_G}$. Then $\mathrm{m}(V \setminus V_1) \leq |A|^{2(100+C_G)} \tilde{\tau}_G$. Also, denote by V_2 the set of $v \in V$ satisfying $\pi_g \pi_{g_1} v = \pi'_g \pi_{g_1} v$ for all $g \in A$ and $g_1 \in A^{C_G}$. Then $\mathrm{m}(V \setminus V_2) \leq \tilde{\tau}_G |A|^{1+C_G}$. Set $W'_H = W_G \cap V_1 \cap V_2$, and $W_H = W'_H \cup \sigma_{e_H}^{-1}(V \setminus \bigcup_{h \in S_H} \sigma_h W'_H)$. Then $\bigcup_{h \in S_H} \sigma_h W_H = V$, verifying condition (ii) in Definition 3.1.

Note that

$$\begin{aligned}
\mathrm{m}(W_H \setminus W'_H) &\leq 1 - \mathrm{m}\big(\bigcup_{h \in S_H} \sigma_h W'_H\big) \\
&\leq 1 - \mathrm{m}\big(\bigcup_{g \in S_G} \pi'_g W'_H\big) \\
&= 1 - \mathrm{m}\big(\bigcup_{g \in S_G} \pi_g W'_H\big) \\
&\leq |S_G|(\mathrm{m}(V \setminus V_1) + \mathrm{m}(V \setminus V_2)) \\
&\leq |S_G|(\tilde{\tau}_G |A|^{2(100+C_G)} + \tilde{\tau}_G |A|^{1+C_G}) \\
&\leq \delta_H.
\end{aligned}$$

Put $\mathcal{V}_{H,j} = \mathcal{V}_{G,j}$ for all $1 \leq j \leq n_G$, and

$$\mathcal{V}_{H,n_H} = \bigcup_{h \in T_H} ((W_H \setminus W'_H) \cup \sigma_h (W_H \setminus W'_H) \cup \sigma_h^{-1}(W_H \setminus W'_H)).$$

Then

$$\mathrm{m}(\mathcal{V}_{H,n_H}) \leq (2|T_H| + 1)\mathrm{m}(W_H \setminus W'_H) \leq 3|T_H|\delta_H,$$

and hence

$$\begin{aligned}
\sum_{j=1}^{n_H} \Upsilon_H(S_{H,j})\mathrm{m}(\mathcal{V}_{H,j}) &= \Upsilon_H(T_H)\mathrm{m}(\mathcal{V}_{H,n_H}) + \sum_{j=1}^{n_G} \Upsilon_H(\kappa(S_{G,j}, X))\mathrm{m}(\mathcal{V}_{G,j}) \\
&= \Upsilon_H(T_H)\mathrm{m}(\mathcal{V}_{H,n_H}) + \frac{1}{2}\sum_{j=1}^{n_G} \Upsilon_G(S_{G,j})\mathrm{m}(\mathcal{V}_{G,j}) \\
&\leq 3\Upsilon_H(T_H)|T_H|\delta_H + \frac{1}{2} \leq 1,
\end{aligned}$$

verifying condition (i) in Definition 3.1. Let $h \in T_H$ and $w_1, w_2 \in W_H$ with $\sigma_h w_1 = w_2$. If $w_1 \notin W'_H$ or $w_2 \notin W'_H$, then (w_1, w_2) is an S_{H,n_H}-edge with both endpoints in $\mathcal{V}_{H,n_H}$. We may thus assume that $w_1, w_2 \in W'_H$. Then

$$w_2 = \sigma_h w_1 = \pi'_{\lambda(h,\varphi(w_1))} w_1 = \pi_{\lambda(h,\varphi(w_1))} w_1 \in \pi_{T_G} w_1$$

and so w_1 and w_2 are connected by a path of length at most C_G in which each edge is an $S_{G,j}$-edge with both endpoints in $\mathcal{V}_{G,j}$ for some $1 \leq j \leq n_G$. It is easily checked that such an edge is also an $S_{H,j}$-edge. This verifies condition (iii) in Definition 3.1. □

3.7. **Property sofic SC under bounded orbit equivalence.**

Proposition 3.18. *Let $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ be free p.m.p. actions which are boundedly orbit equivalent. Suppose that $G \curvearrowright (X, \mu)$ has property sofic SC. Then so does $H \curvearrowright (Y, \nu)$.*

To prove this proposition we may assume that $(X,\mu)=(Y,\nu)$, the actions $G \curvearrowright X$ and $H \curvearrowright X$ are free, and the identity map of X provides a bounded orbit equivalence between the actions $G \curvearrowright X$ and $H \curvearrowright X$. Let $\kappa : G\times X \to H$ and $\lambda : H\times X\to G$ be the associated cocyles.

For each $g\in G$ denote by $\mathscr{P}_g$ the finite Borel partition of X consisting of the sets $X_{g,t} := \{x\in X : gx = tx\}$ for $t\in H$, and likewise for $t\in H$ denote by $\mathscr{P}_t$ the finite Borel partition of X consisting of the sets $X_{g,t}$ for $g\in G$. For every F in $\mathcal{F}(G)$ or $\mathcal{F}(H)$, write ${}_F\mathscr{P} = \bigvee_{g\in F}\mathscr{P}_g$.

The following is a specialization of Lemma 4.2 in [25] to the case of bounded orbit equivalence, which permits a simplification of the statement.

Lemma 3.19. *Let $F\in\overline{\mathcal{F}}(G)$ and set $L=\kappa(F^2,X)\in\overline{\mathcal{F}}(H)$. Let $0<\tau<1$ and $0<\tau'\leq\tau/(60|L|^2)$. Let $\sigma : H\to\mathrm{Sym}(V)$ be an (L,τ')-approximation for H. Let $\varphi\in\mathrm{Hom}_\mu({}_{F^2}\mathscr{P},L,\tau',\sigma)$. Let $\pi' : F^2\to V^V$ be such that*

$$\pi'_g v = \sigma_{\kappa(g,A)} v$$

for all $g\in F^2$, $A\in{}_{F^2}\mathscr{P}$ and $v\in\varphi(A)$. Then there is an (F,τ)-approximation $\pi : G\to\mathrm{Sym}(V)$ for G such that $\rho_{\mathrm{Hamm}}(\pi_g,\pi'_g)\leq\tau/5$ for all $g\in F^2$.

Proof of Proposition 3.18. Let Υ_H be a function $\mathcal{F}(H)\to[0,\infty)$. Define a function $\Upsilon_G : \mathcal{F}(G)\to[0,\infty)$ by $\Upsilon_G(F) = 2\Upsilon_H(\kappa(F,X))$.

Since $G\curvearrowright(X,\mu)$ has property sofic SC, there exists an $S_G\in\overline{\mathcal{F}}(G)$ such that for any $T_G\in\overline{\mathcal{F}}(G)$ there are $C_G, n_G\in\mathbb{N}$, $S_{G,1},\dots,S_{G,n_G}\in\overline{\mathcal{F}}(G)$, a finite Borel partition $\mathscr{C}_G$ of X, an $L_G\in\overline{\mathcal{F}}(G)$, and $0<\tau_G<1$ such that, for any (L_G,τ_G)-approximation $\pi : G\to\mathrm{Sym}(V)$ for G with $\mathrm{Hom}_\mu(\mathscr{C}_G,L_G,\tau_G,\pi)\neq\emptyset$, there are subsets W_G and $\mathcal{V}_{G,j}$ of V for $1\leq j\leq n_G$ satisfying the following conditions:

(i) $\sum_{j=1}^{n_G}\Upsilon_G(S_{G,j})\mathrm{m}(\mathcal{V}_{G,j})\leq 1$,
(ii) $\bigcup_{g\in S_G}\pi_g W_G = V$,
(iii) if $w_1,w_2\in W_G$ satisfy $\pi_g w_1 = w_2$ for some $g\in T_G$ then w_1 and w_2 are connected by a path of length at most C_G in which each edge is an $S_{G,j}$-edge with both endpoints in $\mathcal{V}_{G,j}$ for some $1\leq j\leq n_G$.

Set $S_H = \kappa(S_G,X)\in\overline{\mathcal{F}}(H)$.

Let $T_H\in\overline{\mathcal{F}}(H)$. Set $T_G=\lambda(T_H,X)\in\overline{\mathcal{F}}(G)$. Then we have C_G, n_G, $S_{G,j}$ for $1\leq j\leq n_G$, $\mathscr{C}_G$, L_G, and τ_G as above. Set $C_H = C_G$, $n_H = n_G+1$, $S_{H,j}=\kappa(S_{G,j},X)\in\overline{\mathcal{F}}(H)$ for $1\leq j\leq n_G$, and $S_{H,n_H} = T_H\in\overline{\mathcal{F}}(H)$. Take $0<\delta_H<1/(6\Upsilon_H(T_H)|T_H|)$. Set $U = L_G\cup S_G\cup T_G\cup\bigcup_{j=1}^{n_G}S_{G,j}\in\mathcal{F}(G)$, $\mathscr{C}_H = (\mathscr{C}_G)_{L_G}\vee{}_{U^{2(100+C_G)}}\mathscr{P}\vee{}_{T_H}\mathscr{P}$, $L_H=\kappa(U^{2(100+C_G)},X)\in\mathcal{F}(H)$, and

$$\tilde\tau_G = \min\left\{\tau_G/4, \delta_H/(2|S_G|\cdot|U|^{2(100+C_G)})\right\} > 0,$$
$$\tau_H = \min\left\{\tilde\tau_G/(60|L_H|^2), \tau_G/(2|\kappa(L_G,X)|)\right\} > 0.$$

Let $\sigma : H\to\mathrm{Sym}(V)$ be an (L_H,τ_H)-approximation for H with $\mathrm{Hom}_\mu(\mathscr{C}_H,L_H,\tau_H,\sigma)$ nonempty. Take $\varphi\in\mathrm{Hom}_\mu(\mathscr{C}_H,L_H,\tau_H,\sigma)$. Define $\pi' : U^{2(100+C_G)}\to V^V$ by

$$\pi'_g v = \sigma_{\kappa(g,A)}v$$

for all $g\in U^{2(100+C_G)}$, $A\in\mathscr{P}_g$, and $v\in\varphi(A)$. By Lemma 3.19 there is a $(U^{100+C_G},\tilde\tau_G)$-approximation $\pi : G\to\mathrm{Sym}(V)$ for G such that $\rho_{\mathrm{Hamm}}(\pi_g,\pi'_g)\leq\tilde\tau_G$ for all $g\in U^{100+C_G}$.

Let $g \in L_G$. We have

$$\begin{aligned}\sum_{A\in\mathscr{C}_G} \mathrm{m}(\pi_g\varphi(A)\Delta\varphi(gA)) &\leq \sum_{A\in\mathscr{C}_G} \mathrm{m}(\pi_g\varphi(A)\Delta\pi'_g\varphi(A)) + \sum_{A\in\mathscr{C}_G} \mathrm{m}(\pi'_g\varphi(A)\Delta\varphi(gA)) \\ &\leq 2\rho_{\mathrm{Hamm}}(\pi_g,\pi'_g) + \sum_{A\in\mathscr{C}_G}\sum_{B\in\mathscr{P}_g} \mathrm{m}(\pi'_g\varphi(A\cap B)\Delta\varphi(g(A\cap B))).\end{aligned}$$

For any $A \in \mathscr{C}_G$ and $B \in \mathscr{P}_g$, say $h = \kappa(g, B) \in L_H$, we have $\pi'_g\varphi(A\cap B) = \sigma_h\varphi(A\cap B)$ and $\varphi(g(A\cap B)) = \varphi(h(A\cap B))$, whence

$$\sum_{A\in\mathscr{C}_G} \mathrm{m}(\pi'_g\varphi(A\cap B)\Delta\varphi(g(A\cap B))) = \sum_{A\in\mathscr{C}_G} \mathrm{m}(\sigma_h\varphi(A\cap B)\Delta\varphi(h(A\cap B))) \leq \tau_H.$$

Therefore

$$\begin{aligned}\sum_{A\in\mathscr{C}_G} \mathrm{m}(\pi_g\varphi(A)\Delta\varphi(gA)) &\leq 2\rho_{\mathrm{Hamm}}(\pi_g,\pi'_g) + \sum_{B\in\mathscr{P}_g}\sum_{A\in\mathscr{C}_G} \mathrm{m}(\pi'_g\varphi(A\cap B)\Delta\varphi(g(A\cap B))) \\ &\leq 2\tilde{\tau}_G + \sum_{B\in\mathscr{P}_g} \tau_H \\ &\leq \frac{\tau_G}{2} + |\kappa(g,X)|\tau_H \leq \tau_G.\end{aligned}$$

We also have

$$\sum_{A\in(\mathscr{C}_G)_{L_G}} |\mathrm{m}(\varphi(A)) - \mu(A)| \leq \sum_{B\in\mathscr{C}_H} |\mathrm{m}(\varphi(B)) - \mu(B)| \leq \tau_H \leq \tau_G.$$

Therefore $\varphi \in \mathrm{Hom}_\mu(\mathscr{C}_G, L_G, \tau_G, \pi)$. Then we have W_G and $\mathcal{V}_{G,j}$ for $1 \leq j \leq n_G$ as above.

We now verify conditions (i)-(iii) in Definition 3.1 as referenced in Definition 3.3. Denote by V_1 the set of $v \in V$ satisfying $\pi_{g_1g_2}v = \pi_{g_1}\pi_{g_2}v$ for all $g_1, g_2 \in U^{100+C_G}$. Then $\mathrm{m}(V \setminus V_1) \leq \tilde{\tau}_G|U|^{2(100+C_G)}$. Also denote by V_2 the set of $v \in V$ satisfying $\pi_g\pi_{g_1}v = \pi'_g\pi_{g_1}v$ for all $g \in U$ and $g_1 \in U^{C_G}$. Then $\mathrm{m}(V \setminus V_2) \leq \tilde{\tau}_G|U|^{1+C_G}$. Set $W'_H = W_G \cap V_1 \cap V_2$, and $W_H = W'_H \cup \sigma_{e_H}^{-1}(V \setminus \bigcup_{h\in S_H}\sigma_h W'_H)$. Then $\bigcup_{h\in S_H}\sigma_h W_H = V$, verifying condition (ii) in Definition 3.1. Note that

$$\begin{aligned}\mathrm{m}(W_H \setminus W'_H) &\leq 1 - \mathrm{m}\Big(\bigcup_{h\in S_H}\sigma_h W'_H\Big) \\ &\leq 1 - \mathrm{m}\Big(\bigcup_{g\in S_G}\pi'_g W'_H\Big) \\ &= 1 - \mathrm{m}\Big(\bigcup_{g\in S_G}\pi_g W'_H\Big) \\ &\leq |S_G|(\mathrm{m}(V\setminus V_1) + \mathrm{m}(V\setminus V_2)) \\ &\leq |S_G|(\tilde{\tau}_G|U|^{2(100+C_G)} + \tilde{\tau}_G|U|^{1+C_G}) \leq \delta_H.\end{aligned}$$

Put $\mathcal{V}_{H,j} = \mathcal{V}_{G,j}$ for all $1 \leq j \leq n_G$, and

$$\mathcal{V}_{H,n_H} = \bigcup_{h\in T_H}((W_H\setminus W'_H) \cup \sigma_h(W_H\setminus W'_H) \cup \sigma_h^{-1}(W_H\setminus W'_H)).$$

Then
$$\mathrm{m}(\mathcal{V}_{H,n_H}) \leq (2|T_H|+1)\mathrm{m}(W_H \setminus W'_H) \leq 3|T_H|\delta_H,$$
and hence
$$\begin{aligned}\sum_{j=1}^{n_H} \Upsilon_H(S_{H,j})\mathrm{m}(\mathcal{V}_{H,j}) &= \Upsilon_H(T_H)\mathrm{m}(\mathcal{V}_{H,n_H}) + \sum_{j=1}^{n_G} \Upsilon_H(\kappa(S_{G,j},X))\mathrm{m}(\mathcal{V}_{G,j}) \\ &= \Upsilon_H(T_H)\mathrm{m}(\mathcal{V}_{H,n_H}) + \frac{1}{2}\sum_{j=1}^{n_G} \Upsilon_G(S_{G,j})\mathrm{m}(\mathcal{V}_{G,j}) \\ &\leq 3\Upsilon_H(T_H)|T_H|\delta_H + \frac{1}{2} \leq 1,\end{aligned}$$
verifying condition (i) in Definition 3.1. Let $h \in T_H$ and $w_1, w_2 \in W_H$ with $\sigma_h w_1 = w_2$. If $w_1 \notin W'_H$ or $w_2 \notin W'_H$, then (w_1, w_2) is an S_{H,n_H}-edge with both endpoints in $\mathcal{V}_{H,n_H}$. Thus we may assume that $w_1, w_2 \in W'_H$. Then $w_1 \in \varphi(A)$ for some $A \in \mathscr{P}_h$. Set $g = \lambda(h, A) \in T_G$. Then
$$w_2 = \sigma_h w_1 = \pi'_g w_1 = \pi_g w_1,$$
and so w_1 and w_2 are connected by a path of length at most C_G in which each edge is an $S_{G,j}$-edge with both endpoints in $\mathcal{V}_{G,j}$ for some $1 \leq j \leq n_G$. It is easily checked that such an edge is also an $S_{H,j}$-edge. This verifies condition (iii) in Definition 3.1. □

4. Topological entropy and continuous orbit equivalence

Our energies in this section will be invested in the proof of Theorem 4.1, which in conjunction with Theorem 3.12 and Proposition 3.15 yields Theorem C.

Theorem 4.1. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free continuous actions on compact metrizable spaces, and suppose that they are continuously orbit equivalent. Let $\mathscr{S}$ be a collection of sofic approximations for G, and suppose that the action $G \curvearrowright X$ has property $\mathscr{S}$-SC. Let Π be a sofic approximation sequence for G in $\mathscr{S}$. Then*
$$h(H \curvearrowright Y) \geq h_\Pi(G \curvearrowright X).$$

For the purpose of establishing the theorem we may assume, by conjugating the H-action by a continuous orbit equivalence, that $Y = X$ and that the identity map on X is an orbit equivalence between the two actions. As usual we write κ and λ, respectively, for the cocycle maps $G \times X \to H$ and $H \times X \to G$. As in Section 3.6, we take a dense G_δ subset X_0 of X such that X_0 is $G * H$-invariant and that both G and H act on X_0 freely.

Fix a compatible metric d on X which gives X diameter no bigger than 1. For each $t \in G$ (resp. $t \in H$) we can find an $\eta_t > 0$ such that for any $x, y \in X$ with $d(x,y) \leq \eta_t$ we have $\kappa(t,x) = \kappa(t,y)$ (resp. $\lambda(t,x) = \lambda(t,y)$), and for a nonempty finite subset L of G or H we set $\eta_L = \min_{t\in L} \eta_t > 0$.

Lemma 4.2. *Let $L \in \overline{\mathcal{F}}(H)$ and $0 < \delta, \tau < 1$ with $\tau < \delta^2$. Set $F = \lambda(L^2, X) \in \overline{\mathcal{F}}(G)$ and*
$$\tau' = \min\left\{\eta_{L^2}\tau^{1/2}/(8|F|)^{1/2}, \tau/(22|F|^2)\right\} > 0.$$
Let $\delta_1 > 0$ be such that $(\tau + 7\delta_1)^{1/2} \leq \delta$. Let $\pi : G \to \mathrm{Sym}(V)$ be an (F, τ')-approximation for G. Suppose that $S \in \overline{\mathcal{F}}(G)$ and that W is a subset of V satisfying the following conditions:

(i) $\pi_{g^{-1}}\pi_g w = w$ *for all* $w \in W$ *and* $g \in S$,
(ii) $\pi_g \pi_a \pi_h w = \pi_{gah} w$ *for all* $g, h \in S$, $a \in \lambda(L, X)$, *and* $w \in W$,
(iii) $\mathrm{m}(\bigcup_{g\in S} \pi_g W) \geq 1 - \delta_1$.

Take $0 < \delta_2 \leq \eta_{S\cup\lambda(L,X)}$ *such that for any* $x, y \in X$ *with* $d(x, y) \leq \delta_2$ *one has* $\max_{t\in\kappa(S,X)} d(tx, ty) \leq \delta_1$. *Set*

$$\delta' = \delta_1^{1/2}\delta_2/(|S|^{1/2}|S\lambda(L, X)S|^{1/2}) > 0.$$

Let φ_0 *be a map in* $\mathrm{Map}_d(F, \tau', \pi) \cap \mathrm{Map}_d(S\lambda(L, X)S, \delta', \pi)$ *with* $\varphi_0(V) \subseteq X_0$ *and* φ *a map in* $\mathrm{Map}_d(S\lambda(L, X)S, \delta', \pi)$ *such that*

$$\kappa(g, \varphi_0(w)) = \kappa(g, \varphi(w))$$

for all $w \in W$ *and* $g \in S\lambda(L, X)S$ *satisfying* $\pi_g w \in W$. *Let* $\sigma : H \to \mathrm{Sym}(V)$ *be a map such that* $\rho_{\mathrm{Hamm}}(\sigma_t, \sigma'_t) \leq \tau$ *for all* $t \in L$, *where* $\sigma' : H \to V^V$ *is given by*

$$\sigma'_t v = \pi_{\lambda(t,\varphi_0(v))} v$$

for all $t \in H$ *and* $v \in V$. *Take* $\tilde{\varphi} : V \to X$ *such that* $\tilde{\varphi} = \varphi$ *on* W *and such that for each* $v \in \bigcup_{g\in S} \pi_g W$ *one has*

$$\tilde{\varphi}(v) = \kappa(g, \varphi_0(w))\varphi(w)$$

for some $g \in S$ *and* $w \in W$ *with* $\pi_g w = v$. *Then* $\tilde{\varphi} \in \mathrm{Map}_d(L, \delta, \sigma)$.

Proof. For each $t \in H$ set $V_t = \{v \in V : \sigma_t v = \sigma'_t v\}$. Then $\mathrm{m}(V_t) \geq 1 - \tau$ for all $t \in L$.

Denote by V_φ the set of all $v \in V$ satisfying $d(g\varphi(v), \varphi(\pi_g v)) \leq \delta_2$ for all $g \in S\lambda(L, X)S$. Then

$$\mathrm{m}(V \setminus V_\varphi) \leq |S\lambda(L, X)S|\left(\frac{\delta'}{\delta_2}\right)^2 = \frac{\delta_1}{|S|}.$$

We define V_{φ_0} in the same way, and get $\mathrm{m}(V \setminus V_{\varphi_0}) \leq \delta_1/|S|$. Set $W' = W \cap V_\varphi \cap V_{\varphi_0}$ and $V' = (\bigcup_{g\in S} \pi_g W) \setminus (\bigcup_{g\in S} \pi_g(V \setminus (V_\varphi \cap V_{\varphi_0})))$. Then

$$\mathrm{m}(V') \geq \mathrm{m}\left(\bigcup_{g\in S} \pi_g W\right) - |S| \cdot \mathrm{m}(V \setminus V_\varphi) - |S| \cdot \mathrm{m}(V \setminus V_{\varphi_0}) \geq 1 - 3\delta_1.$$

Let $t \in L$, $v_1 \in V_t \cap V'$, and $v_2 \in V'$ be such that $\sigma_t v_1 = v_2$. Then we can find some $g_1, g_2 \in S$ and $w_1, w_2 \in W$ such that $\pi_{g_j} w_j = v_j$ and $\tilde{\varphi}(v_j) = \kappa(g_j, \varphi_0(w_j))\varphi(w_j)$ for $j = 1, 2$. Since $v_j \in V'$, we actually have $w_j \in W'$. We also have

$$\begin{aligned} w_2 = \pi_{g_2^{-1}} v_2 &= \pi_{g_2^{-1}} \sigma_t v_1 \\ &= \pi_{g_2^{-1}} \sigma'_t v_1 \\ &= \pi_{g_2^{-1}} \pi_{\lambda(t,\varphi_0(v_1))} v_1 \\ &= \pi_{g_2^{-1}} \pi_{\lambda(t,\varphi_0(v_1))} \pi_{g_1} w_1 \\ &= \pi_{g_2^{-1}\lambda(t,\varphi_0(v_1))g_1} w_1. \end{aligned}$$

Observe that

$$d(g_1\varphi_0(w_1), \varphi_0(v_1)) = d(g_1\varphi_0(w_1), \varphi_0(\pi_{g_1} w_1)) \leq \delta_2 \leq \eta_{\lambda(L,X)}$$

and

$$d(\varphi_0(w_2), g_2^{-1}\lambda(t, \varphi_0(v_1))g_1\varphi_0(w_1))$$

$$= d(\varphi_0(\pi_{g_2^{-1}\lambda(t,\varphi_0(v_1))g_1} w_1), g_2^{-1}\lambda(t,\varphi_0(v_1))g_1\varphi_0(w_1)) \leq \delta_2 \leq \eta_S,$$

and hence

$$\kappa(\lambda(t,\varphi_0(v_1)), g_1\varphi_0(w_1)) = \kappa(\lambda(t,\varphi_0(v_1)), \varphi_0(v_1)) = t$$

and

$$\kappa(g_2^{-1}, \lambda(t,\varphi_0(v_1))g_1\varphi_0(w_1)) = \kappa(g_2, g_2^{-1}\lambda(t,\varphi_0(v_1))g_1\varphi_0(w_1))^{-1} = \kappa(g_2, \varphi_0(w_2))^{-1}.$$

Therefore

$$\begin{aligned}
&\kappa(g_2^{-1}\lambda(t,\varphi_0(v_1))g_1, \varphi_0(w_1)) \\
&\qquad = \kappa(g_2^{-1}, \lambda(t,\varphi_0(v_1))g_1\varphi_0(w_1))\kappa(\lambda(t,\varphi_0(v_1)), g_1\varphi_0(w_1))\kappa(g_1,\varphi_0(w_1)) \\
&\qquad = \kappa(g_2,\varphi_0(w_2))^{-1} t \kappa(g_1,\varphi_0(w_1)).
\end{aligned}$$

We then get

$$\begin{aligned}
\kappa(g_2,\varphi_0(w_2))^{-1}\tilde{\varphi}(v_2) &= \varphi(w_2) \\
&\approx_{\delta_2} g_2^{-1}\lambda(t,\varphi_0(v_1))g_1\varphi(w_1) \\
&= \kappa(g_2^{-1}\lambda(t,\varphi_0(v_1))g_1, \varphi(w_1))\varphi(w_1) \\
&= \kappa(g_2^{-1}\lambda(t,\varphi_0(v_1))g_1, \varphi_0(w_1))\varphi(w_1) \\
&= \kappa(g_2,\varphi_0(w_2))^{-1} t\kappa(g_1,\varphi_0(w_1))\varphi(w_1) \\
&= \kappa(g_2,\varphi_0(w_2))^{-1} t\tilde{\varphi}(v_1),
\end{aligned}$$

and consequently $d(\tilde{\varphi}(v_2), t\tilde{\varphi}(v_1)) \leq \delta_1$. We conclude that

$$d_2(t\tilde{\varphi}, \tilde{\varphi}\sigma_t) \leq (\tau + 6\delta_1 + \delta_1^2)^{1/2} \leq (\tau + 7\delta_1)^{1/2} \leq \delta$$

and hence that $\tilde{\varphi} \in \mathrm{Map}_d(L,\delta,\sigma)$. □

Proof of Theorem 4.1. Let $\Pi = \{\pi_k : G \to \mathrm{Sym}(V_k)\}_{k=1}^\infty$ be a sofic approximation sequence in $\mathscr{S}$ with $h_\Pi(G \curvearrowright X) \geq 0$. Let $\varepsilon > 0$. To establish the theorem it is enough to show the existence of a sofic approximation sequence Σ for H such that $h_\Sigma(H \curvearrowright Y) \geq h^\varepsilon_{\Pi,2}(G \curvearrowright X) - 2\varepsilon$.

For each $F \in \mathcal{F}(G)$, since $\kappa : G \times X \to H$ is continuous there exists a finite clopen partition ${}_F\mathscr{P}$ of X such that for every $g \in F$ the map $x \mapsto \kappa(g,x)$ is constant on each member of ${}_F\mathscr{P}$. Define $\Upsilon : \mathcal{F}(G) \to [0,\infty)$ by $\Upsilon(F) = (2/\varepsilon)\log|{}_F\mathscr{P}|$.

Take a decreasing sequence $1 > \delta_1 > \delta_2 > \dots$ converging to 0. Take also a decreasing sequence $1 > \tau_1 > \tau_2 > \dots > 0$ with $\tau_k^2 < \delta_k$ for all k. Choose an increasing sequence $\{L_k\}$ in $\overline{\mathcal{F}}(H)$ with union H.

For each $k \in \mathbb{N}$, set $F_k = \lambda(L_k^2, X) \subseteq G$ and $T'_k = \lambda(L_k, X) \subseteq F_k$.

Since $G \curvearrowright X$ has property $\mathscr{S}$-SC, there is some $S \in \overline{\mathcal{F}}(G)$ such that for each $k \in \mathbb{N}$, there are $C_k, n_k \in \mathbb{N}$, $S_{k,1}, \dots, S_{k,n_k} \in \overline{\mathcal{F}}(G)$, $F_k^\sharp \in \mathcal{F}(G)$, and $\delta_k^\sharp > 0$ such that for any good enough sofic approximation $\pi : G \to \mathrm{Sym}(V)$ in $\mathscr{S}$ with $\mathrm{Map}_d(F_k^\sharp, \delta_k^\sharp, \pi) \neq \emptyset$ there are subsets W' and $\mathcal{V}_j$ of V for $1 \leq j \leq n_k$ satisfying the following conditions:

(i) $\sum_{j=1}^{n_k} \Upsilon(S_{k,j})\mathrm{m}(\mathcal{V}_j) \leq 1$,

(ii) $\bigcup_{g\in S} \pi_g W' = V$,

(iii) if $w_1, w_2 \in W'$ satisfy $\pi_g w_1 = w_2$ for some $g \in T_k := ST'_kS \in \overline{\mathcal{F}}(G)$ then w_1 and w_2 are connected by a path of length at most C_k in which each edge is an $S_{k,j}$-edge with both endpoints in $\mathcal{V}_j$ for some $1 \leq j \leq n_k$.

Take $\varepsilon' > 0$ such that for any $x, y \in X$ with $d(x,y) \leq \varepsilon'$ one has $d(gx, gy) < \varepsilon/8$ for every $g \in S$.

Fix $k \in \mathbb{N}$. Set

$$\tau'_k = \min\left\{\eta_{L^2_k}\tau_k^{1/2}/(8|F_k|)^{1/2}, \tau_k/(22|F_k|^2)\right\} > 0.$$

Let $0 < \delta_{k,1} < 1/2$ be such that $(\tau_k + 7\delta_{k,1})^{1/2} \leq \delta_k$ and $((\varepsilon/4)^2 + \delta_{k,1})^{1/2} < \varepsilon/2$. Take $0 < \delta_{k,2} \leq \eta_{T_k}$ such that for any $x, y \in X$ with $d(x,y) \leq \delta_{k,2}$ one has $\max_{t\in\kappa(S,X)} d(tx, ty) \leq \delta_{k,1}$. Set $\delta'_k = \delta_{k,1}^{1/2}\delta_{k,2}/(|S|^{1/2}|T_k|^{1/2}) > 0$ and $\eta_k = \eta_{\bigcup_{j=1}^{n_k} S_{k,j}}$. By Stirling's formula there is some $0 < \gamma_k < \delta_{k,1}/(3|S|)$ such that for any nonempty finite set V the number of subsets of V with cardinality no bigger than $\gamma_k|V|$ is at most $e^{\varepsilon|V|/2}$. Set $S_k = (\bigcup_{j=1}^{n_k} S_{k,j})^{C_k} \in \mathcal{F}(G)$ and $\delta''_k = \min\{\delta'_k, \min\{\eta_k, \varepsilon/16\}(\gamma_k/|S_k \cup S|)^{1/2}, \tau'_k\} > 0$.

Take an $m_k \geq k$ large enough so that

$$\frac{1}{|V_{m_k}|}\log N_\varepsilon(\mathrm{Map}_d(T_k \cup S_k \cup F_k \cup F_k^\sharp, \min\{\delta''_k/2, \delta_k^\sharp\}, \pi_{m_k}), d_2) \geq \max\{0, h^\varepsilon_{\Pi,2}(G \curvearrowright X) - \varepsilon\}$$

and so that $\pi_{m_k} : G \to \mathrm{Sym}(V_{m_k})$ is an (F_k, τ'_k)-approximation for G and also a good enough sofic approximation for G to guarantee the existence of W' and $\mathcal{V}_1, \ldots, \mathcal{V}_{n_k}$ as above. Denote by $\tilde{V}_{m_k}$ the set of all $w \in V_{m_k}$ satisfying

(iv) $\pi_{m_k,e_G} w = w$,
(v) $\pi_{m_k,g^{-1}}\pi_{m_k,g} w = w$ for all $g \in S$,
(vi) $\pi_{m_k,g}\pi_{m_k,a}\pi_{m_k,h} w = \pi_{m_k,gah} w$ for all $g, h \in S$ and $a \in T'_k$,
(vii) $\pi_{m_k,gh} w = \pi_{m_k,g}\pi_{m_k,h} w$ for all $g, h \in S_k$,
(viii) $\pi_{m_k,g} w \neq \pi_{m_k,h} w$ for all distinct g, h in $T_k \cup S_k$.

Taking m_k sufficiently large, we may assume that $\mathrm{m}(\tilde{V}_{m_k}) \geq 1 - \delta_{k,1}/(3|S|)$.

Take a (d_2, ε)-separated subset Φ of $\mathrm{Map}_d(T_k \cup S_k \cup F_k, \delta''_k/2, \pi_{m_k})$ with maximum cardinality. Since X_0 is dense in X, we may perturb each element of Φ to obtain a $(d_2, \varepsilon/2)$-separated subset Φ_1 of $\mathrm{Map}_d(T_k \cup S_k \cup F_k, \delta''_k, \pi_{m_k})$ with

$$|\Phi_1| = |\Phi| = N_\varepsilon(\mathrm{Map}_d(T_k \cup S_k \cup F_k, \delta''_k/2, \pi_{m_k}), d_2)$$

such that $\varphi(V_{m_k}) \subseteq X_0$ for all $\varphi \in \Phi_1$.

For each $\psi \in \mathrm{Map}_d(T_k \cup S_k \cup F_k, \delta''_k, \pi_{m_k})$, using the fact that $\delta''_k \leq \min\{\eta_k, \varepsilon/16\}(\gamma_k/|S_k \cup S|)^{1/2}$ we have $\mathrm{m}(V_\psi) \geq 1 - \gamma_k$ where

$$V_\psi := \{v \in V_{m_k} : d(g\psi(v), \psi(\pi_{m_k,g}v)) \leq \min\{\eta_k, \varepsilon/16\} \text{ for all } g \in S_k \cup S\}.$$

Thus there is a subset Φ_2 of Φ_1 such that V_φ is the same for all $\varphi \in \Phi_2$ and

$$|\Phi_1| \leq |\Phi_2| e^{\varepsilon|V_{m_k}|/2}.$$

Set $W = W' \cap \tilde{V}_{m_k} \cap V_\varphi \subseteq V_{m_k}$ for $\varphi \in \Phi_2$. Then

$$\mathrm{m}\Big(\bigcup_{g\in S}\pi_{m_k,g}W\Big)$$

$$\geq \mathrm{m}\Big(\bigcup_{g\in S}\pi_{m_k,g}W'\Big) - \mathrm{m}\Big(\bigcup_{g\in S}\pi_{m_k,g}(V\setminus \tilde{V}_{m_k})\Big) - \mathrm{m}\Big(\bigcup_{g\in S}\pi_{m_k,g}(V\setminus V_\varphi)\Big)$$
$$\geq 1-\frac{\delta_{k,1}}{3}-\gamma_k|S| \geq 1-\delta_{k,1}.$$

For each $\psi : V_{m_k} \to X$, define $\Theta(\psi) \in \prod_{j=1}^{n_k} H^{S_{k,j}\times \mathcal{V}_j}$ by $\Theta(\psi)(g_j, v_j) = \kappa(g_j, \psi(v_j))$ for $1 \leq j \leq n_k$ and $(g_j, v_j) \in S_{k,j}\times \mathcal{V}_j$. Then

$$|\Theta(X^{V_{m_k}})| \leq \prod_{j=1}^{n_k} |{}_{S_{k,j}}\mathscr{P}|^{|\mathcal{V}_j|} = \prod_{j=1}^{n_k} e^{(\varepsilon/2)\Upsilon(S_{k,j})|\mathcal{V}_j|} = e^{(\varepsilon/2)\sum_{j=1}^{n_k}\Upsilon(S_{k,j})|\mathcal{V}_j|} \leq e^{\varepsilon|V_{m_k}|/2}.$$

Thus we can find a subset Φ_3 of Φ_2 such that $\Theta(\varphi)$ is the same for all $\varphi \in \Phi_3$ and

$$|\Phi_2| \leq |\Phi_3| e^{\varepsilon|V_{m_k}|/2}.$$

We claim that for any $g \in T_k$ and $w_1, w_2 \in W$ with $\pi_{m_k,g}w_1 = w_2$, the element $\kappa(g, \varphi(w_1)) \in H$ is the same for all $\varphi \in \Phi_3$. If $w_1 = w_2$, then $g = e_G$ and hence $\kappa(g, \varphi(w_1)) = e_H$ for all $\varphi \in \Phi_3$. Thus we may assume that $w_1 \neq w_2$. We can find $l \leq C_k$, $g_1, \dots, g_l \in G$, $w_1 = w_1', w_2', \dots, w_{l+1}' = w_2$ in V_{m_k} such that for each $1 \leq i \leq l$ one has $\pi_{m_k,g_i}w_i' = w_{i+1}'$, $g_i \in S_{k,j_i}$ and $w_i', w_{i+1}' \in \mathcal{V}_{j_i}$ for some $1 \leq j_i \leq n_k$. Since $w_1' \in W \subseteq \tilde{V}_{m_k}$, we have

$$\pi_{m_k,g_ig_{i-1}\dots g_1}w_1' = \pi_{m_k,g_i}\pi_{m_k,g_{i-1}}\dots\pi_{m_k,g_1}w_1' = w_{i+1}'$$

for all $1 \leq i \leq l$. In particular, $\pi_{m_k,g_lg_{l-1}\dots g_1}w_1' = w_{l+1}' = \pi_{m_k,g}w_1'$, and hence

$$g_lg_{l-1}\dots g_1 = g.$$

Note that $w_1' \in W \subseteq V_\varphi$ for all $\varphi \in \Phi_3 \subseteq \Phi_2$. For each $0 \leq i \leq l-1$ and $\varphi \in \Phi_3$, we have $g_ig_{i-1}\dots g_1 \in S_k$, and hence $d(g_ig_{i-1}\dots g_1\varphi(w_1'), \varphi(\pi_{m_k,g_ig_{i-1}\dots g_1}w_1')) \leq \eta_k$, which implies that

$$\kappa(g_{i+1}, g_ig_{i-1}\dots g_1\varphi(w_1')) = \kappa(g_{i+1}, \varphi(\pi_{m_k,g_ig_{i-1}\dots g_1}w_1')).$$

Then

$$\begin{aligned}\kappa(g,\varphi(w_1)) = \kappa(g_lg_{l-1}\dots g_1, \varphi(w_1')) &= \prod_{i=0}^{l-1}\kappa(g_{i+1}, g_ig_{i-1}\dots g_1\varphi(w_1'))\\ &= \prod_{i=0}^{l-1}\kappa(g_{i+1}, \varphi(\pi_{m_k,g_ig_{i-1}\dots g_1}w_1'))\\ &= \prod_{i=0}^{l-1}\kappa(g_{i+1}, \varphi(w_{i+1}'))\\ &= \prod_{i=0}^{l-1}\Theta(\varphi)(g_{i+1}, w_{i+1}')\end{aligned}$$

is the same for all $\varphi \in \Phi_3$. This proves our claim.

Fix one $\varphi_0 \in \Phi_3$. Define $\sigma_k' : H \to V_{m_k}^{V_{m_k}}$ by

$$\sigma_{k,t}'v = \pi_{\lambda(t,\varphi_0(v))}v$$

for all $t \in H$ and $v \in V_{m_k}$. By Lemma 3.17 there is an (L_k, τ_k)-approximation $\sigma_k : H \to \mathrm{Sym}(V_{m_k})$ for H such that $\rho_{\mathrm{Hamm}}(\sigma_{k,t}, \sigma'_{k,t}) \leq \tau_k$ for all $t \in L_k$. For each $\varphi \in \Phi_3$ take a $\tilde{\varphi} : V_{m_k} \to X$ such that $\tilde{\varphi} = \varphi$ on W and such that for each $v \in \bigcup_{g \in S} \pi_{m_k,g} W$ one has

$$\tilde{\varphi}(v) = \kappa(g, \varphi_0(w))\varphi(w)$$

for some $g \in S$ and $w \in W$ with $\pi_{m_k,g} w = v$. We may require that g and w depend only on v, and not on $\varphi \in \Phi_3$. By Lemma 4.2 we have $\tilde{\varphi} \in \mathrm{Map}_d(L_k, \delta_k, \sigma_k)$.

Let φ and ψ be distinct elements in Φ_3. Since $d_2(\varphi, \psi) \geq \varepsilon/2 > ((\varepsilon/4)^2 + \delta_{k,1})^{1/2}$ and $\mathrm{m}\big(\bigcup_{g \in S} \pi_{m_k,g} W\big) \geq 1 - \delta_{k,1}$, we have $d(\varphi(v), \psi(v)) > \varepsilon/4$ for some $v \in \bigcup_{g \in S} \pi_{m_k,g} W$. Then $v = \pi_{m_k,g} w$ for some $g \in S$ and $w \in W$ such that $\tilde{\varphi}(v) = \kappa(g, \varphi_0(w))\varphi(w)$ and $\tilde{\psi}(v) = \kappa(g, \varphi_0(w))\psi(w)$. Using the fact that $w \in W \subseteq V_\varphi = V_\psi$ we have

$$\begin{aligned} &d(g\varphi(w), g\psi(w)) \\ &\quad \geq d(\varphi(\pi_{m_k,g} w), \psi(\pi_{m_k,g} w)) - d(\varphi(\pi_{m_k,g} w), g\varphi(w)) - d(\psi(\pi_{m_k,g} w), g\psi(w)) \\ &\quad \geq \frac{\varepsilon}{4} - \frac{\varepsilon}{16} - \frac{\varepsilon}{16} = \frac{\varepsilon}{8}. \end{aligned}$$

From our choice of ε' we get $d(\tilde{\varphi}(w), \tilde{\psi}(w)) = d(\varphi(w), \psi(w)) > \varepsilon'$. Therefore $\tilde{\Phi}_3 := \{\tilde{\varphi} : \varphi \in \Phi_3\}$ is (d_∞, ε')-separated and $|\tilde{\Phi}_3| = |\Phi_3|$. Thus

$$\begin{aligned} \frac{1}{|V_{m_k}|} \log N_{\varepsilon'}(\mathrm{Map}_d(L_k, \delta_k, \sigma_k), d_\infty) \geq \frac{1}{|V_{m_k}|} \log |\tilde{\Phi}_3| &= \frac{1}{|V_{m_k}|} \log |\Phi_3| \\ &\geq \frac{1}{|V_{m_k}|} \log |\Phi_1| - \varepsilon \\ &\geq h^{\varepsilon}_{\Pi,2}(G \curvearrowright X) - 2\varepsilon. \end{aligned}$$

Now $\Sigma = \{\sigma_k\}_{k \in \mathbb{N}}$ is a sofic approximation sequence for H. For any finite set $L \subseteq H$ and $\delta > 0$, we have $L \subseteq L_k$ and $\delta > \delta_k$ for all large enough k, and hence

$$\begin{aligned} \varlimsup_{k \to \infty} \frac{1}{|V_{m_k}|} \log N_{\varepsilon'}(\mathrm{Map}_d(L, \delta, \sigma_k), d_\infty) &\geq \varlimsup_{k \to \infty} \frac{1}{|V_{m_k}|} \log N_{\varepsilon'}(\mathrm{Map}_d(L_k, \delta_k, \sigma_k), d_\infty) \\ &\geq h^{\varepsilon}_{\Pi,2}(G \curvearrowright X) - 2\varepsilon. \end{aligned}$$

Taking infima over L and δ, we obtain

$$h_\Sigma(H \curvearrowright X) \geq h^{\varepsilon'}_{\Sigma,\infty}(H \curvearrowright X) \geq h^{\varepsilon}_{\Pi,2}(G \curvearrowright X) - 2\varepsilon.$$

□

5. Measure entropy and bounded orbit equivalence

In this final section we establish Theorem 5.2, which in conjunction with Proposition 3.15 yields Theorem A.

For a general reference on the C*-algebra theory and terminology used in the following proof, see [31].

Lemma 5.1. *Let $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ be orbit equivalent free p.m.p. actions and suppose that $H \curvearrowright (Y, \nu)$ is uniquely ergodic. Then there are a zero-dimensional compact metrizable space Z, a continuous action $G * H \curvearrowright Z$, and a $G * H$-invariant Borel probability measure μ_Z on Z of full support such that*

(i) *$G \curvearrowright (Z, \mu_Z)$ is measure conjugate to $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Z, \mu_Z)$ is measure conjugate to $H \curvearrowright (Y, \nu)$,*
(ii) *$H \curvearrowright Z$ is uniquely ergodic,*
(iii) *there is a $G * H$-invariant Borel subset Z_0 of Z with $\mu_Z(Z_0) = 1$ such that $Gz = Hz$ for every $z \in Z_0$.*

If furthermore $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ are boundedly orbit equivalent then we may demand that both $G \curvearrowright Z_0$ and $H \curvearrowright Z_0$ be free and that the cocycles $\kappa : G \times Z_0 \to H$ and $\lambda : H \times Z_0 \to G$ extend to continuous maps $G \times Z \to H$ and $H \times Z \to G$, so that $G \curvearrowright Z$ and $H \curvearrowright Z$ are continuously orbit equivalent.

Proof. We may assume that $(X, \mu) = (Y, \nu)$ and that $Gx = Hx$ for every $x \in X$. Denote by $\mathscr{B}$ the σ-algebra of Borel subsets of X. Denote by φ the mean $f \mapsto \int_X f \, d\mu$ on $L^\infty(X, \mu)$.

Let V be a finite subset of $\mathscr{B}$. We claim that for every $\varepsilon > 0$ there is a finite subset W of $\mathscr{B}$ containing V such that for any mean ψ on $L^\infty(X, \mu)$ satisfying $\psi(s1_A) = \psi(1_A)$ for all $s \in H$ and $A \in W$ one has $|\varphi(1_A) - \psi(1_A)| < \varepsilon$ for all $A \in V$. Suppose to the contrary that for some $\varepsilon > 0$ and every finite subset W of $\mathscr{B}$ containing V there is a mean ψ_W on $L^\infty(X, \mu)$ satisfying $\psi_W(s1_A) = \psi_W(1_A)$ for all $s \in H$ and $A \in W$ and $\max_{A \in V} |\varphi(1_A) - \psi_W(1_A)| \geq \varepsilon$. Then any cluster point ψ of the net $\{\psi_W\}$ (with index directed by inclusion) is H-invariant and $\max_{A \in V} |\varphi(1_A) - \psi(1_A)| \geq \varepsilon$. This contradicts the unique ergodicity of $H \curvearrowright (X, \mu)$, thus verifying our claim.

For any countable subset V of $\mathscr{B}$, writing V as the union of an increasing sequence $\{V_k\}_k$ of finite subsets of V and taking a sequence $\{\varepsilon_k\}_k$ of positive numbers tending to 0, we conclude from above that there is a countable subset W of $\mathscr{B}$ containing V such that for any mean ψ on $L^\infty(X, \mu)$ satisfying $\psi(s1_A) = \psi(1_A)$ for all $s \in H$ and $A \in W$ one has $\varphi(1_A) = \psi(1_A)$ for all $A \in V$.

For a given countable set $\mathscr{A} \subseteq \mathscr{B}$, denote by $\mathscr{A}'$ the $G*H$-invariant subalgebra of $\mathscr{B}$ generated by $\mathscr{A}$, which is again countable. Take a countable subset $\mathscr{A}_1$ of $\mathscr{B}$ such that for any distinct $x, y \in X$ one has $1_A(x) \neq 1_A(y)$ for some $A \in \mathscr{A}_1$. Inductively, having constructed a countable subset $\mathscr{A}_k$ of $\mathscr{B}$, we take a countable subset $\mathscr{A}_{k+1}$ of $\mathscr{B}$ containing $\mathscr{A}_k'$ such that for any mean ψ on $L^\infty(X, \mu)$ satisfying $\psi(s1_A) = \psi(1_A)$ for all $s \in H$ and $A \in \mathscr{A}_{k+1}$, one has $\varphi(1_A) = \psi(1_A)$ for all $A \in \mathscr{A}_k'$.

Now we put $\mathscr{A} = \bigcup_k \mathscr{A}_k$. This is a countable $G * H$-invariant subalgebra of $\mathscr{B}$. For any mean ψ on $L^\infty(X, \mu)$ satisfying $\psi(s1_A) = \psi(1_A)$ for all $s \in H$ and $A \in \mathscr{A}$, one has $\varphi(1_A) = \psi(1_A)$ for all $A \in \mathscr{A}$. Denote by $\mathfrak{A}$ the $G * H$-invariant unital C^*-subalgebra of $L^\infty(X, \mu)$ generated by the functions 1_A for $A \in \mathscr{A}$. Then $\mathfrak{A}$ is the closure of the linear span of the functions 1_A for $A \in \mathscr{A}$ in $L^\infty(X, \mu)$. Thus every state of $\mathfrak{A}$ is determined by its values on the functions 1_A for $A \in \mathscr{A}$. Since every state of $\mathfrak{A}$ extends to a mean of $L^\infty(X, \mu)$, we conclude that $\varphi|_{\mathfrak{A}}$ is the unique H-invariant state on $\mathfrak{A}$.

Define a $G * H$-action on $\{0, 1\}^{\mathscr{A} \times (G*H)}$ by $(sw)_{A,t} = w_{A,s^{-1}t}$ for $w \in \{0, 1\}^{\mathscr{A} \times (G*H)}$, $A \in \mathscr{A}$, and $s, t \in G * H$, and consider the $G * H$-equivariant Borel map $\pi : X \to \{0, 1\}^{\mathscr{A} \times (G*H)}$ given by $\pi(x)_{A,t} = 1_A(t^{-1}x) = 1_{tA}(x)$ for $x \in X$, $A \in \mathscr{A}$, and $t \in G * H$. Since $\mathscr{A}_1 \subseteq \mathscr{A}$, the map π is injective and hence is a Borel isomorphism from X to $\pi(X)$ [20, Corollary 15.2]. Put $\mu_Z = \pi_*\mu$ and $Z = \mathrm{supp}(\mu_Z)$. Then Z is zero-dimensional and μ_Z is a $G * H$-invariant Borel probability measure on Z of full support. Put $Z_0 = \pi(X) \cap Z$. Then Z_0 is $G * H$-invariant with $\mu_Z(Z_0) = 1$, and $Gz = Hz$ for all $z \in Z_0$. The pull-back map $\pi^* : C(Z) \to L^\infty(X, \mu)$ is a $G * H$-equivariant

$*$-homomorphism. From the Stone–Weierstrass theorem we get $\pi^*(C(Z)) = \mathfrak{A}$. Since Z is the support of $\pi_*\mu$, the map π^* is injective and hence is an isomorphism from $C(Z)$ to $\mathfrak{A}$. Thus $C(Z)$ has a unique H-invariant state, which means that $H \curvearrowright Z$ is uniquely ergodic.

Now assume that $G \curvearrowright (X,\mu)$ and $H \curvearrowright (Y,\nu)$ are boundedly orbit equivalent. By passing to suitable invariant subsets we may assume that $G \curvearrowright X$ and $H \curvearrowright X$ are both genuinely free and that the cocycles $\kappa' : G \times X \to H$ and $\lambda' : H \times X \to G$ are both bounded. Adding more sets to $\mathscr{A}_1$, we may assume that for every $t \in G$ (resp. $t \in H$) there is a finite partition $\mathscr{P}$ of X contained in $\mathscr{A}_1$ such that κ' (resp. λ') is constant on $\{t\} \times P$ for every $P \in \mathscr{P}$. Then we can extend κ (resp. λ) continuously to $G \times Z \to H$ (resp. $H \times Z \to G$). □

Theorem 5.2. *Let $G \curvearrowright (X,\mu)$ and $H \curvearrowright (Y,\nu)$ be free p.m.p. actions which are boundedly orbit equivalent. Let $\mathscr{S}$ be a collection of sofic approximations for G. Suppose that G has property $\mathscr{S}$-SC and that the action $H \curvearrowright (Y,\nu)$ is uniquely ergodic. Let Π be a sofic approximation sequence in $\mathscr{S}$. Then*

$$h_\nu(H \curvearrowright Y) \geq h_{\Pi,\mu}(G \curvearrowright X).$$

Proof. Combine Lemma 5.1, Theorem 4.1, and the variational principle (Theorem 10.35 in [24]). □

References

[1] A. Alpeev. The entropy of Gibbs measures on sofic groups. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **436** (2015), 34–48.

[2] A. Alpeev. Random ordering formula for sofic and Rokhlin entropy of Gibbs measures. arXiv:1705.08559.

[3] T. Austin. Behaviour of entropy under bounded and integrable orbit equivalence. *Geom. Funct. Anal.* **26** (2016), 1483–1525.

[4] R. M. Belinskaya. Partitions of Lebesgue space in trajectories defined by ergodic automorphisms. *Functional Anal. Appl.* **2** (1968), 190–199.

[5] L. Bowen. Measure conjugacy invariants for actions of countable sofic groups. *J. Amer. Math. Soc.* **23** (2010), 217–245.

[6] L. Bowen. Sofic entropy and amenable groups. *Ergodic Theory Dynam. Systems* **32** (2012), 427–466.

[7] N.-P. Chung and Y. Jiang. Continuous cocycle superrigidity for shifts and groups with one end. *Math. Ann.* **368** (2017), 1109–1132.

[8] D. B. Cohen. Continuous cocycle superrigidity for the full shift over a finitely generated torsion group. *Int. Math. Res. Not.* **2020**, 1610–1620.

[9] A. del Junco and D. J. Rudolph. Kakutani equivalence of ergodic $\mathbb{Z}^n$ actions. *Ergodic Theory Dynam. Systems* **4** (1984), 89–104.

[10] H. A. Dye. On groups of measure preserving transformations I. *Amer. J. Math.* **81** (1959), 119–159.

[11] J. Feldman. New K-automorphisms and a problem of Kakutani. *Israel J. Math.* **24** (1976), 16–38.

[12] A. Fieldsteel and N. A. Friedman. Restricted orbit changes of ergodic $\mathbf{Z}^d$-actions to achieve mixing and completely positive entropy. *Ergodic Theory Dynam. Systems* **6** (1986), 505–528.

[13] D. Gaboriau. Coût des relations d'équivalence et des groupes. *Invent. Math.* **139** (2000), 41–98.

[14] B. Hayes. Fuglede-Kadison determinants and sofic entropy. *Geom. Funct. Anal.* **26** (2016), 520–606.

[15] B. Hayes. Polish models and sofic entropy. *J. Inst. Math. Jussieu* **17** (2018), 241–275.

[16] B. Hayes. Relative entropy and the Pinsker product formula for sofic groups. *Groups Geom. Dyn.* **15** (2021), 413–463.

[17] J. W. Kammeyer and D. J. Rudolph. Restricted orbit equivalence for ergodic $\mathbb{Z}^d$ actions I. *Ergodic Theory Dynam. Systems* **17** (1997), 1083–1129.

[18] J. W. Kammeyer and D. J. Rudolph. Restricted orbit equivalence for actions of discrete amenable groups. Cambridge Tracts in Mathematics, 146. Cambridge University Press, Cambridge, 2002.

[19] A. B. Katok. Monotone equivalence in ergodic theory. *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), 104–157.

[20] A. S. Kechris. *Classical Descriptive Set Theory.* Graduate Texts in Mathematics, 156. Springer-Verlag, New York, 1995.

[21] D. Kerr. Sofic measure entropy via finite partitions. *Groups Geom. Dyn.* **7** (2013), 617–632.

[22] D. Kerr and H. Li. Bernoulli actions and infinite entropy. *Groups Geom. Dyn.* **5** (2011), 663–672.

[23] D. Kerr and H. Li. Soficity, amenability, and dynamical entropy. *Amer. J. Math.* **135** (2013), 721–761.

[24] D. Kerr and H. Li. *Ergodic Theory: Independence and Dichotomies.* Springer, Cham, 2016.

[25] D. Kerr and H. Li. Entropy, Shannon orbit equivalence, and sparse connectivity. *Math. Ann.* **380** (2021), 1497–1562.

[26] A. N. Kolmogorov. A new metric invariant of transient dynamical systems and automorphisms in Lebesgue spaces. *Dokl. Akad. Nauk SSSR (N.S.)* **119** (1958), 861–864.

[27] A. N. Kolmogorov. Entropy per unit time as a metric invariant of automorphisms. *Dokl. Akad. Nauk SSSR* **124** (1959), 754–755.

[28] X. Li. Continuous orbit equivalence rigidity. *Ergodic Theory Dynam. Systems* **38** (2018), 1543–1563.

[29] A. Mann. *How Groups Grow.* London Mathematical Society Lecture Note Series, 395. Cambridge University Press, Cambridge, 2012.

[30] T. Meyerovitch. Positive sofic entropy implies finite stabilizer. *Entropy* **18** (2016), no. 263, 14 pp.

[31] G. J. Murphy. *C^*-Algebras and Operator Theory.* Academic Press, San Diego, 1990.

[32] D. Ornstein. Bernoulli shifts with the same entropy are isomorphic. *Advances in Math.* **4** (1970), 337–352.

[33] D. S. Ornstein, D. J. Rudolph, and B. Weiss. Equivalence of measure preserving transformations. *Mem. Amer. Math. Soc.* **37** (1982), no. 262, xii+116 pp.

[34] D. S. Ornstein and B. Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc. (N.S.)* **2** (1980), 161–164.

[35] J. Peterson and T. Sinclair. On cocycle superrigidity for Gaussian actions. *Ergodic Theory Dynam. Systems* **32** (2012), 249–272.

[36] S. Popa. On the superrigidity of malleable actions with spectral gap. *J. Amer. Math. Soc.* **21** (2008), 981–1000.

[37] D. J. Rudolph. Restricted orbit equivalence. *Mem. Amer. Math. Soc.* **54** (1985), no. 323, v+150 pp.

[38] D. J. Rudolph and B. Weiss. Entropy and mixing for amenable group actions. *Ann. of Math. (2)* **151** (2000), 1119–1150.

[39] K. Schmidt. Amenability, Kazhdan's property T, strong ergodicity and invariant means for ergodic group-actions. *Ergodic Theory Dynam. Systems* **1** (1981), 223–236.

[40] B. Seward. Positive entropy actions of countable groups factor onto Bernoulli shifts. *J. Amer. Math. Soc.* **33** (2020), 57–101.

[41] Ya. Sinai. On the concept of entropy for a dynamic system. *Dokl. Akad. Nauk SSSR* **124** (1959), 768–771.

[42] A. M. Vershik. *Approximation in Measure Theory.* Ph.D. thesis, Leningrad University, 1973. In Russian.

[43] A. M. Vershik. Theory of decreasing sequences of measurable partitions. *St. Petersburg Math. J.* **6** (1995), 705–761.

David Kerr, Mathematisches Institut, WWU Münster, Einsteinstr. 62, 48149 Münster, Germany
Email address: `kerrd@uni-muenster.de`

Hanfeng Li, Center of Mathematics, Chongqing University, Chongqing 40133, China and Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260-2900, USA
Email address: `hfli@math.buffalo.edu`